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A Rigorous Method for Computing Geodetic Distance From Shoran Observations - and Appendixes A-C (Technical Report) (None)

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U. S. Army Corps of Engineers, Army Map Service, Washington, D. C. 4
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Investigations indicated that the distances measured by means of the Shoran navigation system were short by about 1 part in 20,000. A study was made to discover what portion of this error could be attributed to the method of computation used by the Air Force to obtain geodetic distance from Shoran dial readings and weather observations. The basic theory and assumptions used in computing geodetic distance from Shoran dial readings are outlined, and a hypothetical problem is solved by two methods. Proof is established that mathematical errors due to the use of the Air Force reduction methods are negligible.

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ARMY MAP SERVICE

TECHNICAL REPORT

Number 4

A RIGOROUS METHOD FOR COMPUTING GEODETIC DISTANCE FROM SHORAN OBSERVATIONS

Project 8-35-04-002

1 June 1950

204907

Submitted to

THE CHIEF OF ENGINEERS, U. S. ARMY

by

The Commanding Officer

Army Map Service

Washington 16, D. C.

R E S T R I C T E D

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TABLE OF CONTENTS

Section	Title	Page
I.	Summary	1
II.	Basic Problem and Underlying Assumptions	1
III.	Theory of solution	2
IV.	Solution of Hypothetical Problem by Series	4
V.	Solution of Hypothetical Problem by Numerical Integration	7
VI.	Solution of Hypothetical Problem by Air Force Method	11
VII.	Discussion	11

Appendix A. Algebraic Expressions for Derivatives Used in Section IV

Appendix B. Solution of 153 Mile Line Using Observed Atmosphere

Appendix C. Solution of 100 Mile Line Using Observed Atmosphere

A RIGOROUS METHOD FOR COMPUTING
GEODETIC DISTANCE FROM SHORAN OBSERVATIONS

Report as of 20 January 1950

I. SUMMARY

With respect to the Caribbean Area Shoran Project carried out by the 7th Geodetic Control Squadron, investigations indicate that the measured Shoran distances are, on the whole, short by about 1 part in 20,000. It was one of the primary purposes of this study to discover what portion, if any, of this error could be attributed to the method of computation used by the Air Force to obtain geodetic distance from Shoran dial readings and weather observations. The results of this study indicate that the error arising from this source could be in the neighborhood of 1 part in 150,000 for a half-line distance of 250 miles. On the basis of this result other sources of error must be sought, the most probable of which appears to be the value for the velocity of light. This matter is discussed quite thoroughly by Mr. Aslakson of the U. S. Coast and Geodetic Survey (Transactions American Geophysical Union, August, 1949).

Whether or not this report can be of practical use in future Shoran work it is believed that through it two things of importance have been accomplished:

(1) A rigorous method of unlimited applicability has been developed.

(2) Proof has been established that mathematical errors due to the use of the Air Force reduction method are negligible.

II. BASIC PROBLEM AND UNDERLYING ASSUMPTIONS

The simplicity with which the problem of obtaining geodetic distance from Shoran observations can be stated is deceiving. There are given, essentially, four con/ions with which to work.
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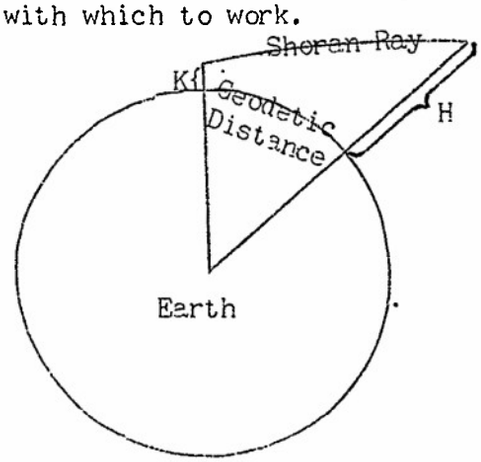


There are known, first, the altitude, H, of the airplane, second, the altitude, K, of a ground station, third, the time, t, required for the Shoran ray to travel from the airplane to the ground station, and fourth, the meteorological conditions existing along the path of the Shoran ray.

The underlying assumptions are as follows:

First, the figure of the earth is represented for the time being by the International Sphere, whose radius, R, is set at 6,371,227 meters or 20,902,933.92 feet.

Second, the Shoran ray is represented by a line which lies in a plane passing through the center of the earth. The geodetic distance is taken as the intersection of this plane with the surface of the earth.

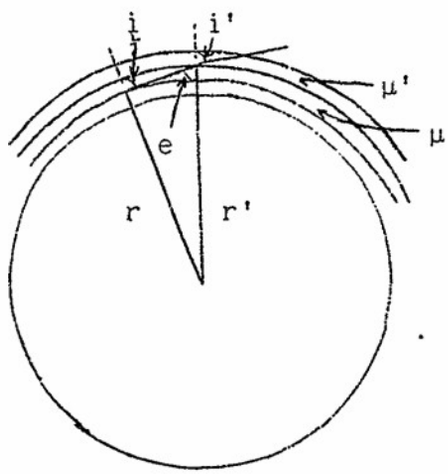
Third, the assumption is made that the Shoran ray is governed by the laws of geometric optics. In justification of this assumption, reference is made to Report No. 447, Radiation Laboratory, Massachusetts Institute of Technology, November, 1943, entitled "The Effect of Atmospheric Refraction on Short Radio Waves."

Fourth, Anderson's 1941 value of 299,776,000 meters per second is used for the velocity of light, V.

Fifth, it is assumed that the velocity of the Shoran ray varies inversely as μ, the index of refraction, which is numerically equal to the square root of the dielectric constant. This point is discussed briefly in an unpublished paper entitled, "Velocity Correction to Shoran Measurements," by Donald A. Rice of the U. S. Coast and Geodetic Survey.

III. THEORY OF SOLUTION

The solution presented here depends on the relation $r\mu \cos \alpha = k$, the proof of which is given below.



The figure represents a spherical earth surrounded by infinitesimally thin spherical layers of atmosphere having different values of μ. A ray, lying in a plane which contains the earth's center, passes through these layers. By the law of sines,

$$\frac{r'}{\sin (180^\circ - i)} = \frac{r}{\sin e} .$$

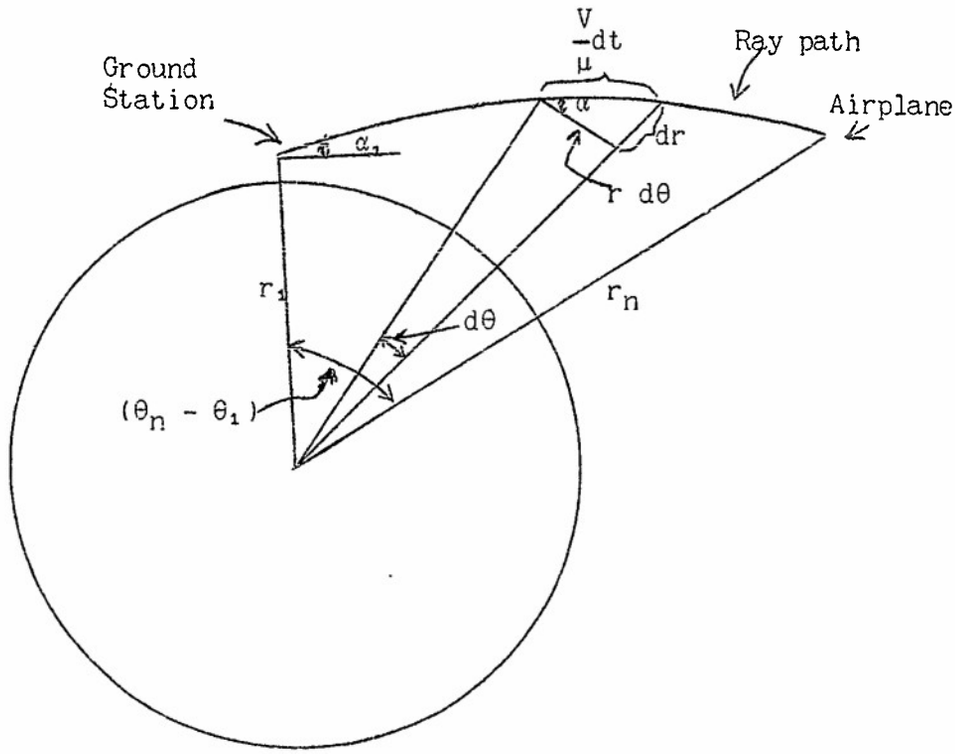
But, by Snell's Law, $\mu' \sin i' = \mu \sin e$, so that

$$\frac{r'}{\sin (180^\circ - i)} = \frac{r}{\frac{\mu'}{\mu} \sin i'}$$

or $r\mu \sin i = r'\mu' \sin i'$.

Since this relation holds between adjacent layers of atmosphere, $r\mu \sin i$ can be represented by a constant for a particular path. In the relation, $r\mu \cos \alpha = k$, α represents the complement of i, and k, a constant value depending on the ray.

The method of solution can be illustrated by means of the figure below.



From the differential triangle constructed on the ray path, the following relations are apparent:

$$\frac{dr}{dt} = \frac{V \sin \alpha}{\mu}$$

$$\frac{d\theta}{dt} = \frac{V \cos \alpha}{r\mu}$$

By Maclaurin's series,

$$r_n = r_1 + t\left(\frac{dr}{dt}\right)_1 + \frac{t^2}{2}\left(\frac{d^2r}{dt^2}\right)_1 + \ldots.$$

and

$$\theta_n = \theta_1 + t\left(\frac{d\theta}{dt}\right)_1 + \frac{t^2}{2}\left(\frac{d^2\theta}{dt^2}\right)_1 + \frac{t^3}{6}\left(\frac{d^3\theta}{dt^3}\right)_1 + \ldots.$$

if t is considered to be zero at the ground station.

Since μ is a function of r, and since $\cos \alpha = \frac{k}{r\mu}$ and $\sin \alpha = \sqrt{1 - \frac{k^2}{r^2\mu^2}}$, both $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ are functions of r if a value is assumed for k.

Expressions for the derivatives of r and θ can be found in Appendix A. An examination of these derivatives indicates that their evaluation at the ground station requires only the values of r_1, μ_1 and its derivatives, and α_1 (the assumption of a value for α_1 is, of course, equivalent to the assumption of a value for k). In practice α_1 can be approximated from the triangle whose sides are $(r_1 + 1/3\ R)$, $(r_n + 1/3\ R)$, and (Vt). (When r_1 and r_n are increased by 1/3 R, the ray path, Vt, can, for approximation purposes, be considered as a straight line.) This value for α_1 is then tested in the "r" series to see if it produces r_n. If it produces a value which differs from the given r_n, this erroneous value of r_n can be used to compute a new value for α_1 in the same manner as before and the first value for α_1 can be adjusted by proportion. Usually this adjusted value of α_1 will, when substituted in the "r" series, produce r_n with sufficient accuracy. When, by means of the "r" series, it is shown that the correct value of α_1 has been found, direct substitution in the "θ" series produces the geodetic arc, $\theta_n - \theta_1$. (It is pointed out that in this discussion θ_1 is always equal to zero)

IV. SOLUTION OF HYPOTHETICAL PROBLEM BY SERIES

The hypothetical problem to be solved has the following conditions. The value of r_1 is 20,903,000.00 feet, the value of r_n is 20,950,702.76 feet, and t = .0016 seconds. The atmosphere assumed is known as the NACA "moist" atmosphere. In it, $(\mu - 1)\ 10^6 = 327.54 - 11.134\ 46970\ h + .12052,34160 h^2$, where h = altitude in thousands of feet. It is pointed out that r_n is specially chosen because it is the value produced by the "r" series when $\alpha_1 = 0$. The choice of this value for α_1 is almost essential in order to evaluate the derivatives of r and θ with reasonable ease, and since the purpose of a solution by series is only to verify the accuracy of the more practical numerical integration method, it is felt that the choice of conditions is justified. When $\alpha_1 = 0$ is used in the derivatives of r and θ, all terms vanish except the first of every other derivative.

With the aid of Appendix A, the following solution can be set down.

$$r_1 = 2.0903\ 00000 \times 10^4 \text{ (in thousands of feet)}$$

$$\mu_1 = 1.0003\ 26805$$

$$\mu_1^{I} = -1.1118\ 54133 \times 10^{-5}$$

$$\mu_1^{II} = 2.4104\ 68320 \times 10^{-7}$$

(Note: In the following expressions the subscript "1" is omitted.)

$$a = \frac{r\mu^{I}}{\mu} = -2.3233\ 49412 \times 10^{-1}$$

$$b = \frac{r^2\mu^{II}}{\mu} = 1.0528\ 74877 \times 10^{2}$$

$$c \text{ etc.} = 0.0$$

$$m = a + 1 = 7.6766\ 50588 \times 10^{-1}$$

$$m^2 = 5.8930\ 96425 \times 10^{-1}$$

$$m^3 = 4.5239\ 24214 \times 10^{-1}$$

$$m^4 = 3.4728\ 58548 \times 10^{-1}$$

$$m^5 = 2.6659\ 92161 \times 10^{-1}$$
$$m^6 = 2.0465\ 89029 \times 10^{-1}$$
$$m^7 = 1.5710\ 94887 \times 10^{-1}$$
$$m^8 = 1.2060\ 74649 \times 10^{-1}$$
$$a^2 = 5.3979\ 52490 \times 10^{-2}$$
$$a^3 = -1.2541\ 32974 \times 10^{-2}$$
$$a^4 = 2.9137\ 89108 \times 10^{-3}$$
$$b^2 = 1.1085\ 45507 \times 10^{4}$$
$$ab = -2.4461\ 96226 \times 10^{1}$$
$$a^2b = 5.6833\ 68563$$
$$m^I = 1.0500\ 11733 \times 10^{2}$$
$$m^{II} = 3.8882\ 89940 \times 10^{2}$$
$$m^{III} = -3.2011\ 61522 \times 10^{4}$$
$$(m^I)^2 = 1.1025\ 24639 \times 10^{4}$$
$$(m^I)^3 = 1.1576\ 63807 \times 10^{6}$$
$$(m^I m^{II}) = 4.0827\ 50058 \times 10^{4}$$
$$m(m^I) = 8.0605\ 73188 \times 10^{1}$$
$$m(m^I)^2 = 8.4636\ 96418 \times 10^{3}$$
$$m(m^I)^3 = 8.8869\ 80545 \times 10^{5}$$
$$m^2(m^I) = 6.1878\ 20390 \times 10^{1}$$
$$m^2(m^I)^2 = 6.4972\ 84009 \times 10^{3}$$
$$m^2(m^I)^3 = 6.8222\ 24442 \times 10^{5}$$
$$m^2(m^I m^{II}) = 2.4060\ 03977 \times 10^{4}$$
$$m^2(m^{II}) = 2.2914\ 06754 \times 10^{2}$$
$$m^3(m^I) = 4.7501\ 73504 \times 10^{1}$$
$$m^3(m^I)^2 = 4.9877\ 37911 \times 10^{3}$$
$$m^3(m^{II}) = 1.7590\ 32901 \times 10^{2}$$
$$m^3(m^{III}) = -1.4481\ 81212 \times 10^{4}$$
$$m^3(m^I m^{II}) = 1.8470\ 05185 \times 10^{4}$$
$$m^4(m^I) = 3.6465\ 42222 \times 10^{1}$$
$$m^4(m^I)^2 = 3.8289\ 12117 \times 10^{3}$$
$$m^4(m^{II}) = 1.3503\ 48096 \times 10^{2}$$
$$m^4(m^{III}) = -1.1117\ 18116 \times 10^{4}$$
$$m^5(m^I) = 2.7993\ 23049 \times 10^{1}$$
$$m^5(m^{II}) = 1.0366\ 15050 \times 10^{2}$$
$$m^6(m^I) = 2.1489\ 42493 \times 10^{1}$$

$$A = \frac{V}{r\mu} = 4.7036\ 01306 \times 10^{1}$$
$$A^2 = 2.2123\ 86525 \times 10^{3}$$
$$A^3 = 1.0406\ 18415 \times 10^{5}$$

$$A^4 = 4.8946\ 54136 \times 10^{6}$$
$$A^5 = 2.3022\ 50159 \times 10^{8}$$
$$A^6 = 1.0828\ 86685 \times 10^{10}$$
$$A^7 = 5.0934\ 67226 \times 10^{11}$$
$$A^8 = 2.3957\ 63910 \times 10^{13}$$
$$A^9 = 1.1268\ 71826 \times 10^{15}$$

$$\frac{1}{2}\frac{d^2r}{dt^2} = 1.7750\ 53320 \times 10^{7}$$
$$\frac{1}{24}\frac{d^4r}{dt^4} = 3.3959\ 07938 \times 10^{11}$$
$$\frac{1}{720}\frac{d^6r}{dt^6} = 2.1854\ 96426 \times 10^{15}$$
$$\frac{1}{40,320}\frac{d^8r}{dt^8} = -1.8851\ 72971 \times 10^{19}$$

$$\frac{d\theta}{dt} = 4.7036\ 01306 \times 10^{1}$$
$$\frac{1}{6}\frac{d^3\theta}{dt^3} = -2.0441\ 54887 \times 10^{4}$$
$$\frac{1}{120}\frac{d^5\theta}{dt^5} = -9.3373\ 65867 \times 10^{8}$$
$$\frac{1}{5040}\frac{d^7\theta}{dt^7} = -1.9632\ 49131 \times 10^{13}$$
$$\frac{1}{362,880}\frac{d^9\theta}{dt^9} = -1.8383\ 04680 \times 10^{17}$$

$$t = 1.6 \times 10^{-3}$$
$$t^2 = 2.56 \times 10^{-6}$$
$$t^3 = 4.096 \times 10^{-9}$$
$$t^4 = 6.5536 \times 10^{-12}$$
$$t^5 = 1.0485\ 76 \times 10^{-14}$$
$$t^6 = 1.6777\ 216 \times 10^{-17}$$
$$t^7 = 2.6843\ 5456 \times 10^{-20}$$
$$t^8 = 4.2949\ 67296 \times 10^{-23}$$
$$t^9 = 6.8719\ 47674 \times 10^{-26}$$

$$
\begin{array}{rr}
r_n = & 20\ 903.00000 \\
+ & 45.44136 \\
+ & 2.22554 \\
+ & .03667 \\
+ & .00081 \\
\hline
 & 20950.70278 \text{ (in thousands of feet)} \\
- & 20902.93392 \\
\hline
 & 47{,}768.84 \text{ feet = altitude of airplane}
\end{array}
$$

$$
\begin{array}{rl}
\theta_n = & .075257\ 62090 \\
 & -.000083\ 72858 \\
 & -.000009\ 79094 \\
 & -.000000\ 52701 \\
 & -.000000\ 01263 \\
\hline
 & .075163\ 56174 \text{ radians} \times \dfrac{20{,}902{,}933.92}{5280} = 297.56420 \text{ miles}
\end{array}
$$

V. SOLUTION OF HYPOTHETICAL PROBLEM BY NUMERICAL INTEGRATION

In the search for a practical, yet rigorous, method for computing geodetic distance from Shoran observations the method of numerical integration, demonstrated in the computation form below, seems to offer the best possibility.

Here again an approximation to α_1 must be made and then tested to see if it will produce r_n. In the demonstration below, however, the purpose is to produce by numerical integration the same value of $R\ (\theta_n - \theta_1)$ as was obtained by the "series" method. Therefore the computation was begun with $\alpha_1 = 0$ as before. In actual use, the trial computation of r_n using an approximated value of α_1 would utilize every column on the form except that for $\frac{1}{r^2\mu^2}$, which is used only in the computation of $\theta_n - \theta_1$.

An examination of the first two lines of the form reveals that the "series" method of Section IV has been used to produce r_2 and r_3. Since the interval, Δt, is small only two derivatives are required on each line. By reference to the values of the derivatives in the solution of Section IV it is easy to determine the limiting value of Δt that can be used in the first two lines of the computation. To limit the error to .01 feet Δt can be found from the relation

$$\frac{(\Delta t)^4}{4!}\frac{d^4r}{dt^4} = .00001 \text{ (in thousands of feet)} = (\Delta t)^4\ 3.3959\times10^{11} = .00001$$

From this equation Δt = about .00007. Of course, this is the limiting value under a particular set of conditions. In the numerical integration below, the interval Δt = .00002 was used to produce r_2, r_3, r_4, and r_5.

Thereafter the interval was doubled to reduce the labor of computation. In the examples of Appendix B and Appendix C, the interval $\Delta t = .00002$ was used throughout. It is obvious that in an erratic atmosphere the intervals must necessarily be small, while in a smoothly changing atmosphere the intervals can be quite large.

On the form below, the first value of r is the starting value r_1. As explained above, the values r_2 and r_3 are found by the "series" method. The value of r_4 is found by integrating the second degree function determined by $\left(\frac{dr}{dt}\right)_1$, $\left(\frac{dr}{dt}\right)_2$, and $\left(\frac{dr}{dt}\right)_3$. The values of r_5, r_6, etc., are found in a similar manner from the $\frac{dr}{dt}$'s from the three preceding lines. A discussion of this technique can be found in Whittaker and Robinson's "Calculus of Observations."

Briefly, however, the step by step relationship can be explained as follows.

The increase in r during any fraction, ε, of the nth time interval, Δt, can be expressed as

$$r_{n+\varepsilon} - r_n = \Delta t \int_0^{\varepsilon} \left(\frac{dr}{dt}\right)_{n+\varepsilon} d\varepsilon$$

$$\text{But } \left(\frac{dr}{dt}\right)_{n+\varepsilon} = E^{\varepsilon}\left(\frac{dr}{dt}\right)_n = \left(1 - \frac{\Delta}{E}\right)^{-\varepsilon}\left(\frac{dr}{dt}\right)_n$$

$$= \left\{1 + \varepsilon\left(\frac{\Delta}{E}\right) + \frac{\varepsilon(\varepsilon+1)}{2}\left(\frac{\Delta}{E}\right)^2\right\}\left(\frac{dr}{dt}\right)_n \text{ to second differences,}$$

where the operator E can be defined by the following,

$$E^n f(x)_o = (1 + \Delta)^n f(x)_o = f(x)_n.$$

When $\varepsilon = 1$

$$r_{n+1} - r_n = \Delta t \int_0^1 \left\{1 + \varepsilon\left(\frac{\Delta}{E}\right) + \frac{\varepsilon(\varepsilon+1)}{2}\left(\frac{\Delta}{E}\right)^2\right\}\left(\frac{dr}{dt}\right)_n d\varepsilon$$

$$= \Delta t \left[\left\{\varepsilon + \frac{\varepsilon^2}{2}\left(\frac{\Delta}{E}\right) + \left(\frac{\varepsilon^3}{6} + \frac{\varepsilon^2}{4}\right)\left(\frac{\Delta}{E}\right)^2\right\}\left(\frac{dr}{dt}\right)_n\right]_0^1$$

$$= \Delta t\left\{\left(\frac{dr}{dt}\right)_n + \frac{1}{2}\Delta\left(\frac{dr}{dt}\right)_{n-1} + \frac{5}{12}\Delta^2\left(\frac{dr}{dt}\right)_{n-2}\right\}$$

Representing $\Delta t\left(\frac{dr}{dt}\right)_n$ by q_n, we have,

$$r_{n+1} - r_n = q_n + .5\Delta q_{n-1} + .417\,\Delta^2 q_{n-2}$$

Because $\Delta^2 q_{n-2}$ is small in value, its coefficient is expressed to only three decimals. When the Δ's in this expression are replaced by their equivalents,

$$r_{n+1} - r_n = 1.917q_n - 1.334q_{n-1} + .417q_{n-2}$$

When t does not contain exactly an integral number of Δt's, the change in r during the fractional portion of the last Δt can be found by using the

general form of the above and integrating between 0 and ε instead of between 0 and 1. However, this resort is seldom required since t can always be divided into an integral number of equal intervals.

A few remarks concerning the actual computation of the attached example may be of help. Except for the last column the computation proceeds from line to line. The "h" on each line is equal to r - R and is obtained merely for use with a graph of μ's against h. The "μ" on each line is obtained from such a graph. The computation of "q" can be accomplished without writing down any intermediate answers. The complement of $\frac{k^2}{r^2\mu^2}$ is gotten first in the top dial. The square root of this result is multiplied by $V\Delta t$ (constant throughout the calculation, except when the interval is changed) and divided by μ. The values of μ^1 required in the first two lines can be found by fitting a polynomial to a few points of the "μ" curve. A number of techniques, any of which will produce good results, can be used to obtain these values. The first few values of r are produced numerically below to indicate more clearly the steps to be taken.

20903.00000 = given starting value of r
20903.00710 = 20903.00000 + .00000 + .00710
20903.02781 = 20903.00710 + .01361 + .00710
20903.06381 = 20903.02781 + 1.917(.02825) - 1.334(.01361) + .417(.00000)
20903.11358 = 20903.06381 + 1.917(.04266) - 1.334(.02825) + .417(.01361)
etc.

Attention is directed to the value of k shown at the top of the computation form. Since $\cos \alpha_1 = 1$ in this particular computation, $k = r_1\mu_1$. In most cases, however, $\cos \alpha_1$ will have a value different from 1 in the expression, $k = r_1\mu_1 \cos \alpha_1$.

The procedure for obtaining θ is relatively simple. Since $\frac{d\theta}{dt} = \frac{V \cos \alpha}{r\mu} = \frac{Vk}{r^2\mu^2}$, it is only necessary to tabulate $\frac{1}{r^2\mu^2}$ for each step. The relation between θ_{n+1} and θ_n is alike in form to that between r_{n+1} and r_n, i.e.,

$$\theta_{n+1} - \theta_n = \Delta t\left(1.917\frac{Vk}{(r\mu)_n^2} - 1.334\frac{Vk}{(r\mu)_{n-1}^2} + .417\frac{Vk}{(r\mu)_{n-2}^2}\right)$$

If $\frac{1}{(r\mu)_n^2}$ is replaced by p_n, then

$$\theta_{n+1} - \theta_n = \Delta tVk(1.917p_n - 1.334p_{n-1} + .417p_{n-2})$$

It is seen that $\theta_4 - \theta_3$, $\theta_5 - \theta_4$, $\theta_6 - \theta_5$, etc., can be obtained immediately from p_1, p_2, p_3, etc. To get values for $\theta_3 - \theta_2$, $\theta_2 - \theta_1$ however, the relationship above can be applied in reverse. That is,

$$\theta_2 - \theta_1 = \Delta tVk(1.917p_2 - 1.334p_3 + .417p_4)$$
$$\text{and } \theta_3 - \theta_2 = \Delta tVk(1.917p_3 - 1.334p_4 + .417p_5)$$

By means of the following arrangement, the form of $\theta_n - \theta_1$ (the total angle produced by t) can be shown.

$$\theta_2 - \theta_1 = \Delta t Vk(+ 1.917p_2 - 1.334p_3 + .417p_4)$$
$$\theta_3 - \theta_2 = \Delta t Vk(+ 1.917p_3 - 1.334p_4 + .417p_5)$$
$$\theta_4 - \theta_3 = \Delta t Vk(+ .417p_1 - 1.334p_2 + 1.917p_3)$$
$$\theta_5 - \theta_4 = \Delta t Vk(+ .417p_2 - 1.334p_3 + 1.917p_4)$$
$$\theta_6 - \theta_5 = \Delta t Vk(+ .417p_3 - 1.334p_4 + 1.917p_5)$$
$$\theta_7 - \theta_6 = \Delta t Vk(+ .417p_4 - 1.334p_5 + 1.917p_6)$$
etc.

From this arrangement

$$\theta_n - \theta_1 = \Delta t Vk(.417p_1 + p_2 + 1.583p_3 + .083p_4 + 1.417p_5 + p_6 + p_7 + p_8 + \ldots + p_{n-3} + .583p_{n-2} + 1.917p_{n-1})$$

This expression is applicable to the computations in Appendix B and C. In the form immediately below, however, a slightly different procedure was followed. In each of the first four intervals Δt was taken as .00002 and in each interval thereafter it was taken as .00004.

The value of q_1 was found by multiplying $\left(\frac{dr}{dt}\right)_1$ by .00002. The values of q_2, q_3, and q_4 were found in the same manner. The value of q_5 was found by multiplying $\left(\frac{dr}{dt}\right)_5$ by .00004. The remaining q's were also found by multiplying by .00004. The following relationships were then used:

$$r_4 - r_3 = 1.917q_3 - 1.334q_2 + .417q_1$$
$$r_5 - r_4 = 1.917q_4 - 1.334q_3 + .417q_2$$
$$r_6 - r_5 = 1.917q_5 - 2(1.334)q_3 + 2(.417)q_1$$
$$r_7 - r_6 = 1.917q_6 - 1.334q_5 + 2(.417)q_3$$
$$r_8 - r_7 = 1.917q_7 - 1.334q_6 + .417q_5$$

For the remaining lines the relationship is similar to that between r_8 and r_7.

The computation of $\theta_n - \theta_1$ in this particular problem is based on the following arrangement:

$$\theta_2 - \theta_1 = .00002\ Vk(+ 1.917p_2 - 1.334p_3 + .417p_4)$$
$$\theta_3 - \theta_2 = .00002\ Vk(+ 1.917p_3 - 1.334p_4 + .417p_5)$$
$$\theta_4 - \theta_3 = .00002\ Vk(+ .417p_1 - 1.334p_2 + 1.917p_3)$$
$$\theta_5 - \theta_4 = .00002\ Vk(+ .417p_2 - 1.334p_3 + 1.917p_4)$$
$$\theta_6 - \theta_5 = .00004\ Vk(+ .417p_1 - 1.334p_3 + 1.917p_5)$$
$$\theta_7 - \theta_6 = .00004\ Vk(+ .417p_3 - 1.334p_5 + 1.917p_6)$$
$$\theta_8 - \theta_7 = .00004\ Vk(+ .417p_5 - 1.334p_6 + 1.917p_7)$$
$$\theta_9 - \theta_8 = .00004\ Vk(+ .417p_6 - 1.334p_7 + 1.917p_8)$$
etc.

Therefore

$$\theta_n - \theta_1 = .00004\ Vk(.6255p_1 + .5p_2 - .334p_3 + .5p_4 + 1.2085p_5 + p_6 + p_7 + \ldots + p_{n-3} + .583p_{n-2} + 1.917p_{n-1})$$

SECTION V

Numerical integration solution of problem presented in section IV.

$K = r_1\mu_1 = 20909.83110$ $\qquad$ T = total time = .0016 seconds $\qquad$ $\frac{d^2r}{dt^2} = \frac{V^2}{r\mu^2}\{(a+1)-\sin^2\alpha(2a+1)\}$

$K^2 = 437.2210366 \times 10^6$ $\qquad$ Δt = .00002 for first four intervals $\qquad$ Δt = .00004 for remaining intervals

h in thousands of feet	r	μ	$r^2\mu^2 \times 10^{-6}$	$q=\frac{V\Delta t \sin\alpha}{\mu}$	$\sin^2\alpha=1-\frac{K^2}{r^2\mu^2}$	$\sin\alpha$	$\mu^I \times 10^6$	$a=\frac{r\mu^I}{\mu}$	a^2	$a+1$	$2a+1$	$(a+1)-\sin^2\alpha(2a+1)$	$\frac{\Delta t^2}{2}\frac{d^2r}{dt^2}$	$\frac{d\theta}{dt} \div VK = \frac{1}{r^2\mu^2}\times 10^{10}$
.06608	20,903.00000	1.000,326,8	437.2210366	.00000	.000000000	.000000	-11.12	-.2324	+.05401	+.7676	+.5352	+.7676	.00710	22.87172657
.07318	903.00710	326,7	2212461	.01361	.000000479	.000692	-11.12	-.2324	+.05401	+.7676	+.5352	+.7676	.00710	171561
.09389	903.02781	326,5	2219378	.02825	—	—	—	—	—	—	—	—	—	167943
.12989	903.06381	326,1	2230942	.04266	—	—	—	—	—	—	—	—	—	161893
.17966	903.11358	325,5	2246515	.11308	—	—	—	—	—	—	—	—	—	153747
.32106	903.25498	324,0	2292555	.17051	—	—	—	—	—	—	—	—	—	129663
.52064	903.45456	321,8	2356816	.22761	—	—	—	—	—	—	—	—	—	096049
.77666	903.71058	319,0	2439441	.28466	—	—	—	—	—	—	—	—	—	052831
1.08982	904.02374	315,5	2539853	.34139	—	—	—	—	—	—	—	—	—	000310
1.45944	904.39336	311,5	2659513	.39859	—	—	—	—	—	—	—	—	—	22.86937725
1.88683	904.82075	307,0	2798969	.45629	—	—	—	—	—	—	—	—	—	864791
2.37218	905.30610	301,8	1956553	.51374	—	—	—	—	—	—	—	—	—	782381
2.91454	905.84846	296,1	3133618	.57145	—	—	—	—	—	—	—	—	—	689791
3.51495	906.44887	289,9	3330599	.62945	—	—	—	—	—	—	—	—	—	586795
4.17352	907.10744	283,2	3547539	.68770	—	—	—	—	—	—	—	—	—	473374
4.89045	907.82437	276,0	3784529	.74614	—	—	—	—	—	—	—	—	—	349484
5.66589	908.59981	268,3	4041628	.80475	—	—	—	—	—	—	—	—	—	215096
6.50002	909.43394	260,3	4320661	.86386	—	—	—	—	—	—	—	—	—	069261
7.39364	910.32756	251,8	4620213	.92312	—	—	—	—	—	—	—	—	—	22.85912722
8.34645	911.28037	243,0	4941914	.98277	—	—	—	—	—	—	—	—	—	744633
9.35921	912.29313	233,9	5286082	1.04282	—	—	—	—	—	—	—	—	—	564832
10.43222	913.36614	224,5	5652839	1.10321	—	—	—	—	—	—	—	—	—	373262
11.56577	914.49969	214,9	6043192	1.16404	—	—	—	—	—	—	—	—	—	169401
12.76041	915.69433	205,1	6457364	1.22527	—	—	—	—	—	—	—	—	—	22.84953141
14.01646	916.95038	195,2	6896373	1.28699	—	—	—	—	—	—	—	—	—	723957
15.33451	918.26843	185.1	7359589	1.34904	—	—	—	—	—	—	—	—	—	482185
16.71471	919.64863	175.1	7849708	1.41172	—	—	—	—	—	—	—	—	—	226428
18.15803	921.09195	165,1	8366261	1.47489	—	—	—	—	—	—	—	—	—	22.83956938
19.66471	922.59863	155,2	8910233	1.53859	—	—	—	—	—	—	—	—	—	673213
21.23537	924.16929	145,4	9481879	1.60280	—	—	—	—	—	—	—	—	—	375129
22.87049	925.80441	135,9	438.0083167	1.66765	—	—	—	—	—	—	—	—	—	061672
24.57083	927.50475	126,7	0714411	1.73311	—	—	—	—	—	—	—	—	—	22.82732692
26.33692	929.27084	117,9	1376724	1.79921	—	—	—	—	—	—	—	—	—	387622
28.16945	931.10337	109,5	2070400	1.86592	—	—	—	—	—	—	—	—	—	026323
30.06898	933.00290	101,7	2797431	1.93334	—	—	—	—	—	—	—	—	—	22.81647773
32.03633	934.97025	094,5	3558167	2.00144	—	—	—	—	—	—	—	—	—	251809
34.07210	937.00602	088,1	4354632	2.07031	—	—	—	—	—	—	—	—	—	22.80837396
36.17717	939.11109	082,5	5187200	2.13991	—	—	—	—	—	—	—	—	—	404358
38.35218	941.28610	077,8	6057025	2.21026	—	—	—	—	—	—	—	—	—	22.79952117
40.59793	943.53185	074,1	6965341	2.28137	—	—	—	—	—	—	—	—	—	480056
42.91517	945.84909	071,7	7915101	2.35340	—	—	—	—	—	—	—	—	—	22.78986664
45.30497	948.23889	070.5	8905896	2.42623	—	—	—	—	—	—	—	—	—	472183
47.76795	950.70187													

$\theta_n - \theta_1 = VK\Delta t[913.725700441640 \times 10^{-10}] = [983,515.0932][20,909.83110][.00004][913.7257004 \times 10^{-10}] = .07516356763$

$S = R[\theta_n - \theta_1] =$ 297.5642211 miles

VI. SOLUTION OF HYPOTHETICAL PROBLEM BY AIR FORCE METHOD

Since the Air Force tables are not constructed to handle a ray which attains the height and length of the one treated in Sections IV and V, that problem has been modified by reducing t from .0016 to .0014 seconds. In the "series" of Section IV it is a simple matter to substitute this new value of t and produce r_n = 20,939,111.80 feet and $R(\theta_n - \theta_1)$ = 260.45172 miles. Since the velocity of propagation is assumed to be 186,219 mps in the Shoran instrument, the dial reading would be 260.70660 miles. Thus the basic values in the Air Force notation are as follows:

$$K = 66.08 \text{ feet}$$
$$H = 36{,}177.88 \text{ feet}$$
$$(H - K) = 36{,}111.80 \text{ feet}$$
$$S = 260.70660 \text{ miles}$$

Of course, the NACA standard "moist" atmosphere is used as before. Since μ is a smoothly changing value, it would seem to conform with Air Force procedure to use intervals of about 1300 feet when computing the velocity correction.

The results of the entire computation appear below:

Dial reading		260.70660
Velocity correction	+	.01641
Reduction to sea level	-	.22602
Slope correction	-	.08971
Curvature correction	+	.04403
Steep slope correction	-	.00002
Assumption I	+	.00010
" VII	-	.00015
" VIII	+	.00200
" IX	+	.00019
" X	-	.00010
Geodetic distance by Air Force method =		260.45333

This result is 8.50 feet in excess of the result obtained by means of series.

VII. DISCUSSION

A comparison of the distance produced by the series solution of Section IV with the distance found through the numerical integration solution of Section V is believed to establish the validity of the latter method. Of course, it is realized that the "μ" curve used in the solution of this particular problem was extremely smooth. It is believed, however, that the method is valid in any case where a reasonable amount of smoothing has been carried out before the "μ" values are used in the integration. In considering the possible roughness that can occur in the "μ" curve, it was thought safer to project on the basis of a second degree curve formed by three values of $\frac{dr}{dt}$, than to project on the basis of a $\frac{dr}{dt}$ curve of higher degree. If small intervals of Δt are used, it is difficult to see how any serious error can occur.

The application of the numerical integration method and the Air Force method to three different problems produced the following distances in miles:

Problem	Air Force Formula	Numerical Integration	Air Force minus Numerical Integration
Section VI	260.45333	260.45172	+.00161
Appendix B	152.52686	152.52683	+.00003
Appendix C	100.49874	100.49872	+.00002

In an attempt to analyze the cause for the difference in the results, a study was made of the height of the ray path used in the problem of Section VI. Below is given a table comparing heights obtained from the numerical integration curve with those obtained from the Air Force formula,

$$h = K + \frac{M_1(H - K)}{M} - .496M_1(M - M_1)$$

M_1 (distance in miles)	Height of ray path in feet		
	Air Force Formula	Numerical Integration	Air Force minus Numerical Integration
0.0000	66.08	66.08	0.00
48	1663.26	1254.12	409.14
70	3159.14	2588.34	570.80
87	4643.90	3963.74	680.16
101	6081.90	5326.01	755.89
113	7469.23	6655.05	814.18
124	8866.44	8014.04	852.40
134	10240.78	9353.70	887.08
144	11714.33	10815.53	898.80
152	12964.59	12059.32	905.27
161	14447.02	13542.46	904.56
169	15832.19	14936.26	895.93
176	17096.30	16216.14	880.16
183	18409.01	17551.54	857.47
190	19770.34	18942.89	827.45
197	21180.27	20390.60	789.67
204	22638.81	21895.16	743.65
210	23927.67	23228.37	699.30
216	25252.25	24601.78	650.47
222	26612.53	26027.31	585.22
228	28008.53	27495.01	513.52
234	29440.24	29003.96	436.28
239	30660.62	30289.71	370.91
244	31905.79	31613.67	292.12
250	33432.73	33243.19	189.54
255	34732.47	34631.98	100.49
260	36057.00	36049.13	7.87
260.4517	36177.88	36177.17	0.71

The velocity correction given in Section VI was obtained from the Air Force values in this table. If the velocity correction is recomputed using the numerical integration ray path heights it is found to be .01516 miles instead of .01641, which appears to account for .00125 miles of the .00161 mile difference occurring on the 260 mile line. The remaining .00036 miles by which the Air Force result exceeds the numerical integration result is probably caused by the difference in radius of curvature of the Shoran beam

between the two methods. Simple calculations indicate that variation in the radius of the Shoran beam has relatively little effect on the computed distance. The difference in distance produced by a 260 mile arc of radius 4R as against 5R is only about .0010 miles. The difference in distance produced by the same arc having a radius of 5R as against 6R is about .0006 miles.

It is believed that greater harm is caused by using erroneous values for the height of the ray path when computing the velocity correction, than is caused by the use of the wrong radius in the geometrical solution. However, in actual practice, the effect of both these errors appears to be negligible.

Although numerical integration cannot be considered as a substitute for the Air Force method, at least not until the availability of the electronic computer, it does make possible the construction of ray path curves for varying conditions. It is hoped that through its use, the technique of computing geodetic distances from Shoran observations will be improved.

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APPENDIX A

Derivatives of r and θ with respect to t

In the derivatives given below, the following substitutions have been used:

$$A = \frac{V}{r\mu}$$

μ^{I}, μ^{II}, μ^{III}, etc., represent the derivatives of μ with respect to r

$$a = \frac{r\mu^{I}}{\mu},\ b = \frac{r^2\mu^{II}}{\mu},\ c = \frac{r^3\mu^{III}}{\mu},\ d = \frac{r^4\mu^{IV}}{\mu},\ \text{etc.}$$

$$m = a + 1$$

$$m^{I} = a + b - a^2$$

$$m^{II} = a + 3b + c - 3a^2 + 2a^3 - 3ab$$

$$m^{III} = a + 7b + 6c + d - 7a^2 + 12a^3 - 6a^4 - 3b^2 - 18ab - 4ac + 12a^2b$$

$$m^{IV} = a + 15b + 25c + 10d + e - 15a^2 + 50a^3 - 60a^4 + 24a^5 - 30b^2 - 75ab - 40ac - 5ad + 120a^2b - 60a^3b + 20a^2c + 30ab^2 - 10bc$$

$$m^{V} = a + 31b + 90c + 65d + 15e + f - 31a^2 + 180a^3 - 390a^4 + 360a^5 - 120a^6 - 270ab - 260ac - 75ad - 6ae - 150bc - 15bd + 780a^2b - 900a^3b + 360a^4b + 300a^2c + 30a^2d - 120a^3c - 270a^2b^2 + 120abc - 195b^2 + 30b^3 - 10c^2 + 450ab^2$$

$$m^{VI} = a + 63b + 301c + 350d + 140e + 21f + g - 63a^2 + 602a^3 - 2100a^4 + 3360a^5 - 2520a^6 + 720a^7 - 903ab - 1400ac - 700ad - 126ae - 7af - 1400bc - 315bd - 21be - 35cd + 4200a^2b + 2800a^2c + 630a^2d + 42a^2e - 5670a^2b^2 - 8400a^3b - 2520a^3c + 2520a^3b^2 - 210a^3d + 7560a^4b - 2520a^5b - 1050b^2 + 630b^3 + 4200ab^2 + 2520abc - 210c^2 + 140ac^2 + 210abd + 840a^4c - 1260a^2bc - 630ab^3 + 210b^2c$$

$$\frac{dr}{dt} = rA \sin \alpha$$

$$\frac{d^2r}{dt^2} = rA^2\{m + \sin^2\alpha(1 - 2m)\}$$

$$\frac{d^3r}{dt^3} = rA^3\{\sin \alpha\ (m^{I} - 6m^2 + 3m) + \sin^3\alpha\ (-2m^{I} + 8m^2 - 6m + 1)\}$$

$$\frac{d^4r}{dt^4} = rA^4\{(mm^{I} - 6m^3 + 3m^2) + \sin^2\alpha\ (m^{II} - 22mm^{I} + 4m^{I} + 48m^3 - 36m^2 + 6m) + \sin^4\alpha\ (-2m^{II} + 28mm^{I} - 8m^{I} - 48m^3 + 44m^2 - 12m + 1)\}$$

$$\frac{d^5r}{dt^5} = rA^5\{\sin \alpha\ (3mm^{II} + (m^{I})^2 - 66m^2m^{I} + 15mm^{I} + 120m^4 - 90m^3 + 15m^2) + \sin^3\alpha\ (m^{III} - 36mm^{II} - 22(m^{I})^2 + 5m^{II} + 388m^2m^{I} - 150mm^{I} + 10m^{I} + 440m^3 - 120m^2 + 10m - 480m^4) + \sin^5\alpha\ (-2m^{III} + 44mm^{II} + 28(m^{I})^2 - 10m^{II} - 368m^2m^{I} + 180mm^{I} - 20m^{I} - 400m^3 + 140m^2 - 20m + 384m^4 + 1)\}$$

$$\frac{d^6r}{dt^6} = rA^6\{(3m^2m^{II} + m(m^{I})^2 - 66m^3m^{I} + 15m^2m^{I} + 120m^5 - 90m^4 + 15m^3) + \sin^2\alpha\ (6mm^{III} + 5m^{I}m^{II} - 192m^2m^{II} - 204m(m^{I})^2 + 33mm^{II} + 16(m^{I})^2 + 2040m^3m^{I} - 876m^2m^{I} + 75mm^{I} + 1980m^4 - 540m^3 + 45m^2 - 2160m^5) + \sin^4\alpha\ (m^{IV} - 54mm^{III} - 80m^{I}m^{II} + 6m^{III} + 896m^2m^{II} + 1092m(m^{I})^2 - 276mm^{II} - 172(m^{I})^2 + 15m^{II} + 3808m^2m^{I} - 570mm^{I} + 20m^{I} - 6864m^3m^{I} + 2100m^3 - 300m^2 + 15m - 6000m^4 + 5760m^5) + \sin^6\alpha\ (-2m^{IV} + 64mm^{III} + 100m^{I}m^{II} - 12m^{III} - 808m^2m^{II} - 1016m(m^{I})^2 + 324mm^{II} + 208(m^{I})^2 - 30m^{II} - 3368m^2m^{I} + 660mm^{I} - 40m^{I} + 5216m^3m^{I} - 1800m^3 + 340m^2 - 30m + 4384m^4 - 3840m^5 + 1)\}$$

$$\frac{d^7r}{dt^7} = rA^7\{\sin \alpha\ (15m^2m^{III} + 18mm^{I}m^{II} + (m^{I})^3 - 468m^3m^{II} - 612m^2(m^{I})^2 + 84m^2m^{II} + 63m(m^{I})^2 + 5076m^4m^{I} - 2268m^3m^{I} + 210m^2m^{I} + 4620m^5 - 1260m^4 + 105m^3 - 5040m^6) + \sin^3\alpha\ (10mm^{IV} + 11m^{I}m^{III} + 5(m^{II})^2 - 456m^2m^{III} - 1152mm^{I}m^{II} - 204(m^{I})^3 + 63mm^{III} + 70m^{I}m^{II} + 7160m^3m^{II} + 12,120m^2(m^{I})^2 - 2436m^2m^{II} - 2772m(m^{I})^2 + 168mm^{II} + 91(m^{I})^2 + 32,200m^3m^{I} - 5376m^2m^{I} + 245mm^{I} - 54,576m^4m^{I} + 14,700m^4 - 2100m^3 + 105m^2 - 42,000m^5 + 40,320m^6) + \sin^5\alpha\ (m^{V} - 76mm^{IV} - 134m^{I}m^{III} - 80(m^{II})^2 + 7m^{IV} + 1820m^2m^{III} + 5376mm^{I}m^{II} + 1092(m^{I})^3 - 462mm^{III} - 700m^{I}m^{II} + 21m^{III} + 9408m^2m^{II} + 11,676m(m^{I})^2 - 1176mm^{II} - 742(m^{I})^2 + 35m^{II} - 20,672m^3m^{II} - 37,608m^2(m^{I})^2 + 19,768m^2m^{I} - 1610mm^{I} + 35m^{I} - 89,152m^3m^{I} + 128,736m^4m^{I} + 7140m^3 - 630m^2 + 21m - 37,800m^4 + 92,064m^5 - 80,640m^6) + \sin^7\alpha\ (-2m^{V} + 88mm^{IV} + 164m^{I}m^{III} + 100(m^{II})^2 - 14m^{IV} - 1576m^2m^{III} - 4848mm^{I}m^{II} - 1016(m^{I})^3 + 532mm^{III} + 840m^{I}m^{II} - 42m^{III} - 8064m^2m^{II} - 10,248m(m^{I})^2 + 1344mm^{II} + 868(m^{I})^2 - 70m^{II} + 14,912m^3m^{II} + 27,840m^2(m^{I})^2 - 16,688m^2m^{I} + 1820mm^{I} - 70m^{I} + 63,168m^3m^{I} - 81,792m^4m^{I} - 5880m^3 + 700m^2 - 42m + 25,984m^4 - 56,448m^5 + 46,080m^6 + 1)\}$$

$$
\begin{aligned}
\frac{d^8 r}{dt^8} = {} & rA^8\{(15m^3m^{III} + 18m^2m^Im^{II} + m(m^I)^3 - 468m^4m^{II} - 612m^3(m^I)^2 + 84m^3m^{II} + 63m^2(m^I)^2 + 5076m^5m^I - 2268m^4m^I + 210m^3m^I + 4620m^6 - 1260m^5 + 105m^4 - 5040m^7) \\
& + \sin^2\alpha\ (45m^2m^{IV} + 81mm^Im^{III} + 33m(m^{II})^2 + 21(m^I)^2m^{II} - 1956m^3m^{III} - 6228m^2m^Im^{II} - 1844m(m^I)^3 + 288m^2m^{III} + 522mm^Im^{II} + 64(m^I)^3 + 30,300m^4m^{II} \\
& + 61,560m^3(m^I)^2 - 10,716m^3m^{II} - 16,236m^2(m^I)^2 + 798m^2m^{II} + 756m(m^I)^2 + 142,920m^4m^I - 25,116m^3m^I + 1260m^2m^I - 234,576m^5m^I + 58,800m^5 - 8400m^4 + 420m^3 \\
& - 168,000m^6 + 161,280m^7) + \sin^4\alpha\ (15mm^V + 21m^Im^{IV} + 21m^{II}m^{III} - 936m^2m^{IV} - 2844mm^Im^{III} - 1602m(m^{II})^2 - 1764(m^I)^2m^{II} + 108mm^{IV} + 144m^Im^{III} + 75(m^{II})^2 \\
& + 20,820m^3m^{III} + 84,120m^2m^Im^{II} + 31,740m(m^I)^3 - 5832m^2m^{III} - 15,768mm^Im^{II} - 2976(m^I)^3 + 336mm^{III} + 420m^Im^{II} + 110,760m^3m^{II} + 194,820m^2(m^I)^2 - 15,372m^2m^{II} \\
& - 18,144m(m^I)^2 + 588mm^{II} + 336(m^I)^2 - 229,536m^4m^{II} - 527,544m^3(m^I)^2 + 243,600m^3m^I - 22,176m^2m^I + 630mm^I - 1,032,336m^4m^I + 1,431,360m^5m^I + 71,400m^4 \\
& - 6300m^3 + 210m^2 - 378,000m^5 + 920,640m^6 - 806,400m^7) + \sin^6\alpha\ (m^{VI} - 102mm^V - 210m^Im^{IV} - 294m^{II}m^{III} + 8m^V + 3348m^2m^{IV} + 11,772mm^Im^{III} + 7036m(m^{II})^2 \\
& + 8652(m^I)^2m^{II} - 720mm^{IV} - 1296m^Im^{III} - 780(m^{II})^2 + 28m^{IV} + 20,496m^2m^{III} + 61,824mm^Im^{II} + 12,768(m^I)^3 - 2184mm^{III} - 3360m^Im^{II} + 56m^{III} - 53,544m^3m^{III} \\
& - 235,680m^2m^Im^{II} - 95,432m(m^I)^3 + 52,696m^2m^{II} + 66,192m(m^I)^2 - 3696mm^{II} - 2352(m^I)^2 + 70m^{II} - 279,168m^3m^{II} - 516,912m^2(m^I)^2 + 481,184m^4m^{II} \\
& + 1,161,120m^3(m^I)^2 + 73,248m^2m^I - 3780mm^I + 56m^I - 594,384m^3m^I + 2,101,056m^4m^I - 2,601,216m^5m^I + 19,600m^3 - 1176m^2 + 28m - 164,640m^4 + 727,552m^5 \\
& - 1,580,544m^6 + 1,290,240m^7) + \sin^8\alpha\ (-2m^{VI} + 116mm^V + 252m^Im^{IV} + 364m^{II}m^{III} - 16m^V - 2808m^2m^{IV} - 10,296mm^Im^{III} - 6248m(m^{II})^2 - 7896(m^I)^2m^{II} + 816mm^{IV} \\
& + 1536m^Im^{III} + 940(m^{II})^2 - 56m^{IV} - 17,088m^2m^{III} - 53,232mm^Im^{II} - 11,264(m^I)^3 + 2464mm^{III} + 3920m^Im^{II} - 112m^{III} + 36,976m^3m^{III} + 168,288m^2m^Im^{II} \\
& + 69,904m(m^I)^3 - 43,568m^2m^{II} - 55,776m(m^I)^2 + 4144mm^{II} + 2688(m^I)^2 - 140m^{II} + 190,976m^3m^{II} + 360,816m^2(m^I)^2 - 290,560m^4m^{II} - 716,928m^3(m^I)^2 - 59,808m^2m^I \\
& + 4200mm^I - 112m^I + 400,736m^3m^I - 1,248,384m^4m^I + 1,421,568m^5m^I - 15,680m^3 + 1288m^2 - 56m + 108,304m^4 - 420,224m^5 + 836,352m^6 - 645,120m^7 + 1)\}
\end{aligned}
$$

$$\frac{d^9r}{dt^9} = rA^9\{\sin\alpha\ (105m^3m^{IV} + 225m^2m^Im^{III} + 84m^2(m^{II})^2 + 81m(m^I)^2m^{II} + (m^I)^4 - 4500m^4m^{III} - 15,696m^3m^Im^{II} - 5532m^2(m^I)^3 + 675m^3m^{III} + 1440m^2m^Im^{II} + 255m(m^I)^3$$

(First term only)

$$+ 69,420m^5m^{II} + 153,396m^4(m^I)^2 - 24,840m^4m^{II} - 42,660m^3(m^I)^2 + 1890m^3m^{II} + 2205m^2(m^I)^2 + 336,780m^5m^I - 60,480m^4m^I + 3150m^3m^I - 545,047m^6m^I + 14,700m^6$$

$$- 2100m^5 + 105m^4 - 42,000m^7 + 40,320m^8)\}$$

$$\frac{d^{10}r}{dt^{10}} = rA^{10}\{105m^4m^{IV} + 225m^3m^Im^{III} + 84m^3(m^{II})^2 + 81m^2(m^I)^2m^{II} + m(m^I)^4 - 4500m^5m^{III} - 15,696m^4m^Im^{II} - 5532m^3(m^I)^3 + 675m^4m^{III} + 1440m^3m^Im^{II} + 255m^2(m^I)^3$$

(First term only)

$$+ 69,420m^6m^{II} + 153,396m^5(m^I)^2 - 24,840m^5m^{II} - 42,660m^4(m^I)^2 + 1890m^4m^{II} + 2205m^3(m^I)^2 + 336,780m^6m^I - 60,480m^5m^I + 3150m^4m^I - 545,040m^7m^I + 132,300m^7$$

$$- 18,900m^6 + 945m^5 - 378,000m^8 + 362,880m^9\}$$

$$\frac{d\theta}{dt} = A$$

$$\frac{d^2\theta}{dt^2} = -2A^2m\sin\alpha$$

$$\frac{d^3\theta}{dt^3} = -2A^3\{m^2 + \sin^2\alpha\ (m^I - 4m^2)\}$$

$$\frac{d^4\theta}{dt^4} = -2A^4\{\sin\alpha\ (4mm^I - 12m^3) + \sin^3\alpha\ (m^{II} - 14mm^I + 24m^3)\}$$

$$\frac{d^5\theta}{dt^5} = -2A^5\{(4m^2m^I - 12m^4) + \sin^2\alpha\ (7mm^{II} + 4(m^I)^2 - 102m^2m^I + 144m^4) + \sin^4\alpha\ (m^{III} - 22mm^{II} - 14(m^I)^2 + 184m^2m^I - 192m^4)\}$$

$$\frac{d^6\theta}{dt^6} = -2A^6\{\sin\alpha\ (18m^2m^{II} + 16m(m^I)^2 - 276m^3m^I + 360m^5) + \sin^3\alpha\ (11mm^{III} + 15m^Im^{II} - 246m^2m^{II} - 292m(m^I)^2 + 2128m^3m^I - 1920m^5) + \sin^5\alpha\ (m^{IV} - 32mm^{III} - 50m^Im^{II}$$

$$+ 404m^2m^{II} + 508m(m^I)^2 - 2608m^3m^I + 1920m^5)\}$$

$$\frac{d^7\theta}{dt^7} = -2A^7\{(18m^3m^{II} + 16m^2(m^I)^2 - 276m^4m^I + 360m^6) + \sin^2\alpha\ (51m^2m^{III} + 113mm^Im^{II} + 16(m^I)^3 - 1158m^3m^{II} - 1832m^2(m^I)^2 + 10,392m^4m^I - 8640m^6) + \sin^4\alpha\ (16mm^{IV}$$

$$+ 26m^Im^{III} + 15(m^{II})^2 - 516m^2m^{III} - 1476mm^Im^{II} - 292(m^I)^3 + 6608m^3m^{II} + 11,844m^2(m^I)^2 - 43,920m^4m^I + 28,800m^6) + \sin^6\alpha\ (m^V - 44mm^{IV} - 82m^Im^{III} - 50(m^{II})^2$$

$$+ 788m^2m^{III} + 2424mm^Im^{II} + 508(m^I)^3 - 7456m^3m^{II} - 13,920m^2(m^I)^2 + 40,896m^4m^I - 23,040m^6)\}$$

$$\frac{d^8\theta}{dt^8} = -2A^8\{\sin\alpha\ (120m^3m^{III} + 312m^2m^Im^{II} + 64m(m^I)^3 - 2736m^4m^{II} - 4896m^3(m^I)^2 + 25{,}152m^5m^I - 20{,}160m^7) + \sin^3\alpha\ (115m^2m^{IV} + 319mm^Im^{III} + 173m(m^{II})^2 + 161(m^I)^2m^{II}$$
$$- 3732m^3m^{III} - 14{,}172m^2m^Im^{II} - 4992m(m^I)^3 + 48{,}404m^4m^{II} + 107{,}264m^3(m^I)^2 - 331{,}440m^5m^I + 201{,}600m^7) + \sin^5\alpha\ (22mm^V + 42m^Im^{IV} + 56m^{II}m^{III} - 972m^2m^{IV}$$
$$- 3312mm^Im^{III} - 1956m(m^{II})^2 - 2352(m^I)^2m^{II} + 17{,}528m^3m^{III} + 75{,}768m^2m^Im^{II} + 30{,}240m(m^I)^3 - 167{,}952m^4m^{II} - 401{,}328m^3(m^I)^2 + 945{,}216m^5m^I - 483{,}840m^7)$$
$$+ \sin^7\alpha\ (m^{VI} - 126m^Im^{IV} - 58mm^V - 182m^{II}m^{III} + 1404m^2m^{IV} + 5148mm^Im^{III} + 3948(m^I)^2m^{II} + 3124m(m^{II})^2 - 18{,}488m^3m^{III} - 84{,}144m^2m^Im^{II} - 34{,}952m(m^I)^3$$
$$+ 145{,}280m^4m^{II} + 358{,}464m^3(m^I)^2 - 710{,}784m^5m^I + 322{,}560m^7)\}$$

$$\frac{d^9\theta}{dt^9} = -2A^9\{(120m^4m^{III} + 312m^3m^Im^{II} + 64m^2(m^I)^3 - 2736m^5m^{II} - 4896m^4(m^I)^2 + 25{,}152m^6m^I - 20{,}160m^8) + \sin^2\alpha\ (465m^3m^{IV} + 1629m^2m^Im^{III} + 831m^2(m^{II})^2 + 1299m(m^I)^2m^{II}$$
$$+ 64(m^I)^4 - 15{,}132m^4m^{III} - 66{,}372m^3m^Im^{II} - 30{,}304m^2(m^I)^3 + 197{,}724m^5m^{II} + 496{,}512m^4(m^I)^2 - 1{,}386{,}960m^6m^I + 806{,}400m^8) + \sin^4\alpha\ (225m^2m^V + 759mm^Im^{IV}$$
$$+ 945mm^{II}m^{III} + 480(m^I)^2m^{III} + 495m^I(m^{II})^2 - 9972m^3m^{IV} - 45{,}756m^2m^Im^{III} - 26{,}028m^2(m^{II})^2 - 57{,}012m(m^I)^2m^{II} - 4992(m^I)^4 + 180{,}828m^4m^{III} + 957{,}048m^3m^Im^{II}$$
$$+ 532{,}896m^2(m^I)^3 - 1{,}752{,}048m^5m^{II} - 4{,}951{,}008m^4(m^I)^2 + 10{,}114{,}560m^6m^I - 4{,}838{,}400m^8) + \sin^6\alpha\ (29mm^{VI} + 64m^Im^V + 98m^{II}m^{IV} + 56(m^{III})^2 - 1686m^2m^V$$
$$- 6726mm^Im^{IV} - 9{,}282mm^{II}m^{III} - 5664(m^I)^2m^{III} - 6660m^I(m^{II})^2 + 40{,}964m^3m^{IV} + 210{,}756m^2m^Im^{III} + 125{,}020m^2(m^{II})^2 + 302{,}820m(m^I)^2m^{II} + 30{,}240(m^I)^4$$
$$- 542{,}760m^4m^{III} - 3{,}124{,}224m^3m^Im^{II} - 1{,}872{,}008m^2(m^I)^3 + 4{,}313{,}504m^5m^{II} + 12{,}853{,}920m^4(m^I)^2 - 21{,}595{,}392m^6m^I + 9{,}031{,}680m^8)\}$$

(Four terms only)

$$\frac{d^{10}\theta}{dt^{10}} = -2A^{10}\{\sin\alpha\ (1050m^4m^{IV} + 4050m^3m^Im^{III} + 1974m^3(m^{II})^2 + 3726m^2(m^I)^2m^{II} + 256m(m^I)^4 - 34{,}200m^5m^{III} - 159{,}336m^4m^Im^{II} - 80{,}832m^3(m^I)^3 + 447{,}960m^6m^{II}$$
$$+ 1{,}192{,}896m^5(m^I)^2 - 3{,}186{,}720m^7m^I + 1{,}814{,}400m^9) + \sin^3\alpha\ (1365m^3m^V + 6060m^2m^Im^{IV} + 7071m^2m^{II}m^{III} + 6477m(m^I)^2m^{III} + 6240mm^I(m^{II})^2 + 1555(m^I)^3m^{II}$$
$$- 60{,}600m^4m^{IV} - 329{,}472m^3m^Im^{III} - 180{,}456m^3(m^{II})^2 - 533{,}664m^2(m^I)^2m^{II} - 81{,}344m(m^I)^4 + 1{,}102{,}620m^5m^{III} + 6{,}606{,}300m^4m^Im^{II} + 4{,}481{,}280m^3(m^I)^3$$
$$- 10{,}767{,}840m^6m^{II} - 34{,}083{,}936m^5(m^I)^2 + 63{,}552{,}960m^7m^I - 29{,}030{,}400m^9) + \sin^5\alpha\ (399m^2m^{VI} + 1593mm^Im^V + 2292mm^{II}m^{IV} + 1239(m^I)^2m^{IV} + 1281m(m^{III})^2$$
$$+ 2895m^Im^{II}m^{III} + 495(m^{II})^3 - 23{,}238m^3m^V - 126{,}654m^2m^Im^{IV} - 166{,}734m^2m^{II}m^{III} - 189{,}228m(m^I)^2m^{III} - 212{,}970mm^I(m^{II})^2 - 76{,}980(m^I)^3m^{II} + 566{,}220m^4m^{IV}$$

(Three terms only)

$$+ 3{,}585{,}480m^3m^Im^{III} + 2{,}071{,}560m^3(m^{II})^2 + 7{,}084{,}920m^2(m^I)^2m^{II} + 1{,}317{,}120m(m^I)^4 - 7{,}540{,}200m^5m^{III} - 50{,}806{,}272m^4m^Im^{II} - 38{,}496{,}624m^3(m^I)^3$$
$$+ 60{,}524{,}256m^6m^{II} + 207{,}124{,}992m^5(m^I)^2 - 309{,}883{,}392m^7m^I + 121{,}927{,}680m^9)\}$$

(Three terms only)

$$\frac{d^{11}\theta}{dt^{11}} = -2A^{11}\{(1050m^5m^{IV} + 4050m^4m^Im^{III} + 1974m^4(m^{II})^2 + 3726m^3(m^I)^2m^{II} + 256m^2(m^I)^4 - 34{,}200m^6m^{III} - 159{,}336m^5m^Im^{II} - 80{,}832m^4(m^I)^3 + 447{,}960m^7m^{II}$$
$$+ 1{,}192{,}896m^6(m^I)^2 - 3{,}186{,}720m^8m^I + 1{,}814{,}400m^{10}) + \sin^2\alpha\ (5145m^4m^V + 26{,}430m^3m^Im^{IV} + 29{,}211m^3m^{II}m^{III} + 35{,}307m^2(m^I)^2m^{III} + 32{,}094m^2m^I(m^{II})^2$$
$$+ 13{,}141m(m^I)^3m^{II} + 256(m^I)^5 - 228{,}600m^5m^{IV} - 1{,}367{,}352m^4m^Im^{III} - 724{,}392m^4(m^{II})^2 - 2{,}525{,}544m^3(m^I)^2m^{II} - 489{,}600m^2(m^I)^4 + 4{,}166{,}220m^6m^{III}$$
$$+ 26{,}804{,}484m^5m^Im^{II} + 20{,}378{,}304m^4(m^I)^3 - 40{,}865{,}760m^7m^{II} - 138{,}873{,}600m^8(m^I)^2 + 245{,}229{,}120m^8m^I - 108{,}864{,}000m^{10}) + \sin^4\alpha\ (3360m^3m^{VI} + 18{,}120m^2m^Im^V$$
$$+ 24{,}591m^2m^{II}m^{IV} + 24{,}792m(m^I)^2m^{IV} + 13{,}476m^2(m^{III})^2 + 54{,}051mm^Im^{II}m^{III} + 8032(m^I)^3m^{III} + 8715m(m^{II})^3 + 10{,}905(m^I)^2(m^{II})^2 - 195{,}900m^4m^V$$
$$- 1{,}289{,}982m^3m^Im^{IV} - 1{,}623{,}048m^3m^{II}m^{III} - 2{,}558{,}898m^2(m^I)^2m^{III} - 2{,}760{,}906m^2m^I(m^{II})^2 - 1{,}799{,}374m(m^I)^3m^{II} - 81{,}344(m^I)^5 + 4{,}782{,}120m^5m^{IV}$$
$$+ 34{,}659{,}408m^4m^Im^{III} + 19{,}490{,}484m^4(m^{II})^2 + 82{,}764{,}936m^3(m^I)^2m^{II} + 21{,}168{,}256m^2(m^I)^4 - 63{,}905{,}520m^6m^{III} - 479{,}294{,}472m^5m^Im^{II} - 425{,}640{,}720m^4(m^I)^3$$
$$+ 516{,}924{,}000m^7m^{II} + 1{,}957{,}670{,}784m^6(m^I)^2 - 2{,}700{,}432{,}000m^8m^I + 1{,}016{,}064{,}000m^{10})\}$$

(Two terms only)

$$\frac{d^{12}\theta}{dt^{12}} = -2A^{12}\{\sin\alpha\ (11{,}340m^5m^V + 62{,}160m^4m^Im^{IV} + 66{,}420m^4m^{II}m^{III} + 90{,}540m^3(m^I)^2m^{III} + 79{,}536m^3m^I(m^{II})^2 + 38{,}484m^2(m^I)^3m^{II} + 1024m(m^I)^5 - 504{,}000m^6m^{IV}$$
$$- 3{,}147{,}840m^5m^Im^{III} - 1{,}631{,}808m^5(m^{II})^2 - 6{,}134{,}976m^4(m^I)^2m^{II} - 1{,}305{,}600m^3(m^I)^4 + 9{,}190{,}800m^7m^{III} + 61{,}042{,}512m^6m^Im^{II} + 48{,}883{,}968m^5(m^I)^3$$
$$- 90{,}293{,}760m^8m^{II} - 317{,}555{,}712m^7(m^I)^2 + 546{,}842{,}880m^9m^I - 239{,}500{,}800m^{11}) + \sin^3\alpha\ (18{,}585m^4m^{VI} + 119{,}490m^3m^Im^V + 154{,}005m^3m^{II}m^{IV} + 213{,}765m^2(m^I)^2m^{IV}$$
$$+ 83{,}115m^3(m^{III})^2 + 438{,}639m^2m^Im^{II}m^{III} + 115{,}883m(m^I)^3m^{III} + 66{,}954m^2(m^{II})^3 + 147{,}231m(m^I)^2(m^{II})^2 + 14{,}421(m^I)^4m^{II} - 1{,}084{,}230m^5m^V - 8{,}040{,}300m^4m^Im^{IV}$$
$$- 9{,}717{,}282m^4m^{II}m^{III} - 18{,}724{,}842m^3(m^I)^2m^{III} - 19{,}441{,}596m^3m^I(m^{II})^2 - 16{,}916{,}502m^2(m^I)^3m^{II} - 1{,}308{,}160m(m^I)^5 + 26{,}495{,}100m^3m^{IV} + 209{,}582{,}364m^5m^Im^{III}$$
$$+ 114{,}907{,}908m^5(m^{II})^2 + 561{,}574{,}692m^4(m^I)^2m^{II} + 173{,}040{,}640m^3(m^I)^4 - 354{,}814{,}920m^7m^{III} - 2{,}856{,}248{,}184m^8m^Im^{II} - 2{,}821{,}100{,}736m^5(m^I)^3 + 2{,}885{,}045{,}760m^8m^{II}$$
$$+ 11{,}736{,}746{,}496m^7(m^I)^2 - 15{,}323{,}575{,}680m^9m^I + 5{,}588{,}352{,}000m^{11})\}$$

$$\frac{d^{13}\theta}{dt^{13}} = -2A^{13}\{(11{,}340m^{6}m^{V} + 62{,}160m^{5}m^{I}m^{IV} + 66{,}420m^{5}m^{II}m^{III} + 90{,}540m^{4}(m^{I})^{2}m^{III} + 79{,}536m^{4}m^{I}(m^{II})^{2} + 38{,}484m^{3}(m^{I})^{3}m^{II} + 1024m^{2}(m^{I})^{5} - 504{,}000m^{7}m^{IV}$$

(Two terms only)

$$- 3{,}147{,}840m^{8}m^{I}m^{III} - 1{,}631{,}808m^{6}(m^{II})^{2} - 6{,}134{,}976m^{5}(m^{I})^{2}m^{II} - 1{,}305{,}600m^{4}(m^{I})^{4} + 9{,}190{,}800m^{8}m^{III} + 61{,}042{,}512m^{7}m^{I}m^{II} + 48{,}883{,}968m^{6}(m^{I})^{3} -$$

$$- 90{,}293{,}760m^{9}m^{II} - 317{,}555{,}712m^{8}(m^{I})^{2} + 546{,}842{,}880m^{10}m^{I} - 239{,}500{,}800m^{12}) + \sin^{2}\alpha\ (67{,}095m^{5}m^{VI} + 477{,}330m^{4}m^{I}m^{V} + 590{,}595m^{4}m^{II}m^{IV} + 980{,}475m^{3}(m^{I})^{2}m^{IV}$$

$$+ 315{,}765m^{4}(m^{III})^{2} + 1{,}921{,}749m^{3}m^{I}m^{II}m^{III} + 657{,}753m^{2}(m^{I})^{3}m^{III} + 280{,}398m^{3}(m^{II})^{3} + 795{,}753m^{2}(m^{I})^{2}(m^{II})^{2} + 125{,}351m(m^{I})^{4}m^{II} + 1024(m^{I})^{8} - 3{,}915{,}450m^{6}m^{V}$$

$$- 31{,}162{,}980m^{5}m^{I}m^{IV} - 36{,}493{,}182m^{5}m^{II}m^{III} - 79{,}316{,}262m^{4}(m^{I})^{2}m^{III} - 79{,}867{,}284m^{4}m^{I}(m^{II})^{2} - 81{,}030{,}588m^{3}(m^{I})^{3}m^{II} - 7{,}855{,}616m^{2}(m^{I})^{5} + 95{,}732{,}100m^{7}m^{IV}$$

$$+ 798{,}194{,}964m^{6}m^{I}m^{III} + 428{,}611{,}548m^{6}(m^{II})^{2} + 2{,}283{,}520{,}716m^{5}(m^{I})^{2}m^{II} + 781{,}820{,}160m^{4}(m^{I})^{4} - 1{,}283{,}409{,}720m^{8}m^{III} - 10{,}780{,}801{,}224m^{7}m^{I}m^{II}$$

$$- 11{,}370{,}567{,}744m^{8}(m^{I})^{3} + 10{,}466{,}092{,}800m^{9}m^{II} + 44{,}577{,}605{,}376m^{8}(m^{I})^{2} - 56{,}261{,}036{,}160m^{10}m^{I} + 20{,}118{,}067{,}200m^{12})\}$$

$$\frac{d^{14}\theta}{dt^{14}} = -2A^{14}\{\sin\alpha\ (145{,}530m^{6}m^{VI} + 1{,}084{,}860m^{5}m^{I}m^{V} + 1{,}309{,}770m^{5}m^{II}m^{IV} + 2{,}362{,}290m^{4}(m^{I})^{2}m^{IV} + 697{,}950m^{5}(m^{III})^{2} + 4{,}515{,}750m^{4}m^{I}m^{II}m^{III} + 1{,}716{,}150m^{3}(m^{I})^{3}m^{III}$$

(One term only)

$$+ 640{,}332m^{4}(m^{II})^{3} + 2{,}025{,}102m^{3}(m^{I})^{2}(m^{II})^{2} + 371{,}274m^{2}(m^{I})^{4}m^{II} + 4096m(m^{I})^{6} - 8{,}493{,}660m^{7}m^{V} - 69{,}872{,}040m^{6}m^{I}m^{IV} - 80{,}327{,}700m^{6}m^{II}m^{III}$$

$$- 184{,}922{,}100m^{5}(m^{I})^{2}m^{III} - 182{,}908{,}872m^{5}m^{I}(m^{II})^{2} - 198{,}537{,}228m^{4}(m^{I})^{3}m^{II} - 20{,}947{,}968m^{3}(m^{I})^{5} + 207{,}711{,}000m^{8}m^{IV} + 1{,}775{,}028{,}600m^{7}m^{I}m^{III}$$

$$+ 941{,}110{,}920m^{7}(m^{II})^{2} + 5{,}226{,}880{,}584m^{6}(m^{I})^{2}m^{II} + 1{,}875{,}222{,}528m^{5}(m^{I})^{4} - 2{,}785{,}784{,}400m^{9}m^{III} - 23{,}863{,}952{,}880m^{8}m^{I}m^{II} - 25{,}965{,}956{,}736m^{7}(m^{I})^{3}$$

$$+ 22{,}743{,}141{,}120m^{10}m^{II} + 99{,}069{,}419{,}520m^{9}(m^{I})^{2} - 123{,}051{,}882{,}240m^{11}m^{I} + 43{,}589{,}145{,}600m^{13})\}$$

$$\frac{d^{15}\theta}{dt^{15}} = -2A^{15}\{(145{,}530m^{7}m^{VI} + 1{,}084{,}860m^{6}m^{I}m^{V} + 1{,}309{,}770m^{6}m^{II}m^{IV} + 2{,}362{,}290m^{5}(m^{I})^{2}m^{IV} + 697{,}950m^{6}(m^{III})^{2} + 4{,}515{,}750m^{5}m^{I}m^{II}m^{III} + 1{,}716{,}150m^{4}(m^{I})^{3}m^{III}$$

(One term only)

$$+ 640{,}332m^{5}(m^{II})^{3} + 2{,}025{,}102m^{4}(m^{I})^{2}(m^{II})^{2} + 371{,}274m^{3}(m^{I})^{4}m^{II} + 4096m^{2}(m^{I})^{6} - 8{,}493{,}660m^{8}m^{V} - 69{,}872{,}040m^{7}m^{I}m^{IV} - 80{,}327{,}700m^{7}m^{II}m^{III}$$

$$- 184{,}922{,}100m^{6}(m^{I})^{2}m^{III} - 182{,}908{,}872m^{6}m^{I}(m^{II})^{2} - 198{,}537{,}228m^{5}(m^{I})^{3}m^{II} - 20{,}947{,}968m^{4}(m^{I})^{5} + 207{,}711{,}000m^{9}m^{IV} + 1{,}775{,}028{,}600m^{8}m^{I}m^{III}$$

$$+ 941{,}110{,}920m^{8}(m^{II})^{2} + 5{,}226{,}880{,}584m^{7}(m^{I})^{2}m^{II} + 1{,}875{,}222{,}528m^{6}(m^{I})^{4} - 2{,}785{,}784{,}400m^{10}m^{III} - 23{,}863{,}952{,}880m^{9}m^{I}m^{II} - 25{,}965{,}956{,}736m^{8}(m^{I})^{3}$$

$$+ 22{,}743{,}141{,}120m^{11}m^{II} + 99{,}069{,}419{,}520m^{10}(m^{I})^{2} - 123{,}051{,}882{,}240m^{12}m^{I} + 43{,}589{,}145{,}600m^{14})\}$$

APPENDIX B

SOLUTION BY AIR FORCE FORMULA OF PROBLEM BASED ON ATMOSPHERE OBSERVED OVER THE LINE FRAN TO ELBOW (CARIBBEAN AREA SHORAN PROJECT)

$$K = 56 \text{ feet}$$
$$H = 27{,}945.08 \text{ feet}$$
$$H - K = 27{,}889.08 \text{ feet}$$
$$S = V_s t = \left(186{,}219 \frac{\text{Mi}}{\text{Sec}}\right)(.00082 \text{ sec})$$
$$= 152.69958 \text{ miles.}$$

Dial reading		152.69958
Velocity correction	+	.01175
Reduction to sea level	−	.10228
Slope correction	−	.09136
Curvature correction	+	.00885
Steep slope correction	−	.00003
Assumption VIII	+	.00035
" IX	+	.00005
" X	−	.00005
Geodetic distance by Air Force method		152.52686

APPENDIX B

Numerical integration solution of problem based on atmosphere observed over the line Fran to Elbow (Caribbean Area Shoran Project)

$K^2 = 437.0509260 \times 10^6$ T = total time = .00082 seconds $\frac{d^2r}{dt^2} = \frac{V^2}{r\mu^2}\{(a + 1) - \sin^2\alpha\ (2a + 1)\}$

Δt = .00002 seconds

h in thousands of feet	r	μ	$r^2\mu^2 \times 10^{-6}$	$q = \frac{V\Delta t \sin\alpha}{\mu}$	$\sin^2\alpha = 1 - \frac{K^2}{r^2\mu^2}$	$\sin\alpha$	$\mu^1 \times 10^6$	$a = \frac{r\mu^1}{\mu}$	a^2	$a + 1$	$2a + 1$	$(a+1) - \sin^2\alpha(2a+1)$	$\frac{\Delta t^2}{2}\frac{d^2r}{dt^2}$	$\frac{d\theta}{dt} \div VK = \frac{1}{r^2\mu^2} \times 10^{10}$
.05600	20,902.98992	1.000,331,1	437.2243738	.39165	.000396702	.0199174	-16.50	-.3448	+.1189	+.6552	+.3104	+.6551	.00606	22.87155200
.45371	903.38763	324.5	.2352420	.40373	.000421549	.0205317	-16.50	-.3448	+.1189	+.6552	+.3104	+.6551	.00606	7098349
.86350	903.79742	317.8	.2465282	.41591										7039314
1.28554	904.21946	313.6	.2605119	.43051										6966174
1.72436	904.65828	310.7	.2763348	.44646										6883420
2.17936	905.11328	306.0	.2912605	.46100										6805364
2.64704	905.58096	300.2	.3057557	.47469										6729564
3.12822	906.06214	291.0	.3178420	.48582										6666365
3.61854	906.55246	260.6	.3117737	.48028										6698095
4.08910	907.02302	265.1	.3353945	.50154										6574589
4.61245	907.54637	255.8	.3491572	.51353										6502634
5.12811	908.06203	250.9	.3664456	.52819										6412252
5.66474	908.59866	243.8	.3826879	.54161										6327346
6.21254	909.14646	242.8	.4047321	.55930										6212120
6.78247	909.71639	241.8	.4277029	.57716										6092064
7.36863	910.30255	240.3	.4509161	.59466										5970753
7.97189	910.90581	220.3	.4586623	.60039										5930275
8.57024	911.50416	204.2	.4696142	.60841										5873047
9.18361	912.11753	201.1	.4925662	.62485										5753124
9.82019	912.75411	201.6	.5196390	.64369										5611687
10.47430	913.40822	200.9	.5463964	.66179										5471914
11.14483	914.07875	195.8	.5699922	.67735										5348671
11.82890	914.76282	188.8	.5924921	.69186										5231164
12.52758	915.46150	178.8	.6129780	.70482										5124186
13.23823	916.17215	186.3	.6492799	.72719										4934640
13.98053	916.91445	181.0	.6757056	.74305										4796682
14.72880	917.66272	172.5	.7003029	.75752										4668284
15.49298	918.42690	167.5	.7271859	.77303										4527971
16.27420	919.20812	163.8	.7566424	.78966										4374246
17.07264	920.00656	160.3	.7869954	.80644										4215864
17.88753	920.82145	156.5	.8177750	.82310										4055278
18.71891	921.65283	152.1	.8487202	.83951										3893852
19.56652	922.50044	147.3	.8799954	.85579										3730726
20.43040	923.36432	143.1	.9124780	.87236										3561328
21.31117	924.24509	139.1	.9458436	.88906										3387352
22.20863	925.14255	135.2	.9799965	.90584										3209297
23.12289	926.05681	130.6	438.0142404	.92235										3030796
24.05338	926.98730	126.3	.0494278	.93901										2847406
25.00078	927.93470	122.5	.0857620	.95590										2658070
25.96522	928.89914	118.7	.1228110	.97263										2465042
26.94653	929.88045	115.3	.1609183	.98993										2266533
27.94508	930.87900													

$\theta_n - \theta_1 = VK\Delta t[936.90566369291] \times 10^{-10} = [983,515.0932][20,905.76299][.00002][936.9056637] \times 10^{-10} = .03852768474$

$S = R[\theta_n - \theta_1] = 152.5268273$ miles

APPENDIX C

SOLUTION BY AIR FORCE FORMULA OF PROBLEM BASED ON ATMOSPHERE OBSERVED OVER THE LINE FRAN TO WIND (CARIBBEAN AREA SHORAN PROJECT)

$$K = 60 \text{ feet}$$
$$H = 13{,}375.47 \text{ feet}$$
$$H - K = 13{,}315.47 \text{ feet}$$
$$S = V_s t = \left(186{,}219 \frac{\text{Mi}}{\text{Sec}}\right)(.00054 \text{ sec})$$
$$= 100.55826 \text{ miles}$$

Dial reading		100.55826
Velocity correction	+	.00184
Reduction to sea level	-	.03232
Slope correction	-	.03162
Curvature correction	+	.00253
Assumption VIII	+	.00005
Geodetic distance by Air Force Method		100.49874

APPENDIX C

Numerical integration solution of problem based on atmosphere observed over the line Fran to Wind (Caribbean Area Shoran Project)

$K = 20907.46626$ $\quad$ T = total time = .00054 seconds $\quad$ $\frac{d^2r}{dt^2} = \frac{V^2}{r\mu^2}\{(a+1) - \sin^2\alpha\,(2a+1)\}$

$K^2 = 437.1221454 \times 10^6$ $\quad$ Δt = .00002 seconds

h in thousands of feet	r	μ	$r^2\mu^2 \times 10^{-8}$	$q=\frac{V\Delta t \sin\alpha}{\mu}$	$\sin^2\alpha=1-\frac{K^2}{r^2\mu^2}$	$\sin\alpha$	$\mu^I x10^6$	$a=\frac{r\mu^I}{\mu}$	a^2	$a+1$	$2a+1$	$(a+1)$ $-\sin^2\alpha(2a+1)$	$\frac{\Delta t^2}{2}\frac{d^2r}{dt^2}$	$\frac{d\theta}{dt} \div VK$ $=\frac{1}{r^2\mu^2}x10^{10}$
.06000	20,902.99392	1,000,327,4	437.2213068	.296134	.000226799	.0150598	-1.96	-.0410	.0017	+.9590	+.9180	+.9588	.00887	22.87171244
.36500	903.29892	326,8	.2335416	.313869	.000254775	.0159617	-3.40	-.0710	.0050	+.9290	+.8580	+.9288	.00859	107243
.68746	903.62138	325,7	.2460698	.331044										041712
1.02686	903.96078	324,3	.2590444	.347937										22.86973849
1.38313	904.31705	322,1	.2720259	.364057										905955
1.75492	904.68884	318,7	.2846073	.379024										840157
2.14095	905.07487	314,7	.2972605	.393503										773987
2.54149	905.47541	310,2	.3100832	.407651										706935
2.95608	905.89000	305,8	.3235611	.422031										636356
3.38540	906.31932	301,1	.3374331	.436296										563931
3.82878	906.76270	296,2	.3516985	.450515										489348
4.28639	907.22031	291,0	.3662972	.464613										413028
4.75800	907.69192	285,7	.3813942	.478756										334109
5.24385	908.17777	280,0	.3967370	.492713										253910
5.74346	908.67738	274,6	.4129179	.507016										169336
6.25777	909.19169	268,1	.4287520	.520634										086581
6.78493	909.71885	258,2	.4421501	.531883										016562
7.32145	910.25537	228,5	.4386204	.528961										035008
7.84304	910.77696	215,4	.4489852	.537552										22.85980843
8.38969	911.32361	210,0	.4671335	.552267										886009
8.95187	911.88579	207,0	.4880311	.568736										776819
9.52957	912.46349	205,4	.5108033	.586153										657845
10.12483	913.05875	206,2	.5364104	.605136										524076
10.74011	913.67403	208,5	.5641686	.625061										379087
11.37553	914.30945	208,4	.5906705	.643506										240677
12.02764	914.96156	205,0	.6149842	.659976										113710
12.69503	915.62895	197,7	.6365246	.674234										001237
13.37547	916.30939													

$$\theta_n - \theta_1 = VK\Delta t[617.26942518 \times 10^{-10}]$$
$$= [983,515.0932][20907.46626][.00002][617.2694252] \times 10^{-10}$$
$$= .02538558612$$

$S = R[\theta_n - \theta_1] = 100.4987176$ miles

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