**Abstract**

Supply and demand in the World oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply-demand expectation process. In order to be able to understand the price balancing process, it is important to know the economic forces and the behavior of energy commodity spot price processes. The relationship between the different energy sources and its utility together with uncertainty also play a role in many important energy issues. The qualitative and quantitative behavior of energy commodities in which the trend in

**Subject Terms**

Delayed Volatility; Stochastic Hybrid System; Risk-Neutral Dynamics; Extended Kalman Filter; Local Lagged Adapted Generalized Method of Moments; Applications to the US Treasury Bill Yield Interest Rate, the US Eurocurrency Exchange Rate and Four Energy Commodity Data Sets
Stochastic Modeling and Analysis of Energy Commodity Spot Price Processes

ABSTRACT
Supply and demand in the World oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply-demand expectation process. In order to be able to understand the price balancing process, it is important to know the economic forces and the behavior of energy commodity spot price processes. The relationship between the different energy sources and its utility together with uncertainty also play a role in many important energy issues. The qualitative and quantitative behavior of energy commodities in which the trend in price of one commodity coincides with the trend in price of other commodities, have always raised the questions regarding their interactions. Moreover, if there is any interaction, then one would like to know the extent of influence on each other. In this work, we undertake the study to shed a light on the above highlighted processes and issues. The presented study systematically deals with the development of stochastic dynamic models and mathematical, statistical and computational analysis of energy commodity spot price and interaction processes.

Below we list the main components of the research carried out in this dissertation.

1. Employing basic economic principles, interconnected deterministic and stochastic models of linear log-spot and expected log-spot price processes coupled with non-linear volatility process are initiated.
2. Closed form solutions of the models are analyzed.
3. Introducing a change of probability measure, a risk-neutral interconnected stochastic model is derived.
4. Furthermore, under the risk-neutral measure, expectation of the square of volatility is reduced to a continuous-time deterministic delay differential equation.
5. The by-product of this exhibits the hereditary effects on the mean-square volatility process.
6. Using a numerical scheme, a time-series model is developed and utilized to estimate the state and parameters of the dynamic model. In fact, the developed time-series model includes the extended GARCH model as special case.
7. Using the Henry Hub natural gas data set, the usefulness of the linear interconnected stochastic models is outlined.

8. Using natural and basic economic ideas, interconnected deterministic and stochastic models in (1) are extended to non-linear log-spot price, expected log-spot price and volatility processes.
9. The presented extended models are validated.
10. Closed form solution and risk-neutral models of (8) are outlined.
11. To exhibit the usefulness of the non-linear interconnected stochastic model, to increase the efficiency and to reduce the magnitude of error, it was essential to develop a modified version of extended Kalman filtering approach. The modified approach exhibits the reduction of magnitude of error. Furthermore, Henry Hub natural gas data set is used to show the advantages of the non-linear interconnected stochastic model.

12. Parameter and state estimation problems of continuous time non-linear stochastic dynamic process is motivated to initiate an alternative innovative approach. This led to introduce the concept of statistic processes, namely, local sample mean and sample variance.
13. Then it led to the development of an interconnected discrete-time dynamic system of local statistic processes and its mathematical model.
14. This paved the way for developing an innovative approach referred as Local Lagged adapted Generalized Method of Moments (LLGMM). This approach exhibits the balance between model specification and model prescription of continuous time dynamic processes.
15. In addition, it motivated to initiate conceptual computational state and parameter estimation and simulation schemes that generates a mean square sub-optimal procedure.
16. The usefulness of this approach is illustrated by applying this technique to four energy commodity data sets, the U. S. Treasury Bill Yield Interest Rate and the U.S. Eurocurrency Exchange Rate data sets for state and parameter estimation problems.
17. Moreover, the forecasting and confidence-interval problems are also investigated.

18. The non-linear interconnected stochastic model (8) was further extended to multivariate interconnected energy commodities and sources with and without external random intervention processes.
19. Moreover, it was essential to extend the interconnected discrete-time dynamic system of local sample mean and variance processes to multivariate discrete-time dynamic system.
20. Extending the LLGMM approach in (15) to a multivariate interconnected stochastic dynamic model under intervention process, the parameters in the multivariate model are estimated. These estimated parameters help in analyzing the short and long term relationship between the energy commodities.
These developed results are applied to the Henry Hub natural gas, crude oil and coal data sets.
Stochastic Modeling and Analysis of Energy Commodity Spot Price Processes

by

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Dedication

To God.
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Real and Simulated Prices (with jump) for Natural gas, Crude oil, and Coal.

Real, Simulated, Forecasted Prices and 95% C.I. with no jump.

Real, Simulated, Forecasted Prices and 95% C.I with jump.
Abstract

Supply and demand in the World oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply-demand expectation process. In order to be able to understand the price balancing process, it is important to know the economic forces and the behavior of energy commodity spot price processes. The relationship between the different energy sources and its utility together with uncertainty also play a role in many important energy issues. The qualitative and quantitative behavior of energy commodities in which the trend in price of one commodity coincides with the trend in price of other commodities, have always raised the questions regarding their interactions. Moreover, if there is any interaction, then one would like to know the extent of influence on each other. In this work, we undertake the study to shed a light on the above highlighted processes and issues. The presented study systematically deals with the development of stochastic dynamic models and mathematical, statistical and computational analysis of energy commodity spot price and interaction processes.

Below we list the main components of the research carried out in this dissertation.

(1) Employing basic economic principles, interconnected deterministic and stochastic models of linear log-spot and expected log-spot price processes coupled with non-linear volatility process are initiated. (2) Closed form solutions of the models are analyzed. (3) Introducing a change of probability measure, a risk-neutral interconnected stochastic model is derived. (4) Furthermore, under the risk-neutral measure, expectation of the square of volatility is reduced to a continuous-time deterministic delay differential equation. (5) The by-product of this exhibits the hereditary effects on the mean-square volatility process. (6) Using a numerical scheme, a time-series model is developed and utilized to estimate the state and parameters of the dynamic model. In fact, the developed time-series model includes the extended GARCH model as special case. (7) Using the Henry Hub natural gas data set, the usefulness of the linear interconnected stochastic models is outlined.
Using natural and basic economic ideas, interconnected deterministic and stochastic models in (1) are extended to non-linear log-spot price, expected log-spot price and volatility processes. (9) The presented extended models are validated. (10) Closed form solution and risk-neutral models of (8) are outlined. (11) To exhibit the usefulness of the non-linear interconnected stochastic model, to increase the efficiency and to reduce the magnitude of error, it was essential to develop a modified version of extended Kalman filtering approach. The modified approach exhibits the reduction of magnitude of error. Furthermore, Henry Hub natural gas data set is used to show the advantages of the non-linear interconnected stochastic model.

(12) Parameter and state estimation problems of continuous time non-linear stochastic dynamic process is motivated to initiate an alternative innovative approach. This led to introduce the concept of statistic processes, namely, local sample mean and sample variance. (13) Then it led to the development of an interconnected discrete-time dynamic system of local statistic processes and (14) its mathematical model. (15) This paved the way for developing an innovative approach referred as Local Lagged adapted Generalized Method of Moments (LLGMM). This approach exhibits the balance between model specification and model prescription of continuous time dynamic processes. (16) In addition, it motivated to initiate conceptual computational state and parameter estimation and simulation schemes that generates a mean square sub-optimal procedure. (17) The usefulness of this approach is illustrated by applying this technique to four energy commodity data sets, the U. S. Treasury Bill Yield Interest Rate and the U.S. Eurocurrency Exchange Rate data sets for state and parameter estimation problems. (18) Moreover, the forecasting and confidence-interval problems are also investigated.

(19) The non-linear interconnected stochastic model (8) was further extended to multivariate interconnected energy commodities and sources with and without external random intervention processes. (20) Moreover, it was essential to extend the interconnected discrete-time dynamic system of local sample mean and variance processes to multivariate discrete-time dynamic system. (21) Extending the LLGMM approach in (15) to a multivariate interconnected stochastic dynamic model under intervention process, the parameters in the multivariate model are estimated. These estimated parameters help in analyzing the short and long term relationship between the energy commodities.

These developed results are applied to the Henry Hub natural gas, crude oil and coal data sets.
1.1 Introduction

In this chapter, we shall provide a number of basic definitions and important results which shall be used in later chapters.

1.2 General Notations

\begin{itemize}
  \item \text{i.e.} : that is.
  \item a.s : almost surely.
  \item \(G := H\) : G is defined by H or G is denoted by H.
  \item \(G(x) \equiv H(x)\) : \(G(x)\) and \(H(x)\) are identically equal.
  \item \(\emptyset\) : the empty set.
  \item \(G^T\) : the transpose of G.
  \item \(a \lor b\) : the maximum of a and b.
  \item \(f : A \rightarrow B\) : the mapping \(f\) from \(A\) to \(B\).
  \item \(\mathbb{Z}\) : set of integers
  \item \(I_a(b) = I(a,b)\) : the set \(\{x \in \mathbb{Z} : a \leq x \leq b\}\).
  \item \(\mathbb{R}\) : the real line.
  \item \(\mathbb{R}^n\) : the \(n\)-dimensional Euclidean space.
  \item \(\mathbb{R}^+\) : the set of all nonnegative real numbers \([0, \infty)\).
  \item \(\mathbb{R}_+\) : the set of all positive real numbers \((0, \infty)\).
  \item \(\mathbb{R}^{n \times m}\) : the space of real \(n \times m\)-matrices.
  \item \(\mathcal{C}\) : the family of all real-valued continuous functions.
\end{itemize}
\[ C_n : \text{the family of all real-valued functions } V(x) \text{ which are continuously } n\text{-times differentiable in } x. \]

\[ C_{n,m} : \text{the family of all real-valued functions } V(t, x) \text{ which are continuously } n\text{-times differentiable in } t \text{ and } m\text{-times differentiable in } x. \]

\[ a \neq b : \text{a is not equal to } b. \]

\[ a \in A : \text{a is an element of } A. \]

\[ \| A \| = \| A \|_2 : \text{the Euclidean norm of } A. \]

\[ trA = traceA : \text{the trace of a square matrix } A. \]

\[ det A : \text{determinant of square matrix } A. \]

\[ V_x : \nabla V = (V_{x_1}, ..., V_{x_n}) = \left( \frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n} \right). \]

\[ V_{xx} : (V_{x_i x_j})_{n \times n} = \left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{n \times n} \]

### 1.3 Stochastic Differential Equation

Given a n-dimensional stochastic process on \( t \geq t_0 \), a typical Itô-Doob type stochastic differential equation is given by

\[
dx = \mu(t,x)dt + \sigma(t,x)dW(t), \quad x(t_0) = x_0, \tag{1.1}
\]

where \( \mu : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n; \sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) and \( W(t) = (W_1(t), ..., W_m(t))^T \) is a standard Wiener process on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P}) \); the filtration function \( (\mathcal{F}_t)_{t \geq 0} \) is right-continuous, and each \( \mathcal{F}_t \) with \( t \geq 0 \) contains all \( \mathcal{P} \)-null sets in \( \mathcal{F}_t \). We say \( \mu \) is the drift coefficient while \( \sigma \) is the diffusion coefficient.

Next, we state the Itô-Doob Lemma.

**Theorem 1.1** Let \( u \) be a continuous map from \( \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( u(t, x) \) has continuous partial derivatives up to second order in \( x \) and to first order in \( t \), then the process \( u(t, x(t)) \) also has an Itô-Doob differential, and

\[
du(t, x) = Lu(t, x)dt + u_x(t, x)dW(t) \tag{1.2}
\]

where \( L \) is a differential operator defined by:

\[
Lu(t, x) = u_t(t, x) + u_x(t, x)\mu(t, x) + \frac{1}{2} \text{trace} \left( \sigma^T(t, u_t(t, x))u_{xx}(t, x)\sigma(t, u_t(t, x)) \right). \tag{1.3}
\]
The following theorem concerns the classical existence, uniqueness and certain other properties of the solution of (1.1).

**Theorem 1.2** [79] Assume that there exist two positive constants $K$ and $\overline{K}$ such that

- (Lipschitz condition): for all $x, y \in \mathbb{R}^n$ and $t \in [t_0, T]$
  \[
  \|\mu(t,x) - \mu(t,y)\|_2 \vee \|\sigma(t,x) - \sigma(t,y)\|_2 \leq K \|x - y\|_2;
  \]

- (Linear growth condition): for all $(t, x) \in [t_0, T] \times \mathbb{R}^n$
  \[
  \|\mu(t,x)\|_2 \vee \|\sigma(t,x)\|_2 \leq \overline{K} (1 + \|x\|_2),
  \]

where $\vee$ is the max symbol. Then there exists a unique solution $x(t)$ to (1.1).

In the following, we state a result that exhibits the existence of non-linear stochastic differential equations.

**Theorem 1.3** [[57], Thm 3.5] Suppose that the local solution of (1.1) exists on every cylinder $[t_0, \infty) \times U_n$, where $U_n = \{x \in \mathbb{R}^n : \|x\| < n\}$. Moreover, suppose that there exists a nonnegative function $V \in C_{1,2}$ such that for some constant $c > 0$

\[
\begin{cases}
  LV \leq cV \\
  V_n = \inf_{\|x\| > n} V(t,x) \to \infty \text{ as } n \to \infty,
\end{cases}
\]

where the $L$-operator is defined in (1.3). Then, for every random variable $x(t_0)$ independent of the process $W_i(t) - W_i(t_0)$, there exists a solution $x(t)$ of the system of stochastic differential equation (1.1) which is almost surely continuous stochastic process and is unique up to equivalence.

### 1.4 Behavior of Delayed Process

Consider a nonlinear delayed integro-differential equation of the form

\[
\frac{dv(t)}{dt} = cv(t) + \beta \int_{-\tau}^{0} v(t + s)ds.
\]

In order to find approximate solution representation, we need to investigate the behavior of (1.5). For this purpose, we present a result regarding its solution process. Our result is based on results of [64] and [62].
DEFINITION 1.4.1 A non-constant solution $v(t)$ of (1.5) is said to be

- **oscillatory** if $v(t)$ has arbitrary large number of zeros on $\mathbb{R}^+ = [0, \infty)$, that is, there exists an unbounded sequence $\{t_n \in \mathbb{R}^+\}$ such that $v(t_n) = 0$.

- **non-oscillatory** if $v(t)$ is not oscillatory, that is, there exist a positive number $T$ such that $v(t)$ is either positive or negative for all $t \geq T$.

Following the definition in [91],

DEFINITION 1.4.2 The stochastic integral with respect to Brownian motion $W(t)_{t \in \mathbb{R}^+}$ of any simple predictable process $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ of the form

$$u(t, w) = \sum_{i=1}^{n} F_i 1_{(t_{i-1}, t_i]}(t), \quad t \in \mathbb{R}^+,$$

is defined by

$$\int_{0}^{\infty} u(t) dW(t) = \sum_{i=1}^{n} F_i (W(t_i) - W(t_{i-1})),$$

where $F_i$ is an $\mathcal{F}_{t_{i-1}}$ measurable random variable for $i = 1, ..., n$, $u(t) \equiv u(t, w)$.

In the following, we state a result that exhibits the existence of solution of system of non linear equations. For the sake of easy reference, we shall state the Implicit function theorem without proof.

THEOREM 1.4 Implicit Function Theorem[2] Let $F = \{F_1, F_2, ..., F_q\}$ be a vector-valued function defined on an open set $S \in \mathbb{R}^{q+k}$ with values in $\mathbb{R}^q$. Suppose $F \in C_1$ on $S$. Let $(u_0; v_0)$ be a point in $S$ for which $F(u_0; v_0) = 0$ and for which the $q \times q$ determinant $\det [D_j F_i(u_0; v_0)] \neq 0$. Then there exists a $k-$ dimensional open set $T_0$ containing $v_0$ and unique vector-valued function $g$, defined on $T_0$ and having values in $\mathbb{R}^q$, such that $g \in C_1$ on $T_0$, $g(v_0) = u_0$, and $F(g(v); v) = 0$ for every $v \in T_0$. 


Chapter 2

Linear Stochastic Modeling of Energy Commodity Spot Price Processes with Delay in Volatility

2.1 Introduction

In real world situations, the expected spot price of energy commodities and its measure of variation are not constant. This is because of the fact that a spot price is subject to random environmental perturbation. Moreover, some statistical studies of stock price [8] raised the issue of market’s delayed response. This indeed causes the price to drift significantly away from the market quoted price. It is well recognized that time-delay models in economics [41, 56, 123] are more realistic than the models without time-delay. Discrete-time stochastic volatility models [9, 38] have been developed in economics. Recently, a survey paper by Hansen and Lunde [46] has estimated these types of models and concluded that the performance of the GARCH(1,1) is better than any other model. Furthermore, Cox-Ingersoll-Ross(CIR) developed a mean reverting interest rate model that was based on the mean-level interest rate as exponentially weighted integral of past history of interest rate and the relationship between level dependent volatility and the square root of the interest rate [19]. Employing the Ornstein Uhlenbeck [126] and Cox-Ingersoll-Ross(CIR) [19] processes, Heston developed a stochastic model for the volatility of stock spot asset. Recently [51], a continuous time stochastic volatility models have been generalized.

In this work, using basic economic principles, we systematically develop both deterministic and stochastic dynamic models for the log-spot price process. In addition, by treating a diffusion coefficient parameter in the non-seasonal log-spot price dynamic system as a stochastic volatility function of log-spot price, a stochastic model for interconnected system of log-spot price, expected log-spot price and hereditary volatility process is developed. Introducing a numerical scheme, a time-series model is developed and it is utilized to estimate the system parameters. The organization of this study is as follows:

In Section 2.2, we develop a stochastic interconnected models for energy commodity spot price and give an illustration by analyzing Henry Hub Natural gas daily Spot price from 1997 to 2011. In
Section 2.3, we obtain closed form solutions of the log of spot and the expected log of spot prices. In Section 2.4, by outlining the risk-neutral dynamics of price process, sufficient conditions are given to ensure that the risk-neutral dynamics of our model is equivalent to the developed model in Section 2.2. Furthermore, it is shown that the mean of the square of volatility under the risk-neutral measure is a deterministic continuous-time delay differential equation. In addition, sufficient conditions are also given to investigate both the oscillatory and non-oscillatory behavior of the expected value of square of volatility [62, 64].

2.2 Model Derivation

We denote $S(t)$ to be the spot price for a given energy commodity at a time $t$. Since the price of energy commodity are non-negative, to minimize ambiguity and for the sake of simplicity, it is expressed as an exponential function of the following form;

$$S(t) = \exp(x_2(t) + f(t)),$$  \hspace{1cm} (2.1)

where $x_2(t)$ stands for the nonseasonal log of the spot price at time $t$, $f(t)$ is the price at $t$ influenced by the seasonality and it is considered as a Fourier series comprising of linear combinations of sine and cosine functions;

$$f(t) = A_0 + \sum_{k=1}^{N} \left( A_k \cos \left( \frac{2\pi kt}{P} \right) + B_k \sin \left( \frac{2\pi kt}{P} \right) \right),$$ \hspace{1cm} (2.2)

where $P, A_0, A_k, B_k, k = 1...N$ are all constant parameters. $P$ is the period which represents the number of trading days in a year. Without loss in generality, we choose $N = 2$. By modeling the seasonal term this way, we are able to account for the peak season high price and off peak season low price of gas.

We present the dynamics for the spot price process.

2.2.1 Deterministic Non-Seasonal Log-Spot Price Dynamic Model

Under the basic economic principle of demand and supply processes, the price of a energy commodity will remain within a given finite upper bound. Let $\kappa > 0$ be the expected upper limit of $x_2(t)$.

In a real world situation, the nonseasonal log of spot price is governed by the spot price dynamic process. This leads to a development of dynamic model for the nonseasonal process $x_2(t)$. In
This case, \( \kappa - x_2(t) \) characterizes the market potential for \( x_2(t) \) per unit of time at a time \( t \). This market potential is influenced by the underlying market forces on the nonseasonal log of spot price, \( x_2(t) \). This leads to the following principle regarding the dynamic of non-seasonal log-spot price process of energy goods. The change in nonseasonal log of spot price of the energy commodity \( \Delta x_2(t) = x_2(t + \Delta t) - x_2(t) \) over the interval of length \( |\Delta t| \) is directly proportional to the market potential price.

\[
\Delta x_2(t) \propto (\kappa - x_2(t))\Delta t. \tag{2.3}
\]

This implies

\[
dx_2(t) = \gamma(\kappa - x_2(t))dt, \tag{2.4}
\]

where \( \gamma \) is a positive constant of proportionality, \( dx_2(t) \) and \( dt \) are differentials of \( x_2(t) \) and \( t \) respectively. From this mathematical model, we note that as the nonseasonal log price, \( x_2(t) \) fall below the expected price \( \kappa \), \( \kappa - x_2(t) \) is positive. Hence \( x_2(t) \) is increasing at the constant rate \( \gamma \) per unit size of \( \kappa - x_2(t) \) per unit time. On the other hand, if the nonseasonal log price \( x_2(t) \) is above the expected price \( \kappa \), then \( \kappa - x_2(t) \) is negative and hence \( x_2(t) \) decreases at the rate \( \gamma \) per unit size per unit time.

From (2.3), we note that the steady-state or equilibrium state nonseasonal log of spot price is given by

\[
x_2^* = \kappa. \tag{2.5}
\]

In the real world situation, the expected price of the nonseasonal log spot price \( \kappa \) is not a constant parameter. Therefore, we consider the expected nonseasonal log of spot price to be the mean of nonseasonal log spot price, \( x_2(t) \), at time \( t \) denoted by \( x_1(t) \). Under this assumption, (2.4) reduces to

\[
dx_2(t) = \gamma(x_1(t) - x_2(t))dt. \tag{2.6}
\]

Moreover, in order to preserve the equilibrium of nonseasonal log spot price \( (\kappa = x_2^*) \), we further assume that the mean of nonseasonal spot price process is operated under the principle described by (2.3).

\[
\Delta x_1(t) \propto (\kappa - x_1(t))\Delta t \tag{2.7}
\]

and hence

\[
dx_1(t) = \mu(\kappa - x_1(t))dt, \tag{2.8}
\]
where $\mu$ is a positive constant of proportionality. From (2.6) and (2.7), the mathematical model for the deterministic nonseasonal spot price process is described by the following system of differential equations:

$$
\begin{cases}
    dx_1(t) = \mu(\kappa - x_1(t))dt, \\
    dx_2(t) = \gamma(x_1(t) - x_2(t))dt.
\end{cases}
$$

(2.9)

### 2.2.2 Stochastic Non-Seasonal Log-Spot Price Dynamic Model

We note that in (2.3), $\kappa$ is not just the time-varying deterministic log of spot price, instead it is a stochastic process describing random environmental perturbations as follows:

$$
\kappa = x_1(t) + e_2(t)
$$

(2.10)

where $x_1(t)$ is the deterministic part and $e_2(t)$ is the stochastic white noise process. From this, (2.4) becomes

$$
\begin{align*}
    dx_2(t) &= \gamma(x_1(t) + e_2(t) - x_2(t))dt \\
          &= \gamma(x_1(t) - x_2(t))dt + \gamma e_2(t)dt \\
          &= \gamma(x_1(t) - x_2(t))dt + \sigma(t, x_2(t))dW_2(t).
\end{align*}
$$

(2.11)

where $\sigma(t, x_2(t))dW_2(t) = \gamma e_2(t)dt$ and $dW_2(t) \sim \mathcal{N}(0, dt)$.

Following the argument used in the derivation of (2.11), the dynamic from (2.7) reduces to

$$
dx_1(t) = \mu(\kappa - x_1(t))dt + \delta dW_1(t)
$$

(2.12)

where $\delta > 0$ is a constant and $dW_1(t) \sim \mathcal{N}(0, dt)$.

From (2.12) and (2.11), the mathematical model for the stochastic nonseasonal spot price process is described by the following system of differential equations:

$$
\begin{cases}
    dx_1 = \mu(\kappa - x_1(t))dt + \delta dW_1(t), \\
    dx_2 = \gamma(x_1 - x_2(t))dt + \sigma(t, x_2(t))dW_2(t).
\end{cases}
$$

(2.13)

### 2.2.3 Continuous Stochastic Volatility Model with Delay

When considering energy commodities, the measure of variation of the spot price under random environmental perturbation is not predictable, because it depends on nonseasonal log of spot price. Bernard and Thomas [8] in their work raised the issue of market’s delayed response. They observed changes in drift returns that leads to two possible explanations. First explanation suggests that a part of the price response to new information is delayed. The second explanation suggests that
researchers fail to adjust fully a raw return for risks, because the capital-asset-pricing model used to calculate the abnormal return is either incomplete or incorrect estimation. In this study, we incorporate the past history of nonseasonal log of spot price in the coefficient of diffusion parameter, that is, the volatility $\sigma(t, x_2(t))$ of the spot price follows the GARCH model \[140\]. It is assumed that the measure of variation of random environmental perturbations of $x_1(t)$ is constant. Under these assumptions, we propose an interconnected mean-reverting non-seasonal stochastic model for mean log-spot price, log-spot price, and volatility as follows:

$$dx_1 = \mu(\kappa - x_1)dt + \delta dW_1(t), \ x_1(t_0) = x_{01}$$
$$dx_2 = \gamma(x_1 - x_2)dt + \sigma(t, x_2)dW_2(t), \ x_2(t_0) = x_{02}$$
$$d\sigma^2(t, x_2) = \left[\alpha + \beta \left(\int_{t - \tau}^t \sigma(s, x_2)e^{-\gamma(t-s)}dW_2(s) + \delta \int_{t - \tau}^t \phi(s, t)dW_1(s)\right)^2 + c\sigma^2(t, x_2)\right]dt, \quad (2.14)$$

where

$$\phi(a, b) = \frac{\gamma}{\mu} \left( e^{-\gamma(b-a)} - e^{-\mu(b-a)} \right), \ \gamma, \mu \text{ are defined in } (2.4 \text{ and } 2.8), \ a, b \in \mathbb{R}. \quad (2.15)$$

For the sake of completeness, we assume the following

$${\bf H}_1: \ x_{2t}(\theta) = x_2(t + \theta), \ \theta \in [-\tau, 0], \ \gamma, \mu, \delta \in \mathbb{R}_+, \ \alpha, \beta, \ c \text{ are in } \mathbb{R}, \ (we \ will \ later \ show \ that \ -2 < c < 0), \ \sigma: [0, T] \times \mathcal{C} \to \mathbb{R} \text{ is a continuous mapping, } \mathcal{C} \text{ is the Banach space of continuous functions defined on } [-\tau, 0] \text{ into } \mathbb{R} \text{ and equipped with the supremum norm; } W_1(t) \text{ and } W_2(t) \text{ are standard Wiener process defined on a filtered probability space } (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P}), \text{ the filtration function } (\mathcal{F})_{t \geq 0} \text{ is right-continuous, and for } t \geq 0, \mathcal{F}_t \text{ contains all } \mathcal{P}-\text{null sets. We know that system } (2.14) \text{ can be re-written as}$$

$$d\mathbf{x} = [\mathbf{A} \mathbf{x} + \mathbf{p}] dt + \sum(t, x_2) \ d\mathbf{W}(t), \ \mathbf{x}(t_0) = \mathbf{x}_0, \quad (2.16)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} -\mu & 0 \\ \gamma & -\gamma \end{bmatrix}, \ \mathbf{p}(t) = \begin{bmatrix} \mu \kappa \\ 0 \end{bmatrix}, \ \sum(t, x_2(t)) = \begin{bmatrix} \delta & 0 \\ 0 & \sigma(t, x_2(t)) \end{bmatrix},$$

$$\mathbf{W}(t) = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \ \mathbf{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}. \quad (2.17)$$
Moreover, (2.16) can be considered as a system of nonlinear Itô-Doob type stochastic perturbed system of the following deterministic linear system of differential equations

$$dx = Ax \, dt.$$  \hspace{1cm} (2.17)

In the following, we present an illustration to justify the structure of log spot price dynamic model.

2.2.4 Illustration

We present an illustrate the above described interconnected stochastic dynamic model for non-seasonal log spot price of energy commodity under the influence of random perturbations on mean-level and delayed volatility.

We consider the Henry Hub Natural Gas Daily spot price from 1997 to 2011.

![Figure 1.: Plot of Henry Hub Daily Natural Gas Spot Prices, 1997-2011](image.png)

We can clearly see that

- Prices appear as being randomly driven and clearly non-negative
- There is a tendency of spot prices to move back to their long term level (mean reversion).
- There are sudden large changes in spot prices (jumps/spikes).
- There is an unpredictability of spot price volatility.

A summary of the statistics is presented in Table 1 below. We find that $\ln \left( \frac{S(t+1)}{S(t)} \right)$ has the smallest variance. Thus, it suggests a good candidate for our modeling. Hence, we use the logarithmic price, rather than the raw price data for our model.
Table 1: Descriptive statistics of Henry Hub daily natural gas spot prices, 1997-2010

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_t$</td>
<td>4.9519</td>
<td>2.4966</td>
<td>1.0391</td>
<td>4.3491</td>
<td>1.05</td>
<td>18.48</td>
</tr>
<tr>
<td>$S_{t+1} - S_t$</td>
<td>-0.0001142</td>
<td>0.3189</td>
<td>-0.7735</td>
<td>191.8911</td>
<td>-8.01</td>
<td>6.50</td>
</tr>
<tr>
<td>$ln(S_t)$</td>
<td>1.4754</td>
<td>0.5048</td>
<td>-0.0465</td>
<td>2.1540</td>
<td>0.0488</td>
<td>2.9167</td>
</tr>
<tr>
<td>$ln(S_{t+1}/S_t)$</td>
<td>2.8485e-5</td>
<td>0.0473</td>
<td>0.4814</td>
<td>22.0473</td>
<td>-0.56</td>
<td>0.5657</td>
</tr>
</tbody>
</table>

2.3 Closed Form Solution

In this section, we find the solution representation of (2.16) in terms of the solution of unperturbed system of differential deterministic (2.17). This is achieved by employing method of variation of constants parameter [70].

**THEOREM 2.1 (Closed Form Solution)**

Let $x(t) = x(t, t_0; x_0)$ and $y(t, t_0; x_0) = \Phi(t, t_0)x_0$ be the solutions of the perturbed and unperturbed system of differential equations (2.16) and (2.17) respectively. Then

$$x(t) = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0, t) & e^{-\gamma(t-t_0)} \end{bmatrix} x_0 + \begin{bmatrix} \kappa (1 - e^{-\mu(t-t_0)}) \\ \omega(t_0, t) \end{bmatrix}$$

$$+ \int_{t_0}^{t} \begin{bmatrix} \delta e^{-\mu(t-s)}dW_1(s) \\ \delta \phi(s, t)dW_1(s) + \sigma(s, x_2(s))e^{-\gamma(t-s)}dW_2(s) \end{bmatrix},$$

(2.18)

where

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix},$$

(2.19)

$$\omega(a, b) = \kappa \left( [1 - e^{-\gamma(b-a)}] - \phi(a, b) \right), \quad a, b \in \mathbb{R}. \quad (2.20)$$

and $\phi$ is defined in (2.15); the fundamental solution, $\Phi(t)$ of (2.17) is given by

$$\Phi(t, t_0) = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0, t) & e^{-\gamma(t-t_0)} \end{bmatrix}$$

(2.21)

**Proof.** The result follows by imitating the eigenvalue type method described in [69, 70]. Therefore

$$x_1(t) = e^{-\mu(t-t_0)}x_{01} + \kappa \left( 1 - e^{-\mu(t-t_0)} \right) + \delta \int_{t_0}^{t} e^{-\mu(t-s)}dW_1(s),$$

(2.22)
\[ x_2(t) = \phi(t_0, t)x_{01} + e^{-\gamma(t-t_0)}x_{02} + \omega(t_0, t) + \delta \int_{t_0}^{t} \phi(s, t) dW_1(s) + \int_{t_0}^{t} \sigma(s, x_2(s)) e^{-\gamma(t-s)} dW_2(s). \tag{2.23} \]

In the following, we present the statistical properties of the solutions (2.22) and (2.23).

**Theorem 2.2** Under the hypothesis of Theorem 2.1, we have

\[
\mathbb{E}[x(t)] = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0, t) & e^{-\gamma(t-t_0)} \end{bmatrix} x_0 + \begin{bmatrix} \kappa (1 - e^{-\mu(t-t_0)}) \\ \omega(t_0, t) \end{bmatrix},
\]

\[
\text{var}[x(t)] = \begin{bmatrix} \delta^2 \int_{t_0}^{t} e^{-2\mu(t-s)} ds \\ \int_{t_0}^{t} \mathbb{E}(\sigma^2(s, x_2(s))) e^{-2\gamma(t-s)} ds + \delta^2 \int_{t_0}^{t} \phi^2(s, t) ds \end{bmatrix}.
\]

Moreover,

\[
\lim_{t \to \infty} \mathbb{E}[x(t)] = \begin{bmatrix} \kappa \\ \kappa \end{bmatrix},
\]

\[
\lim_{t \to \infty} \text{var}[x(t)] = \begin{bmatrix} \frac{\delta^2}{2\mu} \\ \lim_{t \to \infty} \frac{\mathbb{E}(\sigma^2(t, x_2(t)))}{2\gamma} + \frac{\delta^2}{2\mu} \left[ \frac{\gamma}{\mu + \gamma} \right] \end{bmatrix}.
\]

Hence,

\[
\lim_{t \to \infty} \mathbb{E}[x_1(t)] = \lim_{t \to \infty} \mathbb{E}[x_2(t)] = \kappa,
\]

\[
\lim_{t \to \infty} \text{var}(x_1(t)) = \frac{\delta^2}{2\mu}.
\]

**Proof.** From (2.18), we observe that

\[
\mathbb{E}[x(t)] = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0, t) & e^{-\gamma(t-t_0)} \end{bmatrix} x_0 + \begin{bmatrix} \kappa (1 - e^{-\mu(t-t_0)}) \\ \omega(t_0, t) \end{bmatrix}.
\]

Hence,

\[
\mathbb{E}(x_1(t)) = x_{01} e^{-\mu(t-t_0)} + \kappa \left( 1 - e^{-\mu(t-t_0)} \right)
\]

\[
\mathbb{E}(x_2(t)) = x_{02} e^{-\gamma(t-t_0)} + \phi(t_0, t)x_{01} + \omega(t_0, t)
\]

\[
\text{var}(x_1(t)) = \delta^2 \int_{t_0}^{t} e^{-2\mu(t-s)} ds = \frac{\delta^2}{2\mu} \left[ 1 - e^{-2\mu(t-t_0)} \right]
\]

\[
\text{var}(x_2(t)) = \int_{t_0}^{t} \mathbb{E}(\sigma^2(s, x_2(s))) e^{-2\gamma(t-s)} ds + \delta^2 \int_{t_0}^{t} \phi^2(s, t) ds.
\]

The result follows by taking the limits as \( t \to \infty. \)  \( \square \)
From Theorem 2.2, we observe that on the long-run, the mean-level of $x_2(t)$ and $x_1(t)$ are the same and it is given by $\kappa$.

### 2.4 Risk-Neutral Dynamics and Pricing

In order to minimize the risk of usage of mathematical model \((2.16)\), we incorporate the risk neutral measure. From the dynamic nature of \((2.16)\), it is known \([20]\) that this model has affine multi-factor structure. In the following, we present a risk neutral measure induced by this type of model. This indeed leads to a risk neutral dynamic model with respect to \((2.16)\). Christa Cuchiero, \([20]\), showed in their work that the market price of risk $\Theta(t) = (\Theta_1(t), \ldots, \Theta_n(t))$ with respect to the stochastic differential equation \((1.1)\) is given by

\[
\Theta_i(t) = \frac{n(t,x_i) - r(t)}{P(t,T)} P(t,T), \quad i = 1, 2, \ldots, n, \tag{2.24}
\]

where $P(t, T) = G(t, x)$ is the zero-coupon bond price, $r(t)$ is the short-term rate factor for the risk-free borrowing or lending at time $t$ over the interval $[t, t + dt]$, and $\eta(t, x_i), \zeta(t, x)$ are defined by

\[
\eta(t, x) = LG(t, x), \quad \zeta(t, x) = \frac{\partial G(t, x)}{\partial x} \sigma(t, x), \tag{2.25}
\]

where $\frac{\partial G(t, x)}{\partial x}$ is the gradient of $G$, and the $L$-operator is defined in \((1.3)\).

In fact, since our price model $X(t) = (x_1(t), x_2(t))^T \ (2.16)$ is an affine multi-factor model, the short-term rate factor $r(t)$ and the zero-coupon bond price $P(t, T)$ can be represented as

\[
r(t) = g + hX(t) \tag{2.26}
\]

\[
P(t, T) = \exp \left( a(t, T) + B(t, T)X(t) \right),
\]

where $g \in \mathbb{R}$, $h \in \mathbb{R}^2$, $a(t, T)$ and $B(t, T) = (B_1(t, T), B_2(t, T), \ldots, B_n(t, T))$ are arbitrary smooth functions. For $n = 2$, from \((2.24)\) and \((2.25)\), the market price of risk $\Theta(t) = (\theta_1(t), \theta_2(t))$ is given by

\[
\Theta(t) = a + b(t)X(t), \tag{2.27}
\]
where

\[
\begin{align*}
\mathbf{a} &= \begin{bmatrix} a_{1,0} \\ a_{2,0} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\
a_{2,0}(t) &= \frac{1}{2} \mathcal{B}_2^2(t,T) \sigma^2(t,\mathbf{x}) + \frac{\mathrm{d}a(t,T)}{dt} - g, \\
a_{2,1}(t) &= \gamma \mathcal{B}_2(t,T) - h_1, \\
a_{2,2}(t) &= -\gamma \mathcal{B}_2(t,T) - h_2, \\
a_{1,0}(t) &= \frac{1}{2} \mathcal{B}_1^2(t,T) \delta^2 + \mu \kappa \mathcal{B}_1(t,T) + \frac{\mathrm{d}a(t,T)}{dt} - g, \\
a_{1,1}(t) &= -\mu \mathcal{B}_1(t,T) - h_1, \\
a_{1,2}(t) &= 0.
\end{align*}
\]

We incorporate a market price of risk process that gives a risk-neutral dynamics of the same class as (2.16) in the following lemma.

**Lemma 2.1** Let us assume that \( a \) and \( \mathcal{B}_i \), \( i = 1, 2 \) in (2.26) are arbitrary constants. The market price of risk processes reduces to:

\[
\begin{align*}
\theta_1(t) &= a_{1,0} + a_{1,1}x_1(t) + a_{1,2}x_2(t) \tag{2.28} \\
\theta_2(t) &= a_{2,0}(t) + a_{2,1}(t)x_1(t) + a_{2,2}(t)x_2(t). \tag{2.29}
\end{align*}
\]

In addition, let us assume that \( \theta_i \), \( i = 1, 2 \) satisfy the Novikov’s condition [108] with the \( \bar{P} \)-Wiener process:

\[
\begin{align*}
\bar{W}_1(t) &= W_1(t) + \int_{t_0}^{t} \theta_1(u) du \\
\bar{W}_2(t) &= W_2(t) + \int_{t_0}^{t} \theta_2(u) du.
\end{align*}
\]

and

\[
\mathbf{C}_1 : \begin{cases} 
    h_1 + h_2 &= 0, \\
    a_{2,0}(t) &= a_{1,2} = 0, \\
    \bar{\gamma} &= \frac{h_1}{\mathcal{B}_2}, \quad \bar{\mu} = \frac{h_2}{\mathcal{B}_1}, \\
    \bar{\mu} \bar{\kappa} &= \mu \kappa - \delta a_{1,0},
\end{cases}
\]

where \( h_1, h_2 \) are arbitrary real numbers; \( \mu, \kappa \) and \( \delta \) are defined in (2.14), \( \theta_i, a_{i,j}, i = 1, 2, j = 0, 1, 2 \) are defined in (2.27).
Then the risk-neutral dynamics of \( x_1(t) \) and \( x_2(t) \) remain within the same class,

\[
dx = [\bar{A} x + \bar{p}] \, dt + \sum (t, x_2) \, dW(t), \quad x(t_0) = x_0
\]  

(2.32)
satisfying \( \textbf{H}_1 \), where

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -\bar{\mu} & 0 \\ \tilde{\gamma}_1 & -\bar{\gamma} \end{bmatrix}, \quad \bar{p}(t) = \begin{bmatrix} \bar{\mu} \bar{\kappa} \\ 0 \end{bmatrix}, \quad \sum (t, x_2(t)) = \begin{bmatrix} \delta & 0 \\ 0 & \sigma(t, x_2(t)) \end{bmatrix},
\]

\[
\bar{W}(t) = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.
\]

Moreover, it satisfies Hypothesis \( \textbf{H}_1 \). Hence,

\[
dx_1 = \bar{\mu}(\bar{\kappa} - x_1) dt + \delta d\bar{W}_1(t),
\]

\[
dx_2 = \tilde{\gamma} (x_1 - x_2) dt + \sigma(t, x_2) d\bar{W}_2(t).
\]  

(2.33)

Proof. The proof follows by substituting (2.30) and \( \bf{C}_1 \) into (2.16). \( \square \)

**Remark 2** Under the assumption of Lemma 2.1, it is obvious that the solution to (2.32) is given by

\[
x(t) = \begin{bmatrix} e^{-\bar{\mu}(t-t_0)} & 0 \\ \tilde{\phi}(t_0, t) & e^{-\tilde{\gamma}(t-t_0)} \end{bmatrix} x_0 + \begin{bmatrix} \bar{\kappa} \left(1 - e^{-\bar{\mu}(t-t_0)}\right) \\ \bar{\omega}(t_0, t) \end{bmatrix} + \int_{t_0}^t \begin{bmatrix} \delta e^{-\bar{\mu}(t-s)} dW_1(s) \\ \delta \tilde{\phi}(s, t) dW_1(s) + \sigma(s, x_2(s)) e^{-\tilde{\gamma}(t-s)} dW_2(s) \end{bmatrix},
\]

(2.34)

\[
x(t_0) = x_0,
\]

where \( \tilde{\phi} \) and \( \bar{\omega} \) are defined as

\[
\tilde{\phi}(a, b) = \frac{\gamma}{\mu - \gamma} \left( e^{-\gamma(b-a)} - e^{-\bar{\mu}(b-a)} \right)
\]

\[
\bar{\omega}(a, b) = \bar{\kappa} \left( [1 - e^{-\gamma(b-a)}] - \tilde{\phi}(a, b) \right).
\]

(2.35)

In the following, we state a result with regards to (2.34).

**Lemma 2.2** Under the assumption \( \textbf{H}_1 \), (2.34) is equivalent to

\[
x(t) = \begin{bmatrix} e^{-\bar{\mu} \tau} & 0 \\ \tilde{\phi}(0, \tau) & e^{-\tilde{\gamma} \tau} \end{bmatrix} x(t - \tau) + \begin{bmatrix} \bar{\kappa} \left(1 - e^{-\bar{\mu} \tau}\right) \\ \bar{\omega}(0, \tau) \end{bmatrix} + \int_{t-\tau}^t \begin{bmatrix} \delta e^{-\bar{\mu}(t-s)} dW_1(s) \\ \delta \tilde{\phi}(s, t) dW_1(s) + \sigma(s, x_2(s)) e^{-\tilde{\gamma}(t-s)} dW_2(s) \end{bmatrix}.
\]

(2.36)
where $x(t - \tau) = \begin{bmatrix} x_1(t - \tau) \\ x_2(t - \tau) \end{bmatrix}$.

Proof. The proof follows by changing the initial time $t_0$ in (2.34) to $t - \tau$. □

Hence,

\begin{equation}
\begin{aligned}
x_1(t) &= x_1(t - \tau) e^{\bar{\mu} \tau} + \bar{\kappa}(1 - e^{-\bar{\mu} \tau}) + \delta \int_{t-\tau}^{t} e^{-\bar{\mu}(t-s)} d\bar{W}_1(s), \\
\tag{2.37}
x_2(t) &= x_2(t - \tau) e^{-\bar{\gamma} \tau} + \bar{\phi}(0, \tau) x_1(t - \tau) + \bar{\omega}(0, \tau) + \bar{\Delta} \int_{t-\tau}^{t} \sigma(s, x_2(s)) e^{-\bar{\gamma}(t-s)} d\bar{W}_2(s) + \\
&\quad + \delta \int_{t-\tau}^{t} \phi(u, t) dW_1(u).
\end{aligned}
\end{equation}

\begin{rema}
From (2.38), we have
\begin{equation}
\begin{aligned}
\int_{t-\tau}^{t} \sigma(s, x_2(s)) e^{-\bar{\gamma}(t-s)} d\bar{W}_2(s) + \delta \int_{t-\tau}^{t} \phi(u, t) dW_1(u) &= x_2(t) - x_2(t - \tau) e^{-\bar{\gamma} \tau} \\
&\quad - x_1(t - \tau) \bar{\phi}(0, \tau) - \bar{\omega}(0, \tau).
\end{aligned}
\tag{2.39}
\end{equation}
\end{rema}

The dynamics of volatility process under risk-neutral dynamic system is described by

\begin{equation}
\begin{aligned}
d\sigma^2(t, x_2) &= \left[ \alpha + \beta \left[ \int_{t-\tau}^{t} \sigma(s, x_2) e^{-\bar{\gamma}(t-s)} d\bar{W}_2(s) + \delta \int_{t-\tau}^{t} \phi(s, t) dW_1(s) \right] \right]^2 \\
&\quad + c \sigma^2(t, x_2) \, dt. \\
\tag{2.40}
\end{aligned}
\end{equation}

We set
\begin{equation}
u(t) = E_{\bar{\mathcal{P}}}[\sigma^2(t, x_2(t))]. \tag{2.42}\end{equation}

Taking the conditional expectation of both sides under the measure $\bar{\mathcal{P}}$, we obtain the following deterministic delay differential equation

\begin{equation}
\begin{aligned}
\frac{du(t)}{dt} &= \alpha + \beta \delta^2 \int_{t-\tau}^{t} \bar{\phi}(s, t)^2 ds + \beta \int_{t-\tau}^{t} u(s) e^{-2\bar{\gamma}(t-s)} ds + cu(t) \\
&= \alpha + \beta \delta^2 D + \beta \int_{t-\tau}^{t} u(s) e^{-2\bar{\gamma}(t-s)} ds + cu(t)
\end{aligned}
\end{equation}

where
\begin{equation}
D = \int_{t-\tau}^{t} \bar{\phi}(s, t)^2 ds \\
= \left( \frac{\bar{\gamma}}{\bar{\mu} - \bar{\gamma}} \right)^2 \left[ \frac{1}{2\bar{\gamma}} (1 - e^{-2\bar{\gamma} \tau}) - \frac{2}{\bar{\mu} + \bar{\gamma}} (1 - e^{(\bar{\mu} + \bar{\gamma}) \tau}) + \frac{1}{2\bar{\mu}} (1 - e^{-2\bar{\mu} \tau}) \right]
\end{equation}

Hence
\begin{equation}
\frac{du(t)}{dt} = cu(t) + \beta \int_{t-\tau}^{t} u(s) e^{-2\bar{\gamma}(t-s)} ds + \nu, \tag{2.43}
\end{equation}

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where
\[ \nu = \alpha + \beta \delta^2 D. \]

**Remark 4** The equilibrium solution process \( u^*(t) \) of (2.43) satisfies the following integral equation
\[ cu^*(t) + \beta \int_{t-\tau}^{t} u^*(s)e^{-2\gamma(t-s)}ds + \nu = 0, \]  
since \( \frac{du^*(t)}{dt} = 0 \). In particular, \( u^*(t) \) is as follows;
\[ u^*(t) = -\left[ \frac{\nu}{c + \frac{\beta}{2\gamma}(1 - e^{-2\gamma\tau})} \right]. \]

Using the transformation
\[ v(t) = u(t) - u^*(t) \]  
we have
\[ \frac{dv(t)}{dt} = cv(t) + \beta \int_{t-\tau}^{t} v(s)e^{-2\gamma(t-s)}ds + \left[ cu^*(t) + \beta \int_{t-\tau}^{t} u^*(s)e^{-2\gamma(t-s)}ds + \nu \right] \]
\[ = cv(t) + \beta \int_{t-\tau}^{t} v(s)e^{-2\gamma(t-s)}ds. \]
Hence,
\[ \frac{dv(t)}{dt} = cv(t) + \beta \int_{-\tau}^{0} v(t+s)e^{2\gamma s}ds. \]  
(2.46)

In order to find approximate solution representation, we need to investigate the behavior of (2.46). For this purpose, we present a result regarding its solution process. Our result is based on results of [62] and [64]. Using definition 1.4.1, we prove the following Lemma using the definition of oscillatory and non-oscillatory solution of (2.46).

**Lemma 2.3** Under the following transformation
\[ v(t) = e^{ct}z(t), \]  
(2.47)

(2.46) is equivalent to
\[ z'(t) = \beta \int_{-\tau}^{0} e^{(c+2\gamma)s}z(t+s)ds. \]  
(2.48)

Moreover,
\begin{itemize}
  \item (i) for \( \beta < 0 \) and \( \frac{\beta \tau}{c+2\gamma} \left[ e^{-(c+2\gamma)\tau} - 1 \right] \leq \frac{1}{e} \), every solution of (2.46) is non-oscillatory.
\end{itemize}
• (ii) for $\beta < 0$ and $\frac{\beta \tau}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\iota} \right] > \frac{1}{e}$, $\iota \in (0, \tau)$ every solution of (2.46) oscillates.

• (iii) for $\beta > 0$, (2.46) has non-oscillatory solutions.

Proof. To prove (i), suppose that

$$\frac{\beta \tau}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\iota} \right] \leq \frac{1}{e}, \beta < 0,$$(2.49)

We observe that every solution of (2.46) is non-oscillatory if and only if every solution (2.48) is non-oscillatory. Therefore, we only need to show that (2.48) has non-oscillatory solution.

Suppose that a solution of (2.48) has the form

$$z(t) = e^{\lambda t}$$ (2.50)

where $\lambda$ is an arbitrary constant which satisfies the equation

$$\lambda = \beta \int_{-\tau}^{0} e^{(c+2\gamma)s} ds.$$ (2.51)

Define

$$G(\lambda) = \lambda - \beta \int_{-\tau}^{0} e^{(c+2\gamma)s} ds.$$ (2.52)

We show that $G(\lambda)$ has at least one real root. From (2.49), (2.50) and nature of $\beta$, we note that $G(0) > 0$ and for any $s \in [-\tau, 0]$,

$$G(\lambda) \leq \lambda - \beta e^{-\lambda \tau} \int_{-\tau}^{0} e^{(c+2\gamma)s} ds$$
$$= -\beta \int_{-\tau}^{0} e^{(c+2\gamma)s} ds \left[-e + \exp \left[-e\tau \beta \int_{-\tau}^{0} e^{(c+2\gamma)s} ds\right]\right]$$
$$\leq 0 \text{ by (2.49).}$$

Therefore, (2.51) has at least one real root $\lambda^*$ that lies between $\beta \int_{-\tau}^{0} e^{(c+2\gamma)s} ds$ and 0, showing that (2.46) has non-oscillatory solution. □

Next, we outline the proof for Lemma 5 (ii)

Proof. Suppose

$$\frac{\beta \iota}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\iota} \right] > \frac{1}{e}, \beta < 0, \text{ for any } \iota \in (0, \tau).$$ (2.53)

We only need to show that (2.48) oscillates. To verify this, suppose on the contrary that $z(t)$ is a non-oscillatory solution of (2.48). Then for sufficiently large $t_0 > 0$ and without loss in generality,
Define Case 1

Define $w$

Set $w$

Hence, for any $s \in (-\tau, -\tau)$, $t - \tau < t + s < t - \tau < t$, (2.54) yields

$$z(t) < z(t - \tau) < z(t + s).$$

Define

$$w(t) = \frac{z(t - \tau)}{z(t)}, \quad t \geq t_1.$$  

Note that $w(t) > 1$. Dividing (2.54) by $z(t)$ and using (2.55), we have

$$\frac{z'(t)}{z(t)} - \frac{\beta}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\tau} \right] w(t) < \frac{z'(t)}{z(t)} - \frac{\beta}{c + 2\gamma} \int_{-\tau}^{t} e^{(c+2\gamma)s} z(t + s) ds \leq 0.$$

Integrating from $t - \tau$ to $t$, for $t \geq t_1$,

$$\log z(t) - \log z(t - \tau) - \frac{\beta}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\tau} \right] \int_{t-\tau}^{t} w(s) ds \leq 0,$$

and hence

$$\log w(t) \geq \frac{\beta}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\tau} \right] \int_{t-\tau}^{t} w(s) ds, \quad t \geq t_1$$

Set

$$\lim_{t \to \infty} \inf w(t) = K.$$  

Since $w(t) > 1$, $K \geq 1$, hence $K$ is either finite or infinite. We show next that none of these cases is true.

Case 1. Assume $K$ is finite. There exist sequence $\{t_n\}$, $t_n \geq t_1 \ni t_n \to \infty$ and $w(t_n) \to K$ as $n \to \infty$. By integral mean value theorem, $\exists c_n \in (t_n - \tau, t_n)$ such that

$$\log w(t_n) \geq \frac{\beta t_n}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\tau} \right] w(c_n).$$

Define $K_1 = \lim_{n \to \infty} w(c_n)$.

Noting that $K_1 \geq K$ and taking limits of (2.59), we have

$$\frac{\log K}{K} \geq \frac{\beta t}{c + 2\gamma} \left[ e^{-(c+2\gamma)\tau} - e^{-(c+2\gamma)\tau} \right].$$

Since

$$\max_{K \geq 1} \frac{\log K}{K} = \frac{1}{e},$$

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the relation (2.60) implies
\[
\frac{\beta t}{c + 2\gamma} \left[ e^{-(c+2\gamma)t} - e^{-(c+2\gamma)i} \right] \leq \frac{1}{e}
\]  
(2.62)
which contradicts (2.53). Hence \( K \) is not finite.

Case 2. Assume that \( K \) is infinite, from (2.56) and (2.58), we have
\[
\lim_{t \to \infty} \left[ \frac{z(t - \iota)}{z(t)} \right] = \infty.
\]  
(2.63)
Choose \( t_* = t - \alpha, \alpha > 0 \), such that \( t - \iota < t_* < t \) for \( t \geq t_1 \). Integrating both sides of (2.54) from \( t_* \) to \( t \) and \( t - \iota \) to \( t_* \), we have
\[
z(t) - z(t_*) - \beta \int_{-\tau}^{-t} e^{(c+2\gamma)s} \left[ \int_{t_*}^{t} z(u + s) \, du \right] ds \leq 0, \ t \geq t_1
\]  
(2.64)
\[
z(t_*) - z(t - \iota) - \beta \int_{-\tau}^{-t} e^{(c+2\gamma)s} \left[ \int_{t_*}^{t} z(u + s) \, du \right] ds \leq 0, \ t \geq t_1
\]  
(2.65)
respectively. We observe that for any \( u \in [t_*, t], s \in [-\tau, -t], u + s < t + s < t - \iota \), hence, \( z(t - \iota) < z(t + s) < z(u + s) \), and for any \( u \in [t - \iota, t_*], u + s < t + s < t_* - \iota \), hence \( z(t_* - \iota) < z(t_* + s) < z(u + s) \). Hence (2.64) and (2.65) become
\[
z(t) + z(t - \iota) \frac{\beta \alpha}{c + 2\gamma} \left[ e^{-(c+2\gamma)t} - e^{-(c+2\gamma)i} \right] \leq z(t_*), \ t \geq t_1
\]  
(2.66)
\[
z(t_*) + z(t_* - \iota) \frac{\beta(t - \alpha)}{c + 2\gamma} \left[ e^{-(c+2\gamma)t} - e^{-(c+2\gamma)i} \right] \leq z(t - \iota), \ t \geq t_1.
\]  
(2.67)
Dividing (2.66) and (2.67) by \( z(t) \) and \( z(t_*) \) respectively, and using (2.53) and (2.63), we have
\[
\lim_{t \to \infty} \frac{z(t_*)}{z(t)} = \lim_{t_* \to \infty} \frac{z(t_* - \iota)}{z(t_*)} = \infty.
\]  
(2.68)
Dividing (2.66) by \( z(t_*) \) we have
\[
\frac{z(t)}{z(t_*)} + \frac{z(t - \iota)}{z(t_*)} \frac{\beta \alpha}{c + 2\gamma} \left[ e^{-(c+2\gamma)t} - e^{-(c+2\gamma)i} \right] \leq 1, \ t \geq t_1
\]  
(2.69)
which contradicts (2.68) and (2.53).

The proof of Lemma 5(iii) is similar to that of 5(i).

Following Lemma 2.3, the delayed differential equation (2.43) has a non-oscillatory solution if \( \beta < 0 \) and \( \frac{\beta \tau}{c + 2\gamma} \left[ e^{-(c+2\gamma)t} - 1 \right] \leq \frac{1}{e} \). Under these condition, we can describe the asymptotic behaviors of solutions of (2.43). Moreover, we seek a solution in the form \( u(t) = \psi_1 + \psi_2 e^{\rho t} \),
where $\psi_1, \psi_2$ and $\rho$ are arbitrary constants. In this case, the characteristic equation with respect to (2.43) is

$$h(\rho) = \rho - c - \beta \left[ \frac{1 - e^{-(\rho+2\bar{\gamma})}}{\rho + 2\bar{\gamma}} \right] = 0. \quad (2.70)$$

From $u(t_0) = u_0$, we obtain

$$\psi_1 = u^* = -\left[ \frac{\nu}{c + \frac{\mu}{2\bar{\gamma}}(1 - e^{-2\bar{\gamma} \tau})} \right],$$

$$\psi_2 = (u_0 - \psi_1)e^{-\rho t_0}. \quad (2.71)$$

However, using numerical simulation for (2.43), we observe that $u(t)$ is asymptotically stable. From (2.46) and (4.43), the numerical scheme is defined as follows;

$$v_i = (1 + c\Delta t + \beta(\Delta t)^2e^{-2\bar{\gamma}})v_{i-1} + \beta(\Delta t)^2(v_{i-2}e^{-4\bar{\gamma}} + v_{i-3}e^{-6\bar{\gamma}} + \ldots + v_{i-l}e^{-2\bar{\gamma}l})$$

$$u_i = v_i + u^* \quad (2.72)$$

where $v_i = v(t_i)$, $u_i = u(t_i)$ and $\{t_i\}_{i=1}^{m}$ is the time grid with a mesh of constant size $\Delta t$, $l$ is the discrete-time delay analogue of $\tau$.

Solution is shown in Figure (2).

Figure 2.: Solution of (2.43) with parameters in Table 2.
Chapter 3
Parameter Estimation

3.1 Introduction

In this chapter, we find an expression for the forward price of energy commodity. Using the representation of forward price, we apply the Least-Square Optimization and Maximum Likelihood techniques to estimate the parameters defined in (2.2) and (2.34).

3.2 Derivation of Forward Price

Let $F(t, T)$ be the forward price at time $t$ of an energy goods with maturity at time $T$. We define

$$F(t, T) = \mathbb{E}_\mathbb{P} (S(T))$$

(3.1)

where $S(T)$ is defined by (2.1), the expectation here is taken with respect to the risk neutral measure defined in (2.30).

**Remark** 5 At maturity, it is expected that the forward price is equal to the spot price at that time, i.e. $F(T, T) = S(T)$. This is the basic assumption of the risk neutral valuation method.

From (2.34), the forward price $F(t, T)$ can be expressed as

$$F(t, T) = \mathbb{E}_\mathbb{P} (S_T)$$

$$= \mathbb{E}_\mathbb{P} (\exp[f(T) + x_2(T)])$$

$$= \exp \left[ f(T) + e^{-\gamma (T-t)}x_2(t) + \phi(t, T)x_1(t) + \bar{\omega}(t, T) + \Upsilon(t, T) \right] ,$$

(3.2)

where $\Upsilon(t, T)$ is defined by

$$\Upsilon(t, T) = \exp \left[ \frac{\psi_1g(t, T, 2\bar{\gamma}) + (u_0 - \psi_1)e^{\rho T}g(t, T, \rho + 2\bar{\gamma}) + \left[ \frac{\delta_\epsilon}{\mu - \gamma} \right]^2 h(t, T, \bar{\mu}, \bar{\gamma})}{2} \right] ,$$

and

$$h(t, T, \bar{\mu}, \bar{\gamma}) = (g(t, T, 2\bar{\gamma}) - 2g(t, T, \bar{\mu} + \bar{\gamma}) + g(t, T, 2\bar{\mu})) .$$
\[ g(t, T, a) = \frac{1 - e^{-a(T-t)}}{a}, \text{for any } a \in \mathbb{R} \quad (3.3) \]

and \( \psi_1 \) is defined in (2.71). Hence

\[
\log F(t, T) = f(T) + e^{-\gamma(T-t)} x_2(t) + \tilde{\phi}(t, T) x_1(t) + \tilde{\omega}(t, T) + \Upsilon(t, T)
\]

\[
= f(T) + e^{-\gamma(T-t)} (\log S(t) - f(t)) + \tilde{\phi}(t, T) x_1(t) + \tilde{\omega}(t, T) + \Upsilon(t, T) \quad (3.4)
\]

\[
A(t, T) = f(T) + e^{-\gamma(T-t)} (\log S(t) - f(t)) + \tilde{\omega}(t, T) + \Upsilon(t, T) \quad (3.5)
\]

\[
B(t, T) = \tilde{\phi}(t, T)
\]

Define

\[
\begin{align*}
\epsilon_1 &= (\tilde{\mu}, \tilde{\nu}, \delta) \\
\epsilon_2 &= (\gamma, \alpha, \beta, \epsilon, \tau) \\
\epsilon_3 &= (A_0, A_1, A_2, B_1, B_2) \\
\epsilon &= (\epsilon_1, \epsilon_2, \epsilon_3),
\end{align*}
\]

where \( \epsilon \) consists of the risk-neutral parameters in (2.2) and (2.34).

We can represent \( \log F(t, T) \) as \( \log F(t, T; \epsilon) \), \( x_1(t) \equiv x_1(t; \epsilon_1) \), \( x_2(t) \equiv x_2(t; \epsilon_2) \), \( f(t) \equiv f(t; \epsilon_3) \).

In the following section, we use the Least square optimization approach to estimate the parameters \( \gamma, \tilde{\mu}, \tilde{\nu} \) and \( \delta \).

### 3.3 Parameter Estimation Techniques

In this section, we discuss about the estimation of parameters of the stochastic interconnected models for energy commodity’s spot price (2.14). A numerical scheme is used to develop time-series model, and using the Least Squares optimization and Maximum Likelihood techniques, we outline the parameter estimations for our model.

#### 3.3.1 Least Squares Optimization Techniques

For time \( t_i, i \in \{1, 2, \ldots, N\} = I(1, N) \), let \( S(t_i) \) denote the historical spot price of commodity. For fixed \( i \in I(1, N) \), \( \tilde{F}(t_i, T_j^i) \) represent an observe future price at a time \( t_i \) with delivery time \( T_j^i \) for \( j \in I(1, n_i) \). These data values are obtainable from the energy market.

For each given quoted time \( t_i \), we obtain \( x_1(t; \epsilon_1) \) such that it minimizes the sum of squares

\[
\text{sqdiff}(t_i, \epsilon) = \sum_{j=1}^{n_i} \left( \log F(t_i, T_j^i; \epsilon) - \log \tilde{F}(t_i, T_j^i) \right)^2, \quad (3.6)
\]
where \( \log F(t_i, T^j_i; \epsilon) \) is described in (3.4). Differentiating (3.6) with respect to \( x_1(t; \epsilon_1) \) and setting the result to be zero to get the optimal value of \( x_1(t; \epsilon_1) \) as a function of the parameter set, we have

\[
\tilde{x}_1(t_i; \bar{\epsilon}) = \frac{\sum_{j=1}^{n_i} B(t_i, T^j_i) \left( \log \tilde{F}(t_i, T^j_i) - A(t_i, T^j_i) \right)}{\sum_{j=1}^{n_i} \left[ B(t_i, T^j_i) \right]^2}, \quad i \in I(1, m),
\]

(3.7)

Substituting this optimal value into the initial sum of squares (3.6), and summing over the range of initial times \( \{t_i\} \) and performing a nonlinear least-squares optimization as follows:

\[
\text{sqdiff}(\epsilon) = \text{arg min}_{\epsilon} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \left[ A_{t_i, T^j_i} + B_{t_i, T^j_i} x_1(t_i) - \log \tilde{F}(t_i, T^j_i) \right]^2.
\]

(3.8)

With the obtained \( \bar{\epsilon}, \{x_1(t)\}^N_1, \{x_2(t)\}^N_1, \{f(t)\}^N_1 \) and \( \{S_t\}^N_1 \) are easily computed.

In the case of real-world \( \mathcal{P} \)-parameters \( [\gamma, \mu, \kappa, \delta] \) estimation, the estimates of \( \gamma \) and \( \kappa \) are obtained using a linear regression technique associated with the model \( dx_2 = \gamma(\kappa - x_2(t))dt + \sigma dW_2(t) \). (3.7) contains an estimated hidden process \( \tilde{x}_1(t_i) \) which is obtained by the least square minimization approach. This estimated data is used in a regression of a one-factor mean reverting model \( dx_1(t) = \mu(\kappa - x_1(t)) + \delta dW_1(t) \) to obtain estimates for \( \mu \) and \( \delta \). We remark that this procedure is very stable.

### 3.3.2 Maximum Likelihood Approach

Now, by following the approach in [140] and using Maximum Likelihood approach, the time delay and the delay volatility parameters \( \alpha, \beta \) and \( c \) are estimated. Our model contains two sources of randomness, that is, the Wiener process in the equation for log of spot price and another Wiener process in the equation for expected log of spot price. Therefore, the presented model is an extension of GARCH model [140]

An outline of the procedure is given below. From (2.40), we have that

\[
\frac{d\sigma^2(t, x_2)}{dt} = \alpha + \beta \left[ \int_{t-\tau}^{t} \sigma(s, x_2) e^{-\gamma(t-s)} dW_2(s) + \delta \int_{t-\tau}^{t} \partial(u, t) dW_1(u) \right]^2 + c\sigma^2(t, x_2).
\]

(3.9)

We define the discrete-time analogue value \( l \) to the continuous-time delay \( \tau \) as \( l = \left\lfloor \frac{\tau}{\Delta} \right\rfloor \), where \( \left\lfloor . \right\rfloor \) is the floor function, \( \Delta \) is the size of the mesh of the discrete-time grid. Hence, we define

\[
\epsilon_i = \sigma_i \xi_i, \quad \eta_i = \delta \zeta_i,
\]

(3.10)
where \( \xi, \zeta \) are standard normal variate. The discrete-time delayed model corresponding to (3.9) for volatility is described by

\[
\sigma_n^2 = \alpha + \beta \Delta t \left[ \sum_{i=1}^{l} (\varepsilon_{n-i} e^{-\gamma i} + \eta_{n-i} \tilde{\phi}(0, i)) \right]^2 + r \sigma_{n-1}^2, \tag{3.11}
\]

where \( n = 1, 2, 3, 4, \ldots \), and \( r = 1 + c \).

From (2.39), we further note that

\[
\int_{t-\tau}^{t} \sigma(s, x_2(s)) e^{-\gamma (t-s)} d\tilde{W}_2(s) + \delta \tilde{\phi}(s, t) d\tilde{W}_1(s) = x_2(t) - x_2(t - \tau) e^{-\gamma \tau} - x_1(t - \tau) \tilde{\phi}(0, \tau) - \bar{\omega}(0, \tau),
\]

that is,

\[
\sqrt{\Delta t} \sum_{i=1}^{l} \varepsilon_{n-i} e^{-\gamma i} + \eta_{n-i} \tilde{\phi}(0, i) = x_2(n) - x_2(n-1) e^{-\gamma l} - x_1(n-1) \tilde{\phi}(0, l) - \bar{\omega}(0, l). \tag{3.12}
\]

Define

\[
P(n) = \left[ x_2(n) - x_2(n-1) e^{-\gamma l} - x_1(n-1) \tilde{\phi}(0, l) - \bar{\omega}(0, l) \right]^2, \tag{3.13}
\]

This together with (6.1) yields

\[
\sigma_n^2 = \alpha + \beta P_n + r \sigma_{n-1}^2. \tag{3.14}
\]

The solution of difference equation (3.14) is given by

\[
\sigma_n^2 = \begin{cases} 
\alpha F_n(r) + \beta G_n(r) + H_n(r), & n \geq l + 1 \\
\frac{\varepsilon_n^2}{\sigma_n^2}, & n \leq l,
\end{cases} \tag{3.15}
\]

where for \( r = 1 + c \),

\[
F_n = \sum_{i=0}^{n-l-1} r^i, \tag{3.16}
\]

\[
G_n = \sum_{i=0}^{n-l-1} r^i P_{n-i}, \tag{3.17}
\]

\[
H_n = r^{n-l} \sigma_l^2. \tag{3.18}
\]

We observe that the series \( F_n \) in (3.16) converges if \( |r| < 1 \), that is, \( |1 + c| < 1 \). Hence,

\[
-2 < c < 0. \tag{3.19}
\]

From the definition of \( \varepsilon_n \) in (3.10), the probability density function \( f_{\varepsilon_n} \) of \( \varepsilon_n \) is

\[
f_{\varepsilon_n}(y) = \frac{1}{\sqrt{2\pi \sigma_n}} \exp \left[ -\frac{y^2}{2\sigma_n^2} \right]. \tag{3.20}
\]
Thus the likelihood function $L(\alpha, \beta, c)$ of $f_{\varepsilon}$, $n \in I(1, N)$ for arbitrary large positive integer $N$ is

$$L(\alpha, \beta, c) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi \sigma_n}} \exp \left[-\frac{y_n^2}{2\sigma_n^2}\right].$$

(3.21)

By applying the Maximum Likelihood method, we obtain the estimates $\alpha(l)$, $\beta(l)$, and $r(l)$ for $l \in I(1, p)$ for some arbitrary $p$.

### 3.4 Some Results: Natural Gas

In this section, we apply our model to the Henry Hub daily natural gas data set for the period 02/01/2001-09/30/2004 [25]. The data is collected from the United State Energy Information Administration website (www.eia.gov). Using the Henry Hub daily natural gas data set, we present the calibration results of our model. The parameter estimates of our model for the value of $l = 2$ are given. For this purpose, using a combination of direct search method and Nelder-Mead simplex algorithm, we search iteratively to find the parameters that maximizes the likelihood function. All codes are written in Matlab.


<table>
<thead>
<tr>
<th>$\bar{\gamma}$</th>
<th>$\bar{\mu}$</th>
<th>$\bar{\kappa}$</th>
<th>$\delta$</th>
<th>$\tau$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8943</td>
<td>1.0154</td>
<td>1.5627</td>
<td>0.36</td>
<td>0.008</td>
<td>0.433</td>
<td>-0.07</td>
<td>-1.5</td>
</tr>
</tbody>
</table>

Table 2 shows the risk-neutral parameter estimates of Henry Hub daily natural gas data set [25] for the period 02/01/2001-09/30/2004.

The next figure shows the graphs of the real spot natural gas price data set [25] together with the simulated spot price $S(t)$.  

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Figure 3.: Real, Simulated and Forecasted Prices.

Figure 3 (a) shows the graphs of the real spot natural gas price data set [25] together with the simulated spot price $S(t)$. Figure 3 (b) shows and the the graphs of the real spot natural price data set together with the simulated spot price $S(t)$ and the simulated expected spot price $\exp(x_1(t))$. We notice that in Figure 3 (a), the simulated spot price captures the dynamics of the data set. This shows that the simulation agrees with our mathematical model. Another observation is that the simulated mean level seems to move around the value 4.80 which is close to $\exp(\bar{\kappa}) = \exp(1.5627)$. This confirms the fact that $\bar{\kappa}$ is the equilibrium mean level.

Figure 4.: Simulated $\sigma(t, x_2(t))$.

Figure 4 shows the plot of volatility $\sigma(t, x_2(t))$ with time. It is clear from the graph that the solution is non-oscillatory. This is because Lemma 2.3 (i) is satisfied using the parameter estimates in Table 2.
Chapter 4
Non-Linear Stochastic Modeling of Energy Commodity Spot Price Processes with Delay in Volatility

4.1 Introduction

In real world situation, the expected spot price of energy commodities and its measure of variation are not constant. This is because of the fact that a spot price is subject to both deterministic and random environmental perturbations. Moreover, some statistical studies of stock prices [8] raised the issue of market’s delayed response. This indeed causes the price to drift significantly away from the market quoted price. It is well recognized that time-delay models in economics [41] are more realistic than the models without time-delay. Continuous-time and Discreet-time stochastic volatility models [9, 38] have been developed in economics. Elliot et al [37] developed a model for pricing variance swaps and volatility swaps under a continuous-time Markov-modulated version of Heston’s stochastic volatility model. Recently, in a survey work, Hansen and Lunde [46] have estimated these types of models and concluded that the performance of the GARCH(1,1) model is better than any other model. Furthermore, Cox-Ingersoll-Ross(CIR) developed a mean reverting interest rate model that was based on the mean-level interest rate with exponentially weighted integral of its past history, the relationship between level dependent volatility and the square root of the interest rate [19]. Employing the Ornstein Uhlenbeck [126] and Cox-Ingersoll-Ross(CIR) [19] processes, Heston developed a stochastical model for the volatility of stock spot asset.

In this work, using basic economic principles, we systematically develop interconnected stochastic nonlinear dynamic model for the log-spot price, expected log-spot price and volatility processes. The effort is made to utilize the developed interconnected stochastic model to analyze the Henry Hub daily natural gas data set. The by-product of this led to the development of discretized expected square volatility model and a modification of The Kalman filter approach. This has been achieved by treating a diffusion coefficient parameter in the non-seasonal log-spot price dynamic system as a stochastic volatility functional of log-spot price.
The organization of this study is as follows:

In Section 4.2, we developed a stochastic models for energy commodity’s spot price. We extend the linear interconnected deterministic and stochastic models in (2.14) to non-linear interconnected deterministic and stochastic models. In Section 4.3, the derived model is validated. In Section 4.5, by outlining the risk-neutral dynamics and pricing, risk-neutral dynamics of presented model is derived.

4.2 Model Derivation

The principles of demand and supply processes suggest that the price of a energy commodity will remain within a given finite lower and upper bounds. Let \( \kappa_1 \geq 0 \) and \( \kappa_2 > 0 \) be the expected lower and upper limits of the nonseasonal log of spot price, respectively. In a real world situation, the nonseasonal log of spot price is governed by the spot price dynamic process. In the following, we outline the development of dynamic model for the nonseasonal spot price processes. Let \( x_2(t) \) be the nonseasonal log of spot price at a time \( t \). In this case, \( \kappa_2 \) characterizes the fixed cost, \( (x_2(t) + \kappa_1)(\kappa_2 - x_2(t)) \) characterizes the market potential for \( x_2(t) \) per unit of time at a time \( t \). The market potential is induced/generated by the underlying market forces on the nonseasonal log spot price, \( x_2(t) \). This leads to the following principle regarding the dynamic of price \( x_2(t) \) of energy goods.

The change in nonseasonal log spot price of the energy commodity \( \Delta x_2(t) = x_2(t + \Delta t) - x_2(t) \) over the interval of length \( |\Delta t| \) is directly proportional to the product of the market potential price and the length of the interval.

\[
\Delta x_2(t) \propto (x_2(t) + \kappa_1)(\kappa_2 - x_2(t))\Delta t. \tag{4.1}
\]

This implies

\[
dx_2(t) = \gamma(x_2(t) + \kappa_1)(\kappa_2 - x_2(t))dt, \tag{4.2}
\]

where \( \gamma \) is a positive constant of proportionality, \( dx_2 \) and \( dt \) are differentials of \( x_2(t) \) and \( t \), respectively.

We note that (4.2) has a unique non-zero equilibrium \( \kappa_2 \). Moreover, we observe that whenever the price lies above \( \kappa_2 \), there is a tendency for the price to fall and whenever the price is below \( \kappa_2 \), the price rises back. Hence, \( \kappa_2 \) is the equilibrium of (4.2). Hence

\[
\lim_{t \to \infty} x_2(t) = \kappa_2 \tag{4.3}
\]
In the real world situation, the upper price limit of the nonseasonal log spot price $\kappa_2$ is not a constant parameter. In the following, we employ the argument of Bernard and Thomas [8] to incorporate both the response time delay and random environmental perturbations into the measure of variation of the log-spot price process of energy commodity. Therefore, we consider

$$\kappa_2 = x_1(t) + e_2(t), \quad (4.4)$$

where $e_2$ is a white noise process that characterizes the measure of random variation of the log spot price, $x_1(t)$ describes a mean of non-seasonal log spot price process and it is assumed to be governed by a similar differential equation described in (4.2), that is,

$$dx_1(t) = \mu(x_1(t) + \kappa_3)(\kappa_2 - x_1(t))dt, \quad (4.5)$$

where $\mu$ is a positive constant of proportionality.

Moreover, the mean non-seasonal log spot process is subject to random environmental perturbations. By following the argument used in (4.4), we assume that $\kappa_3$ is subject to random perturbations:

$$\kappa_3 = \kappa_0 + e_1, \quad (4.6)$$

where $\kappa_0$ is constants, and $e_1$ is a white noise and it describes the measure of random influence on the mean non-seasonal log-spot price.

Substituting (4.4) and (4.6) into (4.2) and (4.5), respectively, we obtain

$$\begin{cases}
    dx_1(t) = \mu(x_1(t) + \kappa_0)(\kappa_2 - x_1(t))dt + \mu(\kappa_2 - x_1(t))e_1(t)dt, \\
    dx_2(t) = \gamma(x_2(t) + \kappa_1)(x_1(t) - x_2(t))dt + \gamma(x_2(t) + \kappa_1)e_2(t)dt.
\end{cases} \quad (4.7)$$

Using (4.7) and following the argument of Bernard and Thomas [8], we incorporate both the response time delay and random environmental perturbations into the measure of random variations on the log-spot price process of energy commodity. This leads to the establishment of a stochastic model for nonseasonal log spot price and expected log-spot price processes that is described by the following non-linear system of stochastic functional differential equations:

$$\begin{cases}
    dx_1 = \mu(x_1 + \kappa_0)(\kappa_2 - x_1)dt + \delta(\kappa_2 - x_1)dW_1(t), \quad x_1(t_0) = x_{10}, \\
    dx_2 = \gamma(x_2 + \kappa_1)(x_1 - x_2)dt + \sigma(t, x_{2t})(x_2 + \kappa_1)dW_2(t), \quad x_{t_02} = \vartheta_{02},
\end{cases} \quad (4.8)$$

where

$$\begin{cases}
    \mu e_1(t)dt \equiv \delta dW_1(t) \\
    \gamma e_2(t)dt \equiv \sigma(t, x_{2t})dW_2(t),
\end{cases} \quad (4.9)$$
and $\delta > 0$, $x_{2t}$ is a segment of continuous function $x_2$ defined by $x_{2t}(\theta) = x_2(t + \theta), \theta \in [-\tau, 0]$ for $t \geq 0$, $\sigma$ is a functional defined on $[0, T] \times C([-\tau, 0], \mathbb{R})$ into $\mathbb{R}$.

For the sake of validity and completeness of mathematical model (4.8), we assume the following:

$$
\mathbf{H}_1: \quad x_{2t}(\theta) = x_2(t + \theta), \theta \in [-\tau, 0], x_1 = \vartheta \in C([-\tau, 0], \mathbb{R}^2) \text{ defined as } x_1(\theta) = x(t + \theta) = [x_1(t + \theta), x_2(t + \theta)]^T, \quad \tau \geq 0, \gamma > 0, \mu > 0, \kappa_1 \geq 0, \kappa_2 > 0, \kappa_0 \geq 0, \delta > 0, \sigma: [0, T] \times C \rightarrow \mathbb{R}_+.
$$

is a Lipschitz continuous bounded mapping, $C$ is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm; $W_1(t)$ and $W_2(t)$ are standard Wiener processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathcal{P})$, the filtration function $(\mathcal{F})_{t \geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$.

By following the idea of [140], we define the continuous volatility version of the GARCH type model as:

$$
d\sigma^2(t, \vartheta_2) = \left( \alpha + c\sigma^2(t, \vartheta_2) + \beta \left[ \int_{t-\tau}^t \sigma(s, \vartheta_2) dW_3(s) \right]^2 \right) dt, \quad \sigma^2(t_0, \vartheta_{02}) = \sigma^2_0(\vartheta_{02}) \quad (4.10)
$$

where $\alpha, \beta \in \mathbb{R}_+, c < 0$, and $W_3$ is a Wiener process.

From (4.8) and (4.10), the overall stochastic dynamic model for nonseasonal log spot price, expected log-spot price and volatility processes under random perturbation is described by the following non-linear system of stochastic functional differential equations

$$
\begin{align*}
 dx_1 &= \mu(x_1 + \kappa_0)(\kappa_2 - x_1)dt + \delta(\kappa_2 - x_1)dW_1(t), \quad x_1(t_0) = x_{10}, \\
 dx_2 &= \gamma(x_2 + \kappa_1)(x_1 - x_2)dt + \sigma(t, x_{2t})(x_2 + \kappa_1)dW_2(t), \quad x_{102} = \vartheta_{02}, \\
 d\sigma^2(t, \vartheta_2) &= \left( \alpha + c\sigma^2(t, \vartheta_2) + \beta \left[ \int_{t-\tau}^t \sigma(s, \vartheta_2) dW_3(s) \right]^2 \right) dt, \quad \sigma^2(t_0, \vartheta_{02}) = \sigma^2_0(\vartheta_{02}). \quad (4.11)
\end{align*}
$$

### 4.3 Mathematical Model Validation

In this section, we validate the mathematical model derived in Section 2. We note that the classical existence and uniqueness theorem is not directly applicable to (4.8). We need to modify the existence and uniqueness results. The modification is based on Theorem 3.4 [57] and the usage of linear invertible transformation. For this, we first transform the systems of nonlinear stochastic system of differential equations (4.8) into a geometric mean reverting non-linear stochastic systems of differential equations. We show the global existence of solution process of transformed systems of
differential equations. From this, the solution of the geometric mean reverting non-linear stochastic system follows immediately.

**Lemma 4.1** Using the transformation

\begin{align}
a). \quad \begin{cases} y_1 = x_1 - \kappa_2 \\ y_2 = x_2 + \kappa_1, \end{cases} \\
\lambda_1 = \kappa_0 + \kappa_2 \quad \lambda_2 = \kappa_1 + \kappa_2.
\end{align}

we have \( dy_i(t) = dx_i(t) \) and hence, the system of (4.11) is reduced to

\begin{align}
\begin{cases}
\frac{dy_1}{dt} = -\mu y_1 [\lambda_1 + y_1] dt - \delta y_1 dW_1, \quad y_1(t_0) = y_{10}, \\
\frac{dy_2}{dt} = \gamma y_2 [\lambda_2 + y_1 - y_2] dt + \sigma(t, y_{2t} - \kappa_1) y_2 dW_2, \quad y_{02} = \varphi_{02}, \\
\frac{d\sigma^2(t, \varphi_2 - \kappa_1)}{dt} = \left( \alpha + c\sigma^2(t, \varphi_2 - \kappa_1) + \beta \left[ \int_{t_0}^{t} \sigma(s, \varphi_2 - \kappa_1) dW_3(s) \right]^2 \right) dt, \\
\sigma_1(t, \varphi_1(0)) = -\delta \varphi_1(0) \\
\sigma_2(t, \varphi_2) = \sigma(t, \varphi) \varphi_2(0)
\end{cases}
\end{align}

where

\begin{align}
\begin{cases}
\lambda_1 = \kappa_0 + \kappa_2 \\
\lambda_2 = \kappa_1 + \kappa_2.
\end{cases}
\end{align}

In the following, we give the existence and uniqueness conditions for solutions of the IVP (4.14).

We recall that system of stochastic differential equations (4.14) does not satisfy the classical existence and uniqueness conditions. However, it does satisfy the local Lipschitz condition. We construct sequences of functions for the drift and volatility parts of (4.14) such that the classical existence theorem conditions are valid for a sequence of modified rate coefficients defined on a cylinder \([t_0, \infty) \times U_n\) for \(t_0 \in \mathbb{R}, n \in \{1, 2, 3, \ldots\}\), where \(U_n\) are modified sequence of rate functions defined as:

\[ U_n = \{|y|_0 < n\}. \]
where $\varphi = (\varphi_1, \varphi_2)^T$; $|\varphi|_0 = \sup_{-\tau \leq s \leq 0} |\varphi(s)|$, $\varphi_i \in C([-\tau, 0], \mathbb{R}^2)$, for $i = 1, 2$, and

$$
\begin{align*}
b_1^{(n)}(t, \varphi_1(0)) &= \begin{cases} b_1(t, \varphi_1(0)) & \text{for } |\varphi|_0 < n \\
b_1(t, n) & \text{for } |\varphi| \geq n \end{cases} \quad (4.18) \\
s_1^{(n)}(t, \varphi_1(0)) &= \begin{cases} s_1(t, \varphi_1(0)) & \text{for } |\varphi|_0 < n \\
s_1(t, n) & \text{for } |\varphi|_0 \geq n, \end{cases} \quad (4.19) \\
b_2^{(n)}(t, \varphi_2(0)) &= \begin{cases} b_2(t, \varphi_2(0)) & \text{for } |\varphi|_0 < n \\
b_2(t, n) & \text{for } |\varphi|_0 \geq n \end{cases} \quad (4.20) \\
s_2^{(n)}(t, \varphi) &= \begin{cases} s_2(t, \varphi) & \text{for } |\varphi|_0 < n \\
s_2(t, n) & \text{for } |\varphi|_0 \geq n \end{cases} \quad (4.21) \\
b_3^{(n)}(t, \varphi_2) &= b_3(t, \varphi_2) \quad \forall \ n \quad (4.22)
\end{align*}
$$

Using the sequence of functions (4.18 - 4.22), the modified system of stochastic differential equations (4.14) is described by

$$
\begin{align*}
dy_1^{(n)} &= b_1^{(n)}(t, y_1^{(n)})dt + s_1^{(n)}(t, y_1^{(n)})dW_1, \quad y_1(t_0) = \varphi_1(0), \\
dy_2^{(n)} &= b_2^{(n)}(t, y_2^{(n)})dt + s_2^{(n)}(t, y_2^{(n)})dW_2, \quad y_2(t_0) = \varphi_02, \\
d\sigma_t^{(n)} &= b_3^{(n)}(t, y_2^{(n)})dt, \quad \sigma^2(t, 0, \varphi_02) = \sigma_0^2(\varphi_02). 
\end{align*}
\quad (4.23)
$$

Hence, from (4.18)-(4.22) and assumption $H_1$, system (4.23) satisfies the classical existence and uniqueness conditions [57]. Therefore, there exist a sequence of Markov process $y_1^{(n)}$ and $y_2^{(n)}$ corresponding to equation (4.23). Next, we show that the global solution of (4.14) exists. For this purpose, we need to utilize the following concepts.

**Definition 4.3.1** Define $\tau_1^{(n)}$ and $\tau_2^{(n)}$ to be the first exit time of the process $y_1^{(n)}(t)$ and $y_2^{(n)}(t)$ from the set $|y_1| < n$ and $|y_2| < n$ respectively, that is

$$
\tau_i^{(n)} = \inf\{t > 0 : |y_i(t)| \geq n\}, \; i = 1, 2. \quad (4.24)
$$

Define $\tau_1$ and $\tau_2$ to be the (finite or infinite) limit of the monotone increasing sequence $\tau_1^{(n)}$ and $\tau_2^{(n)}$ respectively as $n \to \infty$.

$$
\tau_i = \lim_{n \to \infty} \tau_i^{(n)} = \inf\{t > 0 : |y_i(t)| \notin [0, \infty)\}, \; i = 1, 2. \quad (4.25)
$$

A process $X(t)$ is regular if for any $(s, x) \in I \times \mathbb{R}^l$,

$$
P\{\tau = \infty\} = 1 \quad (4.26)
$$

where $\tau$ is the limit of the first exit time $\tau_n$. 

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Using Theorems 3.4 and 3.5 of [57], we show that the process \( y(t) = \{y_1(t), y_2(t)\} \) is regular. To do this, we cite the Theorem and show that the conditions in the Theorem are satisfied.

**Theorem 4.1 (Theorem 3.5) [57]** Suppose that the local solution of (4.23) exists on every cylinder \([t_0, \infty) \times U_n\) and, moreover, that there exists a nonnegative function \( V \in C_2 \) such that for some constant \( c > 0 \)

\[
\begin{align*}
    LV & \leq cV \quad \text{for } t > t_0, \\
    V_n & = \inf_{|y| > n} V(t, y) \rightarrow \infty \text{ as } n \rightarrow \infty,
\end{align*}
\]

where the \( L \)-operator is given by

\[
L = \frac{\partial}{\partial t} + \sum_{i=1}^{l} b_i^{(n)}(t, y^{(n)}) \frac{\partial}{\partial y^{(n)}} + \frac{1}{2} \sum_{i,j=1}^{l} \sigma_i^{(n)}(t, y^{(n)}) \sigma_j^{(n)}(t, y^{(n)}) \frac{\partial^2}{\partial y_i^{(n)} \partial y_j^{(n)}}.
\]

Then, for every random variable \( x(t_0) \) independent of the process \( W_i(t) - W_i(t_0) \) there exists a solution \( y(t) \) of the system of stochastic differential equation (4.14) which is almost surely continuous stochastic process and is unique up to equivalence.

**Proof.** We utilize the structure of system (4.23) and establish the conclusion of the theorem for the first component of (4.23), followed by the second component by knowing the nature of the third component of (4.23) in Appendix A.1.

We define a new stochastic process \( \tilde{y}_1(t) \) as

\[
\tilde{y}_1 = y_1^{(n)}, \quad \text{for } t \leq \tau_1^{(n)}.
\]

We show that condition (4.26) is satisfied for \( y_1 \), thereby making the process \( y_1(t) \) to be almost surely defined for all \( t > t_0 \).

We define a nonnegative function \( V_1 \) on \( E = [t_0, \infty) \times \mathbb{R}_+ \) into \( \mathbb{R}_+ \) as follows;

\[
V_1(t, y_1) = \int_0^{y_1} (u^2 + 1) \frac{\mu(k_0 + k_2)}{\delta^2} du + \frac{\delta^2}{k_0 + k_2} \left[ \frac{\mu(k_0 + k_2)(1 + k_0 + k_2)}{\delta^2 + \mu(k_0 + k_2)} \right]^\frac{\mu(k_0 + k_2)}{\delta^2} + 1.
\]

It is obvious that \( V_1 \in C_{1,2} \). Moreover, the \( L \)-operator with respect to the first component of system of stochastic differential equation (4.14) satisfy

\[
LV_1 = -\mu(k_0 + k_2)y_1(y_1^2 + 1) \frac{\mu(k_0 + k_2)}{\delta^2} - \mu y_1^2(y_1^2 + 1) \frac{\mu(k_0 + k_2)}{\delta^2} \leq \mu(k_0 + k_2)(y_1^2 + 1)(y_1^2 + 1) \frac{\mu(k_0 + k_2)}{\delta^2} \leq \mu(k_0 + k_2)y_1(y_1^2 + 1) \frac{\mu(k_0 + k_2)}{\delta^2}.
\]
Case 1: If \( \kappa_0 + \kappa_2 - y_1^2 \leq 0 \), then \( LV_1 \leq 0 \leq V_1 \).

Case 2: If \( \kappa_0 + \kappa_2 - y_1^2 > 0 \), then \(-\sqrt{\kappa_0 + \kappa_2} < y_1 < \sqrt{\kappa_0 + \kappa_2}\) and

\[
\mu(\kappa_0 + \kappa_2 - y_1^2)(y_1^2 + 1) \frac{\mu(\kappa_0 + \kappa_2)}{\delta^2} < \frac{\delta^2}{\kappa_0 + \kappa_2} \left[ \frac{\mu(\kappa_0 + \kappa_2)(1 + \kappa_0 + \kappa_2)}{\delta^2 + \mu(\kappa_0 + \kappa_2)} \right] \frac{\mu(\kappa_0 + \kappa_2)}{\delta^2} + 1,
\]

(4.31)

since the function \( f(x) = \mu(\kappa_0 + \kappa_2 - x^2)(x^2 + 1) \frac{\mu(\kappa_0 + \kappa_2)}{\delta^2} \) has a maximum point at \( x = \sqrt{\kappa_0 + \kappa_2 - \frac{\kappa_0 + \kappa_2}{\mu(\kappa_0 + \kappa_2)} \frac{\delta^2}{\mu(\kappa_0 + \kappa_2)}} \).

Hence, \( LV_1 \leq V_1 \).

Thus, in both cases,

\[
LV_1 \leq V_1.
\]

(4.32)

Furthermore,

\[
\begin{aligned}
V_{1_n} &= \inf_{|y_1|>n} V_1(t, y_1) \\
&= \int_0^n (u^2 + 1) \frac{\mu(\kappa_0 + \kappa_2)}{\delta^2} du + \frac{\delta^2}{\kappa_0 + \kappa_2} \left[ \frac{\mu(\kappa_0 + \kappa_2)(1 + \kappa_0 + \kappa_2)}{\delta^2 + \mu(\kappa_0 + \kappa_2)} \right] \frac{\mu(\kappa_0 + \kappa_2)}{\delta^2} + 1 \rightarrow \infty \text{ as } n \rightarrow \infty.
\end{aligned}
\]

(4.33)

To show that \( \bar{y}_1(t) \) is regular, we define a function

\[
W_1(t, y_1) = V_1(t, y_1) \exp\{-(t-t_0)\},
\]

(4.34)

From (4.32), we note that \( LW_1 \leq 0 \). By defining \( \tau_1^{(n)}(t) = \min(\tau_1^{(n)}, t) \) and imitating the argument of Lemma 3.2 of [57], we have

\[
\mathbb{E}\{V_1(\tau_1^{(n)}(t), \bar{y}_1(\tau_1^{(n)}(t))) \leq e^{(t-t_0)}\mathbb{E}V_1(t_0, y_1(t_0)).
\]

Hence

\[
P\{\tau_1^{(n)} \leq t\} \leq \frac{e^{(t-t_0)}\mathbb{E}V_1(t_0, y_1(t_0))}{\inf_{|y_1|>n,u>t_0} V_1(u, y_1)} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ by (4.33)}.
\]

(4.35)

Thus, using (4.26) and (4.35), the global existence and uniqueness follows by letting \( n \rightarrow \infty \).

From (4.29), we conclude the global existence of \( y_1(t) \) of (4.14) which is an almost surely unique continuous stochastic process. Hence, using (4.12)(a), we can also show that there exist the global solution \( x_1(t) \) of sub-system of (4.8) which is an almost surely continuous and unique stochastic process.
For the proof of the global existence of solution \( y_2(t) \) of the second component of (4.14), we show the existence of solution \( \sigma^2(t, x_{2t}) \) of the third component. The existence and uniqueness of \( \sigma^2(t, x_{2t}) \) follows from Theorem A.1 in Appendix A.1.

For the proof of the existence of \( y_2 \), we note that from the boundedness of functional \( \sigma(t, \vartheta_2) \) and the minimal class of functions [72], we have

\[
||\sigma(t, \vartheta_2)|| \leq \eta \sqrt{||\vartheta_2(0)||} ,
\tag{4.36}
\]

for some positive constant \( \eta > 0 \). From the proof of global existence and almost sure stability of first component of (4.14), we assume that

\[
|y_1(t)| \leq M \quad \forall \ t \geq t_0 \tag{4.37}
\]

for some positive constant \( M \). For the proof of the global existence of \( y_2 \), we define a non-negative Lyapunov function

\[
V_2(t, y_2) = \int_0^{y_2} (u^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} du + \max_{y_2 \in [0,2(M+\kappa_1+\kappa_2)]} \left( \gamma (y_2^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} \left[ (M + \kappa_1 + \kappa_2) |y_2| - \frac{1}{2} y_2^2 \right] \right).
\tag{4.38}
\]

The \( L \)-operator with respect to the second component of (4.14) is given by

\[
LV_2 = \gamma y_2 (y_1 - y_2 + \kappa_1 + \kappa_2) (y_2^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} + \gamma \frac{\sigma^2(t, y_{2t} - \kappa_1)}{2\eta^2} (y_2^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} - y_2^3.
\]

Case 1: If \(-\frac{1}{2} y_2^2 + y_2 (y_1 + \kappa_1 + \kappa_2) < 0\), then \( LV_2 < 0 \leq V_2 \).

Case 2: If \(-\frac{1}{2} y_2^2 + y_2 (y_1 + \kappa_1 + \kappa_2) \geq 0\), then \( 0 \leq |y_2| \leq 2 |y_1 + \kappa_1 + \kappa_2| \).

Since continuous functions on closed intervals are bounded, then the function \( f(y_2) = \gamma (y_2^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} \left[ -\frac{1}{2} y_2^2 + y_2 (y_1 + \kappa_1 + \kappa_2) \right] \) is bounded on the interval \( 0 \leq y_2 \leq 2 (M + \kappa_1 + \kappa_2) \). Hence, for \( y_2 \in [0, 2(M + \kappa_1 + \kappa_2)] \),

\[
LV_2 \leq \gamma (y_2^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} \left[ -\frac{1}{2} y_2^2 + y_2 (y_1 + \kappa_1 + \kappa_2) \right] \leq \gamma (y_2^2 + 1) \frac{2\zeta}{\sqrt{-\lambda}} \left[ -\frac{1}{2} y_2^2 + |y_2| (M + \kappa_1 + \kappa_2) \right] \leq V_2.
\]

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Furthermore,
\[
V_{2n} = \inf_{|y_2| > n} V_2(t, y_2) = \int_0^n (u^2 + 1)^{\frac{n}{\gamma^2}} du + \max_{y_2 \in [0, 2]} \left( \gamma(y_2^2 + 1)^{\frac{n}{\gamma^2}} [(M + \kappa_1 + \kappa_2)y_2 - \frac{1}{2}y_2^2] \right).
\]
(4.39)

It follows that \(V_{2n} \to \infty\) as \(n \to \infty\). By defining
\[
W_2(t, y_2) = V_2(t, y_2) e^{-(t-t_0)},
\]
(4.40)
we have
\[
P\{\tau_2^{(n)} \leq t\} \leq \frac{e^{(t-t_0)}E V_2(t_0, y_2(t_0))}{\inf_{|y_2| > n, u > t_0} V_2(u, y_2)} \to 0 \text{ as } n \to \infty \text{ by (4.39).}
\]
(4.41)

Thus, the global existence and uniqueness of solution of the second component of (4.14) follows by letting \(n \to \infty\). Hence, there exist a global solution \((y_1(t), y_2(t))\) of the system of non-linear stochastic equation (4.14).

Using transformation (4.12), it can be easily deduced that there exist a global solution \((x_1(t), x_2(t))\) of the system of non-linear stochastic system (4.8).

### 4.4 Closed Form Solution Under \(\mathcal{P}\)

We observe that the system of stochastic non-linear differential equations (4.8) is a Itô-Doob stochastic Bernoulli type stochastic differential equations [70]
\[
dy = \left[ P(t)y + Q(t)y^n + \frac{n}{2} \Sigma(t) y^{2n-1} \right] dt + \left[ \Upsilon(t)y^n \right] dW(t)
\]
(4.42)
for any \(n \neq 1\), where \(P, Q, \Sigma, \Upsilon\) are continuous functions.

To find solutions \(y_1(t)\) and \(y_2(t)\), we imitate the procedure [70] for finding the implicit-closed form solution processes of first two components of non-linear stochastic differential equations in (4.14).

We consider an Energy/Lyapunov function
\[
V_i(t, y_i) = \frac{1}{y_i(t)}, \text{ for } i = 1, 2, \ y_i(t) \neq 0,
\]
(4.43)
Hence, applying Itô’s formula to (4.43), we have

\[
\begin{align*}
  dV_1 &= \left[ (\mu \lambda_1 + \delta^2) V_1 + \mu \right] dt + \delta V_1 dW_1(t) \\
  dV_2 &= \left[ (-\gamma (\lambda_2 + y_1(t)) + \sigma^2 (t, \varphi_2)) V_2 + \gamma \right] dt - \sigma(t, \varphi_2 - \kappa_1) V_2 dW_2(t).
\end{align*}
\]

Using the techniques described in [70], the implicit solution to system of differential equation (4.44) is given by

\[
\begin{align*}
  V_1(t, y_1) &= \phi_1(t, t_0) c_1 + \mu \int_{t_0}^{t} \phi_1(t, s) ds \\
  V_2(t, y_2) &= \phi_2(t, t_0) c_2 + \gamma \int_{t_0}^{t} \phi_2(t, s) ds
\end{align*}
\]

where

\[
\begin{align*}
  \phi_1(t, t_0) &= \exp \left[ \left( (\mu (\kappa_2 + \kappa_0) + \frac{1}{2} \delta^2) (t - t_0) + \delta (W_1(t) - W_1(t_0)) \right) \right] \\
  \phi_2(t, t_0) &= \exp \left[ \int_{t_0}^{t} \left( -\gamma (y_1(s) + \lambda_2) + \frac{1}{2} \sigma^2 (s, y_2(s) - \kappa_1) \right) ds - \int_{t_0}^{t} \sigma(s, y_2(s) - \kappa_1) dW_2(s) \right],
\end{align*}
\]

and \( c_i, i = 1, 2 \) are constants.

Comparing (4.43) and (4.45), we have

\[
\begin{align*}
  y_1(t) &= \phi_1(t, t_0) c_1 + \mu \int_{t_0}^{t} \phi_1(t, s) ds \left[ \frac{\phi_1(t, t_0)}{x_{10} - \kappa_2} + \mu \int_{t_0}^{t} \phi_1(t, s) ds \right]^{-1} \\
  y_2(t) &= \phi_2(t, t_0) c_2 + \gamma \int_{t_0}^{t} \phi_2(t, s) ds \left[ \frac{\phi_1(t, t_0)}{y_{20} + \kappa_1} + \gamma \int_{t_0}^{t} \phi_2(t, s) ds \right]^{-1} - \kappa_1.
\end{align*}
\]

Hence, using transformation (4.12) together with the initial condition \( y_1(t_0) = y_{10} > 0, y_{2t_0} = \varphi_{02} > 0 \), we have

\[
\begin{align*}
  x_1(t) &= \left[ \frac{\phi_1(t, t_0)}{x_{10} - \kappa_2} + \mu \int_{t_0}^{t} \phi_1(t, s) ds \right]^{-1} + \kappa_2 \\
  x_2(t) &= \left[ \frac{\phi_1(t, t_0)}{y_{20} + \kappa_1} + \gamma \int_{t_0}^{t} \phi_2(t, s) ds \right]^{-1} - \kappa_1.
\end{align*}
\]

**Remark 6** It is obvious from (4.47) that \( y_i > 0 \) for \( i = 1, 2 \). Also, \( \phi_1(t, t_0) \) is a log-normal random variable \( \log N \left( \left( (\mu (\kappa_0 + \kappa_2) + \frac{1}{2} \delta^2) (t - t_0), \delta^2 (t - t_0) \right) \right) \). Hence

\[
\mathbb{E}_P \phi_1(t, t_0) = \exp \left[ (\mu (\kappa_0 + \kappa_2) + \delta^2)(t - t_0) \right].
\]

By Jensen’s inequality, we have

\[
\mathbb{E}_P \left[ |y_1(t)| \right] \geq \left[ \mathbb{E}_P \left( \frac{\phi_1(t, t_0)}{y_{10}} + \mu \int_{t_0}^{t} \phi_1(t, s) ds \right) \right]^{-1} \\
= \left[ \left( \frac{1}{y_{10}} + \frac{\mu}{\mu (\kappa_0 + \kappa_2) + \delta^2} \exp \left[ (\mu (\kappa_0 + \kappa_2) + \delta^2)(t - t_0) \right] \right)^{-1} - \frac{\mu}{\mu (\kappa_0 + \kappa_2) + \delta^2} \right]^{-1}. \\
\]

Hence,

\[
\lim_{t \to \infty} \mathbb{E}_P \left[ |y_1(t)| \right] \geq 0
\]
Also, since
\[ y_1(t) \leq \left[ \frac{\phi_1(t, t_0)}{y_{10}} \right]^{-1}, \]  
we have
\[ \mathbb{E}_P [y_1(t)] \leq y_{10} \exp(-\mu (\kappa_0 + \kappa_2)(t - t_0)). \]  
(4.53)

Hence, by Squeeze theorem, from (4.50) and (4.53),
\[ \lim_{t \to \infty} \mathbb{E}_P [y_1(t)] = 0. \]  
(4.54)

Consequently, using (4.12), we have
\[ \lim_{t \to \infty} \mathbb{E}_P [x_1(t)] = \kappa_2. \]  
(4.55)

This establishes the fact that \( x_1(t) \) describes the mean of non-seasonal log-spot price.

We can also evaluate the area under the curve \( y_i(t) \) from \( t_0 \) to \( t \). To do this, we re-write (4.47) as
\[
\begin{align*}
y_1(t) &= \frac{\phi_1^{-1}(t,t_0)y_{01}}{1+\mu y_{01}\int_{t_0}^{t} \phi_1^{-1}(s,t_0)ds}, \\
y_2(t) &= \frac{\phi_2^{-1}(t,t_0)\phi_{02}}{1+\gamma \phi_{02}\int_{t_0}^{t} \phi_2^{-1}(s,t_0)ds}.
\end{align*}
\]  
(4.56)

It follows immediately that
\[
\begin{align*}
\int_{t_0}^{t} y_1(s)ds &= \frac{1}{\mu} \ln \left[ 1 + \mu y_{01} \int_{t_0}^{t} \phi_1^{-1}(s,t_0)ds \right], \\
\int_{t_0}^{t} y_2(s)ds &= \frac{1}{\gamma} \ln \left[ 1 + \gamma \phi_{02} \int_{t_0}^{t} \phi_2^{-1}(s,t_0)ds \right].
\end{align*}
\]  
(4.57, 4.58)

Hence, applying Fubini’s theorem, using the concavity of logarithmic function, and the facts that
\[
\begin{align*}
\mathbb{E}_P \phi_1^{-1}(t, t_0) &= \exp[-\mu (\kappa_0 + \kappa_2)(t - t_0)], \\
\mathbb{E}_P \phi_2^{-1}(t, t_0) &= \exp[\gamma \int_{t_0}^{t} (\mathbb{E}_P [y_1(s)] + \lambda_2)ds],
\end{align*}
\]  
(4.59)

and \( \mathbb{E}_P [y_1(t)] \leq y_{10} \) (from (4.53)), we have
\[
\begin{align*}
\int_{t_0}^{t} \mathbb{E}_P [y_1(s)]ds &\leq \frac{1}{\mu} \ln \left[ 1 + \mu \mathbb{E}_P [y_{01}] \int_{t_0}^{t} \mathbb{E}_P [\phi_1^{-1}(s,t_0)]ds \right], \\
&\leq \frac{1}{\mu} \ln \left[ 1 + \frac{\mathbb{E}_P [y_{01}]}{\kappa_0 + \kappa_2} \left( 1 - e^{-\mu (\kappa_0 + \kappa_2)(t - t_0)} \right) \right], \\
\int_{t_0}^{t} \mathbb{E}_P [y_2(s)]ds &\leq \frac{1}{\gamma} \ln \left[ 1 + \gamma \mathbb{E}_P [\phi_{02}] \int_{t_0}^{t} \mathbb{E}_P [\phi_2^{-1}(s,t_0)]ds \right], \\
&\leq \frac{1}{\gamma} \ln \left[ 1 + \frac{\mathbb{E}_P [\phi_{02}]}{\gamma y_{10} + \lambda_2} \left( e^{\gamma (y_{10} + \lambda_2)(t - t_0)} - 1 \right) \right].
\end{align*}
\]  
(4.60)

In addition to the above outlined results, we present a few more properties of \( y_1 \) and \( y_2 \).
**Theorem 4.2** If $\gamma - \frac{n^2}{2} > 0$, then

$$
\mathbb{E}_P[y_2(t)] \leq \left[ \frac{2\gamma - n^2}{2\gamma(n_1 + n_2 + M)} + \left( f_0 - \frac{2\gamma - n^2}{2\gamma(n_1 + n_2 + M)} \right) e^{-\gamma(n_1 + n_2 + M)(t-t_0)} \right]^{-1},
$$

where $M$ is defined in (4.37), where $f_0 = \frac{1}{\sqrt{\mathbb{E}[y_2(t)]}}|_{t=t_0}$.

**Proof.** Using the fact that $y_2 > 0$, define the Lyapunov function $v : [t_0, \infty) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$
v(t, y_2) = y_2^2.
$$

Then from (4.14)

$$
dv = Lvdt + 2\sigma(t, y_2)vdW_2(t),
$$

where the operator $L$ is defined as

$$
L = \frac{\partial}{\partial t} + \gamma y_2(n_1 + n_2 + y_1 - y_2) \frac{\partial}{\partial y_2} + \frac{1}{2} \sigma^2(t, y_2) y_2^2 \frac{\partial^2}{\partial y_2^2}.
$$

Using (4.36) and (4.37), the operator $L$ satisfies

$$
Lv \leq -(2\gamma - n^2)v^3 + 2\gamma(n_1 + n_2 + M)v.
$$

Define $u(t) = \mathbb{E}_P(v(t, y_2(t)))$. By applying Theorem 4.8.1 of [66], we obtain

$$
\mathbb{E}[v(t, y_2(t))] \leq u(t, t_0, u_0),
$$

where $u(t, t_0, u_0)$ is a solution of

$$
du(t) = \left[ -(2\gamma - n^2)u^3(t) + 2\gamma(n_1 + n_2 + M)u(t) \right] dt,
$$

Thus,

$$
\mathbb{E}_P(y_2^2(t)) \leq \left[ \frac{2\gamma - n^2}{2\gamma(n_1 + n_2 + M)} + \left( f_0 - \frac{2\gamma - n^2}{2\gamma(n_1 + n_2 + M)} \right) e^{-\gamma(n_1 + n_2 + M)(t-t_0)} \right]^{-2}.
$$

By using Hölder’s inequality, inequality (4.61) follows.

□

**Theorem 4.3** If $4\gamma(n_1 + n_2) > (2\gamma + 3n^2)$, then

$$
\mathbb{E}_P[|y_2(t)|] \geq \frac{1}{\sqrt{\left( \frac{1}{\sqrt{\mathbb{E}[y_2^2(t)]}} - \frac{\beta_1}{\alpha_1} \right) e^{-\alpha_1(t-t_0)} + \frac{\beta_1}{\alpha_1}}},
$$

for

$$
\frac{1}{\sqrt{\mathbb{E}[y_2^2(t)]}} - \frac{\beta_1}{\alpha_1} < 0.
$$

40
where
\[
\begin{align*}
\beta_1 &= \gamma + \frac{3}{2} \eta^2, \\
\alpha_1 &= 2\gamma (\kappa_1 + \kappa_2) - \beta_1.
\end{align*}
\] (4.70)

**Proof.** Using the fact that \( y_2 > 0 \), define the Lyapunov function \( v : [t_0, \infty) \times \mathbb{R}_+ \to \mathbb{R}_+ \) by
\[
v(t, y_2) = \frac{1}{y_2}.
\] (4.71)

Then from (4.14) and (4.36), using the fact that \( \frac{2}{y_2} \leq \frac{1}{y_2} + 1 \), we have
\[
dv = 
\left[-\frac{2}{y_2} \left[\gamma y_2 (\kappa_1 + \kappa_2 + y_1 - y_2) + \frac{3}{2} \sigma^2(t, y_2 t - \kappa_1)\right] dt - \frac{2}{y_2} \sigma(t, y_2 t - \kappa_1) dW_2(t),
\right.
\]
\[
\leq [-\alpha_1 v + \beta_1] dt - 2v \sigma(t, y_2 t - \kappa_1) dW_2(t).
\]

Thus,
\[
d(v e^{\alpha_1 t}) \leq \beta_1 e^{\alpha_1 t} dt - 2v \sigma(t, y_2 t - \kappa_1) dW_2(t).
\]

Hence,
\[
\mathbb{E}_\mathbb{P}[v(t)] \leq \left( v_0 - \frac{\beta_1}{\alpha_1} \right) e^{-\alpha_1 (t-t_0)} + \frac{\beta_1}{\alpha_1}.
\]

Applying Jensen’s inequality, the result follows. \(\square\)

### 4.5 Risk-Neutral Dynamics

In this section, we present a risk-neutral dynamic model corresponding to (4.8).

**Definition 4.5.1** A probability measure \( \mathbb{P} \) is said to be risk-neutral if

- \( \mathbb{P} \) and \( \mathbb{P} \) are equivalent (that is, for every \( A \in \mathcal{F} \), \( \mathbb{P}(A) = 0 \) if and only if \( \mathbb{P}(A) = 0 \), and

- Under \( \mathbb{P} \), the discounted price \( D(t) \) is a martingale.

We shall use this definition to find a risk-neutral dynamics for our model (4.8).

Define the riskless asset
\[
B_i(t) = \exp \left[ \int_{t_0}^t r_i(s) ds \right], \quad t \in [0, T], \quad i = 1, 2
\] (4.72)

where \( r_i, i = 1, 2 \) are the interest rate function.
Using the first two components of (4.14), define the discounted price of \( y_1(t) = x_1(t) - \kappa_2, \)
\( y_2(t) = x_2(t) + \kappa_1, \) by

\[
D_i(t) = \frac{\varphi_i(0)}{B_i(t)} = \exp \left[ - \int_{t_0}^{t} r_i(s) ds \right] y_i(t). \tag{4.73}
\]

Applying Itô's Lemma to (4.73), we have

\[
\begin{align*}
\frac{dD_1}{D_1} &= -\delta D_1 \left[ \frac{\mu(\rho_0 + \rho_2 + y_1) + r_1}{\delta} dt + dW_1(t) \right] \\
\frac{dD_2}{D_2} &= \sigma(t, y_2t - \kappa_1) D_2 \left[ \frac{\gamma(y_1 + \kappa_1 + \kappa_2 - y_2) - r_2}{\sigma(t, y_2t - \kappa_1)} dt + dW_2(t) \right].
\end{align*} \tag{4.74}
\]

Define the market price of risk

\[
\begin{align*}
\theta_1 &= \frac{\mu(\rho_0 + \rho_2 + y_1) + r_1}{\delta} \\
\theta_2 &= \frac{\gamma(y_1 + \kappa_1 + \kappa_2 - y_2) - r_2}{\sigma(t, y_2t - \kappa_1)},
\end{align*} \tag{4.75}
\]

where \( \varphi_i, i = 1, 2 \) are as defined in (4.15).

Using Girsanov’s theorem, we obtain the following result concerning the change of probability measure.

**THEOREM 4.4** Suppose that \( \theta_i, i = 1, 2 \) satisfy the Novikov’s condition [108], with the \( \mathbb{P} \)-Wiener process

\[
\begin{align*}
\dot{W}_1(t) &= W_1(t) + \int_{t_0}^{t} \theta_1(u) du, \\
\dot{W}_2(t) &= W_2(t) + \int_{t_0}^{t} \theta_2(u) du.
\end{align*} \tag{4.76}
\]

Then \( D_i(t) \) is a positive local martingale with respect to \( \mathbb{P} \), and is given by

\[
\begin{align*}
D_1(t) &= D_{10} \exp \left[ -\frac{1}{2} \int_{t_0}^{t} \delta^2 ds - \int_{t_0}^{t} \delta dW_1(s) \right] \\
D_2(t) &= D_{20} \exp \left[ -\frac{1}{2} \int_{t_0}^{t} \sigma^2(s, y_2s - \kappa_1) ds + \int_{t_0}^{t} \sigma(s, y_2s - \kappa_1) d\dot{W}_2(s) \right]. \tag{4.77}
\end{align*}
\]

Substituting (4.76) into (4.14), we notice that first two component of (4.14) reduces to a geometric stochastic equation given by

\[
\begin{align*}
dy_1 &= r_1(t) y_1 dt - \delta y_1 d\dot{W}_1 \\
dy_2 &= r_2(t) y_2 dt + \sigma(t, y_2s - \kappa_1) y_2 d\dot{W}_2 
\end{align*} \tag{4.78}
\]

Using transformation (4.12), (4.78) reduces to

\[
\begin{align*}
dx_1 &= -r_1(\kappa_2 - x_1) dt + \delta(\kappa_2 - x_1) d\dot{W}_1 \\
dx_2 &= r_2(x_2 + \kappa_1) dt + \sigma(t, x_2s - \kappa_1) (x_2 + \kappa_1) d\dot{W}_2 
\end{align*} \tag{4.79}
\]
5.1 Introduction

In this chapter, we present the estimation scheme to estimate the parameters in the interconnected system of nonlinear stochastic differential equation (4.11). We use discretized Scheme for Continuous-Time GARCH Model to develop the Maximum Likelihood techniques. The developed techniques is used to estimate parameters in the model for volatility process in (4.11). Furthermore, modifying the extended Kalman filter technique, we estimate the parameters in the model for log-spot and expected log-spot price in (4.11).

The Kalman Filter is a powerful and widely used technique in state and parameter estimation problems. It is used for finding minimum mean squared error (MMSE) estimation of linear state dynamic systems and observations [115]. Nonlinear state dynamic and observations are estimated by employing the Extended Kalman Filter (EKF) scheme [115]. Moreover, the EKF scheme deals with state and parameter estimation of linearized version of both nonlinear state dynamic and observations [73]. It is well known [78] that the linearized Taylor scheme does not provide sufficiently accurate representation. Moreover, due to its overly crude approximation, the scheme generates problem in convergence [78].

Several other approaches have been made to find a better filter than the EKF scheme. Unlike the usual EKF approach, Magnus [78], Tor Steinar [124] and Luo [75] propose a new set of estimators which are based on polynomial approximations of the nonlinear transformations using the Stirling’s interpolation formula. Under this scheme, derivatives of rate functions are avoided due to interpolation approximation formula. As discussed in [78], the Stirling’s interpolation formula accommodates easy implementation of the filters and enables state estimation when the derivatives are not smooth. It has been remarked that this approach provides a similar, or superior performance than the existing EKF approach. Simon Julier [113, 114, 115] claims that the EKF filtering strategy is difficult to implement, difficult to tune, and only reliable for systems which are almost linear. This
leads to the development of a new linear state and covariance estimator using unscented transformation. The new scheme was claimed to be superior than that of the EKF, and, in fact, the scheme generalizes elegantly to the nonlinear system without the linear step required by the EKF scheme. Higher filters have also been discussed by Jazwinski, [53], Maybeck, [81], and Madsen etal [76].

Our main focus in this paper is to reduce the magnitude of error that occurs during the estimation process of the EKF approach. This error is due to the overly simplified approximation scheme. In the process of the error reduction, we modified the Extended Kalman Filter scheme by incorporating second order polynomial approximation for the expected state variable and covariance. This scheme is applied to study the state and parameter estimation problems of nonlinear system of stochastic differential equation. The drift and diffusion part of the nonlinear differential equations are approximated using the Stirling’s interpolation formula [78]. This modified approach estimates the parameters of a system of nonlinear stochastic differential equation with lesser magnitude of error compared to the usual EKF approach [73]. Although the magnitude of error in the state and covariance of the EKF is reduced, it is however important to note that our scheme is computationally too demanding/computer intensive. An algorithm is developed to implement this scheme. The extended Kalman filter approach is compared with the developed modified extended Kalman filtering approach. The scheme is applied to Henry Hub natural gas data and to estimate parameters. The details are exhibited in the graph.

The organization of this work is as follows:

In Section 5.2, we present the discretized scheme for continuous-time GARCH Model. In Section 5.3, we present a modified EKF scheme. In Section 5.4, we applied the scheme to estimate the parameters for a stochastic dynamic model for Henry Hub Natural gas.

5.2 Discretized Scheme for Continuous-Time GARCH Model (4.10)

In this section, we formulate a discretized scheme and outline a procedure for estimating the parameters \( \alpha, \beta, \tau \) and \( c \) in (4.10). An outline of the procedure is given below:

Define the discrete-time delay value \( l \) to be the analogue of the continuous-time delay \( \tau \). Given the value of \( l \), we define the size of the mesh of the discrete-time grid as \( \Delta = \frac{\tau}{l} \). Furthermore, we define

\[
\varepsilon_i = \sigma_i \xi_i, \quad (5.1)
\]

where \( \xi_i \) is a white noise process. The discrete-time delayed model corresponding to (4.10) for
volatility is described by
\[ \sigma_n^2 = \alpha + \beta \Delta t \left[ \sum_{i=1}^{l} \varepsilon_{n-i} \right]^2 + q\sigma_{n-1}^2. \] (5.2)

\[ \sigma_i^2 = \sigma_0^2 \quad \text{for} \quad i \in [-\tau, 0], \] and \( q = 1 + \epsilon. \)

Since \( \varepsilon_j \) is a normal random variable with mean 0 and variance \( \sigma_j^2 \), we can write
\[ \sum_{i=1}^{l} \varepsilon_{n-i} \equiv \sqrt{\sum_{i=1}^{l} \sigma_{n-i}^2} \epsilon, \] (5.3)

where \( \epsilon \) is a standard normal variable. Hence, (6.1) reduces to
\[ \sigma_n^2 = \alpha + \beta \Delta t \sum_{i=1}^{l} \sigma_{n-i}^2 \epsilon^2 + q\sigma_{n-1}^2, \] (5.4)

Using the fact that \( \epsilon^2 \) is a \( \chi^2(1) \) random variable, we find the probability density function
\[ f(\sigma_n^2|\sigma_{n-i}^2, 1 \leq i \leq l) \] of \( \sigma_n^2 \) given \( \sigma_{n-i}^2, 1 \leq i \leq l \) to be
\[ f(\sigma_n^2|\sigma_{n-i}^2, 1 \leq i \leq l) = \frac{(\sigma_n^2 - \alpha - q\sigma_{n-1}^2)^{-1/2}}{\sqrt{2\pi\beta\Delta t \sum_{i=1}^{l} \sigma_{n-i}^2}} \exp \left[ -\frac{\sigma_n^2 - \alpha - q\sigma_{n-1}^2}{\beta\Delta t \sum_{i=1}^{l} \sigma_{n-i}^2} \right], \alpha + q\sigma_{n-1}^2 < \sigma_n^2 < \infty. \] (5.5)

We define the Likelihood function of \( \sigma_n^2 \) as
\[ \mathcal{L}(\Pi_3) = \log \prod_{n=1}^{N} f(\sigma_n^2|\Pi_3, \sigma_{n-i}^2, 1 \leq i \leq l) \] (5.6)

where \( \Pi_3 = \{\alpha, \beta, q\} \) are the parameters to be estimated. Thus,
\[ \mathcal{L}(\Pi_3) = -\frac{1}{2} \sum_{n=1}^{N} \log \left[ \frac{\sigma_n^2 - \alpha - q\sigma_{n-1}^2}{2\pi\beta\Delta t \sum_{i=1}^{l} \sigma_{n-i}^2} \right] - \sum_{n=1}^{N} \left[ \frac{\sigma_n^2 - \alpha - q\sigma_{n-1}^2}{\beta\Delta t \sum_{i=1}^{l} \sigma_{n-i}^2} \right]. \] (5.7)

Our aim is to find estimators that maximize the Likelihood function (5.7) subject to the constraint (2.49).

Hence, solving for the maximum-likelihood estimators \( \hat{\alpha}, \hat{\beta} \) and \( \hat{q} \) of \( \alpha, \beta \) and \( q \) respectively, we have
\[ \hat{\beta} = \frac{2}{N} \sum_{n=1}^{N} \left[ \frac{\sigma_n^2 - \hat{\alpha} - \hat{q}\sigma_{n-1}^2}{\Delta t \sum_{i=1}^{l} \sigma_{n-i}^2} \right], \]
and \( \hat{\alpha}, \hat{q} \) satisfies

\[
\frac{1}{2} \sum_{n=1}^{N} \left[ \frac{1}{\sigma_n^2 - \hat{\alpha} - \hat{q} \sigma_{n-1}^2} \right] + \sum_{n=1}^{N} \left[ \frac{1}{\beta \Delta t \sum_{i=1}^{l} \sigma_{n-i}^2} \right] = 0,
\]

\[
\frac{1}{2} \sum_{n=1}^{N} \left[ \frac{\sigma_{n-1}^2}{\sigma_n^2 - \hat{\alpha} - \hat{q} \sigma_{n-1}^2} \right] + \sum_{n=1}^{N} \left[ \frac{\sigma_{n-1}^2}{\beta \Delta t \sum_{i=1}^{l} \sigma_{n-i}^2} \right] = 0,
\]

respectively.

To evaluate the parameters, we generate the observation data for \( \sigma(t, \vartheta_2) \) from the discrete version of (4.14) described as

\[
\Delta y_2 = \gamma (\kappa_1 + \kappa_2 + y_1 - y_2) \Delta t + \sigma(t, y_{2t}) \Delta W_2.
\]

(5.8)

We achieve this by using \( y_2 \) as our observation data. We search iteratively to find the parameters that maximize (5.7) using a combination of direct search method and the Nelder-Mead simplex optimization algorithm in Matlab. This completes the parameter estimation problem of (4.10). The parameter estimated are recorded in Table 3.

Table 3: Estimated Parameters of \( \sigma^2(t, \vartheta_2) \) for \( l = 2 \) using the Henry Hub daily natural gas spot prices for the period 01/04/2000-09/30/2004 [24].

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( c )</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.07</td>
<td>1.149</td>
<td>-1.4814</td>
<td>0.005</td>
</tr>
</tbody>
</table>

5.3 Modified Extended Kalman Filter Approach

In this section, we shall be estimating the remaining parameters \( \mu, \gamma, \kappa_0, \kappa_1, \kappa_2 \) and \( \delta \) of (4.8) using the Modified Extended modified Kalman Filter Approach. This is accomplished by approximating the state estimator using a quadratic approximation. The Kalman Filter Approach is modified by employing a second order approximation for state and state variance predictions. To estimate the parameters, we minimized the likelihood function of the prediction error of the measurement process. The approach is described below.
We assume that a dynamic state \( x \in \mathbb{R}^n \) and its observation data \( y \in \mathbb{R}^n \) are described by a general non-linear stochastic dynamic systems.

\[
\begin{align*}
    dx &= f(x; \theta)dt + g(x; \theta)dW(t), \quad x(t_0) = x_0 \\
y(t) &= h(x; \theta) + v(t),
\end{align*}
\]

(5.9)

where \( x_0 \) is a stochastic initial condition satisfying \( E|x_0|^2 < \infty, f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n \times d}, h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \) are continuous functions, \( W : \mathbb{R} \to \mathbb{R}^d \) is a \( d \)-dimensional standard Wiener process on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})\), the filtration function \((\mathcal{F})_{t \geq 0}\) is right-continuous, and each \( \mathcal{F}_t \) with \( t \geq 0 \) contains all \( \mathcal{P} \)-null sets in \( \mathcal{F} \), \( x \) is \( \mathcal{F}_t \) adapted process and non-anticipative, and \( v : \mathbb{R} \to \mathbb{R}^n \) is a \( n \)-dimensional zero mean Gaussian white noise process independent of \( W, \theta \in \Theta \), the parameter space.

Prior to presenting a procedure for the estimation of parameters, we define the following terminologies and notations used throughout this work.

Define

\[
Y_{t_k} = \{y_{t_1}, y_{t_2}, \ldots, y_{t_k}\},
\]

(5.10)
as all observations of the data given up to time \( t_k \).

\[
\begin{align*}
    \hat{y}(t|t_{k-1}) &= \mathbb{E} \left[ y(t)|Y_{t_{k-1}} \right], \\
    \hat{x}(t|t_{k-1}) &= \mathbb{E} \left[ x(t)|Y_{t_{k-1}} \right], \\
P(t|t_{k-1}) &= \mathbb{E} \left[ (x(t) - \hat{x}(t|t_{k-1}))(x(t) - \hat{x}(t|t_{k-1}))^T|Y_{t_{k-1}} \right], \\
R(t|t_{k-1}) &= \mathbb{E} \left[ v(t)v^T(t)|Y_{t_{k-1}} \right], \\
r_{0,2}(t|t_{k-1}) &= \mathbb{E} \left[ (y(t) - \hat{y}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}} \right], \\
r_{1,1}(t|t_{k-1}) &= \mathbb{E} \left[ (x(t) - \hat{x}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}} \right], \\
r_{2,2}(t|t_{k-1}) &= \mathbb{E} \left[ (x(t) - \hat{x}(t|t_{k-1}))(x(t) - \hat{x}(t|t_{k-1}))^T(y(t) - \hat{y}(t|t_{k-1})) \times 
(y(t) - \hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}} \right], \\
r_{1,2}(t|t_{k-1}) &= \mathbb{E} \left[ x(t)(y(t) - \hat{y}(t|t_{k-1}))^T(\hat{y}(t) - \hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}} \right], \\
r_{0,3}(t|t_{k-1}) &= \mathbb{E} \left[ (y(t) - \hat{y}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))^T(\hat{y}(t) - \hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}} \right], \\
r_{1,3}(t|t_{k-1}) &= \mathbb{E} \left[ (x(t) - \hat{x}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))^T(y(t) - \hat{y}(t|t_{k-1})) \times 
(y(t) - \hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}} \right]
\end{align*}
\]

(5.11)
where

\[
\begin{align*}
 r_{0,1}(t|t_{k-1}) &= \mathbb{E} \left[ (y(t) - \hat{y}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))^T | Y_{t_k-1} \right] , \\
 M_{0,2}(t|t_{k-1}) &= \mathbb{E} \left[ (\gamma(t) - \hat{\gamma}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1}))^T | Y_{t_k-1} \right] , \\
 \sigma_{\gamma 2}(t|t_{k-1}) &= \mathbb{E} \left[ (\gamma(t) - \hat{\gamma}(t|t_{k-1}))(y(t) - \hat{y}(t|t_{k-1})) | Y_{t_k-1} \right],
\end{align*}
\]

and

\[
\mathbb{E} \left[ \gamma(t) | Y_{t_k-1} \right] = \mathbb{E} \left[ \gamma(t) | \hat{\gamma}(t|t_{k-1}) \right] + \mathbb{E} \left[ (\gamma(t) - \hat{\gamma}(t|t_{k-1})) | Y_{t_k-1} \right].
\]

Let $\hat{x}(t_k|t_{k-1})$ be the a-priori state estimate at step $k$ given the knowledge of process $Y_{t_{k-1}}$, and $\hat{x}(t_k|t_k)$ be the posterior state estimate at step $k$ given the knowledge of process $Y_{t_k}$. The Extended Kalman Filter approach begins with the goal of computing a-posterior state estimate $\hat{x}(t_k|t_k)$ as a linearized approximation of the form

\[
\begin{align*}
 \hat{x}(t_k|t_k) &= A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1})), \\
 P(t_k|t_k) &= B_0.
\end{align*}
\]

It was shown in Jazwinski [53] that

\[
\begin{align*}
 A_0(t_k|t_{k-1}) &= \hat{x}(t_k|t_{k-1}), \\
 A_1(t_k|t_{k-1}) &= r_{1,1}(t_k|t_{k-1})r_{0,1}^{-1}(t_k|t_{k-1}), \\
 B_0(t_k|t_{k-1}) &= P(t_k|t_{k-1}) - A_1(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1})A_1^T(t_k|t_{k-1}),
\end{align*}
\]

where $A_1$ is the Kalman gain. Instead of approximating the conditional covariance at an observation as a constant, Jazwinski [53] extended it to an approximation of order one.

For the rest of this study, for the sake of simplicity, we write $f(x) = f(x; \theta)$, $g(x) = g(x; \theta)$, and $h(x) = h(x; \theta)$. In this study, we extend the approximate equations for the conditional mean and covariance at an observation to that of order two. To do this, we first state the Taylor’s series expansion of a vector value function $f$ about the vector $\hat{y}$,

\[
f(y) = f(\hat{y}) + \frac{\partial f(\hat{y})}{\partial y} (y - \hat{y}) + \frac{1}{2} \frac{\partial^2 f(\hat{y})}{\partial y^2} \text{diag}(y - \hat{y}, y - \hat{y}, ..., y - \hat{y})(y - \hat{y}),
\]

(5.16)
where \( \frac{\partial f(y)}{\partial y} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}, \quad \frac{\partial^2 f(y)}{\partial y^2} = \begin{pmatrix} \frac{\partial^2 f_1}{\partial y_1^2} & \frac{\partial^2 f_1}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 f_1}{\partial y_1 \partial y_n} \\ \frac{\partial^2 f_2}{\partial y_1^2} & \frac{\partial^2 f_2}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 f_2}{\partial y_1 \partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_n}{\partial y_1^2} & \frac{\partial^2 f_n}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 f_n}{\partial y_1 \partial y_n} \end{pmatrix} \) 

\( \text{diag}(y - \hat{y}, ..., y - \hat{y}) = \mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1}) \).

We note that \( \frac{\partial^2 f(y)}{\partial y^2} \) and \( \mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1}) \) are \( n \times n \) block matrices whose entries are \( n \)-dimensional row vectors and column vectors, respectively. Moreover, \( \frac{\partial^2 f(y)}{\partial y^2} \) is referred to as vector-valued Hessian matrix, and \( \mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1}) \) is a diagonal matrix defined in (5.13).

Following these definitions and notations, we define the a-posterior state estimate \( \hat{x}(t_k|t_k) \) and a-posterior covariance estimate \( P(t_k|t_k) \) as a quadratic approximation of the form

\[
\hat{x}(t_k|t_k) = A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1})) + A_2(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1})) \\
P(t_k|t_k) = B_0 + B_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T,
\]

where \( A_0 \) is an \( n \times 1 \) matrix (column vector), \( A_1 \) is an \( n \times n \) matrix, \( A_2 \) is an \( n \times n \) block matrix whose entries are \( 1 \times n \) matrix (row vector), \( B_0 \) and \( B_1 \) are square \( n \times n \) matrices.

In order to develop an algorithm for \( \hat{x}(t_k|t_k) \) and \( P(t_k|t_k) \), we need to solve for the \( A_i \) and \( B_i \) for \( i = 0, 1, A_2 \). For this purpose, we need to evaluate each quantity in (5.11)-(5.12). We use the multi-dimensional extension of Stirling’s interpolation formula discussed in Magnus [78] and Luo [75] to approximate the state drift, diffusion and the observation functions in (5.9) up to the second order.

Using the second-order polynomials, we define the multidimensional interpolation formula as

\[
\begin{align*}
\mathbf{f}(x) &= \mathbf{f}(\hat{x}) + \Delta_x \mathbf{f}(\hat{x}) + \frac{1}{2} \Delta_x^2 \mathbf{f}(\hat{x}), \\
\mathbf{g}(x) &= \mathbf{g}(\hat{x}) + \Delta_x \mathbf{g}(\hat{x}) + \frac{1}{2} \Delta_x^2 \mathbf{g}(\hat{x}), \\
\mathbf{h}(x) &= \mathbf{h}(\hat{x}) + \Delta_x \mathbf{h}(\hat{x}) + \frac{1}{2} \Delta_x^2 \mathbf{h}(\hat{x}),
\end{align*}
\]

(5.18)

where the operator \( \Delta_x \) and \( \Delta_x^2 \) are described in [78] and are defined by

\[
\begin{align*}
\Delta_x &= \frac{1}{\delta} \left( \sum_{p=1}^{n} \Delta x_p \mu_p \delta_p \right), \\
\Delta_x^2 &= \frac{1}{\delta^2} \left( \sum_{p=1}^{n} \Delta x_p^2 \delta_p^2 + \sum_{p=1}^{n} \sum_{q=1}^{n} \Delta x_p \Delta x_q (\mu_p \delta_p)(\mu_q \delta_q) \right),
\end{align*}
\]

(5.19)
where $\Delta x$, $\delta_p$ and $\mu_p$ are defined by

\begin{align*}
\Delta x &= x - \hat{x}, \\
\delta_p \mathbf{f}(\hat{x}) &= \mathbf{f}(\hat{x} + \frac{h}{2} e_p) - \mathbf{f}(\hat{x} - \frac{h}{2} e_p), \\
\mu_p \mathbf{f}(\hat{x}) &= \frac{1}{2} \left[ \mathbf{f}(\hat{x} + \frac{h}{2} e_p) + \mathbf{f}(\hat{x} - \frac{h}{2} e_p) \right],
\end{align*}

and $h > 0$ is the step size, $e_p$ is the $p$th unit vector.

Using the Cholesky transformation, we transform $x$ to a variable $z$ which is mutually uncorrelated. Following [78], we write

\begin{equation}
\begin{align*}
z &= S_x^{-1} x, \\
\tilde{\mathbf{f}}(z) &= \mathbf{f}(S_x z) = \mathbf{f}(x).
\end{align*}
\end{equation}

From (5.21), (5.18) reduces to

\begin{align*}
\tilde{\mathbf{f}}(z) &= \tilde{\mathbf{f}}(\tilde{z}) + \tilde{D}_x \tilde{\mathbf{f}}(\tilde{z}) + \frac{1}{2} \tilde{D}_x^2 \tilde{\mathbf{f}}(\tilde{z}), \\
\tilde{\mathbf{g}}(z) &= \tilde{\mathbf{g}}(\tilde{z}) + \tilde{D}_x \tilde{\mathbf{g}}(\tilde{z}) + \frac{1}{2} \tilde{D}_x^2 \tilde{\mathbf{g}}(\tilde{z}), \\
\tilde{\mathbf{h}}(z) &= \tilde{\mathbf{h}}(\tilde{z}) + \tilde{D}_x \tilde{\mathbf{h}}(\tilde{z}) + \frac{1}{2} \tilde{D}_x^2 \tilde{\mathbf{h}}(\tilde{z}).
\end{align*}

Let $\sigma_i$ represent the $i$th moment of an arbitrary element in $\Delta z$. We shall use the interpolation approximations (5.22) to evaluate the expressions in (5.11)-(5.12). For this purpose, we prove the following Lemma.

**Assumption B:**

As discussed in Magnus [78], we assume $\Delta z$ to be iid Gaussian. Hence,

\begin{equation}
\sigma_{2i-1} = 0, \; i \in \mathbb{N}.
\end{equation}

**LEMMA 5.1** Under the Assumption $\mathcal{B}$, we have

\begin{align*}
r_{0,2}(t_k|t_{k-1}) &= \frac{\sigma_2^2}{h^2} \sum_{p=1}^{n} \mu_p \delta_p \tilde{\mathbf{h}}(\tilde{z}) \mu_p \delta_p \tilde{\mathbf{h}}(\tilde{z})^T + \frac{\sigma_4^2 - \sigma_2^2}{4h^4} \sum_{p=1}^{n} \delta_p^2 \tilde{\mathbf{h}}(\tilde{z}) \delta_p^2 \tilde{\mathbf{h}}(\tilde{z})^T \\
&\quad + \frac{\sigma_2^2}{4h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}(\tilde{z}) \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T + R
\end{align*}

\begin{align*}
r_{1,1}(t_k|t_{k-1}) &= \frac{\sigma_2}{h} \sum_{p=1}^{n} S_x \left( \mu_p \delta_p \tilde{\mathbf{h}}(\tilde{z}) \right)^T \\
r_{1,2}(t_k|t_{k-1}) &= S_x \left( D(t_k|t_{k-1})_{\{\hat{\mathbf{h}}, \tilde{\mathbf{h}}^T\}} \right)_{1 \leq i \leq n} + \hat{x}(t_k|t_{k-1}) r_{0,2}(t_k|t_{k-1})_{\{\hat{x}, \tilde{\mathbf{h}}^T\}}
\end{align*}
\[ r_{1,3}(t_k|t_{k-1}) = S_x J - 2S_x E(t_k|t_{k-1}) - 2r_{1,1} C(t_k|t_{k-1}) C^T - r_{1,1} C^T C(t_k|t_{k-1}) - S_x D(t_k|t_{k-1}) \{ \tilde{h}^T \tilde{h} \} C^T, \]

\[ r_{2,2}(t_k|t_{k-1}) = S_x (Q_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n} \]

\[ r_{0,3}(t_k|t_{k-1}) = (L_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n} \]

\[ r_{0,4}(t_k|t_{k-1}) = \mathbb{E} \left[ AA^T AA^T | Y_{t_{k-1}} \right] - \mathbb{E} \left[ AA^T AC^T | Y_{t_{k-1}} \right] - \mathbb{E} \left[ AA^T CA^T | Y_{t_{k-1}} \right] + \mathbb{E} \left[ AC^T AC | Y_{t_{k-1}} \right] + \mathbb{E} \left[ AC^T CC | Y_{t_{k-1}} \right] \]

\[ \sigma_{Y,2}(t_k|t_{k-1}) = (F_i)_{1 \leq i \leq n}, \]

where

\[ A = \bar{D}_\Delta \tilde{h}(\tilde{z}) + \frac{1}{2} \bar{D}^2_{\Delta z} \tilde{h}(\tilde{z}) + v \]

\[ C = \frac{\sigma_2}{2\bar{h}^2} \sum_{p=1}^n \delta_p^2 \tilde{h}(\tilde{z}), \]

\[ F_i = \frac{\sigma_2}{\bar{h}^2} \sum_{p=1}^n \left( \mu_p \delta_p \tilde{h}(\tilde{z}) \right) \mu_p \delta_p \tilde{h}_i(\tilde{z}) + \frac{\sigma_4 - \sigma_2^2}{4\bar{h}^4} \sum_{p=1}^n \left( \delta_p^2 \tilde{h}(\tilde{z}) \right) \delta_p^2 \tilde{h}_i(\tilde{z}) \]

\[ + \frac{\sigma_2^2}{4\bar{h}^4} \sum_{p=1}^n \sum_{q=1, q \neq p}^n \left( \mu_p \delta_p \mu_q \delta_q \tilde{h}(\tilde{z}) \right) \mu_p \delta_p \mu_q \delta_q \tilde{h}_i(\tilde{z}) + e_i R_{i,i} \]

\[ r_{0,2}(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h}^T \} = \left( \frac{\sigma_2}{\bar{h}^2} \sum_{p=1}^n \mu_p \delta_p \tilde{h}_i(\tilde{z}) \mu_p \delta_p \tilde{h}^T(\tilde{z}) + \frac{\sigma_4 - \sigma_2^2}{4\bar{h}^4} \sum_{p=1}^n \delta_p^2 \tilde{h}_i(\tilde{z}) \delta_p^2 \tilde{h}^T(\tilde{z}) \right) \]

\[ + \frac{\sigma_2^2}{4\bar{h}^4} \sum_{p,q=1, q \neq p}^n \mu_p \delta_p \mu_q \delta_q \tilde{h}_i \mu_p \delta_p \mu_q \delta_q \tilde{h}^T + R_{i,i} e^T \right)_{1 \leq i \leq n} , \]

\[ \tilde{h} \equiv \tilde{h}(\tilde{z}) = \tilde{h}(\tilde{z}(t_k|t_{k-1})), \] \[ r_{0,2}(t_k|t_{k-1}) = r_{0,2}(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h}^T \}, \] \[ r_{0,2}(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h}^T \}, J(t_k|t_{k-1}), E(t_k|t_{k-1}), D(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h}^T \}, D(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h}^T \}, D(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h}^T \}, Q_{i,j}, E \left[ AA^T AA^T | Y_{t_{k-1}} \right], E \left[ AA^T AC^T | Y_{t_{k-1}} \right], E \left[ AA^T CA^T | Y_{t_{k-1}} \right], E \left[ AA^T CC^T | Y_{t_{k-1}} \right], E \left[ AC^T AA^T | Y_{t_{k-1}} \right], E \left[ AC^T AC^T | Y_{t_{k-1}} \right], E \left[ AC^T CA^T | Y_{t_{k-1}} \right], E \left[ AC^T CC^T | Y_{t_{k-1}} \right], and L_{i,j} are given in Appendix B.2. \]

Proof. The proof is given in Appendix B.3.

We can now use these values to solve for \( A_i, B_i, i = 0, 1, \) and \( A_2. \) The first step in the algorithm is to solve for \( A_i, B_i, i = 0, 1 \) and \( A_2 \) in (5.17). For this purpose, we use the following Lemma by
following the description of the moment propagation procedure across the observations described in Jazwinski [53].

**Lemma 5.2** Under the assumptions in Lemma 5.1, we have

\[
A_0(t_k|t_{k-1}) = r_{1,0}(t_k|t_{k-1}) - A_2(t_k|t_{k-1})\sigma_Y^2(t_k|t_{k-1}),
\]

\[
A_1(t_k|t_{k-1}) = \left[r_{1,1}(t_k|t_{k-1}) - A_2(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1})^T\right] r_{0,2}(t_k|t_{k-1})^{-1},
\]

\[
A_2(t_k|t_{k-1}) = T_1(t_k|t_{k-1})T_2^{-1}(t_k|t_{k-1}),
\]

where

\[
T_1(t_k|t_{k-1}) = r_{1,2}(t_k|t_{k-1}) - r_{1,0}(t_k|t_{k-1})\sigma_Y^2(t_k|t_{k-1}) - r_{1,1}(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1})
\]

\[
T_2(t_k|t_{k-1}) = M_{0,2}(t_k|t_{k-1}) - \sigma_Y^2(t_k|t_{k-1})\sigma_Y^T(t_k|t_{k-1}) - r_{0,3}^T(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1}).
\]

**Proof.** Proof is in Appendix B.4. \(\square\)

**Remark 7** If \(A_2 = 0\), (5.2) reduces to

\[
A_0 = \hat{x}(t_k|t_{k-1})
\]

\[
A_1 = r_{1,1}(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1})^{-1}.
\]

Now, we present a Lemma for finding \(B_0\) and \(B_1\).

**Lemma 5.3** Under the assumptions in Lemma 5.1, we have

\[
B_1 = \left(N_2(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1}) - N_1(t_k|t_{k-1})\right) \left[r_{0,4}(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1}) - r_{0,2}(t_k|t_{k-1})\right]^{-1}
\]

\[
B_0 = N_1(t_k|t_{k-1}) - B_1(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1}),
\]

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where

\[ N_1 = \mathbb{E} \left[ (x(t_k) - \hat{x}(t_k|t_k)) (x(t_k) - \hat{x}(t_k|t_k))^T | Y_{k-1} \right] \]

\[ = P(t_k|t_{k-1}) - r_{1,1}(t_k|t_{k-1}) A_1^T - r_{1,2}(t_k|t_{k-1}) A_2^T - A_1 r_{1,1}(t_k|t_{k-1})^T \]

\[ - A_2 r_{1,2}(t_k|t_{k-1}) + (\hat{x}(t_k|t_{k-1}) - A_0)(\hat{x}(t_k|t_{k-1}) - A_0)^T \]

\[ - (\hat{x}(t_k|t_{k-1}) - A_0) r_{0,2}(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h} \}^T A_2 \]

\[ + A_1 r_{0,2}(t_k|t_{k-1}) A_1^T + A_1 r_{0,2}(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h} \}^T A_2^T \]

\[ - A_2 r_{0,2}(t_k|t_{k-1}) \{ \tilde{h}, \tilde{h} \} (\hat{x}(t_k|t_{k-1}) - A_0) \]

\[ + A_2 r_{0,3}(t_k|t_{k-1}) A_1 + A_2 M_{0,2}(t_k|t_{k-1}) A_2^T \]

\[ N_2 = \mathbb{E} \left[ ((x(t_k) - A_0)(x(t_k) - A_0)^T - (x(t_k) - A_0)(y(t_k) - \hat{y}(t_k|t_{k-1})))^T \right] \]

\[ = \mathbb{E} \left[ ((x(t_k) - A_0)(x(t_k) - A_0)^T - (x(t_k) - A_0)(y(t_k) - \hat{y}(t_k|t_{k-1})))^T A_1^T \right] \]

\[ - (x(t_k) - A_0)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T \mathbb{E} \left[ (\mathbb{E} \left[ (x(t_k) - \hat{x}(t_k|t_k)) (x(t_k) - \hat{x}(t_k|t_k))^T | Y_{k-1} \right] \right] \]

\[ - A_1 (y(t_k) - \hat{y}(t_k|t_{k-1}))(x(t_k) - A_0)^T \]

\[ + A_1 (y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T A_1^T \right] \]

\[ \times \]

\[ \left( (y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right) \]

\[ = r_{2,2} + (\hat{x}(t_k|t_{k-1}) - A_0)(\hat{x}(t_k|t_{k-1}) - A_0)^T r_{0,2} \]

\[ - \mathbb{E} \left[ ((x(t_k) - \hat{x}(t_k|t_k))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T A_1(A_1^T (y(t_k) - \hat{y}(t_k|t_{k-1}) \times \right] \]

\[ \left( (y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right) \]

\[ - \mathbb{E} \left[ (A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(x(t_k) - \hat{x}(t_k|t_{k-1}))^T (y(t_k) - \hat{y}(t_k|t_{k-1}) \times \right] \]

\[ \left( (y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right) \]

\[ - \mathbb{E} \left[ (A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(x(t_k|t_{k-1}) - A_0)(y(t_k) - \hat{y}(t_k|t_{k-1}) \times \right] \]

\[ \left( (y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right) \]

\[ - \mathbb{E} \left[ (A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k|t_{k-1}) - A_0)(y(t_k) - \hat{y}(t_k|t_{k-1}) \times \right] \]

\[ \left( (y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right) \]

\[ + \mathbb{E} \left[ (A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T A_1(y(t_k) - \hat{y}(t_k|t_{k-1}) \times \right] \]

\[ \left( (y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right) \]

\[ \text{, and } \hat{x}(t_k) \text{ is given in (5.17).} \]

\[ \square \]

\[ \text{Proof. The proof is shown in Appendix B.5.} \]

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Remark 8 If $B_1 = 0$ and $A_2 = 0$, then from (5.25) and Remark 7, we have

\[ A_0 = r_{1,0}(t_k|t_{k-1}) = \hat{x}(t_k|t_{k-1}), \]

\[ A_1 = r_{1,1}(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1})^{-1} \]

\[ B_0 = N_1 = P_{t_k|t_{k-1}} - A_1 r_{0,2}(t_k|t_{k-1}) A_1^T. \]

Thus, the presented state and covariance algorithm includes the EKF scheme [82] as a special case.

### 5.3.1 Posterior Prediction of State and Covariance of Nonlinear System

A final step in the recursive algorithm is to predict the state $\hat{x}(t_{j+1}|t_j)$ and state variance $P(t_{j+1}|t_j)$ at the time of the following measurement. Using (5.9), the definition of $P(t|t_{k-1})$ in (5.11), and (5.18), we have

\[
\hat{x}(t_{k+1}|t_k) = \hat{x}(t_k|t_k) + \left[ \tilde{f}(\hat{z}(t_k|t_k)) + \frac{\sigma_t^2}{2T_n^2} \sum_{p=1}^{n} \delta_p^2 \tilde{f}(\hat{z}(t_k|t_k)) \right] \Delta t,
\]

\[
P(t_{k+1}|t_k) = P(t_k|t_k) + \left[ \sum_{p=1}^{n} \mu_p \delta_p \tilde{f}(\hat{z}(t_k|t_k)) \Delta t \right] \Delta t + \sum_{p=1}^{n} \mu_p^T \delta_p^2 \tilde{f}(\hat{z}(t_k|t_k)) \Delta t.
\]

The one step prediction error

\[
\Delta y(k) = y_k - \hat{y}(t_k|t_{k-1}),
\]
is assumed to be normal with mean 0 and variance \( r_{0,2} \). Hence, for \( N \) independent random observations, the Maximum Likelihood approach is equivalent to maximizing

\[
L(\Theta) = -\frac{1}{2} \sum_{k=1}^{N} \left[ \frac{1}{2} \Delta y^T(k)r_{0,2}^{-1}(t_k|t_{k-1}) \Delta y(k) + \log |r_{0,2}(t_k|t_{k-1})| \right],
\]

(5.28)

where \( \Theta \) is the parameter space.

**Remark 9** The presented predicted algorithm extends the algorithm generated by the EKF approach in a systematic way. We further remark that the second order estimation for nonlinear stochastic systems can be extended to higher order estimation. The scheme is highly complex mathematical expressions. Further detailed examination (applicability/computational, feasibility, et.c) is under investigation.

### 5.3.2 Algorithm

We describe the algorithm used in the computation of the estimates for nonlinear log-spot price stochastic differential equation (5.29) in Appendix B.1.

### 5.4 Some Results: Natural Gas

In this section, we give the parameter estimates for the stochastic differential equation (5.9). We consider the nonlinear stochastic differential equation that was developed for describing continuous time stochastic dynamic model of energy commodities log-spot price processes in (4.11),

\[
\begin{align*}
\dot{x}_1 &= \mu(x_1 + \kappa_0)(\kappa_2 - x_1)dt + \delta(\kappa_2 - x_1)dW_1(t), \quad x_1(0) = x_{10}, \\
\dot{x}_2 &= \gamma(x_2 + \kappa_1)(x_1 - x_2)dt + \sigma(t, x_{2t})(x_2 + \kappa_1)dW_2(t), \quad x_2(0) = x_{20}. \\
y(t) &= x(t) + v(t).
\end{align*}
\]

(5.29)

It follows from (5.29) that

\[
\begin{align*}
f(x; \theta) &= \begin{pmatrix} 
\mu(x_1 + \kappa_0)(\kappa_2 - x_1) \\
\gamma(x_2 + \kappa_1)(x_1 - x_2)
\end{pmatrix}, \\
g(x; \theta) &= \begin{pmatrix} 
\delta(\kappa_2 - x_1) & 0 \\
0 & \sigma(t, x_{2t})(x_2 + \kappa_1)
\end{pmatrix},
\end{align*}
\]

and \( x = \{x_1, x_2\}^T \), where \( \mu > 0, \gamma > 0, \kappa_0 \geq 0, \kappa_1 \geq 0, \kappa_2 > 0, \delta > 0, \sigma > 0, v \) is a white noise, and \( W = \{W_1, W_2\}^T \), \( W_1 \) and \( W_2 \) are independent Wiener processes. This model governs the price for energy commodity at time \( t \). \( x_2(t) \) is the nonseasonal log of spot price at a time \( t \) and
$x_1(t)$ describes a mean process of non-seasonal log spot price at time $t$. The model (5.29) follows the principle of demand and supply processes which suggest that the price of an energy commodity will remain within a given finite lower and upper bounds $\kappa_1 > 0$ and $\kappa_2 > 0$, respectively. In this case, $\kappa_2$ characterizes the fixed cost, $(x_1(t) + \kappa_0)(\kappa_2 - x_1)$ characterizes the market potential for $x_1(t)$ per unit of time at a time $t$. We note that the first component of (5.29) has a unique non-zero equilibrium $\kappa_2$. Moreover, we observe that whenever the price $x_1$ lies above $\kappa_2$, there is a tendency for the price to fall and whenever the price is below $\kappa_2$, the price rises back. Hence, $\kappa_2$ is the equilibrium of the first component of (5.29). Furthermore, $\mu$ and $\gamma$ are the rate of mean reversion for $x_1$ and $x_2$ respectively, $\delta$ and $\sigma$ are the volatility for $x_1$ and $x_2$ respectively.

We apply this model to the Henry-Hub natural gas data set [24]. We use the Henry-Hub natural gas spot price data set [24] for the observation data for $x_2$. We generate observation data for $x_1$ from the forward price $F(t, T)$ at time $t$ of an energy good with maturity at time $T$. We define the forward price as

$$F(t, T) = \mathbb{E}_P (x_2(T)).$$

(5.30)

By definition, $x_1(t)$ is the expected log-spot price, which in this case is the observation data $F(t, T)$. We use Henry-Hub natural gas observed future price at a time $t$ with delivery time $T$.

The existence and uniqueness of the solution of (5.29) is given in Chapter 4.

The initial state of the model is $\hat{x}_1(t_1|t_0) = 1.23$, $\hat{x}_2(t_1|t_0) = 1.456$, $P(t_1|t_0) = \begin{pmatrix} 0.1182 & 0 \\ 0 & 0.22 \end{pmatrix}$.

Table (4) shows the parameter estimates of Henry Hub daily natural gas.

Table 4: Estimated Parameters of (5.29) for Henry Hub daily natural gas spot prices (20 run average)

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\gamma$</th>
<th>$\kappa_0$</th>
<th>$\kappa_1$</th>
<th>$\kappa_2$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>1.78</td>
<td>.69</td>
<td>.56</td>
<td>1.5</td>
<td>0.65</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Table 4 shows the estimates of the parameters of (5.29).

Furthermore, we show some of the estimates of the simulations for the modified extended Kalman filter (MEKF) scheme compared with the usual EKF scheme.
Table 5: Simulation estimates for Henry Hub data [24] using the MEKF and EKF scheme.

<table>
<thead>
<tr>
<th>Data</th>
<th>Simulated Data</th>
<th>EKF Error</th>
<th>Modified EKF Error</th>
<th>Real − Simulated</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.6990</td>
<td>0.0359</td>
<td>-0.0015</td>
<td>-0.0028</td>
</tr>
<tr>
<td>21</td>
<td>2.7202</td>
<td>0.0358</td>
<td>-0.0016</td>
<td>-0.0027</td>
</tr>
<tr>
<td>22</td>
<td>2.7590</td>
<td>0.0358</td>
<td>-0.0014</td>
<td>-0.0025</td>
</tr>
<tr>
<td>23</td>
<td>2.8253</td>
<td>0.0358</td>
<td>-0.0012</td>
<td>-0.0023</td>
</tr>
<tr>
<td>24</td>
<td>2.5620</td>
<td>0.0358</td>
<td>-0.0011</td>
<td>-0.0021</td>
</tr>
<tr>
<td>25</td>
<td>2.5798</td>
<td>0.0358</td>
<td>-0.0010</td>
<td>-0.0020</td>
</tr>
<tr>
<td>26</td>
<td>2.5977</td>
<td>0.0358</td>
<td>-0.0009</td>
<td>-0.0018</td>
</tr>
<tr>
<td>27</td>
<td>2.6151</td>
<td>0.0358</td>
<td>-0.0008</td>
<td>-0.0017</td>
</tr>
<tr>
<td>28</td>
<td>2.6321</td>
<td>0.0358</td>
<td>-0.0007</td>
<td>-0.0016</td>
</tr>
<tr>
<td>29</td>
<td>2.6486</td>
<td>0.0359</td>
<td>-0.0006</td>
<td>-0.0015</td>
</tr>
<tr>
<td>30</td>
<td>2.6640</td>
<td>0.0359</td>
<td>-0.0005</td>
<td>-0.0014</td>
</tr>
<tr>
<td>31</td>
<td>2.6718</td>
<td>0.0359</td>
<td>-0.0004</td>
<td>-0.0013</td>
</tr>
<tr>
<td>32</td>
<td>2.6786</td>
<td>0.0359</td>
<td>-0.0003</td>
<td>-0.0012</td>
</tr>
<tr>
<td>33</td>
<td>2.6853</td>
<td>0.0359</td>
<td>-0.0002</td>
<td>-0.0011</td>
</tr>
<tr>
<td>34</td>
<td>2.6918</td>
<td>0.0359</td>
<td>-0.0001</td>
<td>-0.0010</td>
</tr>
<tr>
<td>35</td>
<td>2.6980</td>
<td>0.0359</td>
<td>0.0000</td>
<td>-0.0009</td>
</tr>
<tr>
<td>36</td>
<td>2.7049</td>
<td>0.0359</td>
<td>0.0001</td>
<td>-0.0008</td>
</tr>
<tr>
<td>37</td>
<td>2.7115</td>
<td>0.0359</td>
<td>0.0002</td>
<td>-0.0007</td>
</tr>
<tr>
<td>38</td>
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<td>0.0359</td>
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<td>-0.0006</td>
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<tr>
<td>39</td>
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<td>-0.0005</td>
</tr>
<tr>
<td>40</td>
<td>2.7309</td>
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<td>0.0005</td>
<td>-0.0004</td>
</tr>
<tr>
<td>41</td>
<td>2.7373</td>
<td>0.0359</td>
<td>0.0006</td>
<td>-0.0003</td>
</tr>
<tr>
<td>42</td>
<td>2.7431</td>
<td>0.0359</td>
<td>0.0007</td>
<td>-0.0002</td>
</tr>
<tr>
<td>43</td>
<td>2.7494</td>
<td>0.0359</td>
<td>0.0008</td>
<td>-0.0001</td>
</tr>
<tr>
<td>44</td>
<td>2.7552</td>
<td>0.0359</td>
<td>0.0009</td>
<td>0.0000</td>
</tr>
<tr>
<td>45</td>
<td>2.7610</td>
<td>0.0359</td>
<td>0.0010</td>
<td>0.0001</td>
</tr>
<tr>
<td>46</td>
<td>2.7667</td>
<td>0.0359</td>
<td>0.0011</td>
<td>0.0002</td>
</tr>
<tr>
<td>47</td>
<td>2.7721</td>
<td>0.0359</td>
<td>0.0012</td>
<td>0.0003</td>
</tr>
<tr>
<td>48</td>
<td>2.7777</td>
<td>0.0359</td>
<td>0.0013</td>
<td>0.0004</td>
</tr>
<tr>
<td>49</td>
<td>2.7832</td>
<td>0.0359</td>
<td>0.0014</td>
<td>0.0005</td>
</tr>
<tr>
<td>50</td>
<td>2.7888</td>
<td>0.0359</td>
<td>0.0015</td>
<td>0.0006</td>
</tr>
<tr>
<td>51</td>
<td>2.7942</td>
<td>0.0359</td>
<td>0.0016</td>
<td>0.0007</td>
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<td>52</td>
<td>2.7996</td>
<td>0.0359</td>
<td>0.0017</td>
<td>0.0008</td>
</tr>
<tr>
<td>53</td>
<td>2.8050</td>
<td>0.0359</td>
<td>0.0018</td>
<td>0.0009</td>
</tr>
<tr>
<td>54</td>
<td>2.8017</td>
<td>0.0359</td>
<td>0.0019</td>
<td>0.0010</td>
</tr>
<tr>
<td>55</td>
<td>2.8074</td>
<td>0.0359</td>
<td>0.0020</td>
<td>0.0011</td>
</tr>
</tbody>
</table>

Note: The table contains simulation estimates for Henry Hub data using the MEKF and EKF scheme. The data includes columns for each day's simulated and real values, along with the error in the EKF and Modified EKF schemes, and the difference between real and simulated data.
Table 5 shows the real data sets, estimated simulation results for the Modified EKF scheme and the usual EKF scheme. The estimated error is calculated by subtracting the simulated estimates from the real data set.

We show the graph of the real and simulation results using MEKF scheme.

![Real Price for Natural gas](image1)

![Simulated Price for Natural gas](image2)

![Real, Simulated and Difference plot](image3)

Figure 5.: Real and Simulated price for Natural gas data set [24] using Modified EKF scheme

Furthermore, we show the graph of the real and simulation results using EKF scheme.

![Real Price for Natural gas](image4)

![Simulated Price for Natural gas](image5)
Figure 6.: Real and Simulated price for Natural gas data set [24] using EKF scheme

Figure 5 (a) shows the graph of the real natural gas data set, Figure 5 (b) shows the simulated price using Modified EKF scheme, and Figure 5 (c) shows the combination of the real, simulated and difference of the real and simulated price of the natural gas data set using the modified extended Kalman filter second order estimation scheme.

Figure 6(a) shows the graph of the real natural gas data set, Figure 6 (b) shows the simulated price the usual ordinary EKF scheme, and Figure 6 (c) shows the combination of the real, simulated and difference of the real and simulated price of the natural gas data set [24] using the usual ordinary EKF scheme.

The following graph show the absolute error of the simulation result using the MEKF and EKF scheme.
Figure 7.: Absolute error estimate using natural gas data set [24]

Figure 7 (a) shows the absolute error of the simulations of natural gas data set using the modified extended Kalman filter scheme, Figure 7 (b) shows the absolute error of the simulations of natural gas data set using the usual extended Kalman filter scheme, and Figure 7 (c) shows the comparison of the absolute error for the modified and usual EKF scheme.

**Remark 10** We further remark that all codes are written in Matlab. To compute the maximization $\arg\min L(\Theta)$ in the algorithm, we use the Nelder-Mead Simplex Method developed in Matlab. Maximizing (5.28) is equivalent to minimizing

$$L(\Theta) = \frac{1}{2} \sum_{k=1}^{N} \left[ \frac{1}{2} \Delta y^T(k)r_{0,2}^{-1}(t_k|t_{k-1})\Delta y(k) + \log |r_{0,2}(t_k|t_{k-1})| \right] .$$

**Remark 11** It is clear from Figures 5 and 6 that the presented scheme is superior than the EKF approach. This shows that the modified extended Kalman filter does in fact reduce the magnitude of error tremendously. Furthermore, the modified extended Kalman filter scheme was able to capture the upward price spike in the neighborhood of time $t = 250$ days better than the EKF scheme. Both scheme were not able to capture the upward spike around the time $t = 800$ days. This might be as a result of the kind of model we are using to describe the dynamics of the natural gas data set. The upward spike in price at these region was due to the decline in production of natural gas and the increase in demand for electricity generation. We will like to also mention one disadvantage with this scheme. It is computational intensive.
Chapter 6
Discrete Time Dynamic Model of Statistics Process and Applications

6.1 Introduction

Recently, several models have been developed to investigate the volatility process described by stochastic differential equations [140] and stochastic difference equations [38]. It is well-recognized that volatility is predictable in many asset markets [9]. Moreover, it is observed that the volatility predictability varies significantly. Engle [38] developed a class of discrete-time models where the variance depends on the past history of state of commodity/service. Bollerslev [9] generalized models in [38] to the GARCH(p,q).

Using the concept of moving average, the estimate for the variance of a general statistics from a stationary sequence is obtained [13]. Employing the batched mean, the grand mean of the individual batch mean and introducing ASAP3 [122], it is shown that ASAP3 fits AR(1) time series model to the batch mean, and it provides better technique for points and confidence-interval estimators.

It is well known and well recognized [33, 82, 118] that the Kalman filtering approach for the system parameter and state estimation problems is based on the continuous time coupled system of state dynamic and observation systems. Using the batched mean and the first order iterative process for $\bar{X}_n$ [137], a first order iterative process [137] is developed to estimate the population variance from a given time series data set.

For the past 40 years, researchers [7, 15, 21, 33, 44, 45, 47, 55, 86, 87, 89, 95, 103, 104, 105, 106, 107, 116, 118, 122] have paid lot of attention for estimating continuous-time dynamic models from discrete time data sets. The Generalized Method of Moments (GMM) developed by Hansen [44], and its extensions [21, 45, 47, 55] have played a significant role in the literature related to the parameter and state estimation problems in linear and nonlinear stochastic dynamic processes. Under the continuous-time dynamic and discrete time data collection processes, the GMM and its extensions/generalizations consist of: 1. Stochastic differential equations of Itô-Doob type, 2. Euler-type discretization scheme, 3. the general moment function, 4. minimizing functional or objective criterion function [44, 47].
The most of the existing parameter and state estimation techniques except the Kalman filtering are centered around the usage of either overall data sets [21, 45, 47, 55], or batched data sets [13], or local data set [107] drawn on an interval of finite length $T$. This leads to an overall parameter estimate on the interval of $T$. In this work, the presented approach is focused on the local moving lagged restriction of a finite sequence of a data set drawn at a partition $P$ of finite interval of length $T$ to a subpartition of $P$ of moving subinterval $[t_{k-m_k}, t_{k-1}]$ of the interval. Moreover, using the lagged adaptive process, the present work initiates the technique to estimate the parameter and state at each data point for the given data set. Of course, these parameter estimates depend on the local admissible lagged finite restricted sequence of data. As the sub-partition moves from left to right, the approach provides a more lagged data subsets. In fact, the available lagged data subset at the previous time is a subset of the available lagged data subset at the subsequent times. The characteristics of this approach reduces the local error between a simulated value of the state of the system corresponding to the local available lagged restricted sequence of data under subpartition and predetermined performance criterion. We finally note that at the left end point of data simulation interval, without loss in generality, it is assumed that there is at least three data points that are assumed to be close enough to the true values of solution process of continuous dynamic process. In general, this is assured by the uniqueness and continuous dependence of solution process with respect to the initial data $(t_0, \varphi_0)$ (for delay stochastic differential equation) and $(t_0, y_0)$ (in the absence of delay stochastic differential equation) [70]. Moreover, as the location of data point approaches close to the right end point of the time interval, the local admissible lagged finite restricted sequence approaches to the given data set. We remark that this situation does not affect the computational ability. This is due to the fact that as the longativity of the past history approaches to the given data set, its influence diminishes. In fact, simulation value approaches to the saturation level under the performance criterion.

The presented local lagged adapted GMM method is based on the: 1. development of stochastic mathematical model of continuous time dynamic process [69, 70], 2. utilizing Euler-type discretized scheme [58] for the stochastic model in 1, 3. developing discrete time interconnected dynamic model for statistic process, 4. employing lagged adaptive expectation process [88] for developing generalized moment equations, 5. conceptual computational parameter estimation problem, 6. conceptual computational state simulation scheme, and 7. mean square $\epsilon$-sub optimal procedure.

The present work is motivated by parameter and state estimation problems of continuous time nonlinear stochastic dynamic model of energy commodity markets described in (4.11). The pur-
pose of the parameter and state estimation problems is for model validation rather than model mis-specification [21]. For the continuous-time dynamic model validation, we need to utilize the existing real world data set. Of course, the real world data set is drawn/recorded at discrete-time on a time interval of finite length. In view of this, employing the stochastic numerical approximation scheme [58], we approximate the continuous time stochastic differential equations. In almost real world dynamic modeling problems [64, 69, 70, 88], future states of continuous time dynamic processes are influenced by the states past history and response/reaction time delay processes to present states [64, 88]. Under this assumption and using the concept of lagged adaptive expectation process [47, 88], we formulate a discrete-time observation system. In fact, the discrete-time dynamic models depend on the past history of the state of a system [59]. By using the method of moments [14], and the constructed observation system, we estimate the state and its parameters. This idea leads to the development of interconnected discrete-time dynamic model of local sample mean and variance statistic processes. One of the by-products of the discrete-time sample variance statistic process is that it provides an alternative approach to the GARCH(1,1) model [9, 10]. Furthermore, the usage of the continuous-time stochastic dynamic model [69, 70], lagged expectation process, $m_k$- local lagged generalized method of moments, and interconnected discrete-time dynamic model of local sample mean and variance statistics processes lead to an alternative innovative method of state and parameter estimation problems for continuous-time dynamic models described by stochastic differential equations. The developed method is referred as local lagged adapted generalized method of moments (LLGMM). The numerical approximation process and simulation processes need to be synchronized with the existing data collection process. Using a schedule synchronization process, the concepts of local admissible sample/data observation size, local admissible finite conditional restriction sequence of data set are introduced. We estimate the parameters locally and then determine the local $\epsilon$-sub-optimal simulated state estimates. In fact, our approach is more suitable and robust for forecasting problem. It also provides upper and lower bounds for the forecasted state of the system.

The organization of this study is as follows:

In Section 6.2, we derive a discrete time dynamic model for sample mean and variance processes. We introduce a new concept of parameter and state estimation techniques. This new concept is motivated by the parameter and state estimation problems of continuous time non-liner stochastic dynamic process. In Section 6.3, we construct observation system from a nonlinear stochastic functional differential equations. In addition, using the method of moments [14], in the context of
lagged adaptive expectation process [88], we briefly outline a procedure to estimate the state parameters locally. The conceptual computational and simulation schemes are presented in Section 6.4. Moreover, a conceptual Matlab code and its implementation scheme are designed. The usefulness of computational algorithm is illustrated by applying the code to four energy commodity data sets, U.S. Treasury Bill Yield Interest Rate data set, and U. S. Eurocurrency Exchange Rate data set for the state and parameter estimation problems. Moreover, we compare the usage of GARCH(1,1) model with the presented model. Furthermore, we compare our simulated volatility U.S. Treasury Bill Yield Interest rate data with the simulated work of Chan et al [15].

6.2 Derivation of Discrete Time Dynamic Model for sample mean and variance Processes.

In this section, we use the idea of moving average to derive an algorithm for the mean and variance of sample sequences with respect to a continuous stochastic process. The development of idea and model of statistic for mean and variance processes is motivated by the state and parameter estimation problems of continuous time nonlinear stochastic dynamic model of the energy commodity market described in (4.11). In addition, the problem of price forecasting of energy goods is also addressed. For this purpose, we need to introduce a few definitions and notations.

Let \( \tau \) and \( \gamma \) be finite constant time delays such that \( 0 < \gamma \leq \tau \). Here, \( \tau \) characterizes the influence of the past performance history of state of dynamic process, and \( \gamma \) describes the reaction or response time delay. In general, these time delays are unknown and random variables. These types of delay play a role in developing mathematical models of continuous time [64] and discrete time [59, 88] dynamic processes. Based upon the practical nature of data collection process, it is essential to either transform these time delays into positive integers or design the data collection schedule in relations with these delays. For this purpose, we describe the discrete version of time delays of \( \tau \) and \( \gamma \) as

\[
\begin{align*}
r &= \left\lfloor \frac{\tau}{\Delta t_i} \right\rfloor + 1, \quad \text{and} \quad q &= \left\lfloor \frac{\gamma}{\Delta t_i} \right\rfloor + 1, \\
\end{align*}
\]

(6.1)

respectively. Moreover, for the sake of simplicity, we assume that \( 0 < \gamma < 1 \) (q=1).

**Definition 6.2.1** Let \( x \) be a continuous time stochastic process defined on an interval \([-\tau, T]\) into \( \mathbb{R} \), for some \( T > 0 \). For \( t \in [-\tau, T] \), let \( \mathcal{F}_t \) be an increasing sub-sigma algebra of a complete probability space \((\Omega, \mathcal{F}, \mathcal{P})\) for which \( x(t) \) is \( \mathcal{F}_t \) measurable. Let \( P \) be a partition of \([-\tau, T]\) defined
by
\[ P := \{ t_i = t_0 + i\Delta t \}, \text{ for } i \in I(-r, N), \]  
(6.2)
where \( \Delta t = \frac{T-t_0}{N} \) and \( I(a, b) \) is defined by \( I(a, b) = \{ j \in \mathbb{Z} | a \leq j \leq b \} \).

Let \( \{ x(t_i) \}_{i=-r}^{N} \) be a finite sequence corresponding to the stochastic process \( x \) and partition \( P \) in Definition 6.2.1. We further note that \( x(t_i) \) is \( \mathcal{F}_{t_i} \)-measurable for \( i \in I(-r, N) \). We recall the definition of forward time shift operator \( F \) [11]:
\[ F^i x(t_k) = x(t_{k+i}). \]  
(6.3)
In addition, let us denote \( x(t_i) \) by \( x_i \) for \( i \in I(-r, N) \).

**Definition 6.2.2** For \( q = 1 \) and \( r \geq 1 \), each \( k \in I_0(N) \) and each \( m_k \in I(2, r + k - 1) \), a partition \( P_k \) of closed interval \( [t_{k-m_k}, t_{k-1}] \) is called local at time \( t_k \) and it is defined by
\[ P_k := t_{k-m_k} < t_{k-m_k+1} < \ldots < t_{k-1}. \]  
(6.4)
Moreover, \( P_k \) is referred as the \( m_k \)-point sub-partition of the partition \( P \) in (6.2) of the closed sub-interval \( [t_{k-m_k}, t_{k-1}] \) of \( [-\tau, T] \).

**Definition 6.2.3** For each \( k \in I_0(N) \) and each \( m_k \in I(2, r + k - 1) \), a local finite sequence at a time \( t_k \) of the size \( m_k \) is restriction of \( \{ x(t_i) \}_{i=-r}^{N} \) to \( P_k \) in (6.4) [2], and it is defined by
\[ S_{m_k,k} := \{ F^i x_{k-1} \}_{i=-m_k+1}^{0}. \]  
(6.5)
As \( m_k \) varies from \( 2 \) to \( k+r-1 \), the corresponding local sequence \( S_{m_k,k} \) at \( t_k \) varies from \( \{ x_i \}_{i=k-2}^{k-1} \) to \( \{ x_i \}_{i=-r+1}^{0} \). As a result of this, the sequence defined in (6.5) is also called a \( m_k \)-local moving sequence. Furthermore, the average corresponding to the local sequence \( S_{m_k,k} \) in (6.5) is defined by
\[ \bar{S}_{m_k,k} := \frac{1}{m_k} \sum_{i=-m_k+1}^{0} F^i x_{k-1}. \]  
(6.6)
The average/mean defined in (6.6) is also called the \( m_k \)-local average/mean. Moreover, the \( m_k \)-local variance corresponding to the local sequence \( S_{m_k,k} \) in (6.5) is defined by
\[ S^2_{m_k,k} := \begin{cases} \frac{1}{m_k} \sum_{i=-m_k+1}^{0} \left( F^i x_{k-1} - \frac{1}{m_k} \sum_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2, & \text{for small } m_k \\ \frac{1}{m_k-1} \sum_{i=-m_k+1}^{0} \left( F^i x_{k-1} - \frac{1}{m_k} \sum_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2, & \text{for large } m_k \end{cases} \]  
(6.7)
65
DEFINITION 6.2.4 For each fixed $k \in I(0, N)$, and any $m_k \in I_2(k + r - 1)$, the sequence 
$\{\bar{S}_{i,k}\}_{i=k-m_k}^{k-r}$ is called a $m_k-$local moving average/mean process at $t_k$. Moreover, the sequence 
$\{s_{i,k}^2\}_{i=k-m_k}^{k-r}$ is called a $m_k-$local moving variance process at $t_k$.

DEFINITION 6.2.5 Let $\{x(t_i)\}_{i=1}^N$ be a random sample of continuous time stochastic dynamic process collected at partition $P$ in (6.2). The local sample average/mean in (6.6) and local sample variance in (6.7) are called discrete time dynamic processes of sample mean and sample variance statistics.

DEFINITION 6.2.6 Let $\{x(t_i)\}_{i=1}^N$ be a random sample of continuous time stochastic dynamic process collected at partition $P$ in (6.2). The $m_k$-local moving average and variance defined in (6.6) and (6.7) are called the $m_k$-local moving sample average/mean and local moving sample variance at time $t_k$, respectively. Moreover, $m_k$-local sample average and $m_k$-local sample variance are referred to as local sample mean and local sample variance statistics for the local mean and variance of the continuous time stochastic dynamic system at time $t_k$, respectively.

In the following, we derive a dynamic algorithm described by the interconnected discrete-time local conditional sample average/mean and variance dynamic processes. First, we shall state and prove a change in $\bar{S}_{m_k,k}$ and $s_{m_k,k}^2$ with respect to change in time $t_k$. This fundamental result is motivated by Exercise 5.15 in [14].

DEFINITION 6.2.7 Let $\{E[x(t_i)|\mathcal{F}_{t_i-1}]\}_{i=-r+1}^N$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-$\sigma$ algebra $\mathcal{F}_{t_i}$, $t_i \in P$ in (6.2). The $m_k$-local conditional moving average and variance defined in the context of (6.6) and (6.7) are called the $m_k$-local conditional moving sample average/mean and local conditional moving sample variance, respectively.

LEMMA 6.1 (Discrete Time Dynamic Model of Local Sample Mean and Sample Variance Process). Let $\{E[x(t_i)|\mathcal{F}_{t_i-1}]\}_{i=-r+1}^N$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-$\sigma$ algebra $\mathcal{F}_{t_i}$, $t_i$ belong to partition $P$ in (6.2). Let $\bar{S}_{m_k,k}$ and $s_{m_k,k}^2$ be $m_k$-local conditional sample average and local conditional sample variance at $t_k$ for each $k \in I(0, N)$. Then, an interconnected discrete time dynamic model of local conditional sample mean and sample variance statistics is described by
The proof of Lemma 6.1 for small $m_k$, $m_{k-1} \leq m_k$, is given in C.1. The case for small $m_k$, $m_k \leq m_{k-1}$ is also described in C.2. The proof for large $m_k$, $m_{k-1} \leq m_k$, is given in C.3. \hfill \Box
**Remark 12** The interconnected system (6.8) can be re-written as the one-step Gauss-Sidel dynamic system [62] of iterative process described by

\[
X(k) = A(k, X(k-1))X(k-1) + e(k),
\]

(6.10)

where

\[
X(k) = \begin{pmatrix} X_1(k) \\ X_2(k) \end{pmatrix},
\]

\[
X_1(k) = \bar{S}_{m_{k-p+1},k-p+1},
\]

\[
X_2(k) = \begin{pmatrix} s_{m_{k-p+1},k-p+1}^2 \\ s_{m_{k-p+2},k-p+2}^2 \\ \vdots \\ s_{m_{k-1},k-1}^2 \\ s_{m_k,k}^2 \end{pmatrix},
\]

\[
A(k, X(k-1)) = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k, X(k-1)) & A_{22}(k) \end{pmatrix},
\]

\[
A_{11}(k) = \frac{m_{k-p}}{m_k - p + 1}, A_{12}(k) = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix},
\]

\[
A_{21}(k) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{(m_k-1)m_{k-p}}{m_k \prod_{j=0}^{m_k-j} m_{k-j}} \bar{S}_{m_{k-p},k-p} \end{pmatrix}, \text{ for small } m_k
\]

\[
A_{21}(k) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{m_{k-p}}{\prod_{j=0}^{m_k-p} m_{k-j}} \bar{S}_{m_{k-p},k-p} \end{pmatrix}, \text{ for large } m_k,
\]
\[
\begin{align*}
A_{22}(k) &= \begin{cases}
\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & 0 & 0 & 0 & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & 1
\end{pmatrix}, & \text{for small } m_k; \\
\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & 0 & 0 & 0 & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 0 & 1
\end{pmatrix}, & \text{for large } m_k
\end{cases},
\end{align*}
\]

\[
e(k) = \begin{pmatrix} e_1(k) \\ e_2(k) \end{pmatrix},
\]

\[
e_1(k) = \eta(k - p),
\]

\[
e_2(k) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \varepsilon(k - p + 1) \end{pmatrix}.
\]

**Remark 13** For each \( k \in I(0, N) \), \( p = 2 \) and small \( m_k \), the inter-connected system (6.8) reduces to the following special case:

\[
\begin{align*}
\tilde{S}_{m_{k-1},k-1} &= \frac{m_{k-2} - m_{k-1}}{m_{k-1}} \tilde{S}_{m_{k-2},k-2} + \eta m_{k-2,k-2}, \quad \tilde{S}_{m_0,0} = \tilde{S}_0 \\
\tilde{s}_{m,k}^2 &= \frac{m_{k-1} - m_k - 1}{m_k} \left[ m_{k-1}^2 s_{m_{k-1},k-1}^2 + \frac{m_{k-2} - m_{k-1}}{m_k} s_{m_{k-2},k-2}^2 + \frac{m_{k-2} - m_{k-1}}{m_k} \tilde{s}_{m_{k-2},k-2}^2 \right] \\
&+ \varepsilon m_{k-1,k-1}, \quad s_{m_{k},i}^2 = s_{i}^2, \quad i \in I_{-2}(0),
\end{align*}
\]

where
where \( \varphi_1 = \frac{m_k-1}{m_k} \varphi_1 \), \( \varphi_2 = \frac{m_k-1}{m_k} \varphi_2 \), and \( \varphi_3 = \frac{m_k-2}{m_k-1} \). For small \( m_k \), \( m_{k-1} \leq m_k \), \( \forall k \), we have \( \varphi_1 < 1 \), \( \varphi_2 < 1 \), and \( \varphi_3 \leq 1 \). From 0 < \( \varphi_i \), \( i = 1, 2, 3 \), and the fact that \( \varphi_1 + \varphi_2 = \frac{m_k-1}{m_k} \left[ m_{k-1} + \frac{m_{k-2}}{m_{k-1}} \right] \leq \frac{m_k-1}{m_k} \left[ m_{k-1} + 1 \right] \leq \frac{m_k-1}{m_k} < 1 \), the stability of the trivial solution \( X(k) = 0 \) of the homogeneous solution corresponding to (6.10) follows. Moreover, under the stated condition, the convergence of solutions of (6.10) also follows.

REMARK 14 Define \( \varphi_1 = \frac{m_k-1}{m_k} \varphi_1 \), \( \varphi_2 = \frac{m_k-1}{m_k} \varphi_2 \), and \( \varphi_3 = \frac{m_k-2}{m_k} \). For small \( m_k \), \( m_{k-1} \leq m_k \), \( \forall k \), we have \( \varphi_1 < 1 \), \( \varphi_2 < 1 \), and \( \varphi_3 \leq 1 \). From 0 < \( \varphi_i \), \( i = 1, 2, 3 \), and the fact that \( \varphi_1 + \varphi_2 = \frac{m_k-1}{m_k} \left[ m_{k-1} + \frac{m_{k-2}}{m_{k-1}} \right] \leq \frac{m_k-1}{m_k} \left[ m_{k-1} + 1 \right] \leq \frac{m_k-1}{m_k} < 1 \), the stability of the trivial solution \( X(k) = 0 \) of the homogeneous solution corresponding to (6.10) follows. Moreover, under the stated condition, the convergence of solutions of (6.10) also follows.

REMARK 15 Also, (6.11) can be re-written as

\[
X(k) = A(k, X(k-1))X(k-1) + e(k),
\]

where \( X(k) \), \( A(k) \) and \( e(k) \) are defined by \( X(k) = \begin{pmatrix} X_1(k) \\ X_2(k) \end{pmatrix}, X_1(k) = \bar{S}_{m_{k-1},k-1}, X_2(k) = \begin{pmatrix} s_{m_{k-1},k-1}^2 \\ s_{m_k,k}^2 \end{pmatrix}, \)

\[
A(k) = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{pmatrix}, A_{11}(k) = \frac{m_{k-2}}{m_{k-1}}, A_{12}(k) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, A_{21}(k) = \begin{pmatrix} 0 \\ \frac{(m_k-1)m_{k-2}}{m_k m_{k-1}} \bar{S}_{m_{k-2},k-2} \end{pmatrix},
\]

\[
A_{22}(k) = \begin{pmatrix} 0 \\ \frac{(m_{k-1})m_{k-2}}{m_k m_{k-1}} \end{pmatrix}, e(k) = \begin{pmatrix} e_1(k) \\ e_2(k) \end{pmatrix}, e_1(k) = \eta(k-2), e_2(k) = \begin{pmatrix} 0 \\ \varepsilon(k-1) \end{pmatrix}.
\]

REMARK 16 From Remark 13, we note that the local sample variance statistics at time \( t_k \) depends on the state of the \( m_{k-1} \) and \( m_{k-2} \)-local sample variance statistics at time \( t_{k-1} \) and \( t_{k-2} \), and the \( m_{k-2} \)-local sample mean statistics at time \( t_{k-2} \). We shall later compare the \( m_k \)-local sample variance statistics with the GARCH(p,q) model and show that the \( m_k \)-local sample variance statistics gives a better forecast than the GARCH(p,q) model under the usage of simulating a real data set.
6.3 Parametric Estimation

In this section, we consider a parameter estimation problem in drift and diffusion coefficients of a very general continuous-time nonlinear stochastic dynamic model described by a systems stochastic differential equations. This problem is motivated by the continuous-time dynamic model validation problem described in (4.11) in the context of energy commodity data set. This is achieved by utilizing the lagged adaptive process [88] and the interconnected discrete-time dynamics of local sample mean and variances statistic processes model in Section 6.2 (Lemma 6.1). We consider a general system of stochastic differential equations under the influence of hereditary effects in both the drift and diffusion coefficients described by

\[ dy = f(t, y_t)dt + \sigma(t, y_t)dW(t), y_{t_0} = \varphi_0, \]  

where \( y_t(\theta) = y(t + \theta), \theta \in [-\tau, 0], f, \sigma : [0, T] \times \mathbb{C} \to \mathbb{R}^q \) are Lipschitz continuous bounded functionals; \( \mathbb{C} \) is the Banach space of continuous functions defined on \([-\tau, 0]\) into \( \mathbb{R}^q \) equipped with the supremum norm; \( W(t) \) is standard Wiener process defined on a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\); \( \varphi_0 \in \mathbb{C} \), and \( y_0(t_0 + \theta) \) is \((\mathcal{F}_t)\) measurable; the filtration function \((\mathcal{F}_t)_{t \geq 0}\) is right-continuous, and each \( \mathcal{F}_t \) with \( t \geq t_0 \) contains all \( \mathbb{P} \)-null sets in \( \mathcal{F} \); the solution process \( y(t_0, \varphi_0)(t) \) of (6.13) is adapted and non-anticipating with respect to \((\mathcal{F}_t)_{t \geq 0}\).

Let \( V \in C([-\tau, \infty) \times \mathbb{R}^q, \mathbb{R}^m] \), and its partial derivatives \( V_t, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial y^2} \) exist and are continuous. We apply Itô-Doob stochastic differential formula [70] to \( V \), and we obtain

\[ dV(t, y, y_t) = LV(t, y, y_t)dt + V_y(t, y)\sigma(t, y_t)dW(t) \]  

where the \( L \) operator is defined by

\[
\begin{align*}
LV(t, y, y_t) &= V_t(t, y) + V_y(t, y)f(t, y_t) + \frac{1}{2}tr(V_{yy}(t, y))\sigma(t, y_t)\sigma^T(t, y_t) \\
\sigma(t, y_t) &= \sigma(t, y_t)\sigma^T(t, y_t).
\end{align*}
\]  

(6.15)

For (6.13) and (6.14), we present the Euler-type discretization scheme [58]:

\[
\begin{align*}
\Delta y_i &= f(t_{i-1}, y_{t_{i-1}})\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W_{t_{i-1}}, \quad i \in I_1(N) \\
\Delta V(t_i, y(t_i)) &= LV(t_{i-1}, y(t_{i-1}), y_{t_{i-1}})\Delta t_i + V_y(t_{i-1}, y(t_{i-1}))\sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i)
\end{align*}
\]  

(6.16)

Define \( \mathcal{F}_{t_{i-1}} = \mathcal{F}_{t_{i-1}} \) as the filtration process up to time \( t_{i-1} \). With regard to the continuous time dynamic system (6.13) and its transformed system (6.14), the more general moments of \( \Delta y(t_i) \) are
as follows:

\[
\begin{align*}
E [\Delta y(t_i) | F_{i-1}] & = f(t_{i-1}, y_{t_{i-1}}) \Delta t_i, \\
E [(\Delta y(t_i) - E [\Delta y(t_i) | F_{i-1}]) (\Delta y(t_i) - E [\Delta y(t_i) | F_{i-1}])] & = \sigma(t_{i-1}, y_{t_{i-1}}) \sigma^T(t_{i-1}, y_{t_{i-1}}) \Delta t_i, \\
E [\Delta V(t_i, y(t_i)) | F_{i-1}] & = LV(t_{i-1}, y(t_i), y_{t_{i-1}}) \Delta t_i, \\
E [(\Delta V(t_i, y(t_i)) - E [\Delta V(t_i, y(t_i)) | F_{i-1}]) (\Delta V(t_i, y(t_i)) - E [\Delta V(t_i, y(t_i)) | F_{i-1}])] & = B(t_{i-1}, y(t_{i-1}), y_{t_{i-1}}) \\
\end{align*}
\]

(6.17)

where \( B(t_{i-1}, y(t_{i-1}), y_{t_{i-1}}) = V_y(t_{i-1}, y(t_{i-1})) b(t_{i-1}, y_{t_{i-1}}) V_y(t_{i-1}, y(t_{i-1}))^T \Delta t_i \), and \( T \) stands for the transpose of the matrix.

From (6.16) and (6.17), we have

\[
\begin{align*}
\Delta y_i & = E [\Delta y(t_i) | F_{i-1}] + \sigma(t_{i-1}, y_{t_{i-1}}) \Delta W_{t_{i-1}}, \ i \in I_1(N) \\
\Delta V(t_i, y(t_i)) & = E [\Delta V(t_i, y(t_i)) | F_{i-1}] + V_y(t_{i-1}, y(t_{i-1})) \sigma(t_{i-1}, y_{t_{i-1}}) \Delta W(t_i)
\end{align*}
\]

(6.18)

This provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic systems (6.13) and (6.14). This indeed leads to a formulation of \( m_k \)-local generalized method of moments at \( t_k \).

Example 1:

For \( V(t, y) \) in (6.14) is defined by \( V(t, y) = \|y\|_p^p = \sum_{j=1}^{n} |y_j|^p \). In this case, we have

\[
dV = \left[ p \sum_{j=1}^{n} |y_j|^{p-1} \text{sgn}(y_j) f(t, y_t^j) + \frac{p(p-1)}{2} |y_j|^{p-2} \sigma(t, y_t^j) \right] dt + p \sum_{j=1}^{n} |y_j|^{p-1} \text{sgn}(y_j) \sigma(t, y_t^j) dW^j.
\]

(6.19)

Hence, the discretized form of (6.19) is given by

\[
\Delta V_i = \left[ p \sum_{j=1}^{n} |y_{t_{i-1}}^j|^{p-1} \text{sgn}(y_{t_{i-1}}^j) f(t_{i-1}, y_{t_{i-1}}^j) + \frac{p(p-1)}{2} |y_{t_{i-1}}^j|^{p-2} \sigma(t_{i-1}, y_{t_{i-1}}^j) \right] dt + p \sum_{j=1}^{n} |y_{t_{i-1}}^j|^{p-1} \text{sgn}(y_{t_{i-1}}^j) \sigma(t_{i-1}, y_{t_{i-1}}^j) dW_i^j.
\]

(6.20)
In this special case, (6.18) reduces to

\[
\begin{cases}
\Delta y_i = E[\Delta y(t_i)|F_i-1] + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W_{i-1}, \quad i \in I_1(N) \\
\Delta \left( \sum_{j=1}^{n} |y_{j_t}^j|^p \right) = E\left[ \Delta \left( \sum_{j=1}^{n} |y_{j_t}^j|^p \right) |F_i-1 \right] + p \sum_{j=1}^{n} |y_{j_t}^j|^{p-1} \text{sgn}(y_{j_t}^j)\sigma(t_{i-1}, y_{t_{i-1}})dW_{i}^j.
\end{cases}
\]

(6.21)

**Example 2:**

We consider AR(1) model as another example to exhibit the parameter and state estimation problem. The AR(1) model is of the following type

\[ X_i = \alpha_{i-1}X_{i-1} + e_i, \quad X_0 = x_0, \]

where \( X_i \) are \( F_i \) measurable, and \( e_i \) are independent white noise process and independent of \( x_0 \).

Hence

\[
\begin{cases}
E[X_i|F_{i-1}] = \alpha_{i-1}X_{i-1} \\
E[X_iX_i^T|F_{i-1}] = \alpha_{i-1}X_{i-1}X_{i-1}^T \alpha_{i-1}^T + E[e_ie_i^T|F_{i-1}]
\end{cases}
\]

(6.23)

In the following, we state a result that exhibits the existence of solution of system of non-linear equations. For the sake of easy reference, we shall re-state the Implicit function theorem without proof.

**Theorem 6.1 Implicit Function Theorem**[2] Let \( F = \{F_1, F_2, ..., F_q\} \) be a vector-valued function defined on an open set \( S \subset \mathbb{R}^{q+k} \) with values in \( \mathbb{R}^q \). Suppose \( F \in C_1 \) on \( S \). Let \( (u_0; v_0) \) be a point in \( S \) for which \( F(u_0; v_0) = 0 \) and for which the \( q \times q \) determinant \( \det [D_j F_i(u_0; v_0)] \neq 0 \).

Then there exists a \( k \)--dimensional open set \( T_0 \) containing \( v_0 \) and unique vector-valued function \( g \), defined on \( T_0 \) and having values in \( \mathbb{R}^q \), such that \( g \in C_1 \) on \( T_0 \), \( g(v_0) = u_0 \), and \( F(g(v); v) = 0 \) for every \( v \in T_0 \).

**6.4 Applications for Illustrations**

In the following, we give specific models for different commodities and apply the method of moments to estimate their parameters.
6.4.1 Application 1: Dynamic Model for Energy Commodity Price

We consider a stochastic dynamic model of energy commodities described by the following non-linear stochastic differential equation

$$dy = ay(\mu - y)dt + \sigma(t, y_{i})dydW(t), y_{t_{0}} = \varphi_{0},$$  \hspace{1cm} (6.24)

where $y_{t}(\theta) = y(t + \theta)$; $\theta \in [-\tau, 0]$, $\mu, a \in \mathbb{R}$; the initial process $\varphi_{0} = \{y(t_{0} + \theta)\}_{\theta \in [-\tau, 0]}$ is $\mathcal{F}_{t_{0}}$-measurable and independent of $\{W(t), t \in [0, T]\}$; $W(t)$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t})_{t \geq 0}, \mathcal{P})$ defined in (6.13); $\sigma : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}_{+}$ is a Lipschitz continuous and bounded functional; $\mathcal{C}$ is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm.

We pick a Lyapunov function $V(t, y) = \ln(y)$ in (6.14) for (6.24). Using Itô-differential formula [70], we have

$$d(\ln(y)) = \left[a(\mu - y) - \frac{1}{2}\sigma^{2}(t, y)\right]dt + \sigma(t, y_{i})dW(t).$$  \hspace{1cm} (6.25)

By setting $\Delta t_{i} = t_{i} - t_{i-1}$, $\Delta y_{i} = y_{i} - y_{i-1}$, the combined Euler discretized scheme for (6.24) and (6.25) is

$$\begin{align*}
\Delta y_{i} &= ay_{i-1}(\mu - y_{i-1})\Delta t_{i} + \sigma(t_{i-1}, y_{i-1})y_{i-1}\Delta W(t_{i}), \quad y_{0} = \varphi_{0}, \\
\Delta (\ln(y_{i})) &= \left[a(\mu - y_{i-1}) - \frac{1}{2}\sigma^{2}(t_{i-1}, y_{i-1})\right]\Delta t_{i} + \sigma(t_{i-1}, y_{i-1})\Delta W(t_{i}), \quad y_{0} = \varphi_{0}.
\end{align*}$$  \hspace{1cm} (6.26)

where $\varphi_{0} = \{y_{i}\}_{i=-r}^{0}$ is a given finite sequence of $\mathcal{F}_{0}$-measurable random variables, and it is independent of $\{\Delta W(t_{i})\}_{i=1}^{N}$.

Applying conditional expectation to (6.26) with respect to $\mathcal{F}_{t_{i-1}} \equiv \mathcal{F}_{i-1}$, we obtain

$$\begin{align*}
\mathbb{E}[\Delta y_{i}|\mathcal{F}_{i-1}] &= ay_{i-1}(\mu - y_{i-1})\Delta t \\
\mathbb{E}[\Delta (\ln(y_{i}))|\mathcal{F}_{i-1}] &= \left[a(\mu - y_{i-1}) - \frac{1}{2}\sigma^{2}(t_{i-1}, y_{i-1})\right]\Delta t \\
\mathbb{E}\left[(\Delta (\ln(y_{i})) - \mathbb{E}[\Delta (\ln(y_{i}))|\mathcal{F}_{i-1}])^{2}|\mathcal{F}_{i-1}\right] &= \sigma^{2}(t_{i-1}, y_{i-1})\Delta t.
\end{align*}$$  \hspace{1cm} (6.27)

From (6.27), (6.26) reduces to

$$\begin{align*}
\Delta y_{i} &= \mathbb{E}[\Delta y_{i}|\mathcal{F}_{i-1}] + \sigma(t_{i-1}, y_{i-1})y_{i-1}\Delta W(t_{i}) \\
\Delta (\ln(y_{i})) &= \mathbb{E}[\Delta (\ln(y_{i}))|\mathcal{F}_{i-1}] + \sigma(t_{i-1}, y_{i-1})\Delta W(t_{i}).
\end{align*}$$  \hspace{1cm} (6.28)

(6.28) provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic systems (6.24) and (6.25).
For $k \in I(0, N)$, applying the lagged adaptive expectation process [88], from Definitions 6.2.3 – 6.2.7, and using (6.8) and (6.28), we formulate a local observation/measurement process at $t_k$ as a algebraic functions of $m_k$-local functions of restriction of the overall finite sample sequence $\{y_i\}_{i=-r}^N$ to subpartition $P_k$ in Definition 6.2.2:

$$
\begin{align*}
\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} [\Delta y_i | \mathcal{F}_{i-1}] &= a \left[ \frac{\mu}{m_k} \sum_{i=k-m_k}^{k-1} y_{i+1} - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_i^2 \right] \Delta t, \\
\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}] &= a \left[ \mu - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_i - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_i^2 \right] \Delta t \\
- \frac{1}{2m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} \left[ (\Delta (\ln(y_i))) - \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}] \right]^2 | \mathcal{F}_{i-1} \right], \\
\end{align*}
$$

(6.29)

$$
\hat{\sigma}_{m_k,k}^2 = 
\begin{cases}
\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} \left[ (\Delta (\ln(y_i))) - \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}] \right]^2 | \mathcal{F}_{i-1} \\
\frac{1}{(m_k-1) \Delta t} \sum_{i=k-m_k}^{k-1} \mathbb{E} \left[ (\Delta (\ln(y_i))) - \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}] \right]^2 | \mathcal{F}_{i-1} 
\end{cases}
$$

if $m_k$ is small. 

(6.30)

From (6.30), it follows that the average volatility square $\hat{\sigma}_{m_k,k}^2$ is given by

$$
\hat{\sigma}_{m_k,k}^2 = \frac{s_{m_k,k}^2}{\Delta t},
$$

(6.31)

where $s_{m_k,k}^2$ is the local sample variance statistics for volatility at $t_k$ in the context of $x(t_i) = \Delta (\ln(y_i))$. We define

$$
\begin{align*}
F_1 (\mathbb{E} [\Delta y_i | \mathcal{F}_{i-1}], \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}]; a, \mu) &= \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} [\Delta y_i | \mathcal{F}_{i-1}] \\
&= a \left[ \frac{\mu}{m_k} \sum_{i=k-m_k}^{k-1} y_{i+1} - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_i^2 \right] \Delta t \\
F_2 (\mathbb{E} [\Delta y_i | \mathcal{F}_{i-1}], \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}]; a, \mu) &= \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}] \\
&= a \left[ \mu - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_i - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_i^2 \right] \Delta t + \frac{s_{m_k,k}^2}{2}.
\end{align*}
$$

(6.32)

Then we have

$$
\begin{align*}
F_1 (\mathbb{E} [\Delta y_i | \mathcal{F}_{i-1}], \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}]; a, \mu) &= 0, \\
F_2 (\mathbb{E} [\Delta y_i | \mathcal{F}_{i-1}], \mathbb{E} [\Delta (\ln(y_i)) | \mathcal{F}_{i-1}]; a, \mu) &= 0.
\end{align*}
$$

(6.33)
Let $F = \{F_1, F_2\}$. The determinant of the Jacobian matrix of $F$ is given by

$$JF(\alpha, \mu) = -\frac{m}{m_k} \left[ -\sum_{i=k-m_k}^{k-1} y_i^2 - \frac{1}{m_k} \left( \sum_{i=k-m_k}^{k-1} y_i \right)^2 \right] (\Delta t)^2$$

(6.34)

provided that $a \neq 0$ or the sequence $\{x(t_{i-1})\}_{i=-r+1}^N$ is neither zero nor a constant. This fulfills the hypothesis of Theorem 6.1.

Thus, by the application of Theorem 6.1 (Implicit Function Theorem), we conclude that for every non-constant $m_k$-local sequence $\{x(t_i)\}_{i=k-m_k}$, there exist a unique solution of system of algebraic equations (6.33), $\hat{a}_{m_k, k}$ and $\hat{\mu}_{m_k, k}$ as a point estimates of $a$ and $\mu$, respectively.

We also note that the estimated values of $a$ and $\mu$ change at each time $t_k$. For instance, at time $t_0 = 0$ and the given $F_{-1}$ measurable discrete-time process $y_{-r+1}, y_{-r+2}, \ldots, y_1$, (6.29)-(6.30) reduce to

$$\begin{align*}
\begin{aligned}
\frac{1}{m_0} \sum_{i=-m_0}^{0} \Delta y_i &= a \left[ \frac{\mu}{m_0} \sum_{i=-m_0}^{0} y_i - \frac{1}{m_0} \sum_{i=-m_0}^{0} y_i^2 \right] \Delta t,
\frac{1}{m_0} \sum_{i=-m_0}^{0} \Delta (\ln y_i) &= a \left[ \mu - \frac{1}{m_0} \sum_{i=-m_0}^{0} y_i - 1 \right] \Delta t - \frac{s_{m_0, 0}^2}{2},
\hat{\sigma}_{m_0, 0}^2 &= \frac{s_{m_0, 0}^2}{\Delta t}.
\end{aligned}
\end{align*}$$

(6.35)

The initial solution of algebraic equations (6.35) at time $t_0$ is given by

$$\begin{align*}
\begin{aligned}
\hat{a}_{m_0, 0} &= \left( \frac{1}{m_0} \sum_{i=-m_0}^{0} \Delta (\ln y_i) + \frac{s_{m_0, 0}^2}{2} \right) \left( \frac{1}{m_0} \sum_{i=-m_0}^{0} y_i - 1 \right) \Delta t - \frac{1}{m_0} \sum_{i=-m_0}^{0} \Delta y_i,
\frac{1}{m_0} \sum_{i=-m_0}^{0} \Delta (\ln y_i) &= \frac{1}{m_0} \left( \sum_{i=-m_0}^{0} y_i^2 - \frac{1}{m_0} \left( \sum_{i=-m_0}^{0} y_i \right)^2 \right) \Delta t,
\hat{\mu}_{m_0, 0} &= \frac{1}{m_0 \Delta t} \sum_{i=-m_0}^{0} \Delta (\ln y_i) + \frac{s_{m_0, 0}^2}{2 \Delta t} + \frac{\delta_{m_0, 0}}{m_0} \left( \sum_{i=-m_0}^{0} y_i - 1 \right)
\hat{\sigma}_{m_0, 0}^2 &= \frac{s_{m_0, 0}^2}{\Delta t}.
\end{aligned}
\end{align*}$$

(6.36)

At time $t_1 = 1$ and the given $F_0$ measurable discrete-time process $y_{-r}, y_{-r+1}, \ldots, y_1, y_0$, (6.29)-(6.30) reduce to

$$\begin{align*}
\begin{aligned}
\frac{1}{m_1} \sum_{i=1-m_1}^{0} \Delta y_i &= a \left[ \frac{\mu}{m_1} \sum_{i=1-m_1}^{0} y_i - 1 \right] \Delta t - \frac{1}{m_1} \sum_{i=1-m_1}^{0} y_i^2 \Delta t,
\frac{1}{m_1} \sum_{i=1-m_1}^{0} \Delta (\ln y_i) &= a \left[ \mu - \frac{1}{m_1} \sum_{i=1-m_1}^{0} y_i - 1 \right] \Delta t - \frac{s_{m_1, 1}^2}{2},
\hat{\sigma}_{m_1, 1}^2 &= \frac{s_{m_1, 1}^2}{\Delta t}.
\end{aligned}
\end{align*}$$

(6.37)
The solution of algebraic equations (6.37) is given by
\[
\begin{align*}
\hat{a}_{m,1} &= \frac{1}{m_1} \sum_{i=1-m_1}^{0} \Delta(y_{i}) + \frac{s^2_{m,1}}{\Delta t} \left( \frac{1}{m_1} \sum_{i=1-m_1}^{0} y_{i-1} \right) - \frac{1}{m_1} \sum_{i=1-m_1}^{0} \Delta y_i \\
\hat{\mu}_{m,1} &= \frac{1}{m_1 \Delta t} \sum_{i=1-m_1}^{0} \Delta(y_{i}) + \frac{s^2_{m,1}}{\Delta t} + \frac{\hat{a}_{m,1}}{m_1} \left( \frac{1}{m_1} \sum_{i=1-m_1}^{0} y_{i-1} \right) \\
\hat{\sigma}^2_{m,1} &= \frac{s^2_{m,1}}{\Delta t},
\end{align*}
\]

Likewise, for \(k = 2\), we have
\[
\begin{align*}
\hat{a}_{m,2} &= \frac{1}{m_2} \sum_{i=2-m_2}^{1} \Delta(y_{i}) + \frac{s^2_{m,k}}{\Delta t} \left( \frac{1}{m_2} \sum_{i=2-m_2}^{1} y_{i-1} \right) - \frac{1}{m_2} \sum_{i=2-m_2}^{1} \Delta y_i \\
\hat{\mu}_{m,2} &= \frac{1}{m_2 \Delta t} \sum_{i=2-m_2}^{1} \Delta(y_{i}) + \frac{s^2_{m,k}}{\Delta t} + \frac{\hat{a}_{m,2}}{m_2} \left( \frac{1}{m_2} \sum_{i=2-m_2}^{1} y_{i-1} \right) \\
\hat{\sigma}^2_{m,2} &= \frac{s^2_{m,k}}{\Delta t},
\end{align*}
\]

Hence, from (6.29)-(6.30) and applying the principle of mathematical induction [69], we have
\[
\begin{align*}
\hat{a}_{m,k} &= \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \Delta(y_{i}) + \frac{s^2_{m,k}}{\Delta t} \left( \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_{i-1} \right) - \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \Delta y_i \\
\hat{\mu}_{m,k} &= \frac{1}{m_k \Delta t} \sum_{i=k-m_k}^{k-1} \Delta(y_{i}) + \frac{s^2_{m,k}}{\Delta t} + \frac{\hat{a}_{m,k}}{m_k} \left( \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_{i-1} \right) \\
\hat{\sigma}^2_{m,k} &= \frac{s^2_{m,k}}{\Delta t},
\end{align*}
\]

**Remark 17** We note that without loss in generality, the discrete-time data set \(\{y_{r+i} : i \in I_1(r-1)\}\) is assumed to be close to the true values of the solution process of the continuous-time dynamic process. In fact, this assumption is feasible in view of the uniqueness and continuous dependence of solution process of stochastic functional or ordinary differential equation with respect to the initial data [70].

**Remark 18** If the sample \(\{y_{i}\}_{i=k-m_k-1}^{k-1}\) is a constant sequence, then it follows from (6.40) and the fact that \(\Delta(\ln y_i) = 0\) and \(s^2_{m,k} = 0\), that \(\hat{\mu}_{m,k} \rightarrow \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_{i-1}\), and it follows from (6.29)-(6.30) that \(\hat{a}_{m,k} = 0\).

**Remark 19** As we stated before, estimated parameters \(a, \mu, \sigma^2\) depend upon the time at which data point is drawn. This is what we expected because of the fact that nonlinearity of the dynamic
model generates non stationary solution process. Using this locally estimated parameters of the continuous-time dynamic system, we can find the average of this local parameters over the size of data set as follows:

\[
\begin{align*}
\bar{a} &= \frac{1}{N} \sum_{i=0}^{N} a_{\hat{m}_i,i}, \\
\bar{\mu} &= \frac{1}{N} \sum_{i=0}^{N} \mu_{\hat{m}_i,i} \\
\bar{\sigma}^2 &= \frac{1}{N} \sum_{i=0}^{N} \sigma_{\hat{m}_i,i}^2.
\end{align*}
\]  

(6.41)

\(\bar{a}, \bar{\mu}, \text{and} \bar{\sigma}^2\) are referred to as aggregated parameter estimates of \(a, \mu, \text{and} \sigma^2\) over the given entire finite interval of time, respectively.

**6.4.2 Application 2: Dynamic Model for U.S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate**

We also apply the above presented scheme for estimating parameters of a continuous-time model for U.S. Treasury Bill Yield Interest Rate [128] and U. S. Eurocurrency Exchange Rate [129] processes. By employing dynamic modeling process [69, 70], a continuous time dynamic model of interest rate process under random environmental perturbations can be described by

\[
d\hat{y} = (\beta y + \mu y \delta) dt + \sigma y^\gamma dW(t), \quad y(t_0) = y_0,
\]  

(6.42)

where \(\beta, \mu, \delta, \sigma, \gamma \in \mathbb{R}; \ y(t, t_0, y_0)\) is adapted, non-anticipating solution process with respect to \(\mathcal{F}_t\); the initial process \(y_0\) is \(\mathcal{F}_{t_0}\)-measurable and independent of \(\{W(t), t \in [t_0, T]\}\); \(W(t)\) is a standard Wiener process defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).

For (6.42), we consider the Lyapunov functions \(V_1(t, y) = \frac{1}{2} y^2\), and \(V_2(t, y) = \frac{1}{3} y^3\) as in (6.14). The Itô differentials of \(V_i\), for \(i = 1, 2\), are given by

\[
\begin{align*}
\text{d}V_1 &= [y(\beta y + \mu y \delta) + \frac{1}{2} \sigma^2 y^{2\gamma}] \, dt + \sigma y^\gamma + 1 dW(t) \\
\text{d}V_2 &= [y^2(\beta y + \mu y \delta) + \sigma^2 y^{2\gamma + 1}] \, dt + \sigma y^{\gamma + 2} dW(t).
\end{align*}
\]  

(6.43)

Following the approach in Section 6.3 and illustration 6.4.1, the Euler discretized scheme \((\Delta t = 1)\) for (6.42) is defined by

\[
\begin{align*}
\Delta y_i &= (\beta y_{i-1} + \mu y_{i-1}^\delta) + \sigma y_{i-1}^{\gamma} \Delta W_i \\
\frac{1}{2} \Delta (y_i^2) &= y_{i-1}(\beta y_{i-1} + \mu y_{i-1}^\delta) + 2^\frac{1}{2} \sigma^2 y_{i-1}^{2\gamma} + 2^{\gamma + 1} y_{i-1}^{\gamma + 1} \Delta W_i \\
\frac{1}{3} \Delta (y_i^3) &= y_{i-1}^2(\beta y_{i-1} + \mu y_{i-1}^\delta) + 3^\frac{1}{2} \sigma^2 y_{i-1}^{2\gamma + 1} + 3^{\gamma + 2} y_{i-1}^{\gamma + 2} \Delta W_i.
\end{align*}
\]  

(6.44)
Applying conditional expectation to (6.44) with respect to \( \mathcal{F}_{i-1} \), we obtain

\[
\begin{align*}
E[\Delta y_i | \mathcal{F}_{i-1}] &= \beta y_{i-1} + \mu y_{i-1}^\delta \\
\frac{1}{2}E[\Delta(y_i^2) | \mathcal{F}_{i-1}] &= \beta y_{i-1}^2 + \mu y_{i-1}^{\delta+1} + \frac{1}{2} \sigma^2 y_{i-1}^{2\gamma} \\
\frac{1}{2}E[\Delta(y_i^3) | \mathcal{F}_{i-1}] &= \beta y_{i-1}^3 + \mu y_{i-1}^{\delta+2} + \sigma^2 y_{i-1}^{2\gamma+1} \\
E[(\Delta y_i - E[\Delta y_i | \mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}] &= \sigma^2 y_{i-1}^{2\gamma} \\
\frac{1}{4}E[(\Delta(y_i^2) - E[\Delta(y_i^2)])^2 | \mathcal{F}_{i-1}] &= \sigma^2 y_{i-1}^{2\gamma+2}.
\end{align*}
\]

From (6.45), (6.44) reduces to

\[
\begin{align*}
\Delta y_i &= E[\Delta y_i | \mathcal{F}_{i-1}] + \sigma y_{i-1}^\gamma \Delta W(t_i) \\
\frac{1}{2} \Delta(y_i^2) &= \frac{1}{2} E[\Delta(y_i^2) | \mathcal{F}_{i-1}] + \sigma y_{i-1}^{\gamma+1} \Delta W_i \\
\frac{1}{3} \Delta(y_i^3) &= \frac{1}{3} E[\Delta(y_i^3) | \mathcal{F}_{i-1}] + \sigma y_{i-1}^{\gamma+2} \Delta W_i.
\end{align*}
\]

Following the argument used in (6.29)-(6.30), for \( k \in I(0, N) \), applying the lagged adaptive expectation process [88], from Definitions 6.2.3 – 6.2.7, and using (6.8) and (6.45), we formulate a local observation/measurement process at \( t_k \) as an algebraic functions of \( m_k \)-local functions of restriction of the overall finite sample sequence \( \{y_i\}_{i=-r}^N \) to subpartition \( P_k \) in Definition 6.2.2:

\[
\begin{align*}
\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} E[\Delta y_i | \mathcal{F}_{i-1}] &= \beta \sum_{i=k-m_k}^{k-1} y_{i-1} \\
+ \mu \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1} \\
\frac{1}{2m_k} \sum_{i=k-m_k}^{k-1} \left[ E[\Delta(y_i^2) | \mathcal{F}_{i-1}] - E[\Delta y_i - E[\Delta y_i | \mathcal{F}_{i-1}]^2 | \mathcal{F}_{i-1}] \right] &= \beta \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma} \\
+ \mu \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2} \\
\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \left[ \frac{1}{3} E[\Delta(y_i^3) | \mathcal{F}_{i-1}] - \sigma^2 y_{i-1}^{2\gamma+1} \right] &= \beta \sum_{i=k-m_k}^{k-1} y_{i-1}^{3\gamma} \\
+ \mu \sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+3} \\
\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} E[(\Delta y_i - E[\Delta y_i | \mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}] &= \sigma^2 \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma} \\
\frac{1}{4m_k} \sum_{i=k-m_k}^{k-1} \left[ \frac{1}{3} E[(\Delta(y_i^2) - E[\Delta(y_i^2)])^2 | \mathcal{F}_{i-1}] \right] &= \sigma^2 \sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma+2}.
\end{align*}
\]
Following the approach discussed in Section 6.4.1, the solution of $\sigma_{m_k,k}$ is given by

$$\sigma_{m_k,k} = \left[ \frac{s^{2}_{m_k,k}}{\frac{1}{m_k} \sum_{i=k-m_k}^{k-1} y_{i-1}^{2}} \right]^{1/2}$$

(6.48)

and $\gamma_{m_k,k}$ satisfies the following nonlinear algebraic equation

$$s^{2}_{m_k,k} \sum_{i=k-m_k}^{k-1} y_{i-1}^{2} \gamma_{m_k,k} + 2 = \frac{1}{4} s^{2}_{m_k,k} \sum_{i=k-m_k}^{k-1} y_{i-1}^{2} = 0,$$

(6.49)

where $s^{2}_{m_k,k}$ and $s^{2}_{m_k,k}$ denotes the local moving variance of $\Delta y_i$ and $\Delta(y_i^2)$ respectively.

To solve for the parameters $\beta$, $\mu$ and $\delta$, we define the conditional moment functions

$$F_j \equiv F_j \left( \mathbb{E} \left[ \Delta y_i | \mathcal{F}_{i-1} \right], \mathbb{E} \left[ (\Delta y_i)^2 | \mathcal{F}_{i-1} \right], \mathbb{E} \left[ \Delta(y_i)^3 | \mathcal{F}_{i-1} \right] \right), \quad j = 1, 2, 3$$

as

$$F_1 = \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \mathbb{E} \left[ \Delta y_i | \mathcal{F}_{i-1} \right] - \beta \frac{\sum_{i=k-m_k}^{k-1} y_{i-1}}{m_k} - \mu \frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k}$$

$$F_2 = \frac{1}{2m_k} \sum_{i=k-m_k}^{k-1} \left[ \mathbb{E} \left[ (\Delta y_i)^2 | \mathcal{F}_{i-1} \right] - \mathbb{E} \left[ (\Delta y_i - \mathbb{E} \left[ \Delta y_i | \mathcal{F}_{i-1} \right])^2 | \mathcal{F}_{i-1} \right] \right] - \beta \frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k}$$

$$F_3 = \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \left[ \frac{1}{3} \mathbb{E} \left[ (\Delta y_i)^3 | \mathcal{F}_{i-1} \right] - \sigma^2 y_{i-1}^2 \right] - \beta \frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} - \mu \frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^4}{m_k}.$$

(6.50)

Using (6.47), we have

$$\begin{cases}
F_1 = 0 \\
F_2 = 0 \\
F_3 = 0
\end{cases}$$

(6.51)

Let $F = \{F_1, F_2, F_3\}$. The determinant of the Jacobian matrix of $F$ is given by

$$J F(\beta, \mu, \delta) = -\frac{1}{m_k} \det \begin{pmatrix}
\sum_{i=k-m_k}^{k-1} y_{i-1} & \sum_{i=k-m_k}^{k-1} y_{i-1}^\delta & \sum_{i=k-m_k}^{k-1} \left( \ln y_{i-1} \right) y_{i-1}^\delta \\
\sum_{i=k-m_k}^{k-1} y_{i-1}^2 & \sum_{i=k-m_k}^{k-1} y_{i-1}^\delta + 1 & \sum_{i=k-m_k}^{k-1} \left( \ln y_{i-1} \right) y_{i-1}^\delta + 1 \\
\sum_{i=k-m_k}^{k-1} y_{i-1}^3 & \sum_{i=k-m_k}^{k-1} y_{i-1}^\delta + 2 & \sum_{i=k-m_k}^{k-1} \left( \ln y_{i-1} \right) y_{i-1}^\delta + 2
\end{pmatrix} \neq 0$$

(6.52)
provided $\delta \neq 1$ and the sequence $\{y(t_i)\}_{i=k-m_k}^{k-1}$ is neither zero nor a constant. We want to avoid the case where $\delta = 1$ because this will change the structure of (6.42). Thus, by the application of Theorem 6.1 (Implicit Function Theorem), we conclude that for every non-constant $m_k$-local sequence $\{y(t_i)\}_{i=k-m_k}^{k-1}$, $\delta \neq 1$, there exist a solution of system of algebraic equations (6.51) $\hat{\beta}_{m_k,k}$, $\hat{\mu}_{m_k,k-1}$, $\hat{\delta}_{m_k,k}$ as a point estimates of $\beta$ and $\mu$, and $\delta$ respectively.

The solution of (6.51) is given by

$$\begin{align*}
\hat{\beta}_{m_k,k} &= \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \Delta y_i \sum_{i=k-m_k}^{k-1} y_i^{2,1-1} \left[ \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \Delta (y_i^2) - \sigma_{m_k,k}^2 \right] \sum_{i=k-m_k}^{k-1} y_i^{1-1} \\
\hat{\mu}_{m_k,k} &= \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \Delta y_i - \hat{\mu}_{m_k,k-1} \sum_{i=k-m_k}^{k-1} y_i^{1-1} \sum_{i=k-m_k}^{k-1} \delta_{m_k,k} \\
\hat{\beta}_{m_k,k} &= \frac{1}{m_k} \sum_{i=k-m_k}^{k-1} \Delta y_i - \hat{\beta}_{m_k,k-1} \sum_{i=k-m_k}^{k-1} y_i^{1-1} \sum_{i=k-m_k}^{k-1} \delta_{m_k,k} \\
\hat{\delta}_{m_k,k} &= \sum_{i=k-m_k}^{k-1} y_i^{1-1} \\
\end{align*}$$

where $\delta_{m_k,k}$ satisfies the third equation in (6.47) described by

$$\frac{1}{3m_k} \sum_{i=k-m_k}^{k-1} \Delta (y_i^3) - \frac{\sigma_{m_k,k}^2}{m_k} \sum_{i=k-m_k}^{k-1} y_i^{2,1-1} - \beta \sum_{i=k-m_k}^{k-1} y_i^{3,1-1} - \mu \sum_{i=k-m_k}^{k-1} y_i^{3,2-1} = 0$$

(6.54)
Chapter 7  
Computational and Simulation Algorithms

7.1 Introduction

In this chapter, we outline computational, data organizational and simulation schemes. We introduce the ideas of iterative data process and data simulation time schedules in relation with the real time data observation/collection schedule. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is essential to identify an admissible set of local conditional sample average and sample variance parameters, namely, the size of local conditional sample in the context of a partition of time interval $[-\tau, T]$. Moreover, the discrete time dynamic model of conditional sample mean and sample variance statistics processes in Section 6.2 and the theoretical parameter estimation scheme in Section 6.3 motivates to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below:

7.2 Coordination of Data Observation, Iterative Process, and Simulation Schedules:

Without loss of generality, we assume that the real data observation/collection partition schedule $P$ is defined in (6.2). Now, we present definitions of iterative process and simulation time schedule.

**Definition 7.2.1** The iterative process time schedule in relation with the real data collection schedule is defined by

$$IP = \{F^{-r}t_i : \text{for } t_i \in P\}, \quad (7.1)$$

where $F^{-r}t_i = t_{i-r}$, and $F^{-r}$ is a forward shift operator [11].

The simulation time is based on the order $p$ of the time series model of $m_k$-local conditional sample mean and variance processes in Lemma 6.1.
**Definition 7.2.2** The simulation process time schedule in relation with the real data observation schedule is defined by

\[ SP = \begin{cases} 
F^r t_i : & \text{for } t_i \in P, \text{ if } p \leq r \\
F^p t_i : & \text{for } t_i \in P, \text{ if } p > r.
\end{cases} \tag{7.2} \]

**Remark 20** We note that the initial times of iterative and simulation processes are equal to the real data times \( t_r \) and \( t_p \), respectively. Moreover, iterative and simulation processes time in (7.1) and (7.2), respectively justify Remark 17. In short, \( t_i \) is the scheduled time clock for the collection of the \( i \)-th observation of the state of the system under investigation. The iterative process and simulation process times are \( t_{i+r} \) and \( t_{i+p} \), respectively.

### 7.3 Conceptual Computational Parameter Estimation Scheme

For the conceptual computational dynamic system parameter estimation, we need to introduce a few concepts of local admissible sample/data observation size, \( m_k \)-local admissible conditional finite sequence at \( t_k \in SP \), local finite sequence of parameter estimates at \( t_k \).

**Definition 7.3.1** For each \( k \in I(0, N) \), we define local admissible sample/data observation size \( m_k \) at \( t_k \) as \( m_k \in OS_k \), where

\[ OS_k = \begin{cases} 
I(2, r + k - 1), & \text{if } p \leq r, \\
I(2, p + k - 1), & \text{if } p > r,
\end{cases} \tag{7.3} \]

Moreover, \( OS_k \) is referred as the local admissible set of lagged sample/data observation size at \( t_k \).

**Definition 7.3.2** For each admissible \( m_k \in OS_k \) in Definition 7.3.1, a \( m_k \)-local admissible lagged-adapted finite restriction sequence of conditional sample/data observation at \( t_k \) to subpartition \( P_k \) of \( P \) in Definition 6.2.3 is defined by \( \{ \mathbb{E}[y_i|\mathcal{F}_{i-1}] \}_{i=k-m_k}^{k-1} \). Moreover, a \( m_k \)-class of admissible lagged-adapted finite sequences of conditional sample/data observation of size \( m_k \) at \( t_k \) is defined by

\[ AS_k = \{ \{ \mathbb{E}[y_i|\mathcal{F}_{i-1}] \}_{i=k-m_k}^{k-1} : m_k \in OS_k \} = \{ \{ \mathbb{E}[y_i|\mathcal{F}_{i-1}] \}_{i=k-m_k}^{k-1} \}_{m_k \in OS_k}. \tag{7.4} \]

In the case of energy commodity model, for each \( m_k \in OS_k \), we find corresponding \( m_k \)-local admissible adapted finite sequence of conditional sample/data observation at \( t_k \), \( \{ \mathbb{E}[y_i|\mathcal{F}_{i-1}] \}_{i=k-m_k}^{k-1} \). Using this sequence and (6.40), we compute \( \hat{\alpha}_{m_k, k}, \hat{\mu}_{m_k, k} \) and \( \hat{\sigma}_{m_k, k}^2 \). This leads to a local finite sequence of parameter estimates at \( t_k \) defined on \( OS_k \) as follows: \( \{ (\hat{\alpha}_{m_k, k}, \hat{\mu}_{m_k, k}, \hat{\sigma}_{m_k, k}^2) \} \}_{m_k \in OS_k} =
For any arbitrary small positive number \( \epsilon \), simulated values of \( \{\hat{y}_{m,k}, \hat{\mu}_{m,k}, \hat{\sigma}^2_{m,k}\}\) \( m_k \in S_k \) or \( \{(\hat{a}_{m,k}, \hat{\mu}_{m,k}, \hat{\sigma}^2_{m,k})\} \)

It is denoted by \( (A_k, M_k, S_k)\) = \( \{(\hat{a}_{m,k}, \hat{\mu}_{m,k}, \hat{\sigma}^2_{m,k})\}\) \( m_k \in OS_k \).

### 7.4 Conceptual Computation of State Simulation Scheme

For the development of a conceptual computational scheme, we need to employ the method of induction. The presented simulation scheme is based on the idea of lagged adaptive expectation process [88]. An autocorrelation function (ACF) analysis [11, 14] performed on \( s^2_{m,k}\) suggests that the interconnected discrete time dynamic model of local conditional sample mean and sample variance statistics in (6.8) is of order \( p = 2 \). In view of this, we need to identify the initial data.

We begin with a given initial data \( y_0, \{\hat{s}^2_{m_{0,0}}\} m_0 \in OS_0, \{\hat{s}^2_{m_{-1,-1}}\} m_{-1} \in OS_{-1}, \{\hat{S}^2_{m_{-1,-1}}\} m_{-1} \in OS_{-1} \).

Let \( y_{m,k}^\theta \) be a simulated value of \( E[y_k|\mathcal{F}_{k-1}] \) at time \( t_k \) corresponding to an admissible sequence \( \{E[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1} \in AS_k \). This simulated value is derived from the discretized Euler scheme (6.26) by

\[
y_{m,k}^\theta = y_{m_k-1,k-1} + \hat{a}_{m_k-1,k-1}(\hat{\mu}_{m_k-1,k-1} - y_{m_k-1,k-1})y_{m_k-1,k-1}(1 - \hat{\sigma}_{m_k-1,k-1})y_{m_k-1,k-1}\Delta W_{m_k,k}.
\]  

(7.5)

Let \( \{y_{m,k}^\theta\} m_k \in OS_k \) be a \( m_k \)-local sequence of simulated values corresponding to \( m_k \)-admissible lagged adapted finite sequence of conditional observation belonging to \( AS_k \) and corresponding term of sequence \( (A_k, M_k, S_k) \). Thus, \( \{y_{m,k}^\theta\} m_k \in OS_k \) is the finite sequence corresponding to finite simulated values of \( E[y_k|\mathcal{F}_{k-1}] \) at \( t_k \).

### 7.5 Mean-Square Sub-Optimal Procedure

To find the the best estimate of \( E[y_k|\mathcal{F}_{k-1}] \) using a local admissible finite sequence \( \{y_{m,k}^\theta\} m_k \in OS_k \) of simulation of \( \{E[y_i|\mathcal{F}_{i-1}]\} \), we need to compute a finite sequence of quadratic mean square error corresponding to \( \{y_{m,k}^\theta\} m_k \in OS_k \). The quadratic mean square error is defined below.

**Definition 7.5.1** The quadratic mean square error of \( E[y_k|\mathcal{F}_{k-1}] \) relative to each member of the term of local admissible sequence \( \{y_{m,k}^\theta\} m_k \in OS_k \) of simulated values is defined by

\[
\Xi_{m_k,k,y_k} = (E[y_k|\mathcal{F}_{k-1}] - y_{m_k}^\theta)^2.
\]  

(7.6)

For any arbitrary small positive number \( \epsilon \) and for each time \( t_k \), to find the the best estimate from the admissible simulated values of simulated sequence of \( \{y_{m,k}^\theta\} m_k \in OS_k \) for \( E[y_k|\mathcal{F}_{k-1}] \), we de-
termine the following sub-optimal admissible set of $m_k$-size local conditional sample

$$M_k = \{m_k : \Xi_{m_k,k,y_k} < \epsilon \text{ for } m_k \in OS_k\}. \quad (7.7)$$

Among these collected values, the value that gives the minimum $\Xi_{m_k,k,y_k}$ is recorded as $\hat{m}_k$. If more than one value of $m_k$ exists, then the largest of such $m_k$'s is recorded as $\hat{m}_k$. If condition (7.7) is not met at time $t_k$, the value of $m_k$ where the minimum $\min_{m_k} \Xi_{m_k,k,y_k}$ is attained, is recorded as $\hat{m}_k$. The $\epsilon-$ level sub-optimal estimates of the parameters $\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}$ and $\hat{\sigma}^2_{m_k,k}$ at $\hat{m}_k$ are also recorded as $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma^2_{\hat{m}_k,k}$, respectively. Finally, the simulated value $y^s_{\hat{m}_k,k}$ at time $t_k$ with $\hat{m}_k$ is now recorded as the best estimate for $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at $t_k$. This value is called the $\epsilon-$ sub-optimal simulated value $y^s_{\hat{m}_k,k}$ of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at $t_k$. Similar reasoning can be provided for the estimates of the parameters of the U.S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate model. A detailed flowchart of the conceptual algorithm is as follows:

**Flowchart 1: LLGMM Conceptual Computational Algorithm.**

Moreover, a detailed simulation algorithm is presented in C.4

### 7.6 Applications: Four Energy Commodity Data Sets

Now, we apply the above conceptual computational algorithm for the real time data sets namely daily Henry Hub Natural gas data set for the period 01/04/2000-09/30/2004, daily crude oil data set for the period 01/07/1997 – 06/02/2008, daily coal data set for the period of 01/03/2000 – 10/25/2013, and weekly ethanol data set for the period of 03/24/2005 – 09/26/2013, [22, 24, 23, 136]. Using $\Delta t = 1$, $\epsilon = 0.001$, $r = 5$, and $p = 2$, the $\epsilon-$ level sub-optimal estimates of parameters $\alpha$, $\mu$ and $\sigma^2$ at each real data times are exhibited in Table 6.
Table 6: Estimates $\tilde{m}_k$, $\sigma^2_{\tilde{m}_k}$, $\mu_{\tilde{m}_k}$, and $\sigma_{\tilde{m}_k}$.  

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>Natural gas</th>
<th>$t_k$</th>
<th>Crude oil</th>
<th>$t_k$</th>
<th>Coal</th>
<th>$t_k$</th>
<th>Ethanol</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3 0.0001</td>
<td>2.2231</td>
<td>0.6011</td>
<td>5</td>
<td>3 0.0001</td>
<td>24.4100</td>
<td>0.0321</td>
</tr>
<tr>
<td>6</td>
<td>3 0.0002</td>
<td>2.1260</td>
<td>0.6122</td>
<td>6</td>
<td>3 0.0002</td>
<td>24.7165</td>
<td>0.0341</td>
</tr>
<tr>
<td>7</td>
<td>3 0.0002</td>
<td>2.5131</td>
<td>0.6087</td>
<td>7</td>
<td>4 0.0003</td>
<td>25.9546</td>
<td>0.0537</td>
</tr>
<tr>
<td>8</td>
<td>4 0.0002</td>
<td>2.2494</td>
<td>0.1628</td>
<td>8</td>
<td>5 0.0006</td>
<td>25.5550</td>
<td>0.0467</td>
</tr>
<tr>
<td>9</td>
<td>4 0.0002</td>
<td>2.2658</td>
<td>-0.1497</td>
<td>9</td>
<td>4 0.0006</td>
<td>25.5695</td>
<td>0.0499</td>
</tr>
<tr>
<td>10</td>
<td>4 0.0003</td>
<td>2.1371</td>
<td>0.1968</td>
<td>10</td>
<td>4 0.0004</td>
<td>25.4787</td>
<td>0.0221</td>
</tr>
<tr>
<td>11</td>
<td>4 0.0004</td>
<td>2.5071</td>
<td>-0.2781</td>
<td>11</td>
<td>3 0.0001</td>
<td>25.7742</td>
<td>0.0100</td>
</tr>
<tr>
<td>12</td>
<td>4 0.0000</td>
<td>2.2550</td>
<td>0.3545</td>
<td>12</td>
<td>3 0.0002</td>
<td>26.9477</td>
<td>-0.0157</td>
</tr>
<tr>
<td>13</td>
<td>4 0.0005</td>
<td>2.5122</td>
<td>0.6246</td>
<td>13</td>
<td>3 0.0001</td>
<td>25.8786</td>
<td>-0.0112</td>
</tr>
<tr>
<td>14</td>
<td>4 0.0015</td>
<td>2.4850</td>
<td>0.5604</td>
<td>14</td>
<td>5 0.0005</td>
<td>22.1834</td>
<td>0.0049</td>
</tr>
<tr>
<td>15</td>
<td>3 0.0007</td>
<td>2.5378</td>
<td>0.4846</td>
<td>15</td>
<td>5 0.0004</td>
<td>23.5245</td>
<td>0.0010</td>
</tr>
<tr>
<td>16</td>
<td>3 0.0007</td>
<td>2.5715</td>
<td>0.7737</td>
<td>16</td>
<td>4 0.0002</td>
<td>23.8500</td>
<td>0.0000</td>
</tr>
<tr>
<td>17</td>
<td>5 0.0011</td>
<td>2.5668</td>
<td>0.5984</td>
<td>17</td>
<td>4 0.0002</td>
<td>23.8486</td>
<td>0.0502</td>
</tr>
<tr>
<td>18</td>
<td>4 0.0010</td>
<td>2.5831</td>
<td>0.5423</td>
<td>18</td>
<td>5 0.0004</td>
<td>23.2931</td>
<td>-0.0113</td>
</tr>
<tr>
<td>19</td>
<td>5 0.0007</td>
<td>2.5893</td>
<td>0.4256</td>
<td>19</td>
<td>3 0.0000</td>
<td>24.4715</td>
<td>0.1282</td>
</tr>
<tr>
<td>20</td>
<td>5 0.0006</td>
<td>2.6100</td>
<td>0.0683</td>
<td>20</td>
<td>3 0.0004</td>
<td>24.3878</td>
<td>0.0415</td>
</tr>
</tbody>
</table>

Table 6 shows the estimates of the $\epsilon$-sub optimal size $\tilde{m}_k$, the parameters $\sigma^2_{\tilde{m}_k}$, $\mu_{\tilde{m}_k}$, and $\sigma_{\tilde{m}_k}$ for each of the energy commodity data sets. Moreover, $p \leq r$, and the initial real data time is $t_e = t_0$.

In the following, the graph of $a_{\tilde{m}_k}$ for daily natural gas, daily crude oil, daily coal, and weekly ethanol are exhibited in Figure 8 (a), (b), (c) and (d), respectively.
Figure 8.: The graph of mean reverting rate $a_{\hat{m}_k, k}$ with time $t_k$

Figures 8: (a), (b), (c) and (d) are the graphs of $a_{\hat{m}_k, k}$ against time $t_k$ for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136], respectively. It shows the rate at which the data sets are reverting to the mean level.

Furthermore, we show the graphs of $\mu_{\hat{m}_k, k}$ for each of the data set: Natural gas, Crude Oil, Coal and Ethanol in Figure 9 (a), (b), (c) and (d), respectively.
Figures 9: (a), (b), (c) and (d) are the graphs of $\hat{\mu}_{m,k}$ against time $t_k$ for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136] respectively. The sample means of the real data $y_k$ sets for Natural gas, Crude oil data and Coal data are given by 4.5385, 54.0093, 27.1441 and 2.1391 respectively. It can be seen from Figure 9: (a) that the graph of $\hat{\mu}_{m,k}$ for the Henry Hub Natural gas data set moves around the mean value 4.5385 of the real data set. Also, from Figure 9: (b), the values of $\hat{\mu}_{m,k}$ for the crude oil data moves around the mean value 54.0093 of the crude oil data set. Likewise, from Figure 9: (c), the values of $\hat{\mu}_{m,k}$ for the coal data moves around the mean value 27.1441 of the coal data set. Finally, from Figure 9: (d), the values of $\hat{\mu}_{m,k}$ for the ethanol data moves around the mean value 2.1391 of the ethanol data set. This analysis shows that the parameter $\hat{\mu}_{m,k}$ is close to the respective mean of the data set at time $t_k$.

We remark that $\{\mu_{m,i}\}_{i=0}^N$ and $\{a_{m,i}\}_{i=0}^N$ are discrete-time $\epsilon-$ sub-optimal simulated random samples generated by the scheme described at the beginning of Section 7.6.
Next, we show the graph of $s^2_{m,k,k}$ for each of the data set: Natural gas, Crude oil, Coal and Ethanol in Figures 10 (a), (b), (c) and (d), respectively.

![Graph of $s^2_{m,k,k}$ against $k$ for Natural Gas](image)

![Graph of $s^2_{m,k,k}$ against $k$ for Crude Oil](image)

![Graph of $s^2_{m,k,k}$ against $k$ for Coal](image)

![Graph of $s^2_{m,k,k}$ against $k$ for Ethanol](image)

Figure 10.: Moving Variance $s^2_{m,k,k}$ against $k$ for three commodities

Figures 10: (a), (b), (c) and (d) are graphs of $s^2_{m,k,k}$ against time $t_k$ with initial delay $\tau = 5$ for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136] respectively. We found these estimates using the discrete time dynamic model (6.8) with $p = 2$, with the usage of $p = 2$ because of the autocorrelation and partial autocorrelation function of the series $x$, as described in [80]. Using the third part of (6.53), the volatility square at time $t_k$ can be calculated.

The overall descriptive statistics of data sets regarding the energy commodity prices and estimated parameters are recorded in the Table 7.
Table 7: Descriptive Statistics for $a$, $\mu$, and $\sigma^2$.

<table>
<thead>
<tr>
<th>Data Set</th>
<th>$\bar{Y}$</th>
<th>Std(Y)</th>
<th>$\Delta n(Y)$</th>
<th>var($\Delta n(Y)$)</th>
<th>$\hat{a}$</th>
<th>Std($\hat{a}$)</th>
<th>$\hat{\mu}$</th>
<th>Std($\hat{\mu}$)</th>
<th>$\hat{\sigma}^2$</th>
<th>Std($\hat{\sigma}^2$)</th>
<th>95% C.I. $\hat{\mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nat. Gas</td>
<td>4.5504</td>
<td>1.5090</td>
<td>0.0008</td>
<td>0.0015</td>
<td>0.1867</td>
<td>0.3013</td>
<td>4.5538</td>
<td>2.3565</td>
<td>0.0013</td>
<td>0.0017</td>
<td>(4.4196, 4.6880)</td>
</tr>
<tr>
<td>Crude Oil</td>
<td>54.0093</td>
<td>31.0248</td>
<td>0.0003</td>
<td>0.0006</td>
<td>0.0215</td>
<td>0.0517</td>
<td>54.0037</td>
<td>37.4455</td>
<td>0.0005</td>
<td>0.0008</td>
<td>(51.8978, 56.1636)</td>
</tr>
<tr>
<td>Coal</td>
<td>27.1441</td>
<td>17.8394</td>
<td>0.0003</td>
<td>0.0015</td>
<td>0.0464</td>
<td>0.0879</td>
<td>27.0567</td>
<td>21.3506</td>
<td>0.0014</td>
<td>0.0022</td>
<td>(25.8405, 28.2729)</td>
</tr>
<tr>
<td>Ethanol</td>
<td>2.1391</td>
<td>0.4455</td>
<td>0.0011</td>
<td>0.0020</td>
<td>0.3167</td>
<td>0.8745</td>
<td>2.1666</td>
<td>0.7972</td>
<td>0.0018</td>
<td>0.0030</td>
<td>(2.0919,2.2414)</td>
</tr>
</tbody>
</table>

Table 7 shows the descriptive statistics for $a$, $\mu$, and $\sigma^2$ with time delay $r = 5$. Note that the mean value of the estimated samples $\{a_{\hat{m}_i,k}\}_{i=0}^{N}$ and $\{\sigma_{a_{\hat{m}_i,k}}^2\}_{i=0}^{N}$ are $\bar{a} = \frac{1}{N} \sum_{i=0}^{N} a_{\hat{m}_i,k}$, $\bar{\mu} = \frac{1}{N} \sum_{i=0}^{N} \mu_{\hat{m}_i,k}$ and $\bar{\sigma}^2 = \frac{1}{N} \sum_{i=0}^{N} \sigma_{a_{\hat{m}_i,k}}^2$, respectively. $\bar{a}$, $\bar{\mu}$, and $\bar{\sigma}^2$ are referred to as aggregated parameter estimates of $a$, $\mu$, and $\sigma^2$ over the given entire finite interval of time, respectively. $\bar{\mu}$ is the descriptive statistics of the parameter $\mu$ estimated in column 8, while $\bar{\sigma}^2$ is the descriptive statistics of the parameter $\sigma^2$ estimated in column 10. Moreover, $\bar{\mu}$ is approximately close to the overall descriptive statistics of the mean $\bar{Y}$ of the real data set for each of the energy commodities shown in column 2. Also, $\bar{\sigma}^2$ is approximately close to the overall descriptive statistics of the variance of $\Delta \ln(Y) = \ln(Y_i) - \ln(Y_{i-1})$ in Column 5. Moreover, column 12 shows that the mean of the actual data set in Column 2 falls within the 95% confidence interval of $\hat{\mu}$. This exhibits that the parameter $\mu_{\hat{m}_i,k}$ is the mean level of $y_k$ at time $t_k$.

We have used the the estimated parameters $a_{\hat{m}_i,k}$, $\mu_{\hat{m}_i,k}$, and $\sigma_{\hat{m}_i,k}^2$ in Figures 8, 9, and 10, respectively to simulate the daily natural gas data set, daily crude oil data set, daily coal data set, and weekly ethanol data set.

In fact, developing the code and flowchart described in C.4 and the parameters described in Figures 8, 9 and 10, we simulate the daily Natural Gas data set, daily Crude Oil data set, the daily Coal data set and weekly ethanol data set.

For this purpose, we pick $\epsilon = 0.01$; for each time $t_k$, the estimates of the simulated prices $y_{\hat{m}_i,k}$ are computed by determining the sub-optimal admissible set of $m_k$-size local conditional sample $\mathcal{M}_k$ defined in (7.7). Among these collected values, the value that gives the minimum $\Xi_{m_k,k,y_k}$ is recorded as $\hat{m}_k$. If condition (7.7) is not met at time $t_k$, the value of $m_k$ where the minimum $\min_{m_k} \Xi_{m_k,k,y_k}$ is attained, is recorded as $\hat{m}_k$. The $\epsilon-$ level sub-optimal estimates of the parameters $a_{\hat{m}_i,k}$, $\mu_{\hat{m}_i,k}$ and $\sigma_{\hat{m}_i,k}^2$ at $\hat{m}_k$ are also recorded as $a_{\hat{m}_i,k}$, $\mu_{\hat{m}_i,k}$ and $\sigma_{\hat{m}_i,k}^2$, the value of $y_{\hat{m}_i,k}$ at time $t_k$ and $\hat{m}_k$ corresponding to $a_{\hat{m}_i,k}$, $\mu_{\hat{m}_i,k}$ and $\sigma_{\hat{m}_i,k}^2$ is also recorded as the $\epsilon-$ sub-optimal simulated value $y_{\hat{m}_i,k}$ as an estimate of $y_k$. A detailed algorithm is given in Appendix C.4.

Finally, in Table 8, we show the results of the real, simulated prices using the local lagged adapted generalized method of moment (LLGMM) and the simulated price using the aggregated parameter
estimates \( a, \mu, \) and \( \sigma^2 \) in Table 7, Column 6, 8, and 10, respectively for the energy commodity price. This estimate is derived using the discretized model

\[
y_t^{ag} = y_{t-1}^{ag} + \bar{a}(\mu - y_{t-1}^{ag})\Delta t + \sigma^2\frac{1}{2} y_{t-1}^{ag} \Delta W_t
\]  
(7.8)

For the rest of this study, we define this estimate \( y_t^{ag} \) at time \( t_k \) by the aggregated GMM simulated estimates (AGMM).

Table 8: Real, Simulation using LLGMM prices, and Simulation using AGMM.

<table>
<thead>
<tr>
<th>( t_k )</th>
<th>Natural gas</th>
<th>Crude oil</th>
<th>Coal</th>
<th>Ethanol</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_k )</td>
<td>Real Simulated ( \hat{y}_{t_k}^a ) (LLGMM)</td>
<td>Simulated ( \hat{y}_{t_k}^a ) (AGMM)</td>
<td>Real Simulated ( \hat{y}_{t_k}^a ) (LLGMM)</td>
<td>Simulated ( \hat{y}_{t_k}^a ) (AGMM)</td>
</tr>
<tr>
<td>5</td>
<td>2.216</td>
<td>2.216</td>
<td>5</td>
<td>25.200</td>
</tr>
<tr>
<td>6</td>
<td>2.260</td>
<td>2.253</td>
<td>25.200</td>
<td>25.000</td>
</tr>
<tr>
<td>7</td>
<td>2.344</td>
<td>2.241</td>
<td>25.950</td>
<td>25.612</td>
</tr>
<tr>
<td>8</td>
<td>2.352</td>
<td>2.249</td>
<td>25.450</td>
<td>25.494</td>
</tr>
<tr>
<td>9</td>
<td>2.322</td>
<td>2.329</td>
<td>25.400</td>
<td>25.411</td>
</tr>
<tr>
<td>10</td>
<td>2.383</td>
<td>2.376</td>
<td>25.100</td>
<td>25.099</td>
</tr>
<tr>
<td>13</td>
<td>2.485</td>
<td>2.554</td>
<td>23.850</td>
<td>24.862</td>
</tr>
<tr>
<td>16</td>
<td>2.523</td>
<td>2.478</td>
<td>23.900</td>
<td>24.071</td>
</tr>
<tr>
<td>17</td>
<td>2.610</td>
<td>2.638</td>
<td>24.500</td>
<td>24.544</td>
</tr>
<tr>
<td>20</td>
<td>2.699</td>
<td>2.726</td>
<td>24.200</td>
<td>23.971</td>
</tr>
</tbody>
</table>

Next, we show the graph of the simulated data set using the LLGMM method for each of the commodities in Figure 11.
Figures 11: (a), (b), (c) and (d) show the graph of the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136] respectively. The red line represents the real data set $y_k$, while the blue line represent the simulated data set $\hat{y}_{m,k}$. The root mean square error of the simulation for the Henry Hub Natural gas data set, the Crude Oil data set, the Coal and Ethanol data set are given by 0.021, 0.013, 0.015, and 0.046 respectively. Here, we begin by using a starting delay of $r = 5$. The simulation starts from $t_r = t_5$. It is clear that the graph fits well, but there are still some regions where the simulation does not capture the real data well. Therefore, this gap is analyzed by increasing the magnitude of time delay.

The following graphs show the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136], respectively, for the case where $r = 10$. 
Figures 12: (a), (b), (c) and (d) show the graph of the Real and Simulated Spot Prices for the daily Henry Hub Natural gas data set [24], daily Crude Oil data set [23], daily Coal data set [22], and weekly ethanol data set [136], respectively, for $r = 10$. The red line represents the real data set $y_k$, while the blue line represent the simulated data set $y_{\hat{m}_k,k}$. The root mean square error of the simulation for the Henry Hub Natural gas data set, the Crude Oil data set, the Coal data set, and ethanol data set are given by $0.004$, $0.001$, $0.002$ and $0.006$, respectively.

**Remark 21** Several other delays were tested and it was found that as the delay $r$ increases, the root mean square error decreases, significantly. Moreover, the curve fitting appears to be better. For example, for starting delay of 20, the root mean square error of the simulation for the Henry Hub natural gas data set, the crude oil data set, coal data set and ethanol data set are given by $2 \times 10^{-4}$, $10^{-5}$, $10^{-4}$, and $5 \times 10^{-4}$, respectively. Furthermore, the simulation results show that the price of a commodity is affected by its volatility $\sigma_{\hat{m}_k,k}^2$, the rate and mean level parameters $\alpha_{\hat{m}_k,k}$ and $\mu_{\hat{m}_k,k}$, respectively.
In Figure 13, we show a comparison between the real data set, simulated price using the local lagged adaptive generalized method (LLGMM) and the simulated price (AGMM) using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\bar{\sigma}^2$ in Table 7, Column 6, 8, and 10, respectively.

![Figure 13](image)

Figure 13.: Real, Simulated Prices using (LLGMM), and Simulated Prices using AGMM.

Figures 13: (a), (b), (c) and (d) show the graph of the Real, simulated prices using the local lagged adaptive generalized method (LLGMM), and the simulated price using the average of the parameters for Henry Hub Natural gas data set [24], daily Crude Oil data set [23], daily Coal data set [22], and weekly ethanol data set [136], respectively, for $r = 5$. The red line represents the real data set $y_k$, the blue line represent the simulated prices using LLGMM method, while the black line represent the simulated price (AGMM) using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\bar{\sigma}^2$ in Table 7, Column 6, 8, and 10, respectively. From these simulated graphs, it is clear that the LLGMM simulation results are more realistic than the AGMM simulation results. This exhibits the power of LLGMM over the AGMM.

**Remark 22** A code similar to the flowchart described in C.4 is designed to exhibit the flowchart algorithm. All the codes for the parameter estimation, simulations and forecasting are written and
tested using Matlab program. Due to the online control nature of \( m_k \) in our model, it is worth mentioning that the running times for each of the four commodities: Natural gas, Crude oil, Coal and Ethanol depend on the robustness of the data. The average running time for each data set is 25 minutes.

In reference to Remark 16, we compare the estimates \( s^2_{\hat{m}_k,k} \) with the estimate derived from the usage of a GARCH(1,1) model described in [9] which is defined by

\[
\begin{align*}
  z_t | \mathcal{F}_{t-1} & \sim \mathcal{N}(0, h_t), \\
  h_t & = \alpha_0 + \alpha_1 h_{t-1} + \beta_1 z_{t-1}^2, \quad \alpha_0 > 0, \quad \alpha_1, \beta_1 \geq 0.
\end{align*}
\]

(7.9)

The parameters \( \alpha_0, \alpha_1, \) and \( \beta_1 \) of the GARCH(1,1) conditional variance model (7.9) for \( x \) for the four commodities natural gas, crude oil, coal, and ethanol are estimated. The estimates of the parameters are given in Table 9.

Table 9: Parameter estimates for GARCH(1,1) Model (7.9).

<table>
<thead>
<tr>
<th>Data Set</th>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \beta_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Gas</td>
<td>( 6.863 \times 10^{-5} )</td>
<td>0.853</td>
<td>0.112</td>
</tr>
<tr>
<td>Crude Oil</td>
<td>( 9.622 \times 10^{-5} )</td>
<td>0.917</td>
<td>0.069</td>
</tr>
<tr>
<td>Coal</td>
<td>( 3.023 \times 10^{-5} )</td>
<td>0.903</td>
<td>0.081</td>
</tr>
<tr>
<td>Ethanol</td>
<td>( 4.152 \times 10^{-4} )</td>
<td>0.815</td>
<td>0.019</td>
</tr>
</tbody>
</table>

Table 9 shows the parameter estimates for GARCH(1,1) Model

We later show a side by side comparison of \( s^2_{\hat{m}_k,k} \) and the volatility described by GARCH(1,1) model described in (7.9) with coefficients in Table 9.
Figures 14: (a), (b) and (c) are graphs of $\hat{s}_{m,k}^2$ and GARCH(1,1) model against time $t_k$ for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22] and weekly ethanol data set [136] respectively. The blue line shows the graph of estimates for $\hat{s}_{m,k}^2$ and the red line shows the graph of estimates for GARCH(1,1) model. The GARCH model does not clearly estimate volatility as heavily evidenced in Figure 14 (d). In fact, it demonstrated insensitivity. The presented analysis suggests that the GARCH model is ineffective in comparison with the framework of moving average process.

We also compare the simulations in Figure 11 with the simulations using the GARCH(1,1) model (7.9) as the conditional variance. The following figure exhibits the comparison.
Figures 15: (a), (b), (c) and (d) are graphs of the simulations using \( s_{mk}^2 \) and GARCH(1,1) model to estimate the volatility process for the daily Henry Hub Natural gas data set [24], daily Crude oil data set [23], daily Coal data set [22], and weekly Ethanol data set [136], respectively. The blue line shows the graph of estimates for the simulations using GARCH(1,1) model to simulate the volatility, the green line is our simulated estimates described in Figure 11, and the red line shows the real data set. It can be seen that the GARCH model fails to capture most of the spikes in the data set. Moreover, the GARCH model creates significant errors by over-and-under estimating the variance. These spikes were perfectly captured in Figure 11 where we use the discrete-time dynamic model of local sample variance statistics process to estimate the volatility process. This illustrates that the dynamic statistic model works better than the GARCH volatility model.
7.7 Applications: U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate Data Sets

In this subsection, we apply the conceptual computational algorithm discussed above to estimate the parameters in (6.42) using the real time Treasury bill yield data sets [128] and the US dollar Eurocurrency data set [129] collected from Forex database.

Table 10: Estimates $\tilde{m}_k, \beta \tilde{m}_k, \mu \tilde{m}_k, \delta \tilde{m}_k, \sigma \tilde{m}_k, \gamma \tilde{m}_k$ for U. S. Treasury Bill Interest Rate.

<table>
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<th>$t_k$</th>
<th>$\tilde{m}_k$</th>
<th>$\beta \tilde{m}_k$</th>
<th>$\mu \tilde{m}_k$</th>
<th>$\delta \tilde{m}_k$</th>
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Table 11: Estimates $\hat{m}_{k}, \beta \hat{m}_{k}, \mu \hat{m}_{k}, \delta \hat{m}_{k}, \sigma \hat{m}_{k}, \gamma \hat{m}_{k}$ for U.S. Eurocurrency Exchange Rate.

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</table>

Tables 10-11 show the estimates for the $\epsilon$- sub-optimal size $\hat{m}_{k}$, the parameters $\beta \hat{m}_{k}, \mu \hat{m}_{k}, \delta \hat{m}_{k}, \sigma \hat{m}_{k}$, and $\gamma \hat{m}_{k}$ for the U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate, respectively. The initial real data time is $t_r = t_5$. 

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Using $\epsilon = 1 \times 10^{-3}$, $r = 5$, and $p = 2$, the $\epsilon-$ level sub-optimal estimates of parameters $\beta$, $\mu$, $\delta$, $\sigma$ and $\gamma$ for each Treasury bill real data set and U.S. Eurocurrency rate data sets are exhibited in Tables 10 and 11, respectively.

Next, we show the graphs of $\hat{\beta}_{m,k}$, $\hat{\mu}_{m,k}$, $\hat{\delta}_{m,k}$, $\hat{\sigma}_{m,k}$, and $\hat{\gamma}_{m,k}$ for the Monthly Treasury bill data set and Monthly U. S. Eurocurrency data set.

Figure 16.: $\beta_{m,k}$, $\mu_{m,k}$, $\delta_{m,k}$, $\sigma_{m,k}$, and $\gamma_{m,k}$ for Interest rate.
Figures 16: (a), (b), (c) and (d) are the graphs of the parameters $\beta_{\hat{m}k,k}$, $\mu_{\hat{m}k,k}$, $\delta_{\hat{m}k,k}$, $\sigma_{\hat{m}k,k}$, and $\gamma_{\hat{m}k,k}$ against time $t_k$ for the U.S. Treasury bill yield respectively.

The next figures show the graphs of the parameters $\beta_{\hat{m}k,k}$, $\mu_{\hat{m}k,k}$, $\delta_{\hat{m}k,k}$, $\sigma_{\hat{m}k,k}$, and $\gamma_{\hat{m}k,k}$ for the Eurocurrency Exchange rate respectively.

Figure 17: $\beta_{\hat{m}k,k}$, $\mu_{\hat{m}k,k}$, $\delta_{\hat{m}k,k}$, $\sigma_{\hat{m}k,k}$, and $\gamma_{\hat{m}k,k}$ for US Eurocurrency.
Figures 17: (a), (b), (c) and (d) are the graphs of the parameters $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ against time $t_k$ for the US Eurocurrency exchange rate respectively.

The overall descriptive statistics of data sets regarding U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate are recorded in the following table.

Table 12: Descriptive Statistics for $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for Interest rate data set.

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<th>$\beta$</th>
<th>Std($\beta$)</th>
<th>$\bar{\mu}$</th>
<th>Std($\mu$)</th>
<th>$\bar{\delta}$</th>
<th>Std($\delta$)</th>
<th>$\bar{\sigma}$</th>
<th>Std($\sigma$)</th>
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Table 13: Descriptive Statistics for $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for US Eurocurrency Exchange Rate data.

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Tables 12 and 13 show the descriptive statistics for $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ with time delay $r = 5$ for the U.S. Treasury Bill Yield interest rate data set and the U. S. Eurocurrency exchange rate data set, respectively.

In Table 14, we show the result for the real, simulated data using the local lagged adapted generalized method of moment (LLMM), and the simulated price (AGMM) using the aggregated parameter estimates $\tilde{\beta}$, $\tilde{\mu}$, $\tilde{\delta}$, $\tilde{\sigma}$ and $\tilde{\gamma}$ in Table 12 and 13 for the U. S. Treasury Bill Yield interest rate and U. S. Eurocurrency exchange rate respectively. The simulated price using the aggregated parameter (AGMM) satisfies the discrete model

$$y_i^{ag} = y_{i-1}^{ag} + (\tilde{\beta}y_{i-1}^{ag} + \tilde{\mu}(y_{i-1}^{ag})^{\tilde{\delta}}) + \tilde{\sigma}(y_{i-1}^{ag})^{\tilde{\gamma}}\Delta W_i.$$ (7.10)

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>Interest Rate Data</th>
<th>$t_k$</th>
<th>Eurocurrency Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LLGMM</td>
<td>AGMM</td>
<td>Simulated</td>
</tr>
<tr>
<td>5</td>
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<td>0.0357</td>
<td>0.0400</td>
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<tr>
<td>6</td>
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<td>0.0357</td>
<td>0.0390</td>
</tr>
<tr>
<td>7</td>
<td>0.0384</td>
<td>0.0365</td>
<td>0.0390</td>
</tr>
<tr>
<td>8</td>
<td>0.0381</td>
<td>0.0378</td>
<td>0.0388</td>
</tr>
<tr>
<td>9</td>
<td>0.0393</td>
<td>0.0387</td>
<td>0.0399</td>
</tr>
<tr>
<td>10</td>
<td>0.0393</td>
<td>0.0406</td>
<td>0.0401</td>
</tr>
<tr>
<td>11</td>
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<td>0.0389</td>
<td>0.0503</td>
</tr>
<tr>
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<tr>
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<td>...</td>
<td>...</td>
</tr>
<tr>
<td>390</td>
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<td>0.0503</td>
<td>0.0303</td>
</tr>
<tr>
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<tr>
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<td>0.0503</td>
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<tr>
<td>393</td>
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<tr>
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<tr>
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<td>0.0514</td>
<td>0.0496</td>
<td>0.0184</td>
</tr>
<tr>
<td>403</td>
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<td>0.0513</td>
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</tr>
<tr>
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<td>0.0252</td>
</tr>
<tr>
<td>405</td>
<td>0.0509</td>
<td>0.0491</td>
<td>0.0228</td>
</tr>
</tbody>
</table>
Next, we show the graphs of the simulated data set for the Treasury bill yield interest rate and US Eurocurrency exchange rate data.

![Figure 18.](image1)

Figures 18(a) and (b) show the real and simulated price for U.S. Treasury bill yield interest rate and U.S. Eurocurrency exchange rate respectively.

In the work of Chan et al [15], they compared the ex post volatility (defined by the absolute value of the change in Treasury bill yield data set) with the simulated volatility (defined by the square root of the conditional variance implied by the estimates of the the solution of (6.42)). It is calculated as $\sigma_{m_k,k} \left( y_{m_k,k} \right)^{\delta_{m_k,k}}$. In order to compare our work with Figure 1 of Chan et al [15], we use our approach/scheme to compute the Ex post volatility and simulated volatility for the U.S. Treasury bill yield interest rate data set [128].

![Figure 19.](image2)

Figure 19.: Ex Post Volatility and Simulated Volatility for Interest Rate.
Figures 19 shows the Ex post volatility and simulated volatility for the U.S. Treasury bill yield interest rate data set [128]. We compare our work with Figure 1 of Chan et al [15]. Their model does not clearly estimate the volatility. It demonstrated insensitivity in the sense that it was unable to capture most of the spikes in the interest rate ex post volatility data set.

Finally, in Figure 20, we show the graphs of comparison of the real price, simulated price using the LLGMM method and the simulated price AGMM method.

![Figure 20. Real, Simulation using LLGMM, and Simulated Price using AGMM for U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate.](image)

Figures 20 (a) and (b) show the real, simulated price using LLGMM, and simulated price using the average parameters $\bar{\beta}$, $\bar{\mu}$, $\bar{\delta}$, $\bar{\sigma}$ and $\bar{\gamma}$ in Table 12 and 13 for U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate respectively.
Chapter 8
Forecasting

8.1 Introduction

In this chapter, we shall sketch an outline about forecasting problem. An \( \epsilon \)-sub-optimal simulated value \( y_{m,k}^f \) at time \( t_k \) is used to define a forecast \( y_{m,k}^f \) for \( y_k \) at the time \( t_k \) for each of the Energy commodity model, and the U. S. Interest rate and U.S. Eurocurrency exchange rate.

8.2 Forecasting, Prediction and Confidence Interval for Energy Commodity Model

In the context of Illustration 6.4.1, we begin forecasting from time \( t_k \). Using the data set up to time \( t_{k-1} \), we compute \( \hat{m}_i \), \( \sigma_i^2 \), \( a_i \hat{m}_i \), and \( \mu \hat{m}_i \) for \( i \in \{0, k-1\} \). We assume that we have no information about the real data set \( \{y_i\}^N_{i=1} \). Under these considerations, imitating the computational procedure outlined in Section 6.4 and using (6.40), we find the estimate of the forecast \( y_{m,k}^f \) at time \( t_k \) as follows:

\[
y_{m,k}^f = y_{m,k-1,k-1} + a_{m,k-1}y_{m,k-1,k-1} \Delta t + \sigma_{m,k-1,k-1} \Delta W_k
\]

where the estimates \( \sigma_{m,k-1,k-1} \), \( a_{m,k-1,k-1} \) and \( \mu_{m,k-1,k-1} \) are defined in (6.40) with respect to the known past data set up to the time \( t_{k-1} \). We note that \( y_{m,k}^f \) is the \( \epsilon \)-sub-optimal estimate for \( y_k \) at time \( t_k \).

To determine \( y_{m,k+1,k+1} \), we need \( \sigma_{m,k}^2 \), \( a_{m,k} \), and \( \mu_{m,k} \). Since we only have information of real data up to time \( t_{k-1} \), we use the forecasted estimate \( y_{m,k}^f \) as the estimate of \( y_k \) at time \( t_k \), and to estimate \( \sigma_{m,k}^2 \), \( a_{m,k} \), and \( \mu_{m,k} \). Hence, we can write \( a_{m,k} \) as \( a_{m,k} \equiv a_{m,k}^f \). We can also re-write \( \mu_{m,k} \equiv \mu_{m,k}^f \). To find \( y_{m,k+2,k+2} \), we use the estimates

\[
a_{m,k+1,k+1} = a_{m,k+1,k+2}y_{m,k+2} + \ldots + y_{m,k+1}y_{m,k}^f
\]
\[
\mu_{m,k+1,k+1} = \mu_{m,k+1,k+2}y_{m,k+2} + \ldots + y_{m,k+1}y_{m,k}^f
\]
Continuing this process in this manner, we use the estimates

\[
\alpha \hat{m}_{k+i-1,k+i-1} = \alpha m_{k+i-1,k+i} y_{k-m_{k+i-1,k+i-1} + 1} \ldots y_{k-1} y_{f_{m_{k+i-1,k+i-1}}} y_{f_{m_{k+i-1,k+i-1}}} \ldots y_{f_{m_{k+i-1,k+i-1}}},
\]

\[
\mu \hat{m}_{k+i-1,k+i-1} = \mu m_{k+i-1,k+i} y_{k-m_{k+i-1,k+i-1} + 1} \ldots y_{k-1} y_{f_{m_{k+i-1,k+i-1}}} y_{f_{m_{k+i-1,k+i-1}}} \ldots y_{f_{m_{k+i-1,k+i-1}}}
\]

to estimate \( y_{f_{m_{k+i-1,k+i-1}}} \)

**Prediction/Confidence Interval for Energy Commodities**

In order to be able to assess the future certainty, we also discuss about the prediction/confidence interval. We define the \( 100(1 - \alpha)\% \) confidence interval for the forecast of the state \( y_{f_{m_{k+i}},i} \) at time \( t_i \), \( i \geq k \), as

\[
y_{f_{m_{k+i}},i} = y_{f_{m_{k+i-1,k+i-1}}} + a_{m_{k+i-1,k+i-1}} (\mu_{m_{k+i-1,k+i-1}} - y_{f_{m_{k+i-1,k+i-1}}}) \Delta t + \sigma_{m_{k+i-1,k+i-1}} y_{f_{m_{k+i-1,k+i-1}}} \Delta W_k.
\]

It is clear that the 95% confidence interval for the forecast at time \( t_i \) is

\[
\left( y_{f_{m_{i-1},i-1}} - 1.96 \left( \frac{s_{m_{i-1},i-1}}{y_{f_{m_{i-1},i-1}}} \right)^{1/2} y_{f_{m_{i-1},i-1}} + 1.96 \left( \frac{s_{m_{i-1},i-1}}{y_{f_{m_{i-1},i-1}}} \right)^{1/2} y_{f_{m_{i-1},i-1}} \right)
\]

where the lower end denotes the lower bound of the state estimate and the upper end denotes the upper bound of the state estimate.
Figure 21.: Real, Simulated and Forecasted Prices for daily Henry Hub natural gas, daily crude oil, daily coal, and weekly ethanol data set.

Figures 21: (a), (b), (c) and (d) show the graph of the forecast and 95 percent confidence limit for the daily Henry Hub Natural gas data set [24], daily Crude Oil data set [23], daily Coal data set [22], and weekly Ethanol data set [136], respectively. Figure 21: (a), (b), (c) and (d) show two region: the simulation region $S$ and the forecast region $F$. For the simulation region $S$, we plot the real data set together with the simulated data set as described in Figure 11. For the forecast region $F$, we plot the estimate of the forecast as explained in Section 8. The upper and the lower simulated sketches in Figure 21 (a), (b), (c) and (d) are corresponding to the upper and lower ends of the 95% confidence interval. For details, see Figure 22.

Next, we show a graph of the upper, least upper bound, lower and greatest lower bounds for the estimates of the forecast for the energy commodity processes after running the simulations for 25 times.
Figure 22.: Bounds for daily Henry Hub natural gas, daily crude oil, daily coal, and weekly ethanol data set.

Figures 22: (a), (b), (c) and (d) show the bounds for the daily Henry Hub Natural gas data set [24], daily Crude Oil data set [23], daily Coal data set [22], and weekly Ethanol data set [136], respectively. These bounds are derived after 25 run time (simulations).

8.3 Forecasting and Prediction/Confidence Interval for U. S. Treasury Bill and U. S. Eurocurrency rate

Following the same procedure explained in Section 8.2, we show the graph of the real, simulated, forecast and 95 percent confidence limit for the Interest rate and US dollar Eurocurrency rate.

Figure 23.: Real, Simulated, Forecast and 95% Confidence Limit for Interest rate and US Eurocurrency data.
Figure 23(a) shows the real, simulated, forecast and 95 percent confidence limit for the Interest rate data sets and Figure 23(b) shows the real, simulated, forecast and 95 percent confidence limit for the US Eurocurrency data.

Lastly, we show some bounds for the U. S. Treasury Bill Interest Rate and U. S. Eurocurrency rate.

![Bounds for Interest rate](image1)

![Bounds for Exchange Rate](image2)

Figure 24.: Bound for U. S. Treasury Bill and U. S. Eurocurrency rate.
Chapter 9

A Two-scale Network Dynamic Model for Energy Commodity Process

9.1 Introduction

Understanding the economy evolution in response to structural changes in energy commodity network system is important to professional economists. The relationship between the different energy sources and their uses provide insights into many important energy issues. The qualitative and quantitative behavior of energy commodities in which the trend in price of one commodity coincides with the trend in prices of other commodities, have always raised the question of whether there is any relationship between prices of energy commodities. If there is any relationship, then what comes to mind is the extent to which one commodity influences the other. Petroleum, natural gas, coal, nuclear fuel, and renewable energy are termed as primary energy components of the energy goods network system because other sources of energy depend on them. Natural gas is usually found near petroleum. This is because of the fact that natural gas and crude oil are rivals in production and substitutes in consumption. As a result of this, energy theory suggests that the two prices should be related. The electric power sector uses primary energy such as coal to generate electricity, which makes electricity a secondary rather than a primary energy source. According to the US Energy Information Administration (EIA), the major energy goods consumed in the United States are petroleum (oil), natural gas, coal, nuclear, and renewable energy. The majority of users are residential and commercial buildings, industry, transportation, and electric power generators. The pattern of fuel usage varies widely by sector [130]. For example, 71% of total petroleum oil provides 93% of the energy used for transportation; 23% of total petroleum oil provides 17% of energy used for residential and commercial use; 5% of total petroleum oil provides 40% of energy used for industrial use; but only 1% of total petroleum oil provides about 1% of the energy used to generate electric power, whereas coal provides 46% of the energy used to generate electric power and natural gas provides 20% of the energy used to generate electric power. This analysis suggests that the strength of interactions between coal and electricity will be stronger than when compared with the strength of interactions between crude oil and electricity, or natural gas and electricity.
Energy price forecasts are highly uncertain. We might expect the price of the natural gas and crude oil to follow the same trend because they are often found mixed with oil in oil wells, and also of the fact that natural gas is often used in petroleum refining and exploration. Recently, Ramberg et al [94] showed that the cointegration relationship between natural gas and crude oil does not appear to be stable through time. They claimed that though there is cointegration between the two energy prices, but there are also recurrent exogeneous factors such as seasonality, episodic heat waves, cold waves and supply interruption from hurricane affecting the trends in the prices. Brown and Yücel [12] also observed that the price of natural gas pulled away from oil prices in 2000, 2002, 2003 and late 2005. Oil prices are influenced by several factors, including some that have mainly short-term impacts and other factors, such as expectations about future world demand for petroleum, other liquids and production decisions of the Organization of the Petroleum Exporting Countries (OPEC) [130]. Supply and demand in the World oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply and demand expectation process. For the petroleum and other liquids, the key determinants of long-term supply and prices can be summarized in four broad categories [130]: the economics of non-OPEC supply, OPEC investment and production decisions, the economics of other liquids supply, and World demand for petroleum and other liquids. According to the US Energy Information Administration (EIA) [130] and following the decline of natural gas prices since 2008, real average delivered price for electricity has dropped gradually to 9.8 cents per kilowatthour (kWh) from 2009 to 2012. Retail electricity price is influenced by the fuel price, and particularly by the natural gas price [130]. However, the relationship between retail electricity price and natural gas price is complex. Many factors influence the degree to which and the time frame over which they are linked. A few notable factors are a share of natural gas generation in a region, the level of costs associated with the electricity transmission and distribution systems, the mix of competitive versus cost-of-service pricing, and the number of customers who purchase power directly from wholesale power markets. As a result of this, it can take time for changes in fuel price to affect electricity price. The question that we are now faced is whether the price of electricity depends on the prices of more than one energy commodities, rather than depending on only one commodity (coal or natural gas).

An understanding of how changes in price of one energy commodity are expressed in terms of other energy commodity is needed. This would prove to be useful in predicting price behavior over the long run, and further facilitates profit maximizing process. To check if there is really indeed a relationship between energy commodities; the need to be able to create a model which explains
such commodity prices relationship over short and long time interval arises. The relationships between energy commodities have been addressed in [4, 12, 39, 40, 49, 93, 94, 131, 132]. The error correction model [4, 12, 39, 49, 93, 94] is the most commonly used model by authors to describe the relationship between energy commodities. Moreover, Hartley et al [49] have concluded that the U. S. natural gas and crude oil remain linked in their long-term movements. In addition, it is exhibited that there is strong evidence of stable relationship between these two energy commodities which are affected by short run seasonal fluctuations and other factors. The rule of thumb [49] has long been used in the energy industry to relate the natural gas prices to crude oil prices. The rule denoted by the 10-to-1 rule states that the price of natural gas is one tenth of the price of crude oil prices. Similarly, 6-to-1 rule states that the price of natural gas is one sixth of the price of crude oil. It has been examined by Brown et al. [12] that these two rules do not perform well when used to assess the relationship between U.S natural gas price and West Texas Intermediate (WTI) crude oil price for the past 20+ years. Moreover, their error correcting model specify the relationship between the two commodities. Using this model, they show that when certain factors are taken into account, movements in crude oil prices can shape the price of natural gas. Vezzoli [131] in his work applies an ordinary least squares (OLS) regression on log of natural gas and crude oil prices. Using this model, he was able to conclude that there is a relationship between natural gas and crude oil prices. Bachmeir et al. [4] showed that the crude oil, coal and natural gas in the United States have weak cross-cointegration using the error correction model. Ramberg et al [94] shows that any simple formula between natural gas and crude oil prices will leave a portion of the natural gas price unexplained. Furthermore, the relationship between natural gas and crude oil using a vector error correction model [12, 94] under the cointegration between the two energy commodities and other factors such as recurrent exogeneous factors are presented. Villar et al. [132] lists some economic factors linking natural gas and crude oil prices, while testing for market integration in the United Kingdom during the time when natural gas was deregulated. Asche et al. [40] have integrated the prices of the energy commodities: natural gas, electricity, and crude oil.

The most of the work is centered around the relationship between prices of energy commodities. In this work, we are interested in an inter-dependence of certain energy commodities. Moreover, we develop a hybrid system of multivariate continuous stochastic network dynamic system.

In this chapter, we further extend the non-linear interconnected stochastic model (4.11) to multivariate interconnected energy commodities and sources with and without external random intervention processes.
9.2 Model Derivation

We denote \( p = [p_1, p_2, \ldots, p_n]^T \) to be a vector of \( n \) energy commodity prices which are considered to have long-run or short-run relationship with each other. Let \( p_j(t) \) be the price of the \( j \)-th energy commodity at time \( t \). The economic principles of demand and supply processes suggest that the price of an energy commodity will remain within a given finite expected lower and upper bounds. Therefore, \( u_j \in \mathbb{R}_+ = (0, \infty) \) and \( l_j \geq 0 \) stand for the expected upper and lower limits of the \( j \)-th energy commodity spot prices, respectively. In the absence of interactions between the energy commodities \( p_j, j \in I(1, n) \), where \( I(a, b) = \{ z \in \mathbb{Z} | a \leq z \leq b \} \), the market potential for the \( j \)-th commodity per unit of time at time \( t \) can be characterized by \((u_j - p_j)(l_j + p_j)\). This simple idea leads to the following economic principle regarding the dynamic of the price \( p_j \) of the \( j \)-th energy commodity. The change in spot price of the \( j \)-th energy commodity \( \Delta p_j(t) = p_j(t + \Delta t) - p_j(t) \) over the interval of length \(|\Delta t|\) is directly proportional to the market potential price.

\[
\Delta p_j(t) \propto (u_j - p_j)(l_j + p_j) \Delta t. \quad (9.1)
\]

This implies that

\[
dp_j = \alpha_j (u_j - p_j) (l_j + p_j) dt, \quad (9.2)
\]

for some constant \( \alpha_j \). From this deterministic mathematical model, if \( \alpha_j > 0 \), we note that as the price falls below the expected price \( u_j \), the price of the \( j \)-th commodity rises, and as the price lies above \( u_j \), there is a tendency for the price to fall. Similar argument follows if \( \alpha_j < 0 \). Hence

\[
\lim_{t \to \infty} p_j(t) = u_j, \quad (9.3)
\]

which implies that \( u_j \) is the equilibrium state of (9.2).

In a real world situation, the expected upper price limit \( u_j \) of the \( j \)-th commodity is not a constant parameter. It varies with time, and moreover it is subject to random environmental perturbations. Therefore, we consider

\[
u_j = y_j + \xi_j, \quad (9.4)
\]

where \( \xi_j \) is a white noise process that characterizes the measure of random fluctuation of the upper price limit of the \( j \)-th commodity; here \( y_j \) stands for the mean of the energy spot price process of the \( j \)-th commodity at time \( t \). It is further assumed that \( y_j \) is governed by a similar dynamic forces described in (9.2), that is,

\[
\dy_j = \mu_j (u_j - y_j)(v_j + y_j) dt, \quad (9.5)
\]
where $\mu_j > 0$ is defined as the mean reversion rate of the mean of the $j$-th commodity, $v_j \geq 0$ is defined as the lower limit of the mean of the $j$-th commodity. By following the argument used in (9.4), we incorporate the effects of random environmental perturbations into the lower limit $v_j$ of the mean of the $j$-th commodity:

$$v_j = v_j + e_j,$$

(9.6)

where $v_j \geq 0$, and $e_j$ is a white noise process describing the measure of random influence on the mean price of the $j$-th commodity.

Substituting expressions in (9.4) and (9.6) into (9.2) and (9.5), respectively, we obtain

$$\begin{align*}
\left\{ \begin{array}{l}
dy_j = \mu_j (u_j - y_j) (v_j + y_j) dt + \mu_j (u_j - y_j) e_j(t) dt \\
dp_j = \alpha_j (y_j - p_j) (l_j + p_j) dt + \alpha_j (l_j + p_j) \xi_j(t) dt. 
\end{array} \right.
\end{align*}$$

(9.7)

In the absence of interactions and using (9.7), the system of stochastic model for isolated expected spot and spot prices processes are described by the following non-linear system of stochastic differential equations:

$$\begin{align*}
\left\{ \begin{array}{l}
dy_j = \mu_j (u_j - y_j) (v_j + y_j) dt + \delta_{j,j} (u_j - y_j) dW_{j,j}(t), \quad y_j(t_0) = y_{j0}, \\
dp_j = \alpha_j (y_j - p_j) (l_j + p_j) dt + \sigma_{j,j} (l_j + p_j) dZ_{j,j}(t), \quad p_j(t_0) = p_{j0}, \quad j \in I(1,n),
\end{array} \right.
\end{align*}$$

(9.8)

where

$$\begin{align*}
\left\{ \begin{array}{l}
\mu_j e_j(t) dt = \delta_{j,j} dW_{j,j}(t), \quad j=1,2,...,n, \\
\alpha_j \xi_j(t) dt = \sigma_{j,j} dZ_{j,j}(t), \quad j=1,2,...,n,
\end{array} \right.
\end{align*}$$

and $\delta_{j,j}, \sigma_{i,j}$ are non-negative for $j = 1, 2, ..., n$.

In the presence of interactions, for each $j \in I(1,n)$, we consider both deterministic and stochastic interaction functions. For each $j \in I(1,n)$, we define the $j$-th aggregate interaction functions $g_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and $h_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ for the $j$th commodity of the mean energy spot price process $y_j(t)$ and the energy spot price process $p_j(t)$ in a energy commodity market network system, respectively. Moreover, we assume that these functions have the following structural forms:

$$\begin{align*}
\left\{ \begin{array}{l}
g_j(t,y) = g_j(t, k_{j,1} y_1, k_{j,2} y_2, ..., k_{j,n} y_n) \\
h_j(t,p) = h_j(t, \gamma_{j,1} p_1, \gamma_{j,2} p_2, ..., \gamma_{j,n} p_n),
\end{array} \right.
\end{align*}$$

(9.9)

where $k_{j,i}$ and $\gamma_{j,i}$ are elements of the $n \times n$ interconnection matrices $E_g$ and $E_h$, respectively. In (9.9), $k_{j,i} \in \mathbb{R}_+$ and $\gamma_{j,i} : \mathbb{R}_+ \to [0,1]$ represent a degree of interaction of the $j$-th commodity with $i$-th commodity in the energy commodity market network system.
For the matrix $E_g$, $k_{j,i} = 0$ with fixed $i \in I(1, n)$ if the $i$-th commodity in the energy market network system does not influence the $j$-th commodity. Likewise, for the matrix $E_h$, $\gamma_{j,i} = 0$ with fixed $i \in I(1, n)$, the $j$-th commodity in the energy market network system sub-component of $p$ is totally unaffected by the influence of the $i$-th commodity.

Finally, we introduce interactions in the diffusion coefficients with respect to the $j$-th commodity of the energy market network system under random environmental perturbations as: $\psi_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\Lambda_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ for each $j \in I(1, n)$. The diffusion part is of the form

$$
\begin{align*}
\psi_j(t, y) \cdot e_j(t) dt &= \sum_{l=1}^{\infty} \psi_{j,l}(t, y) dW_{j,l}(t) \\
\Lambda_j(t, p) \cdot \xi_j(t) dt &= \sum_{l=1}^{\infty} \Lambda_{j,l}(t, p) dZ_{j,l}(t),
\end{align*}
$$

(9.10)

where $e_j$ and $\xi_j$ are $n$-dimensional white noise processes; $\cdot$ stands for dot product.

We assume that the interaction functions (9.9) and (9.10) have the following forms;

$$
\begin{align*}
g(t, y) &= \gamma(t, y) G(t, y) \\
h(t, p) &= \lambda(t, p) H(t, p), \\
\psi(t, y) &= \gamma(t, y) \Psi(t, y), \\
\Lambda(t, p) &= \lambda(t, p) \Phi(t, p),
\end{align*}
$$

where $g(t, y) = [g_1(t, y), \ldots, g_j(t, y), \ldots, g_n(t, y)]^T$, $h(t, p) = [h_1(t, p), \ldots, h_j(t, p), \ldots, h_n(t, p)]^T$ are defined in (9.9), $\psi(t, y) = (\psi_{j,l}(t, y))_{n \times n}$, and $\Lambda(t, p) = (\Lambda_{j,l}(t, p))_{n \times n}$. $\gamma(t, y) = diag(u_1 - y_1, \ldots, u_j - y_j, \ldots, u_n - y_n)$ and $\lambda(t, p) = diag(l_1 + p_1, \ldots, l_j + p_j, \ldots, l_n + p_n)$; $G$, and $H$ are $n \times 1$ column vectors; $\Psi = diag(\psi_1, \ldots, \psi_j, \ldots, \psi_n)$ and $\Phi = diag(\Lambda_1, \ldots, \Lambda_j, \ldots, \Lambda_n)$ are block diagonal matrices; $\psi_j = [\psi_{j,1}, \ldots, \psi_{j,j}, \ldots, \psi_{j,n}]$, $\Lambda_j = [\Lambda_{j,1}, \ldots, \Lambda_{j,j}, \ldots, \Lambda_{j,n}]$. We also assume that $G, H, \Psi$ and $\Phi$ satisfy the local Lipschitz condition. This assumption implies that $g, h, \psi$ and $\Lambda$ also satisfy local Lipschitz condition.

Thus, the interconnected energy commodity network system is described by

$$
\begin{align*}
dy_j &= (u_j - y_j) \left[ (\mu_j (v_j + y_j) + G_j(t, y)) dt + \sum_{l=1}^{\infty} \psi_{j,l}(t, y) dW_{j,l}(t) \right], \\
y_j(t_0) &= y_{j0}, \\
dp_j &= (l_j + p_j) \left[ (\alpha_j (y_j - p_j) + H_j(t, p)) dt + \sum_{l=1}^{\infty} \Phi_{j,l}(t, p) dZ_{j,l}(t) \right], \\
p_j(t_0) &= p_{j0}, \quad j \in I(1, n),
\end{align*}
$$

(9.11)
where the parameters \( \mu_j > 0; \alpha_j > 0; u_j > 0; v_j \geq 0; l_j \geq 0; \delta_{j,l} > 0; \sigma_{j,l} > 0 \); and for \( j \neq l, \delta_{j,l} \geq 0; \sigma_{j,l} \geq 0; j, l \in I(1, n) \); for \( j \in I(1, n) \), \( W_j \) and \( Z_j \) are \( n \)-dimensional independent Wiener processes defined on a filtered probability space \( (\Omega, F, (F_t)_{t \geq 0}, \mathbb{P}) \); for \( l \neq i \), \( \mathbb{E}[dW_{j,l}dW_{k,i}] = 0 \), and for \( l = i \), \( \mathbb{E}[dW_{j,l}dW_{k,i}] = dt \); the filtration function \( (F_t)_{t \geq 0} \) is right-continuous; for each \( t \geq 0 \), each \( F_t \) contains all \( \mathbb{P} \)-null sets in \( F \); the \( n \)-dimensional random vectors \( y(t_0) \) and \( p(t_0) \) are \( F_{t_0} \)-measurable.

The network system of stochastic differential equations in (9.11) can be written as follows:

\[
\begin{align*}
    dy &= a(t, y)dt + \varphi(t, y)dW(t), \quad y(t_0) = y_0 \\
    dp &= b(t, y, p)dt + \sigma(t, p)dZ(t), \quad p(t_0) = p_0,
\end{align*}
\]

where

\[
\begin{align*}
    a(t, y) &= \begin{pmatrix}
        (u_1 - y_1) [\mu_1 (v_1 + y_1) + G_1(t, y)] \\
        (u_2 - y_2) [\mu_2 (v_2 + y_2) + G_2(t, y)] \\
        \vdots \\
        (u_n - y_n) [\mu_n (v_n + y_n) + G_n(t, y)]
    \end{pmatrix}, \\
    b(t, y, p) &= \begin{pmatrix}
        (l_1 + p_1) [\alpha_1 (y_1 - p_1) + H_1(t, p)] \\
        (l_2 + p_2) [\alpha_2 (y_2 - p_2) + H_2(t, p)] \\
        \vdots \\
        (l_n + p_n) [\alpha_n (y_n - p_n) + H_n(t, p)]
    \end{pmatrix}, \\
    \varphi(t, y) &= \text{diag}(A_1(y), \ldots, A_j(y), \ldots, A_n(y)), \quad \sigma(t, p) = \text{diag}(B_1(p), \ldots, B_j(p), \ldots, B_n(p)),
\end{align*}
\]

and

\[
\begin{align*}
    A_j(y) &= (u_j - y_j) \begin{pmatrix}
        \Psi_{j,1} & \Psi_{j,2} & \ldots & \Psi_{j,j-1} & \delta_{j,j} + \Psi_{j,j} & \Psi_{j,j+1} & \ldots & \Psi_{j,n}
    \end{pmatrix}, \\
    B_j(p) &= (l_j + p_j) \begin{pmatrix}
        \Phi_{j,1} & \Phi_{j,2} & \ldots & \Phi_{j,j-1} & \sigma_{j,j} + \Phi_{j,j} & \Phi_{j,j+1} & \ldots & \Phi_{j,n}
    \end{pmatrix};
\end{align*}
\]

\( W = [W_1, \ldots, W_n]^T, \) and \( Z = [Z_1, \ldots, Z_n]^T \) are block matrices; \( W_j = [W_{j,1}, \ldots, W_{j,2}, \ldots, W_{j,n}]^T, Z_j = [Z_{j,1}, \ldots, Z_{j,2}, \ldots, Z_{j,n}]^T; \) and \( \varphi(t, y), \sigma(t, p) \) are \( n \times n \) block matrix with each entries having order \( 1 \times n \).

In the next section, we outline the model validation problems of (9.12), namely, the existence and uniqueness of solution process.
9.3 Mathematical Model Validation

In this section, we validate the mathematical model derived in Section 2. We note that the classical existence and uniqueness theorem [57, 66] is not directly applicable to (9.12). We need to modify the existence and uniqueness results. The modification is based on Theorem 3.4 [57]. We show the global existence of solution process of system of differential equations (9.12).

We note that the rate functions $a, b, \Upsilon, \text{ and } \sigma$ in stochastic system of differential equations (9.12) do not satisfy the classical existence and uniqueness conditions [57]. However these rate functions do satisfy the local Lipschitz condition. Therefore, we construct sequences of functions for the drift and diffusion coefficients of interconnected dynamic system (9.12) so that the classical conditions for existence and uniqueness theorem are applicable. The construction of modification scheme is as follows: First, we define a cylindrical subset $[t_0, \infty) \times U_m$ of $[0, \infty) \times \mathbb{R}^n$ for $t_0 \in [0, \infty), m \in I(1, \infty)$, where $U_m$ is an $n$-dimensional sphere with radius $m$ defined by

$$U_m = \mathbb{B}(x_0, m) = \{x \in \mathbb{R}^n : ||x - x_0|| < m\},$$

for any $m \in I(1, \infty)$. We note that $U_m$ is inscribed in $n$-dimensional parallelepiped $\mathbb{R}(x-x_0, m) = [-m, m] \times \ldots \times [-m, m]$ in $\mathbb{R}^n$.

The developed stochastic network model (9.12) can be written as:

$$\begin{cases} 
  dy = a_m(t, y)dt + \Upsilon_m(t, y)dW(t), \ y(t_0) = y_0 \\
  dp = b_m(t, y, p)dt + \sigma_m(t, p)dZ(t), \ p(t_0) = p_0,
\end{cases}$$

(9.13)

where

$$\begin{cases} 
  a_m(t, y) = a(t, q(y, m)) \\
  \Upsilon_m(t, y) = \Upsilon(t, q(y, m)), \\
  b_m(t, y, p) = b(t, q(y, m), q(p, m)), \\
  \sigma_m(t, p) = \sigma(t, q(p, m)).
\end{cases}$$

(9.14)

Here, for each $j \in I(1, n)$ and $x \in \mathbb{R}^n$, we define $q_j(x, m) = \max \{-m, \min\{x_j - x_{0j}, m\}\}$. Hence, $q(x, m) = [q_1(x, m), \ldots, q_j(x, m), \ldots, q_n(x, m)]^T$, and it is denoted by $x^{(m)}$.

**Remark 23** We observe that $q(x, m)$ satisfies global Lipschitz condition on $\mathbb{R}^n$ with a Lipschitz constant 1. This together with the local Lipschitz condition assumption on the drift and diffusion coefficients of network system of stochastic differential equations (9.12), the modified rate coefficient
functions in (9.13) satisfy the classical existence and uniqueness conditions [57, 66]. Thus, its solution is denoted by \((y_m, p_m)\), for \(m \in I(1, \infty)\). Moreover, it is assumed that \((y, p)\) is non-negative whenever \(y_0, p_0 \in \mathbb{R}_+^n\).

Now we apply Theorems 3.4 and 3.5 of [57] in the context of modified system of stochastic differential equations (9.13) and Remark 23 to establish the global existence of solution of stochastic differential equations in (9.13). For this purpose, we outline the argument used in the proof of these theorems.

In addition to the local Lipschitz conditions on the drift and diffusion coefficients, we further impose the following hypothesis on the coefficients:

\[(H_1)\]
\[
\begin{align*}
|g_j(t, y)| &\leq a_{1,j} + \kappa_j ||y||, \\
|h_j(t, p)| &\leq a'_{1,j} + \gamma_j ||p||, \\
|\psi_{j,l}(t, y)| &\leq a_{2,j} + \tilde{\delta}_{j,l} ||y||, \\
|A_{j,l}(t, p)| &\leq a'_{2,j} + \tilde{\sigma}_{j,l} ||p||.
\end{align*}
\] (9.15)

where for \(i \in I(1, 2)\), \(a_{i,j}, a'_{i,j}\) are non-negative; \(\kappa_j, \gamma_j, \tilde{\delta}_{j,l}, \tilde{\sigma}_{j,l} \in \mathbb{R}_+^+\). From (9.13), we further remark that dynamic of mean of spot price vector \(y\) is decoupled with the dynamic of spot price \(p\).

Now we first apply Theorems 3.4 and 3.5 of [57] in the context of modified system of stochastic differential equations (9.13) and hypothesis \((H_1)\) to establish the global existence of solution of the completely decoupled sub-system of stochastic differential equations in (9.13). For this purpose, we outline the argument used in the proof of these theorems.

**DEFINITION 9.3.1** Let \(\tau_m\) be the first exit time of the solution process \(y_m\) from the set \(\mathbb{B}(y_0, m)\). Define \(\tau\) to be the (finite or infinite) limit of the monotone increasing sequence \(\tau_m\) as \(m \rightarrow \infty\).

\[
\tau = \lim_{m \rightarrow \infty} \tau_m.
\]

We wish to show that

\[
P(\tau = \infty) = 1. \quad (9.16)
\]

In the following, we present a result that is parallel to Theorem 3.5 [57] in the context of the completely decoupled sub-system of stochastic differential equation (9.12). For this purpose, it is enough to exhibit the global existence result for the transformed system (9.13).
LEMMA 9.1 For $m \in I(1, \infty)$, and $y_0 \in \mathbb{R}_+^n$, let $y_m(t) = y_m(t, t_0, y_0)$ be the solution of the completely decoupled sub-system of (9.13), and let the hypothesis $(H_1)$ be satisfied. Let $V_1$ be a function defined on $[t_0, \infty) \times \mathbb{R}_+^n$ into $\mathbb{R}_+$, it is defined by:

$$V_1(t, y) = \ln(||y||^2 + e), \quad (9.17)$$

Then there exists some constant $c_1 > 0$ such that

$$\begin{align*}
LV_1 & \leq c_1 V_1 \\
V_{1,m} &= \inf_{||y|| > m} V_1(t, y) \to \infty \text{ as } m \to \infty,
\end{align*} \quad (9.18)$$

where $L$ is the differential operator with respect to (9.12); $e = \exp(1)$.

Moreover, the global existence of solution of the completely decoupled sub-system of (9.12) follows.

Proof.

It is obvious that $V_1 \in C_{1,2}$ on $[t_0, \infty) \times \mathbb{R}_+^n \to \mathbb{R}_+$. In fact, $\frac{\partial V_1(t, y)}{\partial y_j} = \frac{2y_j}{||y||^2 + e}$, $\frac{\partial^2 V_1(t, y)}{\partial y_j^2} = \frac{4y_j^2}{(||y||^2 + e)^2} - \frac{4y_j y_j}{(||y||^2 + e)^2}$ exist and are continuous functions defined on $[t_0, \infty) \times \mathbb{R}_+^n \to \mathbb{R}$. Moreover, the $L$-operator with respect to the completely decoupled component is as follows:

$$LV_1(t, y) = \sum_{j=1}^{n} \left[ \mu_j (u_j - y_j)(v_j + y_j) + g_j(t, y) \right] \frac{\partial V_1(t, y)}{\partial y_j}$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \left[ \delta_{j,j}(u_j - y_j) + \psi_{j,j}(t, y) \right] + \sum_{l \neq j}^{n} \left[ \psi_{j,l}(t, y) \right] + \sum_{j=1}^{n} \left[ \psi_{j,j}(t, y) \right] \frac{\partial^2 V_1(t, y)}{\partial y_j^2}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j \neq i} \left[ \psi_{i,i}(t, y) \psi_{j,j}(t, y) + 2[\delta_{i,j}(u_i - y_i) + \psi_{i,j}] \psi_{j,j} \right] \frac{\partial^2 V_1(t, y)}{\partial y_i y_j}$$

$$= \sum_{j=1}^{n} \mu_j \left[ \left( u_j - \left( \frac{u_j - v_j}{2} \right) \right)^2 + \left( \frac{u_j + v_j}{2} \right)^2 \right] \frac{2y_j}{||y||^2 + e} + \sum_{j=1}^{n} \left[ \frac{2g_j(t, y) y_j}{||y||^2 + e} \right]$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \left[ \delta_{j,j}(u_j - y_j) + \psi_{j,j} \right] \left( \frac{2}{||y||^2 + e} - \frac{4y_j^2}{(||y||^2 + e)^2} \right)$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{l \neq j} \left[ \psi_{j,l}(t, y) \right] \left( \frac{2}{||y||^2 + e} - \frac{4y_j^2}{(||y||^2 + e)^2} \right)$$

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\[ -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \sum_{i \neq j, j} \psi_{i,j}(t, y_i) \psi_{j,i}(t, y_j) + 2[\delta_{i,j}(u_i - y_i) + \psi_{i,i}] \psi_{j,i} \right] \left( \frac{4y_i y_j}{(||y||^2 + e)^2} \right) \]

\[ \leq 2 \sum_{j=1}^{n} \mu_j \left( \frac{u_j + v_j}{2} \right)^2 \frac{y_j}{(||y||^2 + e)} + \sum_{j=1}^{n} 2g_j(t, y) y_j \left( \frac{\delta_{j,i}(u_j - y_j) + \psi_{j,i}(t, y)}{||y||^2 + e} \right) \]

\[ + \sum_{j=1}^{n} \sum_{i \neq j} \frac{\psi_{i,j}^2(t, y)}{(||y||^2 + e)} \]

\[ -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \sum_{i \neq j, j} \psi_{i,j}(t, y_i) \psi_{j,i}(t, y_j) + 2[\delta_{i,i}(u_i - y_i) + \psi_{i,i}] \psi_{j,i} \right] \left( \frac{4y_i y_j}{(||y||^2 + e)^2} \right) \]

\[ \leq 2 \sum_{j=1}^{n} \mu_j \left( \frac{u_j + v_j}{2} \right)^2 + \sum_{j=1}^{n} \left( a_{1,j} + \kappa_j ||y|| \right) \frac{y_j^2}{(||y||^2 + e)} \]

\[ + \sum_{j=1}^{n} \sum_{i \neq j} \delta_{i,j}^2(u_j + 1)^2 \sum_{j=1}^{n} \left( a_{2,j} + \delta_{j,i} \right)^2 \sum_{j=1}^{n} \left( a_{2,j} + \delta_{j,i} \right)^2 \]

\[ + \sum_{i=1}^{n} \sum_{j \neq i} \left( a_{2,i} + \delta_{i,i} \right) \left( a_{2,j} + \delta_{j,i} \right) \]

\[ + \sum_{i=1}^{n} \sum_{j \neq i} \left( a_{2,i} + \delta_{j,i} \right) \left( \delta_{i,i}(u_i + 1) + (a_{2,i} + \delta_{i,i}) \right) \]

\[ \leq c_1 V_1(t, y), \]

where \( c_1 = 1 + 2 \sum_{j=1}^{n} \mu_j \left( \frac{u_j + v_j}{2} \right)^2 + \sum_{j=1}^{n} \left( a_{1,j} + \kappa_j ||y|| \right) \frac{y_j^2}{(||y||^2 + e)} \]

\[ + \sum_{j=1}^{n} \sum_{i \neq j} \left( a_{2,j} + \delta_{j,i} \right)^2 \sum_{j=1}^{n} \left( a_{2,j} + \delta_{j,i} \right)^2 \]

\[ + \sum_{i=1}^{n} \sum_{j \neq i} \left( a_{2,i} + \delta_{i,i} \right) \left( a_{2,j} + \delta_{j,i} \right) \]

\[ + \sum_{i=1}^{n} \sum_{j \neq i} \left( a_{2,i} + \delta_{j,i} \right) \left( \delta_{i,i}(u_i + 1) + (a_{2,i} + \delta_{i,i}) \right) \]

Furthermore, \( \inf_{||y|| > m} V_1(t, y) \rightarrow \infty \) as \( m \rightarrow \infty \).

To show that \( P(\tau = \infty) = 1 \), we define a function

\[ V(t, y) = V_1(t, y) \exp\{-c_1(t - t_0)\}. \] (9.19)

It is obvious that \( LV \leq 0 \). By defining \( \tau_m(t) = \min(\tau_m, t) \); \( \mathcal{V}(t) = y_m(t) \) for \( t < \tau_m \); and imitating the argument of Lemma 3.2 [57], we have

\[ E\{V_1(\tau_m(t), \mathcal{V}(\tau_m(t)))\} \leq e^{c_1(t-t_0)} E V_1(t_0, y(t_0)). \]

Hence

\[ P(\tau_m \leq t) \leq \frac{e^{c_1(t-t_0)} E V_1(t_0, y(t_0))}{\inf_{||y|| > m, u > t_0} V_1(u, y)} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty \quad \text{by} \quad (9.18). \] (9.20)
The global existence and uniqueness of solution of the first component of (9.13) follows by letting \( m \to \infty \). Hence, from this and the fact that the solution process of transformed system (9.13) is almost surely identical with the solution process of the original system (9.12), we conclude the global existence and uniqueness of (9.12).

Now by following the idea of Lemma 9.1, we present a global existence and uniqueness of solution of the system of stochastic differential equations governing the sub-system \( \mathbf{p} \) in (9.12). We simply state a Lemma without the full proof.

**Lemma 9.2** For \( m \in I(1, \infty) \), and \( y_0, p_0 \in \mathbb{R}^n_+ \), let \( p_m(t) = p_m(t, t_0, p_0) \) be the solution of the system of stochastic differential equations governing the sub-system \( \mathbf{p} \) described in (9.13), and let the hypothesis \((H_1)\) be satisfied. Let \( V_2 \) be a nonnegative function on \([t_0, \infty) \times \mathbb{R}^n_+ \) into \( \mathbb{R}_+ \) defined by:

\[
V_2(t, \mathbf{p}) = \ln(||\mathbf{p}||^2 + e) + \sum_{j=1}^{n} \frac{\alpha_j}{2} \int_{t}^{T} (y_j(s) + l_j)^2 \, ds,
\]

Then there exist a constant \( c > 0 \) such that

\[
egin{align*}
    LV_2 & \leq cV_2 \\
    V_{2,m} &= \inf_{||\mathbf{p}|| > m} V_2(t, \mathbf{p}) \to \infty \text{ as } m \to \infty.
\end{align*}
\]

(9.22)

where \( L \) is the differential operator with respect to (9.12); \( e = \exp(1) \).

Moreover, the global existence of solution of the system of stochastic differential equations governing the sub-system \( \mathbf{p} \) described in (9.12) follows.

**Proof.** It is obvious that \( V_2 \in C_{1,2} \) on \([t_0, \infty) \times \mathbb{R}^n_+ \to \mathbb{R}_+ \). In fact, \( \frac{\partial V_2(t, \mathbf{p})}{\partial t} = -\sum_{j=1}^{n} \frac{\alpha_j}{2} (y_j(t) + l_j)^2 \), \( \frac{\partial V_2(t, \mathbf{p})}{\partial p_j} = \frac{2p_j}{||\mathbf{p}||^2 + e} \), \( \frac{\partial^2 V_2(t, \mathbf{p})}{\partial p_j^2} = \frac{2}{||\mathbf{p}||^2 + e} - \frac{4p_j^2}{(||\mathbf{p}||^2 + e)^2} \), \( \frac{\partial^2 V_2(t, \mathbf{p})}{\partial p_j \partial p_j} = -\frac{4p_j}{(||\mathbf{p}||^2 + e)^2} \) exist and are continuous functions defined on \([t_0, \infty) \times \mathbb{R}^n_+ \to \mathbb{R} \). Moreover, the \( L \)-operator with respect to the system of stochastic differential equations governing the sub-system \( \mathbf{p} \) described in (9.12) is as follows:
\[ LV_2(t, p) = -\sum_{j=1}^{n} \frac{\alpha_j}{2} (y_j(t) + l_j)^2 + \sum_{j=1}^{n} (\alpha_j (y_j - p_j)(l_j + p_j) + h_j(t, p)) \left( \frac{2p_j}{(||p||^2 + e)} \right), \]

\[ + \frac{1}{2} \sum_{j=1}^{n} \left[ \sigma_{j,j}(l_j + p_j) + \Lambda_{j,j}(t, p) \right]^2 + \sum_{l \neq j} \Lambda_{j,l}(t, p)^2 \right] \left( \frac{2}{(||p||^2 + e)} \right) - \frac{4p_j^2}{(||p||^2 + e)^2} \right] \]

\[ + \frac{1}{2} \sum_{j=1}^{n} \sum_{l \neq j} \Lambda_{j,l}(t, p)^2 \left( \frac{2}{(||p||^2 + e)} - \frac{4p_j^2}{(||p||^2 + e)^2} \right) \]

\[ - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \sum_{l \neq i} \Lambda_{i,l} \Lambda_{j,l} + 2[\sigma_{i,i}(l_i + p_i) + \Lambda_{i,i}] \Lambda_{j,i} \right] \frac{4p_i p_j}{(||p||^2 + e)^2} \]

\[ \leq -\sum_{j=1}^{n} \frac{\alpha_j}{2} (y_j + l_j)^2 + \sum_{j=1}^{n} \frac{\alpha_j}{2} (y_j + l_j)^2 \frac{p_j}{(||p||^2 + e)} + \sum_{j=1}^{n} \frac{p_j}{(||p||^2 + e)} + \sum_{j=1}^{n} \left[ a_{1,j} + \gamma_j ||p||^2 + p_j^2 \right] \]

\[ + \sum_{j=1}^{n} \left[ \sigma_{j,j}(l_j + p_j) + \Lambda_{j,j}(t, p) \right]^2 \]

\[ + \sum_{j=1}^{n} \sum_{l \neq j} \left( a_{2,j}^2 + \sigma_{j,l}^2 \right)^2 + 2 \sum_{i=1}^{n} \sum_{l \neq i} \sum_{l \neq i,j} \left( a_{2,i}^2 + \sigma_{i,l} \right)^2 + \left( a_{2,j}^2 + \sigma_{j,l} \right) \]

\[ + 4 \sum_{i=1}^{n} \sum_{j \neq i} \left( a_{2,i}^2 + \sigma_{i,j} \right) \sigma_{i,j}(l_i + 1) + (a_{2,i}^2 + \sigma_{i,i}) \]

\[ \leq cV_2(t, p), \]

where \( c = 1 + \sum_{j=1}^{n} [a_{1,j}^2 + \gamma_j]^2 + 2 \sum_{j=1}^{n} [\sigma_{j,j}^2 (l_j + 1)^2 + (a_{2,j}^2 + \sigma_{j,j})^2] + \sum_{j=1}^{n} \sum_{l \neq j} \left( a_{2,j}^2 + \sigma_{j,l} \right)^2 + \sum_{i=1}^{n} \sum_{j \neq i} \left( a_{2,i}^2 + \sigma_{i,j} \right) \sigma_{i,j}(l_i + 1) + (a_{2,i}^2 + \sigma_{i,i}). \]

Furthermore, \( \inf_{||p||>m} V_2,m(t, p) \to \infty \) as \( m \to \infty. \) \( \square \)
Following the final argument used in proving the global existence of \( y \) in Lemma 9.1, we conclude that there exists a unique global solution to the interconnected system of stochastic differential (9.12).

In the next section, we discuss about the case where we incorporate jump process into the system \((y, p)\).

### 9.4 Energy Commodity Model With and Without Jumps

Due to random interventions that affects the price of energy commodities, we introduce random interventions described by a continuous jump in our model. We follow the approach discussed in [120, 138]. In their work, Wu [120, 138] investigated a class of stochastic hybrid dynamic processes.

Let \( K \geq 0 \) be the number of jumps on the interval \([t_0, T]\), for \( T > 0 \). For \( K \neq 0 \), let \( T_1, ..., T_K \) be the jump times over a time interval \([t_0, T]\) such that \( t_0 = T_0 \leq T_1 < ... < T_K \leq T \), where \( T_i \) denotes the time at which the \( i \)-th jump occurred in the system \((y, p)\). For \( K = 0 \), no jump has occurred on the interval \([t_0, T]\). We denote the \( i \)-th sub-interval by \( T_{i-1} \leq t < T_i \). Knowing the global existence and uniqueness solution process of system (9.12) on the interval \([t_0, T]\), \( T > 0 \) in Section 9.3, for \( i \in I(1, K^*) \) and \( K \neq 0 \), we consider the solution process on each subinterval \([T_{i-1}, T_i]\), between jumps, where \( K^* = K \) if \( T_K = T \), and \( K^* = K + 1 \) if \( T_K < T \). For \( i \in I(1, K^*) \) and \( K = 0 \), we have \( I(1, K) = \emptyset \) or \( I(1, K^*) = \{1\} \). In this case, we consider the solution process on \([t_0, T]\). The interconnected system is governed by the following system of continuous time stochastic process:

\[
\begin{align*}
\{ dy^{i-1} &= a^{i-1}(t, y)dt + \mathbf{Y}^{i-1}(t, y)dW(t), \quad y(T_{i-1}) = y^{i-1}, \quad t \in [T_{i-1}, T_i] \\
\{ dp^{i-1} &= b^{i-1}(t, y, p)dt + \sigma^{i-1}(t, p)dZ(t), \quad p(T_{i-1}) = p^{i-1}, \quad t \in [T_{i-1}, T_i], \quad i \in I(1, K^*) \\
\{ y^i &= \Pi^i y^{i-1}(T_i^{-}, T_{i-1}, y^{i-1}), \\
\{ p^i &= \Theta^i p^{i-1}(T_i^{-}, T_{i-1}, y^{i-1}, p^{i-1}), \\
\end{align*}
\]

(9.23)

where

\[
\Pi^i = \text{diag} (\pi_1^i, \pi_2^i, ..., \pi_n^i), \\
\Theta^i = \text{diag} (\theta_1^i, \theta_2^i, ..., \theta_n^i),
\]

\((y^{i-1}(t, T_{i-1}, y^{i-1}), p^{i-1}(t, T_{i-1}, y^{i-1}, p^{i-1}))\) is the solution of system of equation (9.12); for \( K \neq 0 \) and \( i \in I(1, K^*) \), \( \Pi^i \) and \( \Theta^i \) are jump coefficient matrices. These jump coefficients are estimated
by \( \hat{\pi}_j = \lim_{t \to T_i} \frac{y_j(T_i)}{y_j(t, T_i, y_{i-1})} \); \( \hat{\theta}_j = \lim_{t \to T_i} \frac{p_j(T_i)}{p_j(t, T_i, y_{i-1}, p^{i-1})} \) for \( j \in I(1, n) \). Following the approach in [120, 138], the solution of (9.23) takes the following representation:

\[
\begin{align*}
\mathbf{y}(t, t_0, \mathbf{y}_0) &= \begin{cases} 
\mathbf{y}^0(t, t_0, \mathbf{y}_0), & \mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \leq t < T_1 \\
\mathbf{y}^1(t, T_1, \mathbf{y}^1), & \mathbf{y}^1 = \mathbf{y}(T_1), \quad T_1 \leq t < T_2,
\end{cases} \\
\vdots \\
\mathbf{y}^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}), & \mathbf{y}^{i-1} = \mathbf{y}(T_{i-1}), \quad T_{i-1} \leq t < T_i, \\
\vdots \\
\mathbf{y}^K(t, T_K, \mathbf{y}^K), & \mathbf{y}^K = \mathbf{y}(T_K), \quad T_K \leq t \leq T,
\end{align*}
\]

\[
\begin{align*}
\mathbf{p}(t, t_0, \mathbf{y}_0, \mathbf{p}_0) &= \begin{cases} 
\mathbf{p}^0(t, t_0, \mathbf{y}_0, \mathbf{p}_0), & \mathbf{p}(t_0) = \mathbf{p}_0, \quad t_0 \leq t < T_1 \\
\mathbf{p}^1(t, T_1, \mathbf{y}^1, \mathbf{p}^1), & \mathbf{p}^1 = \mathbf{p}(T_1), \quad T_1 \leq t < T_2, \\
\vdots \\
\mathbf{p}^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}), & \mathbf{p}^{i-1} = \mathbf{p}(T_{i-1}), \quad T_{i-1} \leq t < T_i, \\
\vdots \\
\mathbf{p}^K(t, T_K, \mathbf{y}^K, \mathbf{p}^K), & \mathbf{p}^K = \mathbf{p}(T_K), \quad T_K \leq t \leq T,
\end{cases}
\]

(9.24)

and \( I(1, 0) = \{ i \in \mathbb{Z} : 1 \leq i \leq 0 \} = \emptyset \) and \( I(1, K^*) = \{ 1 \} \).

**Remark 24** For no jump, that is \( K = 0 \), (9.23) and (9.24) reduce to

\[
\begin{align*}
\begin{cases} 
\text{d} \mathbf{y} &= \mathbf{a}(t, \mathbf{y}) \text{d}t + \mathbf{Y}(t, \mathbf{y}) \text{d} \mathbf{W}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \\
\text{d} \mathbf{p} &= \mathbf{b}(t, \mathbf{y}, \mathbf{p}) \text{d}t + \mathbf{\sigma}(t, \mathbf{p}) \text{d} \mathbf{Z}(t), \quad \mathbf{p}(t_0) = \mathbf{p}_0, \quad t_0 \leq t \leq T
\end{cases}
\end{align*}
\]

(9.25)

and

\[
\begin{align*}
\begin{cases} 
\mathbf{y}(t, t_0, \mathbf{y}_0), & \mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \leq t < T, \\
\mathbf{p}(t, t_0, \mathbf{y}_0, \mathbf{p}_0), & \mathbf{p}(t_0) = \mathbf{p}_0, \quad t_0 \leq t < T,
\end{cases}
\end{align*}
\]

(9.26)

respectively.

### 9.5 Multivariate Discrete Time Dynamic Model for Local Sample Mean and Covariance Process

In this section, we use the idea of moving average to derive an algorithm for the mean and covariance of sample sequences with respect to a continuous time stochastic process. The development of idea and model of statistic for mean and covariance processes is motivated by the state and parameter estimation problems of continuous time nonlinear stochastic dynamic model of the energy
commodity market network (4.11). For this purpose, we need to introduce a few definitions and notations.

For each \( i \in I(1, K^*) \), let \( \tau_{i-1} \) and \( \gamma_{i-1} \), be finite constant time delays such that \( 0 < \gamma_{i-1} \leq \tau_{i-1} \). If \( K = 0 \), then there is no jump. Hence, we simply write \( \tau_{i-1} = \tau \) and \( \gamma_{i-1} = \gamma \). Here \( \tau_{i-1} \) characterize the influence of the past performance history of state of dynamic process. \( \gamma_{i-1} \) describe the reaction or response time delays. In general, these time delays are unknown and random variables. These types of delays play an important role in developing mathematical models of continuous time [64] and discrete time [59, 88] dynamic processes. Based upon the practical nature of data collection process, it is essential to either transform these time delays into positive integers or design a suitable data collection schedule or discretization process. For this purpose, for each \( i \in I(1, K^*) \), we describe the discrete version of time delays of \( \tau_i \) and \( \gamma_i \) as follows:

\[
\begin{align*}
\{ \quad \tau_{i-1} &= \left\lceil \frac{\tau_i - 1}{\Delta t_{i-1}} \right\rceil + 1, \quad \text{and} \quad \eta_{i-1} = \left\lceil \frac{\gamma_i - 1}{\Delta t_{i-1}} \right\rceil + 1, \quad \text{for} \quad i \in I(1, K^*). \quad (9.27) \end{align*}
\]

Moreover, for the sake of simplicity, we assume that \( 0 < \gamma_i - 1 < 1, (\eta_{i-1} = 1) \).

**Definition 9.5.1** Let \( x = [x_1, x_2, ..., x_n]^T \) be a continuous time multivariate stochastic process defined on an interval \([t_0 - \tau, T]\) into \( \mathbb{R}^n \), for some \( T > 0 \). For \( t \in [t_0 - \tau, T] \), let \( \mathcal{F}_t \) be an increasing sub-sigma algebra of a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) for which \( x(t) \) is \( \mathcal{F}_t \) measurable. For each \( i \in I(1, K^*) \), let \( \mathbb{P} \) and \( \mathbb{P}^{i-1} \) be a partition of \([t_0 - \tau, T]\) and \([T_{i-1} - \tau_{i-1}, T_i]\), respectively. The partition \( \mathbb{P}^{i-1} \) is derived by decomposing the partition \( \mathbb{P} \). For each \( i \in I(1, K^*) \), the partitions \( \mathbb{P} \) and \( \mathbb{P}^{i-1} \) are defined as follows:

\[
\begin{align*}
\mathbb{P} &= \{ t_k : t_k = t_0 + k \Delta t, \quad k \in I(-r, N) \}, \\
\mathbb{P}^{i-1} &= \{ t_k^{i-1} : t_k^{i-1} = T_{i-1} + k \Delta t, \quad k \in I(-r, N_{i-1}) \}, 
\end{align*}
\]

where \( \Delta t = \frac{T - t_0}{N} = \frac{T_i - T_{i-1}}{N_{i-1}} \) and \( T_0 = t_0 \).

**Remark 25** We define \( S_i = \sum_{l=0}^{i} N_{l-1} \) with \( S_0 = 0 \). For \( K \neq 0 \), we note that we can write \( \mathbb{P} \) as

\[
\{ t_0 < t_1 < ... < t_{N_0} < t_{N_0+N_1} < ... < t_{N_0+N_1+N_2} < ... < t_{S_{i-1}} < t_{S_{i-1}+1} < ... < t_{S_i} < ... < T \}. \]

From this, it follows directly that the jump times \( T_i \) are contained in \( \mathbb{P} \). For any \( t_k^{i-1} \in \mathbb{P}^{i-1}, k \in [0, N_{i-1}] \), we have \( t_k^{i-1} \in \mathbb{P} \). Hence, there is a relationship between elements of \( \mathbb{P}^{i-1} \) with \( \mathbb{P} \) that is described by \( t_k^{i-1} = t_{S_{i-1}+k}, \) for \( k \in I(0, N_i) \). In fact, the relationship between set of jump times \( \{T_1, T_2, ..., T_K\} \) and the partition \( \mathbb{P} \) defined in (9.28) is as: \( T_i \rightarrow t_{S_i} \), where the
$N_{j-1}$’s are the size of partition $\mathbb{P}^{i-1}$ of the sub-interval $[T_{j-1}, T_j]$. It follows that $S_{K^*} = N$. Using these facts, and noting that if $K = 0$, then $\mathbb{P}^{i-1} = \mathbb{P}$, $N_{i-1} = N$, $\tau_{i-1} = \tau$, $\gamma_{i-1} = \gamma$, $r_{i-1} = r$, $\eta_{i-1} = \eta$, $t_{i-1}^k = t_k$. Moreover, (9.28) can be written as:

\[ \mathbb{P}^{i-1} = \{ t_{i-1}^k : t_{i-1}^k = T_{i-1} + k\Delta t, k \in I(-r_{i-1}, N_{i-1}) \}. \quad (9.29) \]

For each $i \in I(1, K^*)$, let $\{ x^{i-1}(t_{i-1}^k) \}_{k=\tau_{i-1}}^{N_{i-1}}$ be a finite sequence corresponding to the stochastic process $x$ and partition $\mathbb{P}^{i-1}$ defined in (9.29). We simply write $x(t_{i-1}^k) \equiv x^{i-1}(t_{i-1}^k)$. We further recall that $x(t_{i-1}^k)$ is $\mathcal{F}_{t_{i-1}^k}$ measurable for $k \in I(-r_{i-1}, N_{i-1})$. We also recall the definition of forward time shift operator $F$ [11]:

\[ F^l x(t_{k-1}^{i-1}) = x(t_{k+l}^{i-1}), \quad l \in I(0, \infty). \quad (9.30) \]

**Definition 9.5.2** For $q_{i-1} = 1$ and $r_{i-1} \geq 1$, each $k \in I(0, N_{i-1})$, and each $m_{i-1}^{k} \in I(2, r_{i-1} + S_{i-1} + k - 1)$, a partition $P_{i-1}^{k}$ of closed interval $[t_{i-1}^{k-m_{i-1}^{k}}, t_{i-1}^{k}]$ is called local at time $t_{i-1}^{k}$, and it is defined by

\[ P_{i-1}^{k} := t_{i-1}^{k-m_{i-1}^{k}} < t_{i-1}^{k-m_{i-1}^{k}+1} < \ldots < t_{i-1}^{k}. \quad (9.31) \]

Moreover, $P_{i-1}^{k}$ is referred as the $m_{i-1}^{k}$ point sub-partition of the partition $\mathbb{P}^{i-1}$ in (9.29) of the closed sub-interval $[t_{i-1}^{k-m_{i-1}^{k}}, t_{i-1}^{k}]$ of $[-\tau_{i-1}, \tau]$. We note that for $K = 0$, that is, there is no jump, we have $P_{i-1}^{k} = P_k$, $m_{i-1}^{k} = m_k$, $t_{i-1}^{k-m_{i-1}^{k}} = t_{i-1}^{k-m_{i-1}^{k}}$, and $t_{i-1}^{k} = t_{k-1}$, where $P_k$ is referred as the $m_k$ point sub-partition of the partition $\mathbb{P}$ in (9.28) of the closed sub-interval $[t_{k-m_{k}}, t_{k}]$ of $[t_{0} - \tau, \tau]$ for $k \in I(0, N)$.

**Definition 9.5.3** For each $i \in I(1, K^*)$, $k \in I(0, N_{i-1})$ and $m_{i-1}^{k} \in I(2, r_{i-1} + S_{i-1} + k - 1)$, a local finite sequence at $t_{i-1}^{k}$ of the size $m_{i-1}^{k}$ is restriction $[2]$ of $\{ x(t_{i-1}^{k}) \}_{k=\tau_{i-1}}^{N_{i-1}}$ to $P_{i-1}^{k}$ in (9.31). This restriction sequence is defined by

\[ S_{m_{i-1}^{k}, k} := \{ F^l x(t_{i-1}^{k-l}) \}_{l=-m_{i-1}^{k}+1}^{0}. \quad (9.32) \]

As $m_{i-1}^{k}$ varies from 2 to $r_{i-1} + S_{i-1} + k - 1$, the corresponding respective local sequence $S_{m_{i-1}^{k}, k}$ at $t_{i-1}^{k}$ varies from $\{ x(t_{i-1}^{k}) \}_{l=k-2}^{k-1}$ to $\{ x(t_{i-1}^{k}) \}_{l=-r_{i-1} + S_{i-1} + k - 1+1}^{k-1}$. As a result of this, the sequence defined in (9.32) is also called a $m_{i-1}^{k}$-local moving sequence. Furthermore, the average corresponding to the local sequence $S_{m_{i-1}^{k}, k}$ in (9.32) is defined by

\[ \overline{S}_{m_{i-1}^{k}, k} = \frac{1}{m_{i-1}^{k}} \sum_{l=-m_{i-1}^{k}+1}^{0} F^l x(t_{i-1}^{k-l}). \quad (9.33) \]

The average/mean defined in (9.33) is also called the $m_{i-1}^{k}$-local average/mean.
For \( i \in I(1, K^*) \), and \( k \in I(0, N_{i-1}) \), the \( m_{k}^{i-1} \)-local covariance matrix corresponding to the local sequence \( S_{m_{k}^{i-1}, k} \) in (9.32) is defined by

\[
\sum m_{k}^{i-1}, k = \begin{pmatrix}
    s_{1,1}^{m_{k}^{i-1}, k} & s_{1,2}^{m_{k}^{i-1}, k} & \ldots & s_{1,n}^{m_{k}^{i-1}, k} \\
    s_{2,1}^{m_{k}^{i-1}, k} & s_{2,2}^{m_{k}^{i-1}, k} & \ldots & s_{2,n}^{m_{k}^{i-1}, k} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{n,1}^{m_{k}^{i-1}, k} & s_{n,2}^{m_{k}^{i-1}, k} & \ldots & s_{n,n}^{m_{k}^{i-1}, k}
\end{pmatrix}
\]  

(9.34)

where \( s_{j,l}^{m_{k}^{i-1}, k} \equiv s_{j,l}^{m_{k}^{i-1}, k}(x) \), \( j, l \in I(1, n) \) is the local sample covariance statistic between \( x_{j} \) and \( x_{l} \) at \( t_{k}^{i-1} \) described by

\[
s_{j,l}^{m_{k}^{i-1}, k} := \begin{cases}
    \frac{1}{m_{k}^{i-1}} \sum_{a=-m_{k}^{i-1}+1}^{0} \left( F^{a} x_{j}(t_{k}^{i-1}) - \frac{1}{m_{k}^{i-1}} \sum_{b=-m_{k}^{i-1}+1}^{0} F^{b} x_{j}(t_{k}^{i-1}) \right) \times \\
    \left( F^{a} x_{l}(t_{k}^{i-1}) - \frac{1}{m_{k}^{i-1}} \sum_{b=-m_{k}^{i-1}+1}^{0} F^{b} x_{l}(t_{k}^{i-1}) \right), & \text{for small } m_{k}^{i-1} \\
    \frac{1}{m_{k}^{i-1}} \sum_{a=-m_{k}^{i-1}+1}^{0} \left( F^{a} x_{j}(t_{k}^{i-1}) - \frac{1}{m_{k}^{i-1}} \sum_{b=-m_{k}^{i-1}+1}^{0} F^{b} x_{j}(t_{k}^{i-1}) \right) \times \\
    \left( F^{a} x_{l}(t_{k}^{i-1}) - \frac{1}{m_{k}^{i-1}} \sum_{b=-m_{k}^{i-1}+1}^{0} F^{b} x_{l}(t_{k}^{i-1}) \right), & \text{for large } m_{k}^{i-1}
\end{cases}
\]  

(9.35)

In the following, we derive a interconnected discrete-time local conditional sample average/mean and covariance dynamic processes. This fundamental result is motivated by Exercise 5.15 in [14]. Denoting \( X(k) \equiv X_{k}^{i-1} \) for \( i \in I(1, K^*) \) and \( k \in I(1, N_{i-1}) \), we state and prove the following Lemma.

**Definition 9.5.4** Let \( \{ \mathbb{E}[x_{j}(t_{k}^{i-1})|\mathcal{F}_{t_{k}^{i-1}}] \}_{k=-r_{i-1}}^{N_{i-1}} \) be a conditional random sample of continuous time stochastic dynamic process with respect to sub-\( \sigma \) algebra \( \mathcal{F}_{t_{k}^{i-1}} \), \( t_{k}^{i-1} \in \mathbb{R}^{i-1} \) in (9.29). The \( m_{k}^{i-1} \)-local conditional moving average and covariance defined in the context of (9.33) and (9.34) are called the \( m_{k} \)-local conditional moving sample average/mean and local conditional moving sample variance, respectively.

**Lemma 9.3 (Multivariate Discrete Time Dynamic Model of Local Sample Mean and Sample Covariance Process).** Let \( \{ \mathbb{E}[x_{j}(t_{k}^{i-1})|\mathcal{F}_{t_{k}^{i-1}}] \}_{k=-r_{i-1}}^{N_{i-1}} \) be a conditional random sample defined in Definition (9.5.4). Let \( \bar{S}_{m_{k}^{i-1}, k} \) and \( \sum m_{k}^{i-1}, k \) be \( m_{k}^{i-1} \)-local conditional sample average and local
conditional sample covariance at $t_{k}^{i-1}$. Then, an interconnected multivariate discrete time dynamic model of local conditional sample mean and sample covariance statistics is described by

$$
\bar{S}_{m_{k-d_{i-1}+1}^{i-1},k-d_{i-1}+1} = \frac{m_{k-d_{i-1}+1}^{i-1}}{m_{k-d_{i-1}+1}^{i-1}} \bar{S}_{m_{k-d_{i-1}}^{i-1},k-d_{i-1}} + \eta m_{k-d_{i-1}}^{i-1},k-d_{i-1}, \bar{S}_{m_{T_{i-1}}^{i-1},T_{i-1}} = \bar{S}_{T_{i-1}}
$$

$$
\sum m_{i-1,k} = \left\{ \begin{array}{l}
d_{i-1} \sum_{j=1}^{d_{i-1}} \left[ \frac{m_{k-j}^{i-1}-1}{\prod_{l=0}^{m_{k-j}^{i-1}-1}} \sum m_{k-j}^{i-1},k-j \right] \\
+ \frac{d_{i-1}^{i-1}}{m_{k-j}^{i-1}} \bar{S}_{m_{k-d_{i-1}}^{i-1},k-d_{i-1}} \bar{S}_{m_{k-d_{i-1}}^{i-1},k-d_{i-1}}^{T} + \varepsilon m_{k-j}^{i-1},k-1, for small m_{k-j}^{i-1}, m_{k-j}^{i-1} \leq m_{k-j}^{i-1} \\
+ \varepsilon m_{k-j}^{i-1},k-1, for large m_{k-j}^{i-1}, m_{k-j}^{i-1} \leq m_{k-j}^{i-1} \\
\sum m_{j-1,j} = \sum_{j} m_{j-1,j}, i \in I(1,K^{*}), j \in I(-d_{i-1},0), initial conditions (9.36)
\end{array} \right.
$$

where

$$\eta = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \vdots \\ \eta^n \end{pmatrix},$$

$$\varepsilon_{m_{k-j}^{i-1},k} = \begin{pmatrix} \varepsilon_{1,1}^{m_{k-j}^{i-1},k} & \varepsilon_{1,2}^{m_{k-j}^{i-1},k} & \varepsilon_{1,3}^{m_{k-j}^{i-1},k} & \cdots & \varepsilon_{1,n}^{m_{k-j}^{i-1},k} \\ \varepsilon_{2,1}^{m_{k-j}^{i-1},k} & \varepsilon_{2,2}^{m_{k-j}^{i-1},k} & \varepsilon_{2,3}^{m_{k-j}^{i-1},k} & \cdots & \varepsilon_{2,n}^{m_{k-j}^{i-1},k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n,1}^{m_{k-j}^{i-1},k} & \varepsilon_{n,2}^{m_{k-j}^{i-1},k} & \varepsilon_{n,3}^{m_{k-j}^{i-1},k} & \cdots & \varepsilon_{n,n}^{m_{k-j}^{i-1},k} \end{pmatrix},$$

$$\varepsilon_{m_{k-j}^{i-1},k} = \begin{pmatrix} \varepsilon_{1,1}^{m_{k-j}^{i-1},k} & \varepsilon_{1,2}^{m_{k-j}^{i-1},k} & \varepsilon_{1,3}^{m_{k-j}^{i-1},k} & \cdots & \varepsilon_{1,n}^{m_{k-j}^{i-1},k} \\ \varepsilon_{2,1}^{m_{k-j}^{i-1},k} & \varepsilon_{2,2}^{m_{k-j}^{i-1},k} & \varepsilon_{2,3}^{m_{k-j}^{i-1},k} & \cdots & \varepsilon_{2,n}^{m_{k-j}^{i-1},k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n,1}^{m_{k-j}^{i-1},k} & \varepsilon_{n,2}^{m_{k-j}^{i-1},k} & \varepsilon_{n,3}^{m_{k-j}^{i-1},k} & \cdots & \varepsilon_{n,n}^{m_{k-j}^{i-1},k} \end{pmatrix}.$$
\[
\eta_{m_{k-d_i-1}^{i-1},k-d_i-1}^j = \frac{1}{m_{k-d_i-1}^{i-1} \sum_{i=1}^{d_i-1} F^x_j(k - d_{i-1})}
\]
\[
- F^{m_{k-d_i-1}^{i-1}} x_j(k - d_{i-1})
\]
\[
- F^{m_{k-d_i-1}^{i-1}} x_j(k - d_{i-1}) + 1
\]

\[
\varepsilon_{m_{k-1}^{i-1},k-1}^{j,l} = \frac{m_{i-1}^{i-1} - 1}{m_{i-1}^{i-1}} \sum_{i=1}^{d_i-1} \left[ \frac{F^{\epsilon + 1} x_j(k - 1) F^{\epsilon + 1} x_l(k - 1)}{\prod_{a=0}^{i-1} m_{k-a}^{i-1}} \right]
\]
\[
- \sum_{i=1}^{d_i-1} \frac{F^{-\epsilon + 1 - m_{i-1}^{i-1}} x_j(k - 1) F^{-\epsilon + 1 - m_{i-1}^{i-1}} x_l(k - 1)}{\prod_{a=0}^{i-1} m_{k-a}^{i-1}}
\]
\[
+ \frac{m_{i-1}^{i-1} - 1}{m_{i-1}^{i-1}} \sum_{i=1}^{d_i-1} \left[ \frac{-\epsilon + 2 - m_{i-1}^{i-1}}{\prod_{a=0}^{i-1} m_{k-a}^{i-1}} \right]
\]
\[
- \frac{d_i-1}{m_{i-1}^{i-1}} \sum_{i=1}^{d_i-1} \left[ \frac{-\epsilon + 2 - m_{i-1}^{i-1}}{\prod_{a=0}^{i-1} m_{k-a}^{i-1}} \right]
\]
\[
- \frac{1}{m_{k}^{i-1}} \sum_{i=1}^{d_i-1} \left[ \frac{-\epsilon + 2 - m_{i-1}^{i-1}}{\prod_{a=0}^{i-1} m_{k-a}^{i-1}} \right]
\]
\begin{align*}
\frac{d_{i-1}}{m_{k-1}^{-1} - 1} \sum_{i=1}^{d_{i-1}} e_{i, l}^j m_{k-1}^{-1} k^{-1} = & \sum_{i=1}^{d_{i-1}} F^{-t+1} x_j(k-1) F^{-t+1} x_l(k-1) \\
& - \sum_{i=1}^{d_{i-1}} F^{-t+1-m_{k-1}^{-1}} x_j(k-1) F^{-t+1-m_{k-1}^{-1}} x_l(k-1) \\
& - \sum_{i=1}^{d_{i-1}} F^{-t+2-m_{k-1}^{-1}} x_j(k-1) F^{-t+2-m_{k-1}^{-1}} x_l(k-1) \\
& - \frac{1}{m_{k-1}^{-1} - 1} \sum_{v, s = -m_{k-1}^{-1}, v \neq s}^{0} F^{v} x_j(k-1) F^{s} x_l(k-1) \\
& + \sum_{i=1}^{d_{i-1}} \left[ F^{-t+1} x_j(k-1) F^{-t+1} x_l(k-1) \right] \\
& + \sum_{i=1}^{d_{i-1}} \left[ F^{-t+1} x_j(k-1) F^{-t+1} x_l(k-1) \right] \\
& + \sum_{i=1}^{d_{i-1}} \left[ F^{-t+1} x_j(k-1) F^{-t+1} x_l(k-1) \right].
\end{align*}

### 9.6 Parametric Estimation

In this section, we consider a parameter estimation problem in drift and diffusion coefficients of (9.23). This is achieved by utilizing the lagged adaptive process [88] and the interconnected discrete-time dynamics of local sample mean and variances statistic processes model in Section 9.5 (Lemma 9.3). For each $i \in I(1, K^*)$, we consider a general interconnected hybrid system described by the system of stochastic differential equations:

\begin{align*}
\begin{cases}
\begin{aligned}
\frac{d x^{i-1}}{t} &= f^{-1}(t, x) dt + \sigma^{i-1}(t, x) dW(t), \quad x(T_{i-1}) = x^{i-1}, \quad t \in [T_{i-1}, T_{i}), \\
x^i &= \Gamma^i x^{i-1}(T_{i-1}, T_{i-1}, x^{i-1}),
\end{aligned}
\end{cases}
\end{align*}

where $\Gamma^i = \text{diag}(\Gamma^i_1, \Gamma^i_2, \ldots, \Gamma^i_j, \ldots, \Gamma^i_n)$ is the jump coefficient matrix; the jump times $T_i$’s are defined in (9.23). For each $j \in I(1, n)$, the estimate of the jump coefficient $\Gamma_j^i$ is given by $\Gamma_j^i = \lim_{t \to T_i^-} x_{j,T_{i-1},x^{i-1}}^{-1}$. 

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Let \( V \in C[[-\tau, \infty) \times \mathbb{R}^n, \mathbb{R}^m] \), and its partial derivatives \( V_t, \frac{\partial V}{\partial x} \) and \( \frac{\partial V^{-1}}{\partial x} \) exist and are continuous on each interval \([T_{i-1}, T_i]\). We apply Itô-Doob stochastic differential formula [70] to \( V \), and we obtain
\[
\begin{cases}
    dV(t, x^{i-1}) &= LV(t, x^{i-1})dt + V_x(t, x^{i-1})\sigma(t, x^{i-1})dW(t), \quad x(T_{i-1}) = x^{i-1}, \quad t \in [T_{i-1}, T_i), \\
    V(T_i, x^i) &= V(T_i, \Gamma^i x^{i-1}(T_i^-, T_{i-1}, x^{i-1})).
\end{cases}
\]
\[ (9.39) \]
where the \( L \) operator is defined by
\[
\begin{align*}
    LV(t, x^{i-1}) &= V_t(t, x^{i-1}) + V_x(t, x^{i-1})f(t, x^{i-1}) + \frac{1}{2} tr(V_{xx}(t, x^{i-1}))c(t, x^{i-1}) \\
    c(t, x^{i-1}) &= \sigma^{i-1}(t, x^{i-1})\sigma^{i-1}(t, x^{i-1})^T.
\end{align*}
\]
\[ (9.40) \]
For (9.38) and (9.39), we present the Euler-type discretization scheme [58]:
\[
\begin{align*}
    \Delta x^{i-1}(t_k^{i-1}) &= f(t_k^{i-1}, x(t_k^{i-1}))\Delta t_k^{i-1} \\
    &\quad + \sigma^{i-1}(t_k^{i-1}, x(t_k^{i-1}))\Delta W(t_k^{i-1}), \quad k \in I(1, N_{i-1}) \\
    x^i &= \Gamma^i x^{i-1}(T_i^-, T_{i-1}, x^{i-1}), \\
    \Delta V(t_k^{i-1}, x^{i-1}(t_k^{i-1})) &= LV(t_k^{i-1}, x^{i-1}(t_k^{i-1}))\Delta t_k^{i-1} \\
    &\quad + V_x(t_k^{i-1}, x^{i-1}(t_k^{i-1}))\sigma^{i-1}(t_k^{i-1}, x(t_k^{i-1}))\Delta W(t_k^{i-1}) \\
    V(T_i, x^i) &= V(T_i, \Gamma^i x^{i-1}(T_i^-, T_{i-1}, x^{i-1})).
\end{align*}
\]
\[ (9.41) \]
Define \( \mathcal{F}_{t_{i-1}}^{i-1} \equiv \mathcal{F}_{k-1}^{i-1} \) as the filtration process up to time \( t_{k-1}^{i-1} \). With regard to the continuous time dynamic system (9.38) and its transformed system (9.39), the more general moments of \( \Delta x(t_k^{i-1}) \) are as follows:
\[
\begin{align*}
    E \left[ \Delta x(t_k^{i-1}) | \mathcal{F}_{k-1}^{i-1} \right] &= f(t_k^{i-1}, x^{i-1}(t_k^{i-1}))\Delta t_k^{i-1}, \\
    E \left[ (\Delta x^{i-1}(t_k^{i-1}) - E \left[ \Delta x^{i-1}(t_k^{i-1}) | \mathcal{F}_{k-1}^{i-1} \right]) \right] &= \sigma(t_k^{i-1}, x^{i-1}(t_k^{i-1}))\times \\
    E \left[ (\Delta V(t_k^{i-1}, x^{i-1}(t_k^{i-1})) | \mathcal{F}_{k-1}^{i-1}) \right] &= LV(t_k^{i-1}, x^{i-1}(t_k^{i-1}))\Delta t_k^{i-1} \\
    E \left[ (\Delta V(t_k^{i-1}, x^{i-1}(t_k^{i-1})) | \mathcal{F}_{k-1}^{i-1}) \right] &= B(t_k^{i-1}, x^{i-1}(t_k^{i-1})) \\
    E \left[ (\Delta V(t_k^{i-1}, x^{i-1}(t_k^{i-1})) | \mathcal{F}_{k-1}^{i-1}) \right] &= B(t_k^{i-1}, x^{i-1}(t_k^{i-1})) \\
    \end{align*}
\]
\[ (9.42) \]
where \( B(t_k^{i-1}, x(t_k^{i-1})) = V_x(t_k^{i-1}, x^{i-1}(t_k^{i-1}))c(t_k^{i-1}, x^{i-1}(t_k^{i-1}))V_x(t_k^{i-1}, x(t_k^{i-1}))^T \Delta t_k^{i-1} \) and \( T \) stands for the transpose of the matrix.
From (9.41)-(9.43), we have

\[
\begin{align*}
\Delta x^{i-1}(t_k^{-1}) = & \quad E\left[\Delta x^{i-1}(t_k^{-1})|F_k^{i-1}\right] \\
+ & \sigma^{i-1}(t_k^{-1}, x^{i-1}(t_k^{-1})) \Delta W(t_k^{-1}), \quad k \in I(1, N_i-1) \\
\Delta V(t_k^{-1}, x^{i-1}(t_k^{-1})) = & \quad E\left[\Delta V(t_k^{-1}, x^{i-1}(t_k^{-1}))|F_k^{i-1}\right] \\
+ & V_x^{i-1}(t_k^{-1}, x^{i-1}(t_k^{-1})) \sigma^{i-1}(t_k^{-1}, x^{i-1}(t_k^{-1})) \Delta W(t_k) \\
\end{align*}
\]

(9.44)

This provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic system (9.39). This indeed leads to a formulation of \(m_k^{i-1}\)-local generalized method of moments at \(t_k^{-1}\).

In the following, we state a result that exhibits the existence of solution of system of non linear equations. For the sake of easy reference, we shall re-state the Implicit function theorem without proof.

**Theorem 9.1 Implicit Function Theorem**[2] Let \(F = \{F_1, F_2, \ldots, F_q\}\) be a vector-valued function defined on an open set \(S \in \mathbb{R}^{q+k}\) with values in \(\mathbb{R}^q\). Suppose \(F \in C_1\) on \(S\). Let \((u_0; v_0)\) be a point in \(S\) for which \(F(u_0; v_0) = 0\) and for which the \(q \times q\) determinant of the Jacobian matrix \(\det [J_F(v_0)] \neq 0\). Then there exists a \(k\)-dimensional open set \(T_0\) containing \(v_0\) and unique vector-valued function \(g\), defined on \(T_0\) and having values in \(\mathbb{R}^q\), such that \(g \in C_1\) on \(T_0\), \(g(v_0) = u_0\), and \(F(g(v); v) = 0\) for every \(v \in T_0\).

**9.6.1 Illustration:**

For each \(j, l \in I(1, n)\) and each \(i \in I(1, K^*)\), we consider a special case of (9.12).

\[
\begin{align*}
dy_j &= \left(u_j^{i-1} - y_j\right) \left[\kappa_{j,j}^{i-1} y_j + \sum_{l \neq j}^{n} \kappa_{j,l}^{i-1} y_l\right] dt + \delta_{j,j}^{i-1} \left(u_j^{i-1} - y_j\right) dW_{j,j}(t) \\
&\quad + \left(u_j^{i-1} - y_j\right) \sum_{l \neq j}^{n} \delta_{j,l}^{i-1} y_l dW_{j,l}(t), \quad y_j(T_{i-1}) = y_j^{i-1}, \quad t \in [T_{i-1}, T_i), \\
y_j^i &= \pi_j^{i} y_j^{i-1} (T_i^{-}, T_{i-1}, y^{i-1}) \\
dp_j(t) &= p_j \left[\gamma_{j,j}^{i-1} (y_j - p_j) + \beta_j^{i-1} + \sum_{l \neq j}^{n} \gamma_{j,l}^{i-1} p_l(t)\right] dt + \sigma_{j,j}^{i-1} p_j dZ_{j,j}(t) \\
&\quad + p_j \sum_{l \neq j}^{n} \sigma_{j,l}^{i-1} p_l dZ_{j,l}(t), \quad p_j(T_{i-1}) = p_j^{i-1}, \quad t \in [T_{i-1}, T_i), \\
p_j^i &= \theta_j^{i} p_j^{i-1} (T_i^{-}, T_{i-1}, y^{i-1}, p^{i-1}).
\end{align*}
\]

(9.45)
Here, $\kappa_{j,l}^{i-1}$, $u_j^{i-1}$, $\beta_j^{i-1}$, $\gamma_{j,l}^{i-1}$, $\delta_{j,l}^{i-1}$, $\sigma_{j,l}^{i-1}$ are the system parameters on the jump subinterval $[T_{i-1}, T_i)$; $u_j^{i-1}$, $\kappa_{j,l}^{i-1}$, $\gamma_{j,l}^{i-1}$, $\delta_{j,l}^{i-1}$ and $\sigma_{j,l}^{i-1}$ are positive; and for $l \neq j$, $\kappa_{j,l}^{i-1}$, $\gamma_{j,l}^{i-1}$, $\delta_{j,l}^{i-1}$ are nonnegative. $W$ and $Z$ are independent standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$ with the properties described in (9.12). It follows that the interconnected system of stochastic differential equations (9.45) has $4n^2 + 2n$ parameters. Also,

$$
\left\{ \kappa_{j,l}^{i-1} \right\}_{l \neq j} > 0 \text{ if } y_l \text{ is cooperating with } y_j,
\left\{ \kappa_{j,l}^{i-1} \right\}_{l \neq j} < 0 \text{ if } y_l \text{ is competing with } y_j, \tag{9.46}
\left\{ \kappa_{j,l}^{i-1} \right\}_{l \neq j} = 0 \text{ if there is no interaction between } y_l \text{ and } y_j, j, l \in I(1, n),
$$

and likewise,

$$
\left\{ \gamma_{j,l}^{i-1} \right\}_{l \neq j} > 0 \text{ if } p_l \text{ is cooperating with } p_j,
\left\{ \gamma_{j,l}^{i-1} \right\}_{l \neq j} < 0 \text{ if } p_l \text{ is competing with } p_j, \tag{9.47}
\left\{ \gamma_{j,l}^{i-1} \right\}_{l \neq j} = 0 \text{ if there is no interaction between } p_l \text{ and } p_j, j, l \in I(1, n).
$$

**Remark 27** For the case $K = 0$, (9.45) reduce to

$$
\begin{cases}
    dy_j &= (u_j - y_j) \left[ \kappa_{j,j} y_j + \sum_{l \neq j} \kappa_{j,l} y_l \right] dt + \delta_{j,j} (u_j - y_j) dW_{j,j} (t) \\
    &+ (u_j - y_j) \sum_{l \neq j} \delta_{j,l} y_l dW_{j,l} (t), y_j (t_0) = y_{j0}, t \in [t_0, T], \\
\end{cases}
\begin{cases}
    dp_j (t) &= p_j \left[ \gamma_{j,j} (y_j - p_j) + \beta_j + \sum_{l \neq j} \gamma_{j,l} p_l (t) \right] dt + \sigma_{j,j} p_j dZ_{j,j} (t) \\
    &+ p_j \sum_{l \neq j} \sigma_{j,l} p_l dZ_{j,l} (t), p_j (t_0) = p_{j0}, t \in [t_0, T], \tag{9.48}
\end{cases}
$$

where for $j, l \in I(1, n)$, the parameters $\kappa_{j,l}$, $u_j$, $\beta_j$, $\gamma_{j,j}$, $\delta_{j,l}$ and $\sigma_{j,l}$ are the system parameters on the interval $[t_0, T]$; $u_j$, $\kappa_{j,j}$, $\gamma_{j,j}$, $\delta_{j,l}$ and $\sigma_{j,j}$ are positive; and for $l \neq j$, $\kappa_{j,l}$, $\gamma_{j,l}$, $\delta_{j,l}$ are nonnegative.

For each $j \in I(1, n)$, we pick a Lyapunov function

$$
\begin{align*}
V_1 (t, y_j) &= (y_j)^9, \\
V_2 (t, p_j) &= (p_j)^9, \tag{9.49}
\end{align*}
$$
in (9.39) for (9.45). Using Itô-differential formula [70], we have

\[
\begin{align*}
\mathcal{D} V_{1j} &= \left[ q (y_j)^{q-1} \left( u_{j}^{i-1} - y_j \right) \left( \kappa^{i-1}_{j,j} y_j + \sum_{l \neq j}^{n} \kappa^{i-1}_{j,l} y_l \right) \\
&\quad + \frac{1}{2} q (q-1) (y_j)^{q-2} \left( u_{j}^{i-1} - y_j \right)^2 \left( \delta^{i-1}_{j,j} \right)^2 \right] dt \\
&\quad + q (y_j)^{q-1} \left( u_{j}^{i-1} - y_j \right) \left[ \delta^{i-1}_{j,j} dW_{j,j}(t) + \sum_{l \neq j}^{n} \delta^{i-1}_{j,l} y_l dW_{j,l}(t) \right], \\
\mathcal{D}^i_{1j} &= \left( \pi_j^i \right)^q y_j \left( T_i^--T_{i-1}, y^{i-1} \right)^q, \text{ if } t = T_i,
\end{align*}
\]

\[
\begin{align*}
dV_{2j} &= (p_j)^q \left[ q \left( \gamma^{i-1}_{j} (y_j - p_j) + \beta^{i-1}_{j} + \sum_{l \neq j}^{n} \gamma^{i-1}_{j,l} p_l \right) \\
&\quad + \frac{1}{2} q (q-1) \left( \left( \sigma^{i-1}_{j,j} \right)^2 + \sum_{l \neq j}^{n} \left( \sigma^{i-1}_{j,l} \right)^2 \left( p_l \right)^2 \right) \right] dt \\
&\quad + q (p_j)^q \left[ \sigma^{i-1}_{j,j} dZ_{j,j}(t) + \sum_{l \neq j}^{n} \sigma^{i-1}_{j,l} p_l dZ_{j,l}(t) \right], \quad p_j \left( T_{i-1} \right) = p^i_{j-1}, \quad t \in [T_{i-1}, T_i),
\end{align*}
\]

\[
\begin{align*}
\mathcal{D}^i_{2j} &= \left( \theta_j^i \right)^q p_j \left( T_i^--T_{i-1}, y^{i-1}, \mathbf{p}^{i-1} \right)^q, \quad \text{if } t = T_i,
\end{align*}
\]

(9.50)

By setting \( \Delta t^{i-1}_k = t^{i-1}_k - t^{i-1}_{k-1} = \Delta t \); \( \Delta \mathbf{y}(t^{i-1}_k) = \mathbf{y}(t^{i-1}_k) - \mathbf{y}(t^{i-1}_{k-1}) \) and \( \Delta \mathbf{p}(t^{i-1}_k) = \mathbf{p}(t^{i-1}_k) - \mathbf{p}(t^{i-1}_{k-1}) \), the combined Euler discretized scheme for (9.50) is

\[
\begin{align*}
\Delta (y_j)^q (t^{i-1}_k) &= \left[ q (y_j)^{q-1} \left( u_{j}^{i-1} - y_j(t^{i-1}_{k-1}) \right) \left( \kappa^{i-1}_{j,j} y_j(t^{i-1}_{k-1}) + \sum_{l \neq j}^{n} \kappa^{i-1}_{j,l} y_l(t^{i-1}_{k-1}) \right) \\
&\quad + \frac{1}{2} q (q-1) (y_j)^{q-2} \left( u_{j}^{i-1} - y_j(t^{i-1}_{k-1}) \right)^2 \left( \delta^{i-1}_{j,j} \right)^2 \right] \Delta t \\
&\quad + q (y_j)^{q-1} \left( u_{j}^{i-1} - y_j(t^{i-1}_{k-1}) \right) \left[ \delta^{i-1}_{j,j} \Delta W_{j,j}(t^{i-1}_k) \right] \\
&\quad + \sum_{l \neq j}^{n} \delta^{i-1}_{j,l} y_l(t^{i-1}_k) \Delta W_{j,l}(t^{i-1}_k), \\
\mathcal{D} (y_j)^q &= \left( \pi_j^i \right)^q y_j^{i-1} \left( T_i^- - T_{i-1}, y^{i-1} \right)^q, \quad \text{if } t = T_i,
\end{align*}
\]

(9.51)
\[
\Delta (p_j)^q (t_k^{-1}) = (p_j)^q (t_k^{-1}) \left[ q \left( \gamma_j^{-1} (y_j(t_j^{-1}) - p_j(t_k^{-1})) + \beta_j^{-1} + \sum_{l \neq j} \gamma_{j,l}^{-1} p_l(t_k^{-1}) \right) 
+ \frac{1}{2} q(q-1) \left( \sigma_{j,j}^{-1} \right)^2 + \sum_{l \neq j} \left( \sigma_{j,l}^{-1} \right)^2 p_l^2(t_k^{-1}) \right] \Delta t 
+ q(p_j)^q (t_k^{-1}) \left[ \sigma_{j,j}^{-1} \Delta Z_{j,j}(t_k^{-1}) + \sum_{l \neq j} \sigma_{j,l}^{-1} p_l(t_k^{-1}) \Delta Z_{j,l}(t_k^{-1}) \right], 
\]
\[
p_j(T_{i-1}) = p_j^{-1} (t_k^{-1}) \in [T_{i-1}, T_i), q \in I(1, n + 1), 
\]
\[
\left( \theta_j^q \right)^q p_j^{-1} (T_i^{-}, T_{i-1}, y^{i-1}, p^{i-1})^q, \text{ if } t = T_i. 
\]

(9.52)

where \( \{y(t_k^{-1})\}_{k=1}^{N_i-1}, \{p(t_k^{-1})\}_{k=1}^{N_i-1} \) are given finite sequence of \( F_{T_k^{-1}} \) measurable random vectors, and are independent of \( \{\Delta W(t_k^{-1})\}_{k=0}^{N_i-1}, \{\Delta Z(t_k^{-1})\}_{k=0}^{N_i-1} \), respectively. We define \( \Delta (y_j)^q (t_k^{-1}) = (y_j)^q (t_k^{-1}) - (y_j)^q (t_k^{-1}) \) and \( \Delta (p_j)^q (t_k^{-1}) = (p_j)^q (t_k^{-1}) - (p_j)^q (t_k^{-1}) \).

Applying conditional expectations to (9.51)-(9.52) with respect to \( F_{T_{k-1}} \equiv F_{T_k^{-1}} \), we obtain

\[
\mathbb{E} \left[ \Delta (y_j)^q (t_k^{-1}) | F_{T_{k-1}} \right] = 
\left[ q (y_j)^q (t_k^{-1}) (u_j^{-1} - y_j(t_k^{-1})) \left( \kappa_{j,j}^{-1} y_j(t_k^{-1}) + \sum_{l \neq j} \kappa_{j,l}^{-1} y_l(t_k^{-1}) \right) 
+ \frac{q(q-1)}{2 \Delta t} (y_j)^{q-2} (t_k^{-1}) \mathbb{E} \left[ (\Delta y_j(t_k^{-1}) - \mathbb{E}[\Delta y_j(t_k^{-1}) | F_{T_{k-1}}])^2 | F_{T_{k-1}} \right] \right] \Delta t 
\]

for \( t_k^{-1} \in [T_{i-1}, T_i) \),

\[
\mathbb{E} \left[ (y_j)^q (t_k^{-1}) | F_{T_{k-1}} \right] = \left( \pi_j^q \right)^q (y_j)^q (T_i^{-}, T_{i-1}, y^{i-1}), \text{ if } t_k^{-1} = T_i, 
\]

(9.53)

\[
\mathbb{E} \left[ \Delta (p_j)^q (t_k^{-1}) | F_{T_{k-1}} \right] = 
\left[ q (p_j)^q (t_k^{-1}) \left( \gamma_j^{-1} (y_j(t_k^{-1}) - p_j(t_k^{-1})) + \beta_j^{-1} + \sum_{l \neq j} \gamma_{j,l}^{-1} p_l(t_k^{-1}) \right) 
+ \frac{q(q-1)}{2 \Delta t} (p_j)^{q-2} (t_k^{-1}) \mathbb{E} \left[ (\Delta p_j(t_k^{-1}) - \mathbb{E}[\Delta p_j(t_k^{-1}) | F_{T_{k-1}}])^2 | F_{T_{k-1}} \right] \right] \Delta t, 
\]

for \( t_k^{-1} \in [T_{i-1}, T_i) \),

\[
\mathbb{E} \left[ (p_j)^q | F_{T_{k-1}} \right] = \left( \theta_j^q \right)^q (p_j)^q (T_i^{-}, T_{i-1}, y^{i-1}, p^{i-1}) \text{, if } t_k^{-1} = T_i, q \in I(1, n + 1), 
\]

(9.54)

\[
\mathbb{E} \left[ (\Delta (y_j)^q (t_k^{-1}) - \mathbb{E}[\Delta (y_j)^q (t_k^{-1}) | F_{T_{k-1}}]) \right] = 
q^2 (y_j)^{q-2} (t_k^{-1}) (u_j^{-1} - y_j(t_k^{-1})) (u_i^{-1} - y_i(t_k^{-1})) \left[ \delta_{j,j}^{-1} \delta_{i,i}^{-1} y_j(t_k^{-1}) \right] \Delta t, 
\]

(9.55)

\[
t_k^{-1} \in [T_{i-1} - \tau_{i-1}, T_i), 
\]

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\[
\begin{align*}
\mathbb{E} \left[ (\Delta (p_j)^q (t_k^{i-1}) - \mathbb{E}[\Delta (p_j)^q (t_k^{i-1}) | F_{k-1}^{i-1}]) \right] \\
(\Delta (p_j)^q (t_k^{i-1}) - \mathbb{E}[\Delta (p_j)^q (t_k^{i-1}) | F_{k-1}^{i-1}]) | F_{k-1}^{i-1} = \\
q^2 (p_j p_i)^q (t_k^{i-1}) \left[ 2\sigma_{i,j}^{i-1} \sigma_{i,j}^{i-1} y_j(t_k^{i-1}) + \sum_{r=1,j,l \neq r}^{n} \sigma_{i,r}^{i-1} \sigma_{i,r}^{i-1} y_r^{2}(t_k^{i-1}) \right], \\
j \neq i, q \in I(1, 2n)
\end{align*}
\]

where $F_{k-1}^{i-1}$ is the filtration up to time $t_k^{i-1}$. From (9.53)-(9.56), (9.51) reduces to

\[
\begin{align*}
\Delta (y_j)^q (t_k^{i-1}) &= \mathbb{E} [\Delta (y_j)^q (t_k^{i-1}) | F_{k-1}^{i-1}] \\
&\quad + q (y_j)^q (t_k^{i-1}) \left( u_j^{i-1} - y_j(t_k^{i-1}) \right) \left[ \delta_{j,j}^{i-1} \Delta W_{j,j}(t_k^{i-1}) \right] \\
&\quad + \sum_{l \neq j}^{n} \delta_{j,j}^{i-1} y_l \Delta W_{j,l}(t_k^{i-1})], \quad y_j(T_i - 1) = y_j^{i-1}, \quad t_k^{i-1} \in [T_{1-1}, T_i], \\
\left( y_j^i \right)^q &= \left( \pi_j^i \right)^q (y_j)^q (T_i^{T_i-1}, y^{i-1}), \text{ if } t_k^{i-1} = T_i \\
\Delta (p_j)^q (t_k^{i-1}) &= \mathbb{E} [\Delta (p_j)^q (t_k^{i-1}) | F_{k-1}^{i-1}] \\
&\quad + q (p_j)^q (t_k^{i-1}) \left[ \sigma_{i,j}^{i-1} \Delta Z_{j,j}(t_k^{i-1}) + \sum_{l \neq j}^{n} \sigma_{i,j}^{i-1} p_l(t_k^{i-1}) \Delta Z_{j,l}(t_k^{i-1}) \right], \\
p_j(T_i - 1) = p_j^{i-1}, \quad t_k^{i-1} \in [T_{1-1}, T_i], \quad q \in I(1, n + 1), \quad j \in I(1, n), \quad (9.57)
\end{align*}
\]

(9.57) provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic systems (9.45) and (9.50).

For $k \in I(0, N_{i-1})$, applying the lagged adaptive expectation process [88], from Definitions 9.5.1 – 9.5.3, and using (9.53)-(9.57), we formulate a local observation/measurement process at $t_k^{i-1}$ as a algebraic functions of $m_{k-1}^{i-1}$-local functions of restriction of the finite sample sequence \{y(t_k^{i-1})\}_{i=-r_{i-1}}^{N_{i-1}} and \{p(t_k^{i-1})\}_{i=-r_{i-1}}^{N_{i-1}} to subpartition $P_k^{i-1}$ in Definition 9.5.2:
\[
\sum_{i=k-m_k^{-1}}^{k-1} \operatorname{E} \left[ \Delta (y_j)^q (t_i^{-1}) | \mathcal{F}_{i-1}^{-1} \right] = \frac{1}{m_k^{-1}} \sum_{i=k-m_k^{-1}}^{k-1} q^2 (y_j y_l)^{q-1} (t_i^{-1}) \left( u_j^{-1} - y_j (t_i^{-1}) \right) \left( u_l^{-1} - y_l (t_i^{-1}) \right) \times \\
\left[ \delta_{j,l}^{-1} \delta_{l,j}^{-1} y_j (t_i^{-1}) + \delta_{j,l}^{-1} \delta_{l,j}^{-1} y_l (t_i^{-1}) + \sum_{j \neq l \neq r} n \delta_{j,r}^{-1} \delta_{l,r}^{-1} y_r^2 (t_i^{-1}) \right],
\]

and
\[
\sum_{i=k-m_k^{-1}}^{k-1} \operatorname{E} \left[ \Delta (p_j)^q (t_i^{-1}) | \mathcal{F}_{i-1}^{-1} \right] = \frac{1}{m_k^{-1}} \sum_{i=k-m_k^{-1}}^{k-1} q^2 (p_j p_l)^{q-1} (t_i^{-1}) \left[ \sigma_{j,j}^{-1} \sigma_{l,l}^{-1} p_j (t_i^{-1}) + \sigma_{l,l}^{-1} \sigma_{j,j}^{-1} p_l (t_i^{-1}) \right] \\
+ \sum_{j \neq l \neq r} \sigma_{j,r}^{-1} \sigma_{l,r}^{-1} p_r^2 (t_i^{-1}) \right], \quad j \neq l, \quad q \in (1, 2n).
\]

For each \( i \in I(1, K^*) \) and each \( j \neq \tau \in I(1, n) \), we define
\[
F_{1q} \left( u_j^{i-1}, \left\{ \kappa_{j,i}^{i-1} \right\}_{t=1}^{n} \right) = F_{1q} \left( \operatorname{E} \left[ \Delta (y_j)^q (t_i^{-1}) | \mathcal{F}_{i-1}^{-1} \right] ; u_j^{i-1}, \left\{ \kappa_{j,i}^{i-1} \right\}_{t=1}^{n} \right),
\]
\[
F_{2q} \left( \left\{ \delta_{j,i}^{i-1} \right\}_{t=1}^{n} \right) = F_{2q} \left( \operatorname{E} \left[ \Delta (y_j)^q (t_i^{-1}) | \mathcal{F}_{i-1}^{-1} \right] ; \left\{ \delta_{j,i}^{i-1} \right\}_{t=1}^{n} \right),
\]
\[
G_{1q} \left( \beta_{j,i}^{i-1}, \left\{ \gamma_{j,i}^{i-1} \right\}_{t=1}^{n} \right) = G_{1q} \left( \operatorname{E} \left[ \Delta (p_j)^q (t_i^{-1}) | \mathcal{F}_{i-1}^{-1} \right] ; \beta_{j,i}^{i-1}, \left\{ \gamma_{j,i}^{i-1} \right\}_{t=1}^{n} \right),
\]
\[
G_{2q} \left( \left\{ \sigma_{j,i}^{i-1} \right\}_{t=1}^{n} \right) = G_{2q} \left( \operatorname{E} \left[ \Delta (p_j)^q (t_i^{-1}) | \mathcal{F}_{i-1}^{-1} \right] ; \left\{ \sigma_{j,i}^{i-1} \right\}_{t=1}^{n} \right),
\]
by
\[
F_{1q} \left( \left\{ u_{ij}^{i-1}, \left\{ \kappa_{ji}^{i-1} \right\}_{t=1}^{n} \right\} \right) = \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} \left\{ \left[ q \left( y_{ij} \right)^{q-1} \left( t_{i-1}^{i-1} \right) \left( u_{ij}^{i-1} - y_{ij} \left( t_{i-1}^{i-1} \right) \right) \times \left( \sum_{t=1}^{n} \kappa_{ji} \delta_{t}^{i-1} \right) \Delta t \right] + \frac{q(q-1)}{2\Delta t} \left( y_{ij} \right)^{q-2} \left( t_{i-1}^{i-1} \right) \delta_{t}^{j,i-1} \delta_{ij} \left( \Delta y_{ij} \right) \Delta t \right\} \\
- \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} \mathbb{E} \left[ \Delta \left( y_{ij} \right)^{q} \left( t_{i-1}^{i-1} \right) | \mathcal{F}_{i-1}^{i-1} \right], \quad q \in I(1, n+1),
\]

\[
F_{2q} \left( \left\{ \delta_{ji}^{i-1}, \delta_{ij}^{i-1} \right\}_{t=1}^{n} \right) = \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} q^2 \left( y_{ij} \right)^{q-1} \left( t_{i-1}^{i-1} \right) \left( u_{ij}^{i-1} - y_{ij} \left( t_{i-1}^{i-1} \right) \right) \times \left[ \delta_{ji}^{i-1} \delta_{ij}^{i-1} y_{ij} \left( t_{i-1}^{i-1} \right) + \delta_{ji}^{i-1} \delta_{ij}^{i-1} y_{ij} \left( t_{i-1}^{i-1} \right) \right] \\
+ \sum_{t=1}^{n} \delta_{ji}^{i-1} \delta_{ij}^{i-1} y_{ij} \left( t_{i-1}^{i-1} \right) - \tilde{\delta}_{m_k^{i-1}, l, i} \left( \Delta \left( y_{ij} \right) \right), \quad j \neq l \in I(1, n), \quad q \in I(1, 2n)
\]

\[
G_{1q} \left( \left\{ \beta_{ji}^{i-1}, \left\{ \gamma_{ji}^{i-1} \right\}_{t=1}^{n} \right\} \right) = \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} \left\{ \left[ q \left( p_{ij} \right)^{q} \left( t_{i-1}^{i-1} \right) \left( \gamma_{ji}^{i-1} \left( y_{ij} \left( t_{i-1}^{i-1} \right) - p_{ij} \left( t_{i-1}^{i-1} \right) \right) \right) \right] \right. \\
+ \beta_{ji}^{i-1} + \sum_{t \neq j}^{n} \gamma_{ji}^{i-1} p_{t} \left( t_{i-1}^{i-1} \right) \left( \Delta p_{ij} \right) \Delta t \right\} \\
- \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} \mathbb{E} \left[ \Delta \left( p_{ij} \right)^{q} \left( t_{i-1}^{i-1} \right) | \mathcal{F}_{i-1}^{i-1} \right], \quad q \in I(1, n+1),
\]

\[
G_{2q} \left( \left\{ \sigma_{ji}^{i-1}, \sigma_{ij}^{i-1} \right\}_{t=1}^{n} \right) = \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} q^2 \left( p_{ij} \right)^{q} \left( t_{i-1}^{i-1} \right) \left[ \sigma_{ji}^{i-1} \sigma_{ij}^{i-1} p_{t} \left( t_{i-1}^{i-1} \right) \right] \\
+ \sigma_{ji}^{i-1} \sigma_{ij}^{i-1} p_{t} \left( t_{i-1}^{i-1} \right) + \sum_{t=1}^{n} \sigma_{ji}^{i-1} \sigma_{ij}^{i-1} p_{t}^2 \left( t_{i-1}^{i-1} \right) \\
- \tilde{\sigma}_{m_k^{i-1}, l, i} \Delta \left( p_{ij} \right)^{q}, \quad j \neq l \in I(1, n), \quad q \in I(1, 2n). \tag{9.60}
\]

For every \( j \in I(1, n) \), we have
\[
\left\{ \begin{array}{l}
F_{1q} \left( \left\{ u_{ij}^{i-1}, \left\{ \kappa_{ji}^{i-1} \right\}_{t=1}^{n} \right\} \right) = 0, \quad q \in I(1, n+1), \\
F_{2q} \left( \left\{ \delta_{ji}^{i-1}, \delta_{ij}^{i-1} \right\}_{t=1}^{n} \right) = 0, \quad q \in I(1, 2n), \\
G_{1q} \left( \left\{ \beta_{ji}^{i-1}, \left\{ \gamma_{ji}^{i-1} \right\}_{t=1}^{n} \right\} \right) = 0, \quad q \in I(1, n+1), \\
G_{2q} \left( \left\{ \sigma_{ji}^{i-1}, \sigma_{ij}^{i-1} \right\}_{t=1}^{n} \right) = 0, \quad q \in I(1, 2n).
\end{array} \right. \tag{9.61}
\]

Let us define \( F_1 = \{ F_{1q} \}_{q \in I(1, n+1)} \), \( F_2 = \{ F_{2q} \}_{q \in I(1, 2n)} \), \( G_1 = \{ G_{1q} \}_{q \in I(1, n+1)} \), and \( G_2 = \{ G_{2q} \}_{q \in I(1, 2n)} \).
Here, in this subsection, we present an illustration regarding the natural gas, crude oil and coal [26, 27, 28].

9.6.2 Illustration: Application to Energy Commodity

In this subsection, we present an illustration regarding the natural gas, crude oil and coal [26, 27, 28]. Here, \( j \in I(1, 3) \) and \( i \in I(1, K^*) \). Moreover, (9.45) reduces to

\[
\begin{align*}
\begin{cases}
    dy_j &= \left( u_j^{i-1} - y_j \right) \left[ \kappa_{ji}^{i-1} y_j + \sum_{l \neq j} 3 \kappa_{ji}^{i-1} y_l \right] dt + \delta_{ji}^{i-1} \left( u_j^{i-1} - y_j \right) dW_{j,t}(t) \\
    &+ \left( u_j^{i-1} - y_j \right) \sum_{l \neq j} 3 \delta_{ji}^{i-1} y_l dW_{j,l}(t), \quad y_j(T_{i-1}) = y_j^{i-1}, \quad t \in [T_{i-1}, T_i), \\
    y_j &= \pi_j^{i-1} \left( T_{i-1}^-, T_{i-1}, y^{i-1} \right), \\
    dp_j(t) &= p_j \left[ \gamma_{ji}^{i-1} (y_j - p_j) + \beta_{ji}^{i-1} + \sum_{l \neq j} 3 \gamma_{ji}^{i-1} p_l(t) \right] dt + \sigma_{ji}^{i-1} p_j dZ_{j,t}(t) \\
    &+ p_j \sum_{l \neq j} 3 \sigma_{ji}^{i-1} p_l dZ_{j,l}(t), \quad p_j(T_{i-1}) = p_j^{i-1}, \quad t \in [T_{i-1}, T_i), \\
    p_j &= \theta_j^{i} \left( T_{i-1}^-, T_{i-1}, y^{i-1}, p^{i-1} \right).
\end{cases}
\end{align*}
\]

(9.62)

For each \( j \in I(1, 3) \), following the argument used in Illustration 9.6.1, we have

\[
F_1 q \left( u_j^{i-1}, \{ \kappa_{ji}^{i-1} \}^{i-1} \right) = \frac{1}{m_k^{i-1}} \sum_{t=k-m_k^{i-1}}^{k-1} \left\{ q (y_j)^{q-1} \left( t_{i-1}^{i-1} \right) \left( u_j^{i-1} - y_j(t_{i-1}^{i-1}) \right) \times \left( \sum_{t=1}^{3} \kappa_{ji}^{i-1} y_j(t_{i-1}^{i-1}) \right) \right\} + \frac{q(q-1)}{2\Delta t} (y_j)^{q-2} \left( t_{i-1}^{i-1} \right) \delta_{ji}^{i-1} \left( \Delta y_j \right) \Delta t
\]

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Thus, for each \( j \), \( q \in I(1, 4) \),

\[
F_{2q} \left( \left\{ \delta_{j,t}^{-1}, \delta_{t,t}^{-1} \right\}_{t=1}^{3} \right) = - \frac{1}{m_{k}^{-1}} \sum_{t=k-m_{k}^{-1}}^{k-1} \frac{q^{2} (y_{j} y_{t})^{q-1} (t_{i}^{-1}) \left( u_{j}^{t-1} - y_{j} (t_{i}^{-1}) \right)}{t_{t-1}^{-1}} \\
\left( u_{i}^{t-1} - y_{j} (t_{i}^{-1}) \right) \left[ \delta_{j,j}^{-1} \delta_{i,j}^{-1} y_{j} (t_{i}^{-1}) + \delta_{i,l}^{-1} \delta_{j,l}^{-1} y_{l} (t_{i}^{-1}) \right] \\
\left( \delta_{j,j}^{-1} \delta_{i,j}^{-1} y_{j} (t_{i}^{-1}) + \delta_{i,l}^{-1} \delta_{j,l}^{-1} y_{l} (t_{i}^{-1}) \right) + 3 \delta_{j,l}^{-1} \delta_{i,l}^{-1} y_{l} (t_{i}^{-1}) \right] - \frac{3 q \delta_{j,l}^{-1} \delta_{i,l}^{-1} y_{l} (t_{i}^{-1})}{m_{k}^{-1}, k}, (\Delta (y)^{q}),

j \neq l \in I(1,3), \ q \in I(1, 6).
\]

\[
G_{1q} \left( \left\{ \beta_{j}, \gamma_{j,t}^{-1} \right\}_{t=1}^{3} \right) = \frac{1}{m_{k}^{-1}} \sum_{t=k-m_{k}^{-1}}^{k-1} \left\{ q \left( p_{j} \right)^{q} (t_{i}^{-1}) \left( \gamma_{j,j}^{-1} (y_{j} (t_{i}^{-1}) - p_{j} (t_{i}^{-1}) \right) \right. \\
+ \beta_{j}^{-1} + 3 \gamma_{j,t}^{-1} p_{k} (t_{i}^{-1}) \right) \\
+ \frac{q (q-1)}{2 \Delta t} p_{j}^{-1} (t_{i}^{-1}) \delta_{j,l}^{-1} \delta_{i,l}^{-1} (\Delta p_{j}) \right] \Delta t \right\} \\
- \frac{1}{m_{k}^{-1}} \sum_{t=k-m_{k}^{-1}}^{k-1} \left[ \Delta \left( p_{j} \right)^{q} (t_{i}^{-1}) | \mathcal{F}_{t-1}^{i-1}, q \in I(1, 4) \right)
\]

\[
G_{2q} \left( \left\{ \sigma_{j,t}^{-1}, \sigma_{t,t}^{-1} \right\}_{t=1}^{3} \right) = \frac{1}{m_{k}^{-1}} \sum_{t=k-m_{k}^{-1}}^{k-1} \left[ q^{2} (p_{j} p_{l})^{q} (t_{i}^{-1}) \left[ \sigma_{j,j}^{-1} \sigma_{i,j}^{-1} p_{j} (t_{i}^{-1}) \right] \\
+ \sigma_{i,l}^{-1} \sigma_{j,l}^{-1} p_{l} (t_{i}^{-1}) + 3 \sum_{t=k-m_{k}^{-1}}^{k-1} \sigma_{j,l}^{-1} \sigma_{i,l}^{-1} p_{l} (t_{i}^{-1}) \right] \\
- \delta_{j,l}^{-1} \delta_{i,l}^{-1} \delta_{j,l}^{-1} (\Delta p)^{q}, j \neq l \in I(1,3), q \in I(1, 6).
\]

and for \( j \neq l \in I(1,3) \), we have

\[
\left\{ \begin{array}{l}
\left\{ F_{1q} \left( \left\{ u_{j}^{-1}, \left\{ \kappa_{j,t}^{-1} \right\}_{t=1}^{3} \right\}_{t=1}^{3} \right) = 0, q \in I(1, 4), \\
F_{2q} \left( \left\{ \delta_{j,t}^{-1}, \delta_{i,t}^{-1} \right\}_{t=1}^{3} \right) = 0, q \in I(1, 6), \\
G_{1q} \left( \left\{ \beta_{j}^{-1}, \left\{ \gamma_{j,t}^{-1} \right\}_{t=1}^{3} \right\}_{t=1}^{3} \right) = 0, q \in I(1, 4), \\
F_{2q} \left( \left\{ \delta_{j,t}^{-1}, \delta_{i,t}^{-1} \right\}_{t=1}^{3} \right) = 0, q \in I(1, 6).
\end{array} \right.
\]

(9.63)

We also \( F_{1} = \{ F_{1q} \}_{q \in I(1, 4)} \), \( F_{2} = \{ F_{2q} \}_{q \in I(1, 3)} \), \( G_{1} = \{ G_{1q} \}_{q \in I(1, 4)} \), and \( G_{2} = \{ G_{2q} \}_{q \in I(1, 3)} \).

Thus, for each \( j \in I(1, 3) \), the determinant of the Jacobian matrix \( JF_{1} \left( \left\{ u_{j}^{-1}, \left\{ \kappa_{j,t}^{-1} \right\}_{t=1}^{n} \right\}_{t=1}^{n} \right) \) is given
by \( \frac{1}{(m_k^{-1})^d} \text{det} \left[ \sum_{t=1}^{k-1} \mathcal{J}_t \right] \), where \( \mathcal{J}_t \) is defined by \( \mathcal{J}_t = \)

\[
\begin{pmatrix}
\sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) (u_j - y_j(t_{i-1}^t)) y_1(t_{i-1}^t) & \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) (u_j - y_j(t_{i-1}^t)) y_2(t_{i-1}^t) & \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) (u_j - y_j(t_{i-1}^t)) y_3(t_{i-1}^t) \\
2 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_2(t_{i-1}^t) & 2 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_3(t_{i-1}^t) & 2 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_1(t_{i-1}^t) \\
3 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_3(t_{i-1}^t) & 3 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_1(t_{i-1}^t) & 3 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_2(t_{i-1}^t) \\
4 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_1(t_{i-1}^t) & 4 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_2(t_{i-1}^t) & 4 \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_3(t_{i-1}^t)
\end{pmatrix}
\]

and \( \text{det} \left[ \sum_{t=1}^{k-1} \mathcal{J}_t \right] = \)

\[
\begin{pmatrix}
\sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_1(t_{i-1}^t) \right] y_2(t_{i-1}^t) y_3(t_{i-1}^t) & \sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_2(t_{i-1}^t) \right] y_1(t_{i-1}^t) y_3(t_{i-1}^t) & \sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_3(t_{i-1}^t) \right] y_1(t_{i-1}^t) y_2(t_{i-1}^t) \\
\sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_2(t_{i-1}^t) \right] y_1(t_{i-1}^t) y_3(t_{i-1}^t) & \sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_3(t_{i-1}^t) \right] y_1(t_{i-1}^t) y_2(t_{i-1}^t) & \sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_1(t_{i-1}^t) \right] y_2(t_{i-1}^t) y_3(t_{i-1}^t) \\
\sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_3(t_{i-1}^t) \right] y_1(t_{i-1}^t) y_2(t_{i-1}^t) & \sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_1(t_{i-1}^t) \right] y_2(t_{i-1}^t) y_3(t_{i-1}^t) & \sum_{t=1}^{k-1} \left[ \sum_{t=1}^{n} \kappa_j y_j(t_{i-1}^t) y_2(t_{i-1}^t) \right] y_1(t_{i-1}^t) y_3(t_{i-1}^t)
\end{pmatrix}
\]

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+ \sum_{i=k-m_k}^{k-1} \left[ \sum_{r=1}^{n} \kappa_{j,r} y_r(t_{i-1}^l) \right] y_j^2(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j^2(t_{i-1}^l) y_1(t_{i-1}^l) \times
\left(\sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_2(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_3(t_{i-1}^l) \right)
- \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_3(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_2(t_{i-1}^l)
+ \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_2(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_1(t_{i-1}^l) \times
\left(\sum_{i=k-m_k}^{k-1} \left[ \sum_{r=1}^{n} \kappa_{j,r} y_r(t_{i-1}^l) \right] \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_2(t_{i-1}^l) \right)
- \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_2(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} \left[ \sum_{r=1}^{n} \kappa_{j,r} y_r(t_{i-1}^l) \right] y_j(t_{i-1}^l)
+ \sum_{i=k-m_k}^{k-1} \left[ \sum_{r=1}^{n} \kappa_{j,r} y_r(t_{i-1}^l) \right] y_j^2(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j^2(t_{i-1}^l) y_2(t_{i-1}^l) \times
\left(\sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_3(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_1(t_{i-1}^l) \right)
- \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_1(t_{i-1}^l) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^l)) y_j(t_{i-1}^l) y_3(t_{i-1}^l)
\[ + \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^2(t_{i-1}^t) y_j(t_{i-1}^t) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^2(t_{i-1}^t) y_j(t_{i-1}^t) \times 
\]
\[ - \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j(t_{i-1}^t) \sum_{i=k-m_k}^{k-1} \left[ \sum_{\tau=1}^{n} \kappa_{j,\tau} y_{\tau}(t_{i-1}^t) \right] y_j(t_{i-1}^t) \]
\[ + \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^2(t_{i-1}^t) y_j(t_{i-1}^t) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^3(t_{i-1}^t) y_j(t_{i-1}^t) \times 
\]
\[ - \sum_{i=k-m_k}^{k-1} \left[ \sum_{\tau=1}^{n} \kappa_{j,\tau} y_{\tau}(t_{i-1}^t) \right] \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j(t_{i-1}^t) y_j(t_{i-1}^t) \]
\[ + \sum_{i=k-m_k}^{k-1} \left[ \sum_{\tau=1}^{n} \kappa_{j,\tau} y_{\tau}(t_{i-1}^t) \right] y_j^2(t_{i-1}^t) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^3(t_{i-1}^t) y_j(t_{i-1}^t) \times 
\]
\[ - \sum_{i=k-m_k}^{k-1} \left[ \sum_{\tau=1}^{n} \kappa_{j,\tau} y_{\tau}(t_{i-1}^t) \right] \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j(t_{i-1}^t) y_j(t_{i-1}^t) \]
\[ + \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^2(t_{i-1}^t) y_j(t_{i-1}^t) \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j^3(t_{i-1}^t) y_j(t_{i-1}^t) \times 
\]
\[ - \sum_{i=k-m_k}^{k-1} \left[ \sum_{\tau=1}^{n} \kappa_{j,\tau} y_{\tau}(t_{i-1}^t) \right] \sum_{i=k-m_k}^{k-1} (u_j - y_j(t_{i-1}^t)) y_j(t_{i-1}^t) y_j(t_{i-1}^t) \]
\[
\begin{align*}
&+ \sum_{i=k-m_i^{-1}}^{k-1} (u_j - y_j(t_i^{-1}) - y_{j-1}(t_i^{-1}))y_j(t_i^{-1}) - y_{j-1}(t_i^{-1}))y_j(t_i^{-1})y_{j-1}(t_i^{-1}) \\
&- \sum_{i=k-m_i^{-1}}^{k-1} (u_j - y_j(t_i^{-1}) - y_{j-1}(t_i^{-1}))y_j(t_i^{-1}) - y_{j-1}(t_i^{-1}))y_j(t_i^{-1})y_{j-1}(t_i^{-1}) \\
\end{align*}
\]

For \( j \neq l \), the determinant of the Jacobians \( JF_2 \left( \left\{ \delta_{i,\tau}^{j-1}, \sigma_{j,\tau}^{i-1} \right\}_{\tau=1}^{3} \right) \) and \( JG_2 \left( \left\{ \sigma_{j,\tau}^{i-1}, \sigma_{i,\tau}^{i-1} \right\}_{\tau=1}^{3} \right) \) can be derived in a similar way. For each \( j \in I(1, 3) \), it follows that the determinant of \( JF_1 \left( u_j^{i-1}, \left\{ \kappa_{j,\tau}^{i-1} \right\}_{\tau=1}^{3} \right) \) is not zero provided that all parameters \( \left\{ \kappa_{j,\tau}^{i-1} \right\}_{\tau=1}^{3} \) are not zero or provided the sequence \( \left\{ y_j^{i-1} (t_\tau^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1} \) is neither zero nor constant. To show this, suppose that the determinant of \( JF_1 \left( u_j^{i-1}, \left\{ \kappa_{j,\tau}^{i-1} \right\}_{\tau=1}^{3} \right) \) is zero. This is equivalent to either one of the following:

- The rows of the matrix are dependent vectors in \( \mathbb{R}^4 \).
- The columns of the matrix are dependent vectors in \( \mathbb{R}^4 \).
- Either one of the rows or columns of the matrix is a zero vector.

This is equivalent to saying either all parameters \( \left\{ \kappa_{j,\tau}^{i-1} \right\}_{\tau=1}^{3} \) are zero, or the sequence \( \left\{ y_j^{i-1} (t_\tau^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1} \) is zero or a constant.

Likewise, determinants of the Jacobians \( JF_2 \left( u_j^{i-1}, \left\{ \delta_{j,\tau}^{j-1}, \sigma_{j,\tau}^{j-1} \right\}_{\tau=1}^{3} \right) \), \( JG_1 \left( \beta_j^{j-1}, \left\{ \gamma_{j,\tau}^{j-1} \right\}_{\tau=1}^{3} \right) \) and \( JG_2 \left( \left\{ \sigma_{j,\tau}^{j-1}, \sigma_{i,\tau}^{j-1} \right\}_{\tau=1}^{3} \right) \) are non-zero if \( \left\{ \delta_{j,\tau}^{j-1}, \delta_{\tau,\tau}^{j-1} \right\}_{\tau=1}^{3}, \left\{ \gamma_{j,\tau}^{j-1} \right\}_{\tau=1}^{3} \) and \( \left\{ \sigma_{j,\tau}^{j-1}, \sigma_{i,\tau}^{j-1} \right\}_{\tau=1}^{3} \) are not zero or provided the sequence \( \left\{ y_j^{i-1} (t_\tau^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1}, \left\{ y_i^{i-1} (t_\tau^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1}, \left\{ p_j^{i-1} (t_\tau^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1} \) and \( \left\{ p_i^{i-1} (t_\tau^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1} \) are neither zero nor constant for \( j \neq l \in I(1, 3) \).

**Remark 28** If the sample \( \left\{ y_j^{i-1} (t_k^{-1}) \right\}_{\tau=k-m_i^{-1}-1}^{k-1} \) is a constant sequence, it follows from (9.51) (q=1) and the fact that \( \Delta \left( y_j^{i-1} (t_k^{-1}) \right) = 0 \) and \( s_{m_k^{-1}}^{i-1} (\Delta y_j) = 0 \), that \( u_j^{i-1} (m_k^{-1}, t_k^{-1}) \rightarrow \frac{1}{m_k^{-1}} \sum_{i=k-m_i^{-1}}^{k-1} y_j^{i-1} (t_k^{-1}) \). It also follows from (9.58) that \( \left\{ \kappa_{j,\tau}^{i-1} \right\}_{\tau=1}^{3} (m_k^{-1}, t_k^{-1}) \rightarrow 0 \).
Chapter 10
Computational and Simulation Algorithms

10.1 Introduction

In this chapter, we outline computational, data organizational and simulation schemes. We introduce the ideas of iterative data process and data simulation time schedules in relation with the real time data observation/collection schedule. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is essential to identify an admissible set of local conditional sample average and sample covariance parameters, namely, the size of local conditional sample in the context of a partition of time interval \([T_{i-1} - \tau_{i-1}, T_i]\). Moreover, the discrete time dynamic model of conditional sample mean and sample covariance statistic processes in Section 9.5 and the theoretical parameter estimation scheme in Section 9.6 motivates to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below:

10.2 Coordination of Data Observation, Iterative Process, and Simulation Schedules:

Without loss of generality, we assume that the real data observation/collection partition schedules \(\mathcal{P}^{i-1}, i \in I(1, K^*)\) are defined in (9.29). Now, we present definitions of iterative process and simulation time schedule.

**Definition 10.2.1** The iterative process time schedule in relation with the real data collection schedule is defined by

\[
\{ I_{\mathcal{P}^{i-1}} = \{ F^{-r_{i-1}}t_k^{i-1} : \text{for} \ t_k^{i-1} \in \mathcal{P}^{i-1} \}, \text{for} \ i \in I(1, K^*), \ k \in I(-r_{i-1}, N_{i-1}) \},
\]

where \(F^{-r_{i-1}}t_k^{i-1} = t_k^{i-1}_{k-r_{i-1}}\) is a forward shift operator [11].

The simulation time is based on the order \(d_{i-1}\) of the time series model of \(m_k^{i-1}\)-local conditional sample mean and covariance processes in Lemma 9.3.
Remark 29 For the case where $K = 0$, we have $I\mathbb{P}_{i-1} = I\mathbb{P}$, where $\mathbb{P}^{i-1} = \mathbb{P}$ is defined in (9.28). This is the iterative time schedule in the absence of jumps.

Definition 10.2.2 The simulation process time schedule in relation with the real data observation schedule is defined by

$$S^{\mathbb{P}^{i-1}} = \begin{cases} \{F_{t^{i-1}, t^{i-1}}: & \text{for } t^{i-1} \in \mathbb{P}^{i-1}, \text{ if } d_{i-1} \leq r_{i-1} \\ \{F_{t^{i-1}, t^{i-1}}: & \text{for } t^{i-1} \in \mathbb{P}^{i-1}, \text{ if } d_{i-1} > r_{i-1}, k \in I(-r_{i-1}, N_{i-1}) \end{cases}$$

(10.2)

Remark 30 For each $i \in I(1, K^*)$, the initial times of iterative and simulation processes are equal to the real data times $t^{i-1}_{r_{i-1}}$ and $t^{i-1}_{d_{i-1}}$, whenever $d_{i-1} \leq r_{i-1}$ and $d_{i-1} > r_{i-1}$, respectively. The iterative process and simulation process times with jump are $t^{i-1}_{k+r_{i-1}}$ and $t^{i-1}_{k+d_{i-1}}$, $i \in I(1, K^*)$, respectively.

10.3 Conceptual Computational Parameter Estimation Scheme

For the conceptual computational dynamic system parameter estimation, we need to introduce a few concepts of local admissible sample/data observation size $m^{i-1}_k$-local admissible conditional finite sequence at $t^{i-1}_k \in S^{\mathbb{P}^{i-1}}$, local finite sequence of parameter estimates at $t^{i-1}_k$.

Definition 10.3.1 For each $i \in I(1, K^*)$, and $t^{i-1}_k \in I(T_{i-1} - r_{i-1}, T_i)$, we define local admissible sample/data observation size $m^{i-1}_k$ at $t^{i-1}_k$ as $m^{i-1}_k \in OS^{i-1}_k$, where

$$OS^{i-1}_k = \begin{cases} I(2, r_{i-1} + S_{i-1} + k - 1), & \text{if } d_{i-1} \leq r_{i-1}, \\ I(2, d_{i-1} + S_{i-1} + k - 1), & \text{if } d_{i-1} > r_{i-1}, k \in I(0, N_{i-1}) \end{cases}$$

(10.3)

Moreover, $OS^{i-1}_k$ is referred as the local admissible set of lagged sample/data observation size at $t^{i-1}_k$.

Remark 31 We note that if $K = 0, S_{i-1} = 0$, the point $t^{i-1}_k = t_k \in [t_0, T]$. Thus, (10.3) reduces to

$$OS^{i-1}_k = \begin{cases} I(2, r + k - 1), & \text{if } d \leq r, \\ I(2, d + k - 1), & \text{if } d > r, k \in I(0, N) \end{cases}$$

Definition 10.3.2 For each $i \in I(1, K^*)$, $m^{i-1}_k \in OS^{i-1}_k$ in Definition 10.3.1 and $k \in I(0, N_{i-1})$, a $m^{i-1}_k$-local admissible lagged-adapted finite restriction sequence of conditional sample/data observation at time $t^{i-1}_k$ to subpartition $P^{i-1}_{k}$ of $\mathbb{P}^{i-1}$ in Definition 9.5.2 is defined by
\[
\left( \{E[y_{i,-1}(t_{i,-1}) | F_{t-1}^{i-1}]\}_{t=k-m_{k}^{-1}}^{k-1}, \{E[p_{i,-1}(t_{i,-1}) | F_{t-1}^{i-1}]\}_{t=k-m_{k}^{-1}}^{k-1} \right).
\]
Moreover, a \( m_{k}^{-1} \)-class of admissible lagged-adapted finite sequences of conditional sample/data observation of size \( m_{k}^{-1} \) at \( t_{k}^{-1} \) is defined by
\[
\mathcal{A}s_{k}^{-1} = \left\{ \left\{ E[y_{i,-1}(t_{i,-1}) | F_{t-1}^{i-1}] \right\}_{t=k-m_{k}^{-1}}^{k-1} : m_{k}^{-1} \in OS_{k}^{-1} \right\}, \quad (10.4)
\]

In the case of energy commodity model, for each \( i \in I(1, K^{*}) \), \( m_{k}^{-1} \in OS_{k}^{-1} \), we find corresponding \( m_{k}^{-1} \)-local admissible adapted finite sequence of conditional sample/data observation at \( t_{k}^{-1} \), \( \left\{ E[y_{i,-1}(t_{i,-1}) | F_{t-1}^{i-1}] \right\}_{t=k-m_{k}^{-1}}^{k-1}, \left\{ E[p_{i,-1}(t_{i,-1}) | F_{t-1}^{i-1}] \right\}_{t=k-m_{k}^{-1}}^{k-1} \). For \( i \in I(1, K^{*}) \), using this sequence and solutions of (9.63), we compute
\[
\left\{ u_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \tilde{\gamma}_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \kappa_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \gamma_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \delta_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \sigma_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \right\} \in [0, N_{i-1}], \text{ for } j, l \in I(1, n).
\]

This leads to a local finite sequence of parameter estimates at \( t_{k}^{-1} \) defined on \( OS_{k}^{-1} \) as follows:
\[
\left\{ \hat{u}_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \tilde{\hat{\gamma}}_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \kappa_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \gamma_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \delta_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \sigma_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}) \right\} \in OS_{k}^{-1}.
\]

The above defined collection is denoted by
\[
(U_{k}, B_{k}, K_{k}, \gamma_{k}, \delta_{k}, \sigma_{k}) = \left\{ \left\{ \hat{u}_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \tilde{\hat{\gamma}}_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \kappa_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \gamma_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \delta_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}), \sigma_{i,j}^{-1}(m_{k}^{-1}, t_{i,k}^{-1}) \right\} \in OS_{k}^{-1} : \right\}
\]

for \( j \in I(1, n), i \in I(1, K^{*}) \).

10.4 Conceptual Computation of State Simulation Scheme

For the development of a conceptual computational scheme, we need to employ the method of induction. The presented simulation scheme is based on the idea of lagged adaptive expectation process [88]. For \( j, l \in I(1, n) \), an autocorrelation function (ACF) analysis [14, 11] performed on \( \left( s_{m_k^{-1}, j}^{i-1}(y), s_{m_k^{-1}, j}^{i-1}(p) \right) \) suggests that the interconnected discrete time dynamic model of local conditional sample mean and sample variance statistics in Lemma 9.3 is of order \( d_{i-1} = 2 \). In view of this, we need to identify the initial data. We begin with a given initial data \( \left( y_{i-1}(T_{i-1}), p_{i-1}(T_{i-1}) \right), \left( \{ y_{i-1}(T_{i-1}) \}_{k=0}^{m_{k}^{-1}} \in OS_{k}^{-1}, \{ y_{i-1}(T_{i-1}) \}_{k=0}^{m_{k}^{-1}} \in OS_{k}^{-1} \right) \),
\[
\left\{ \sum_{m_{i-1}^{(1)}}^{(1)}(y) \right\}_{m_{i-1}^{(1)} \in OS_{i-1}^{(1)}}, \left\{ \sum_{m_{i-1}^{(1)}}^{(1)}(p) \right\}_{m_{i-1}^{(1)} \in OS_{i-1}^{(1)}}, \left\{ \tilde{S}_{m_{i-1}^{(1)}}^{(1)}(y) \right\}_{m_{i-1}^{(1)} \in OS_{i-1}^{(1)}}, \left\{ \tilde{S}_{m_{i-1}^{(1)}}^{(1)}(p) \right\}_{m_{i-1}^{(1)} \in OS_{i-1}^{(1)}}. \]

Let \((y^{s}(m_{i-1}^{(1)}, t_{i-1}^{(1)}), p^{s}(m_{i-1}^{(1)}, t_{i-1}^{(1)}))\) be a simulated value of \((\mathbb{E}[y^{i-1}(t_{i-1}^{(1)})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[p^{i-1}(t_{i-1}^{(1)})|\mathcal{F}_{k-1}^{i-1}] )\) at time \(t_{i-1}^{(1)}\) corresponding to an admissible sequence \(\{\mathbb{E}[y^{i-1}(t_{i-1}^{(1)})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[p^{i-1}(t_{i-1}^{(1)})|\mathcal{F}_{k-1}^{i-1}]\}_{k=i-k_{m}} \in \mathcal{A}_{S_{i-1}^{(1)}}.\) For \(q = 1\), and \(j \in I(1, n)\), the simulated value \((y^{s}_{j}(m_{i-1}^{(1)}, t_{i-1}^{(1)}) \equiv y^{i-1,s}_{j}(m_{i-1}^{(1)}, t_{i-1}^{(1)}), p^{s}_{j}(m_{i-1}^{(1)}, t_{i-1}^{(1)}) \equiv p^{i-1,s}_{j}(m_{i-1}^{(1)}, t_{i-1}^{(1)}))\) is generated from the discretized Euler scheme (9.51)-(9.52) as follows:

\[
\begin{align*}
y^{s}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) &= y^{s}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) + \left( \sum_{i=1}^{n} \kappa_{j,i}^{(1)}(m_{k-1}^{(1)}, t_{k-1}^{(1)})y^{i-1}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) \right) \Delta t + \left( u^{i-1}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) - y^{i-1}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) \right) \Delta t, \\
\pi^{2}_{j}y^{i-1,s}_{j}(T_{-}^{i}), &= \pi^{2}_{j}y^{i-1,s}_{j}(T_{-}^{i}), \\
p^{s}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) &= p^{s}_{j}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) + \left( \sum_{i=1}^{n} \kappa_{j,i}^{(1)}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) \sum_{l \neq j}^{n} \kappa_{j,l}^{(1)}(m_{k-1}^{(1)}, t_{k-1}^{(1)})p^{i-1,s}_{l}(m_{k-1}^{(1)}, t_{k-1}^{(1)}) \Delta t \right)
\end{align*}
\]

To find the simulated value \(y^{i,s}_{j}(T_{i})\) and \(p^{i,s}_{j}(T_{i})\), we need to estimate \(\hat{\pi}_{j}^{i}\) and \(\hat{\theta}_{j}^{i}\) by first simulating

\[
\lim_{t \to T_{i}} y^{i-1}_{j}(t, T_{i-1}, y^{i-1,s}) \equiv y^{i-1,s}_{j}(m_{N_{i-1}^{(1)}}, t_{N_{i-1}^{(1)}})
\]

and

\[
\lim_{t \to T_{i}} p^{i-1}_{j}(t, T_{i-1}, y^{i-1,s}, p^{i-1,s}) \equiv p^{i-1,s}_{j}(m_{N_{i-1}^{(1)}}, t_{N_{i-1}^{(1)}})
\]

as follows:
\[ y_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) = y_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \]
\[ + \left( u_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) - y_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \right) \times \]
\[ \sum_{l=1}^{n} \delta_{j,l}^{i-1}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) y_{l}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \Delta t \]
\[ + \delta_{j,1}^{i-1}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \Delta W_{j,1}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \]
\[ + \sum_{l \neq j}^{n} \delta_{j,l}^{i-1}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) y_{l}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \Delta W_{j,l}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \]}

\[ p_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) = (p_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \]
\[ + p_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \left[ y_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \right] \left( y_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \right) \]
\[ + \sum_{l \neq j}^{n} \gamma_{j,l}^{i-1}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) p_{l}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \Delta Z_{j,l}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \]
\[ + \sum_{l \neq j}^{n} \sigma_{j,l}^{i-1}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) p_{l}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \Delta Z_{j,l}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1}) \]}

From this, we calculate \( \hat{\pi}_{j}^{i} \) and \( \hat{\theta}_{j}^{i} \) as:
\[ \hat{\pi}_{j}^{i} = \frac{\mathbb{E}[y_{j}^{s}(T_{i})|F_{j}^{i-1}]}{y_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1})} \]
\[ \hat{\theta}_{j}^{i} = \frac{\mathbb{E}[p_{j}^{s}(T_{i})|F_{j}^{i-1}]}{p_{j}^{s}(m_{N_{i-1}}^{-1}, t_{N_{i-1}}^{-1})} \] (10.6)

Thus, \( y_{j}^{s}(T_{i}) = \hat{\pi}_{j}^{i} y_{j}^{i-1,s}(T_{i-1}, T_{i-1}, y_{j}^{i-1,s}) \) and \( p_{j}^{s}(T_{i}) = \hat{\theta}_{j}^{i} p_{j}^{i-1,s}(T_{i-1}, T_{i-1}, y_{j}^{i-1,s}, p_{j}^{i-1,s}) \).

Let \( \{y^{s}(m_{k}^{-1}, t_{k}^{-1})\}_{m_{k}^{-1} \in \mathcal{O}S_{k}^{-1}}, \{p^{s}(m_{k}^{-1}, t_{k}^{-1})\}_{m_{k}^{-1} \in \mathcal{O}S_{k}^{-1}} \) be a \( m_{k}^{-1} \)-local sequence of simulated values corresponding to \( m_{k}^{-1} \)-admissible lagged adapted finite sequence of conditional observation belonging to \( \mathcal{A}S_{k}^{-1} \), and corresponding term of sequence \( (\mathcal{U}_{k}, \mathcal{B}_{k}, \mathcal{K}_{k}, \gamma_{k}, \delta_{k}, \sigma_{k}) \).

Thus, for each \( i \in I(I, K^{*}) \), \( \{y^{s}(m_{k}^{-1}, t_{k}^{-1})\}_{m_{k}^{-1} \in \mathcal{O}S_{k}^{-1}}, \{p^{s}(m_{k}^{-1}, t_{k}^{-1})\}_{m_{k}^{-1} \in \mathcal{O}S_{k}^{-1}} \) are the finite sequence correspondence of simulated values of \( (\mathbb{E}[y^{i-1}(t_{k}^{-1})|F_{k}^{-1}], \mathbb{E}[p^{i-1}(t_{k}^{-1})|F_{k}^{-1}]) \) at \( t_{k}^{-1} \).

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10.5 Mean-Square Sub-Optimal Procedure

To find the best estimate of \( \mathbb{E}[y(t_{i-1}^k)|\mathcal{F}_{k-1}^1], \mathbb{E}[p(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \) using a local admissible finite sequence \( \{y^*(m_{i-1}^k, t_{i-1}^k), p^*(m_{i-1}^k, t_{i-1}^k)\}_{m_{i-1}^k \in OS_{i-1}^1} \), we need to compute a finite sequence of quadratic mean square error corresponding to
\[
\{(y^*(m_{i-1}^k, t_{i-1}^k))_{m_{i-1}^k \in OS_{i-1}^1}, \{p^*(m_{i-1}^k, t_{i-1}^k))_{m_{i-1}^k \in OS_{i-1}^1}\}.
\]
The quadratic mean square error is defined below.

**Definition 10.5.1** For each \( i \in I(1, K^*), \) the quadratic mean square error of \( \mathbb{E}[y(t_{i-1}^k)|\mathcal{F}_{k-1}^1], \mathbb{E}[p(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \) relative to each member of the term of local admissible sequence \( \{y^*(m_{i-1}^k, t_{i-1}^k))_{m_{i-1}^k \in OS_{i-1}^1}, \{p^*(m_{i-1}^k, t_{i-1}^k))_{m_{i-1}^k \in OS_{i-1}^1}\} \) of simulated values is defined by
\[
\Xi_{m_{i-1}^k, t_{i-1}^k} = \|y^*(m_{i-1}^k, t_{i-1}^k) - \mathbb{E}[y(t_{i-1}^k)|\mathcal{F}_{k-1}^1]\|^2 + \|p^*(m_{i-1}^k, t_{i-1}^k) - \mathbb{E}[p(t_{i-1}^k)|\mathcal{F}_{k-1}^1]\|^2.
\]
(10.7)

For any arbitrary small positive number \( \varepsilon \) and for each time \( t_{i-1}^k \) to find the the best estimate from the admissible simulated values of simulated sequence of
\[
\{y^*(m_{i-1}^k, t_{i-1}^k))_{m_{i-1}^k \in OS_{i-1}^1}, \{p^*(m_{i-1}^k, t_{i-1}^k))_{m_{i-1}^k \in OS_{i-1}^1}\}
\]
for \( \mathbb{E}[y(t_{i-1}^k)|\mathcal{F}_{k-1}^1], \mathbb{E}[p(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \), we determine the following sub-optimal admissible set of \( m_{i-1}^k \)-size local conditional sample
\[
\mathcal{M}_{t_{i-1}^k} = \{m_{i-1}^k \in OS_{i-1}^1 : \Xi_{m_{i-1}^k, t_{i-1}^k} < \varepsilon\}, \text{ for } i \in I(1, K^*).
\]
(10.8)

Among these collected values, the value that gives the minimum \( \Xi_{m_{i-1}^k, t_{i-1}^k} \) for \( k \in [0, N_i-1] \) are recorded as \( \hat{m}_{i-1}^k \). If more than one value exist, then the largest of such \( m_{i-1}^k \)-s is recorded as \( \tilde{m}_{i-1}^k \). If condition (10.8) is not met at time \( t_{i-1}^k \), the value of \( m_{i-1}^k \) where the minimum
\[
\min_{m_{i-1}^k} \Xi_{m_{i-1}^k, t_{i-1}^k}
\]
is attained is recorded as \( \hat{m}_{i-1}^k \). The \( e^- \) level sub-optimal estimates of the parameters \( \{\hat{u}^i_{j-1}(m_{i-1}^k, t_{i-1}^k), \hat{k}_{j-1}^i(m_{i-1}^k, t_{i-1}^k), \hat{\beta}^i_{j-1}^i(m_{i-1}^k, t_{i-1}^k), \hat{\sigma}^i_{j-1}^i(m_{i-1}^k, t_{i-1}^k), \hat{\gamma}^i_{j-1}^i(m_{i-1}^k, t_{i-1}^k)\} \) are recorded as \( \{\hat{u}^i_{j-1}(\hat{m}_{i-1}^k, t_{i-1}^k), \hat{u}^i_{j-1}(\tilde{m}_{i-1}^k, t_{i-1}^k), \hat{u}^i_{j-1}(\hat{m}_{i-1}^k, t_{i-1}^k), \hat{u}^i_{j-1}(\tilde{m}_{i-1}^k, t_{i-1}^k), \hat{u}^i_{j-1}(\hat{m}_{i-1}^k, t_{i-1}^k)\} \). Finally, the simulated value \( y^*(\hat{m}_{i-1}^k, t_{i-1}^k), p^*(\hat{m}_{i-1}^k, t_{i-1}^k) \) at time \( t_{i-1}^k \) with \( \hat{m}_{i-1}^k \) is now recorded as the best estimate for \( \mathbb{E}[y(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \) and \( \mathbb{E}[p(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \). The value \( y^*(\hat{m}_{i-1}^k, t_{i-1}^k), p^*(\hat{m}_{i-1}^k, t_{i-1}^k) \) of \( \mathbb{E}[y(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \) and \( \mathbb{E}[p(t_{i-1}^k)|\mathcal{F}_{k-1}^1] \) at \( t_{i-1}^k \).
10.6 Illustration: Application of Conceptual Computational Algorithm to Energy Commodity Data Set

In this subsection, we apply the above conceptual computational algorithm to study the relationship between three energy commodities by setting \( n = 3 \) in (9.45). The three energy commodities are daily Henry Hub Natural gas data set, daily crude oil data set, and daily coal data set for the period of 05/04/2009 – 01/03/2014, [26, 27, 28]. Thus, for each pair \((y_1, p_1), (y_2, p_2)\), and \((y_3, p_3)\), the drift and diffusion coefficient function of the stochastic dynamic equation governing \((y_j, p_j)\), for \( j \in I(1, 3) \) have 4 and 3 parameters each to be estimated, respectively. Thus, there are 42 parameters to be estimated in total. Using \( \Delta t = 1, \epsilon = 0.001, \) for each \( j \in I(1, 3) \), the \( \epsilon \)- level sub-optimal estimates of parameters \( u_{j}^{i-1}(\hat{m}_{k}^{i-1}, k), \beta_{j}^{i-1}(\hat{m}_{k}^{i-1}, k), \kappa_{j,l}^{i-1}(\hat{m}_{k}^{i-1}, k), \gamma_{j,l}^{i-1}(\hat{m}_{k}^{i-1}, k), \delta_{j,l}^{i-1}(\hat{m}_{k}^{i-1}, k), \sigma_{j,l}^{i-1}(\hat{m}_{k}^{i-1}, k), l \in I(1, 3) \), at each real data times are exhibited below.

10.6.1 Illustration: Relationship between Natural Gas, Crude Oil and Coal: Without Incorporating Jump Process.

In this subsubsection, we analyze the relationship between Natural Gas, Crude Oil, and Coal without the jump process. For \( j, l \in I(1, 3) \), the stochastic dynamic system governing the three energy commodities is described in (9.48) of Remark (27). Here, \((y_1, p_1)\) denotes the mean spot and the spot price process of Natural gas, \((y_2, p_2)\) denotes the mean spot and the spot price process of Crude oil, and \((y_3, p_3)\) denotes the mean spot and the spot price process of Coal.

Using the discretized scheme (10.5), we apply the above conceptual computational algorithm for the real time data sets namely daily Henry Hub Natural gas data set, daily crude oil data set, and daily coal data set. Using \( r = 10 \), and \( d = 2 \), the \( \epsilon \)- level sub-optimal estimates of the parameters at each real data times are described below.

The parameters corresponding to the natural gas data set are \( u_1(\hat{m}_k, k), \beta_1(\hat{m}_k, k), \kappa_{1,1}(\hat{m}_k, k), \kappa_{1,2}(\hat{m}_k, k), \kappa_{1,3}(\hat{m}_k, k), \gamma_{1,1}(\hat{m}_k, k), \gamma_{1,2}(\hat{m}_k, k), \gamma_{1,3}(\hat{m}_k, k), \delta_{1,1}(\hat{m}_k, k), \delta_{1,2}(\hat{m}_k, k), \delta_{1,3}(\hat{m}_k, k), \sigma_{1,1}(\hat{m}_k, k), \sigma_{1,2}(\hat{m}_k, k), \sigma_{1,3}(\hat{m}_k, k). \) The parameters corresponding to the crude oil data set are \( u_2(\hat{m}_k, k), \beta_2(\hat{m}_k, k), \kappa_{2,1}(\hat{m}_k, k), \kappa_{2,2}(\hat{m}_k, k), \kappa_{2,3}(\hat{m}_k, k), \gamma_{2,1}(\hat{m}_k, k), \gamma_{2,2}(\hat{m}_k, k), \gamma_{2,3}(\hat{m}_k, k), \delta_{2,1}(\hat{m}_k, k), \delta_{2,2}(\hat{m}_k, k), \delta_{2,3}(\hat{m}_k, k), \sigma_{2,1}(\hat{m}_k, k), \sigma_{2,2}(\hat{m}_k, k), \sigma_{2,3}(\hat{m}_k, k). \) The parameters corresponding to coal data set are \( u_3(\hat{m}_k, k), \beta_3(\hat{m}_k, k), \kappa_{3,1}(\hat{m}_k, k), \kappa_{3,2}(\hat{m}_k, k), \kappa_{3,3}(\hat{m}_k, k), \gamma_{3,1}(\hat{m}_k, k), \gamma_{3,2}(\hat{m}_k, k), \gamma_{3,3}(\hat{m}_k, k), \delta_{3,1}(\hat{m}_k, k), \delta_{3,2}(\hat{m}_k, k), \delta_{3,3}(\hat{m}_k, k), \sigma_{3,1}(\hat{m}_k, k), \sigma_{3,2}(\hat{m}_k, k), \sigma_{3,3}(\hat{m}_k, k). \)
The following table gives the parameter estimates u1 (m̂k , k), κ1,1 (m̂k , k), κ1,2 (m̂k , k), κ1,3 (m̂k , k),
u2 , κ2,1 (m̂k , k), κ2,2 (m̂k , k), κ2,3 (m̂k , k), u3 (m̂k , k), κ3,1 (m̂k , k),
κ3,2 (m̂k , k), κ3,3 (m̂k , k) for the decoupled system for y in the case where jump is not incorporated
into the system.
Table 15: Estimates m̂k , u1 (m̂k , k), κ1,1 (m̂k , k), κ1,2 (m̂k , k), κ1,3 (m̂k , k), u2 (m̂k , k), κ2,1 (m̂k , k),
κ2,2 (m̂k , k), κ2,3 (m̂k , k), u3 (m̂k , k), κ3,1 (m̂k , k), κ3,2 (m̂k , k), κ3,3 (m̂k , k) (without jump).
tk

Natural gas
m̂k

u1

κ1,1

Crude oil

κ1,2

κ1,3

×10−16

×10−18

u2

κ2,1

Coal

κ2,2

×10−18

κ2,3

u3

×10−18

κ3,1

κ3,2

×10−18

×10−18

κ3,3

11

1

4.1593

0.0211

0

0

57.7000

0

0

0

16.7407

0

0

0

12

3

4.2000

0.0111

0

0

58.6313

0.0011

0.0310

-0.0012

16.2395

0

0

-0.0376

13

5

4.0616

0.0679

-0.0054

-0.0035

58.5378

-0.0035

0.0205

0.0032

16.2680

0

0

0.1069

14

5

4.0616

-0.0242

-0.0179

0

61.4809

0.0020

0.0098

0

15.5249

0

0

-0.0294

15

8

4.0910

0.6416

-0.2898

0

58.9282

-0.0036

0.0128

0.0071

16.8286

0

0

0.0513

16

8

4.0160

0.2101

0

0

59.6867

-0.0051

0.0080

0.0071

17.0888

0

0

0.0415

17

8

4.9575

0.1876

0

0

60.6244

0.0024

0.0052

0

17.4120

-0.0003

0.0001

0.0555

18

8

4.9575

-0.1947

0

0

61.0700

0

0

0

17.2374

0

-0

0

19

6

4.7336

-1.4476

5.8820

0

61.9414

0

0.0043

-0.0086

16.8438

0.0001

0.0001

0.0768

20

6

2.5646

0.3319

0.7261

0

62.7899

0

0.0053

0.0082

18.3022

-0.0083

0.0027

0.0558

...

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...

...

...

...

495

8

3.9654

0.0591

-0

0

108.2457

0.0038

0.0049

-0.0023

33.1313

0.0027

0.0009

0.0363

496

5

4.0421

0.0616

0.0001

0.0017

107.5186

0

0

0

33.4224

-0.0005

0.0003

0.0214

497

6

4.0514

0.0127

-0.0002

0.0020

109.8836

0

0

-0.0001

33.3388

0.0002

0

0.0443

498

7

4.1646

0.0442

-0.0012

-0.0053

107.8013

-0.0021

0.0033

0.0038

33.2862

-0.0002

0.0006

0.0343

499

6

4.1226

0.0352

-0.0020

0

108.1554

-0.0005

0.0032

0.0039

33.2862

0.0010

0.0001

0.0068

500

6

4.2625

0.0733

-0.0002

0

110.5101

-0.0032

0.0033

0.0016

36.1647

0.0003

0.0003

0.0079

501

8

3.1551

0

0

-0.0009

110.3071

0.0014

0.0025

0

34.7467

0

0

0

502

4

4.1564

0.0914

-0.0002

0

111.1186

0

0.0013

-0.0031

49.4050

0.0026

-0.0002

0.0211

503

5

4.5799

0.0467

0.0004

0

112.0057

0

0.0027

-0.0043

34.7207

-0.0001

-0.0001

0.0216

504

4

4.3061

0.0236

0.0002

0.0007

112.3186

0

0.0021

0.0015

34.4483

0.0019

0.0003

0.0170

505

9

4.4325

-0.0015

-0.0018

0.0030

106.3345

0

0.0043

0.0001

33.7160

0

-0.0006

0.0265

...

...

...

...

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...

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...

...

...

...

...

...

...

1102

7

3.5429

-0.0286

-0.0006

-0.0028

110.3777

0.0006

0.0045

0

5.2399

0

0.0013

0.0008

1103

4

3.5601

0.1028

0.0001

0.0001

111.1585

-0.0003

0.0083

0

5.4824

0

0.0077

0.0485

1104

4

3.5314

0.0809

0.0018

0.0090

109.0996

-0.0007

0.0095

0.0013

11.0949

-0.0018

0.0005

0.1175

1105

4

3.4439

0.1551

-0.0008

-0.0015

106.5667

0.0033

0.0073

-0.0020

4.8300

-0.0012

-0.0003

0.1283

1106

6

3.8206

0.2258

0.0004

0

104.7497

0

0

0.0027

4.8300

0

0.0008

0

1107

4

3.6917

0.2132

-0.0001

-0.0008

105.1229

0.0011

0.0039

0

4.3586

-0.0005

0.0004

0.1418

1108

5

3.7871

0

0

0

105.3595

0.0006

0.0027

-0.0009

4.8000

0.0006

-0.0001

0.1265

1109

4

3.8445

-0.0405

-0.0011

0.0011

102.9022

-0.0044

0.0037

0.0039

5.0279

0

0

0

1110

5

3.8399

0.0212

0.0004

0

102.8313

-0.0020

0.0045

0.0018

4.6817

0.0021

0.0041

0.0536

153


Table 15 shows the estimates of the $\epsilon$-sub-optimal size $\hat{m}_k$, $j \in I(1,3)$, the parameters $u_1(\hat{m}_k, k)$, $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ for each of the energy commodity data sets. Moreover, $d \leq r$ and the initial real data time is $t_r = t_{10}$.

The following table gives the drift coefficient’s parameter estimates $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ for the decoupled dynamical system for $y$ in the case where jump is not incorporated into the dynamical system.

Figure 25.: The graph of mean level $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (without jump).

Figures 25: (a), (b) and (c) are the graphs of $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, and $u_3(\hat{m}_k, k)$ against time $t_k$ for the daily Henry Hub natural gas price [27], daily crude oil price [28], and daily coal price [26] data set, respectively. By plotting the real data sets (shown in Figure 31), it is easily seen that the graphs of $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ are similar to the graph of the real Henry Hub Natural gas, Crude Oil, and Coal data set, respectively. We expect this to happen because $u_j$, $j \in I(1,3)$ are the expected equilibrium spot price processes described in (9.3). This analysis shows that the parameters $u_j$, $j \in I(1,3)$ are statistic process for the respective mean of the data sets at time $t_k$. 
The graph of the parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, and $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled dynamical system for $y$ (with no jump incorporated into the dynamical system) are given below:
The graph of interaction coefficients \( \kappa_{1,1}(\hat{m}_k, k) \), \( \kappa_{1,2}(\hat{m}_k, k) \), \( \kappa_{1,3}(\hat{m}_k, k) \), \( \kappa_{2,1}(\hat{m}_k, k) \), \( \kappa_{2,2}(\hat{m}_k, k) \), \( \kappa_{2,3}(\hat{m}_k, k) \), \( \kappa_{3,1}(\hat{m}_k, k) \), \( \kappa_{3,2}(\hat{m}_k, k) \), \( \kappa_{3,3}(\hat{m}_k, k) \) (without jump).

Figures 26 (a) – (i) show the graph of the \( \epsilon \)-sub-optimal interaction coefficient parameters \( \kappa_{1,1}(\hat{m}_k, k) \), \( \kappa_{1,2}(\hat{m}_k, k) \), \( \kappa_{1,3}(\hat{m}_k, k) \), \( \kappa_{2,1}(\hat{m}_k, k) \), \( \kappa_{2,2}(\hat{m}_k, k) \), \( \kappa_{2,3}(\hat{m}_k, k) \), \( \kappa_{3,1}(\hat{m}_k, k) \), \( \kappa_{3,2}(\hat{m}_k, k) \), \( \kappa_{3,3}(\hat{m}_k, k) \). The interaction coefficients \( \kappa_{j,j}, j \neq l \) are negligible, because each estimate is \( < < 10^{-15} \). Thus, this shows that the model describing the mean spot price, \( y_j \), is mainly characterized by the market potential \( \kappa_{j,j} (u_j - y_j) y_j, j \in I(1, n) \).

The table below shows the estimates of the diffusion coefficient’s parameters for the model governing \( y \).
Table 16: Estimates $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$ (without jump).

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The graph of the diffusion coefficient’s parameter for the decoupled dynamical system for $y$ without jump incorporated into the dynamical system are given below:
Figure 27.: The graph of interaction coefficients $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$ (without jump).

Figures 27 (a) – (i) show the graph of the $\epsilon$-sub-optimal interaction measure of fluctuation coefficient parameters $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$, respectively.

The following table gives the drift coefficient’s parameter estimates $\beta_1(\hat{m}_k, k)$, $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\beta_3(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, and $\gamma_{3,3}(\hat{m}_k, k)$ for the dynamical system for $p$ (without incorporating jump process in the model describing the system $p$).
Table 17: Estimates $\beta_1(m_k, k), \gamma_1,1(m_k, k), \gamma_1,2(m_k, k), \gamma_1,3(m_k, k), \beta_2(m_k, k), \gamma_2,1(m_k, k), \gamma_2,2(m_k, k), \gamma_2,3(m_k, k), \beta_3(m_k, k), \gamma_3,1(m_k, k), \gamma_3,2(m_k, k), \gamma_3,3(m_k, k)$ (without jump).

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Table 17 shows the estimates of the parameters $\beta_1(m_k, k), \gamma_1,1(m_k, k), \gamma_1,2(m_k, k), \gamma_1,3(m_k, k), \beta_2(m_k, k), \gamma_2,1(m_k, k), \gamma_2,2(m_k, k), \gamma_2,3(m_k, k), \beta_3(m_k, k), \gamma_3,1(m_k, k), \gamma_3,2(m_k, k), \gamma_3,3(m_k, k)$ at the $\epsilon$-sub-optimal size $m_k$ and time $t_k$, for each of the energy commodity data sets. Moreover, $p \leq r$, and the initial real data time is $t_r = t_{10}$. 

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Figure 28.: The graph of interaction coefficients $\gamma_{1,1}(\hat{m}_k, k), \gamma_{1,2}(\hat{m}_k, k), \gamma_{1,3}(\hat{m}_k, k), \gamma_{2,1}(\hat{m}_k, k), \gamma_{2,2}(\hat{m}_k, k), \gamma_{2,3}(\hat{m}_k, k), \gamma_{3,1}(\hat{m}_k, k), \gamma_{3,2}(\hat{m}_k, k), \gamma_{3,3}(\hat{m}_k, k)$ (without jump).

Figures 28 (a) – (i) show the graph of the ε-sub-optimal interaction coefficient parameters $\gamma_{1,1}(\hat{m}_k, k), \gamma_{1,2}(\hat{m}_k, k), \gamma_{1,3}(\hat{m}_k, k), \gamma_{2,1}(\hat{m}_k, k), \gamma_{2,2}(\hat{m}_k, k), \gamma_{2,3}(\hat{m}_k, k), \gamma_{3,1}(\hat{m}_k, k), \gamma_{3,2}(\hat{m}_k, k), \gamma_{3,3}(\hat{m}_k, k)$ without jump. According to (9.47), the estimate $\gamma_{j,l}(\hat{m}_k, k), j \neq l$, is positive if commodity $p_i$ is cooperating with commodity $p_j$, and negative if commodity $p_i$ is competing with commodity $p_j$. There is no interaction between the two commodities if $\gamma_{j,l}(\hat{m}_k, k) = 0$.

It is apparent from the graph of $\gamma_{1,3}(\hat{m}_k, k)$ that coal and natural gas are competing and cooperating depending on the time period. It is also apparent graph of $\gamma_{1,2}(\hat{m}_k, k)$ that natural gas and crude oil are also either cooperating or competing, depending on the time period.

The next figure shows the graph of the parameter estimates $\beta_1(\hat{m}_k, k), \beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ in the drift coefficient of the model describing the system $\mathbf{p}$.
Figure 29.: The graph of $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (without jump).

Figures 29: (a), (b) and (c) are the graphs of $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ against time $t_k$ for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively (without jump).
Table 18: Estimates $\sigma_{1,1}(\hat{m}_k, k), \sigma_{1,2}(\hat{m}_k, k), \sigma_{1,3}(\hat{m}_k, k), \sigma_{2,1}(\hat{m}_k, k), \sigma_{2,2}(\hat{m}_k, k), \sigma_{2,3}(\hat{m}_k, k), \sigma_{3,1}(\hat{m}_k, k), \sigma_{3,2}(\hat{m}_k, k), \sigma_{3,3}(\hat{m}_k, k)$ (without jump).

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164
Table 18 gives the $\epsilon$-sub-optimal estimates of the parameters $\sigma_{1,1}(\hat{m}_k, k), \sigma_{1,2}(\hat{m}_k, k), \sigma_{1,3}(\hat{m}_k, k), \sigma_{2,1}(\hat{m}_k, k), \sigma_{2,2}(\hat{m}_k, k), \sigma_{2,3}(\hat{m}_k, k), \sigma_{3,1}(\hat{m}_k, k), \sigma_{3,2}(\hat{m}_k, k), \sigma_{3,3}(\hat{m}_k, k)$ for each of the energy commodity data sets.
Figure 30.: The graph of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (without jump).

Figures 30: (a), (b) and (c) are the graphs of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$ against time $t_k$ for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively.
Table 19: Real and simulated estimates (without jump) for Natural gas, Crude oil, and Coal.

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Table 19 shows the Real and simulated estimates for the spot price processes $p_j(t), j \in I(1, 3)$ corresponding to the natural gas, crude oil and coal prices.
The next figure shows the graph of the real and simulated prices for Natural gas, Crude oil, and Coal data set.

![Real and Simulated Prices for Natural gas, Crude oil, and Coal](image)

Figure 31.: Real and Simulated Prices (without jump) for Natural gas, Crude oil, and Coal.

Figures 31: (a), (b), and (c) show the graph of the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [27], daily crude oil data set [28], and daily coal data set [26], respectively. The red line represents the real data set $p(t_k)$, while the blue line represent the simulated data set $p^*(\hat{\theta}_{k}, k)$. Here, we begin by using a starting delay of $r = 10$. The simulation starts from $t_r = t_{10}$. The spikes in the graph is as a result of jump. The estimates at the jump times are not fitted properly. To reduce magnitude of error, we increase the magnitude of time delay. We later compare this result with the case where jump is incorporated into the system.

10.6.2 Relationship between Natural Gas, Crude Oil and Coal: With Jump Incorporated.

In this subsubsection, we analyze the relationship between Natural Gas, Crude Oil, and Coal with the jump process. Here, we apply the above conceptual computational algorithm in Section 10 for the real time data sets namely daily Henry Hub Natural gas data set, daily crude oil data set, and
daily coal data set for the period of 05/04/2009 – 01/03/2014, [26, 27, 28]. For \( i \in I(1, K^*) \), \( K \neq 0 \), we use \( \Delta r_{i-1} = 1; \epsilon = 0.001; r_{i-1} = 10 \) and \( d_{i-1} = 2 \). The \( \epsilon \)- level sub-optimal estimates of the parameters at each real data times are described below for each commodity data sets. We also note that there are \( K = 15 \) jumps in the system.

The parameters corresponding to the model governing natural gas price data set are \( u_{1i}^{i-1}(\hat{m}_k, t_k) \), \( \beta_{i1}^{i-1}(\hat{m}_k, t_k), \gamma_{1i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{1i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{12i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{13i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{14i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{11i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{12i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{13i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{14i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{11i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{12i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{13i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{14i}^{i-1}(\hat{m}_k, t_k) \). The parameters corresponding to the model governing crude oil price data set are \( u_{2i}^{i-1}(\hat{m}_k, t_k) \), \( \beta_{2i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{21i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{22i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{23i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{21i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{22i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{23i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{21i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{22i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{23i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{21i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{22i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{23i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{24i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{33i}^{i-1}(\hat{m}_k, t_k) \), while the parameters corresponding to the model governing coal price data set are \( u_{3i}^{i-1}(\hat{m}_k, t_k) \), \( \beta_{3i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{31i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{32i}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{33i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{31i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{32i}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{33i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{31i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{32i}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{33i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{31i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{32i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{33i}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{34i}^{i-1}(\hat{m}_k, t_k) \).

For the sake of simplicity and in order to be able to compare our results in this subsection with the results in subsection 10.6.1, for each \( j, l \in I(1, n) \), we re-write the parameters \( u_{j}^{i-1}(\hat{m}_k, t_k) \), \( \beta_{j}^{i-1}(\hat{m}_k, t_k) \), \( \kappa_{jl}^{i-1}(\hat{m}_k, t_k) \), \( \gamma_{jl}^{i-1}(\hat{m}_k, t_k) \), \( \delta_{jl}^{i-1}(\hat{m}_k, t_k) \), \( \sigma_{jl}^{i-1}(\hat{m}_k, t_k) \) after they have been estimated as \( u_{j}(\hat{m}_k, k), \beta_{j}(\hat{m}_k, k), \kappa_{jl}(\hat{m}_k, k), \gamma_{jl}(\hat{m}_k, k), \delta_{jl}(\hat{m}_k, k), \) and \( \sigma_{jl}(\hat{m}_k, k) \).

First, we give results for the jump times of the system \( \{T_i\}_{i \in I(1, K^*)} \).

**Table 20:** Result for the jump times of the system \( (y, p) \)

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<th>61</th>
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<th>722</th>
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<th>870</th>
<th>930</th>
<th>1113</th>
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</table>

Table 20 shows the result for the jump times of the system \( (y, p) \). These results are derived by recording the times at which an entry of a commodity differ by twice the standard deviation or more from the mean of that commodity. These times are now combined into a single array and sorted out in an increasing order. It follows from Table 20 that \( K = 15 \).

We give the estimate of the jump coefficient matrices \( \Pi_j \) and \( \Theta_j \) defined in (9.23) in the following table.
Table 21: Estimates $\pi_i^1$, $\pi_i^2$, $\pi_i^3$, $\theta_i^1$, $\theta_i^2$, and $\theta_i^3$.

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<th>$T_i$</th>
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The following table gives the drift coefficient’s parameter estimates $u_1(\hat{m}_k, k)$, $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled dynamical system for $y$ with jump.
The following figures show the parameter estimates $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, and $u_3(\hat{m}_k, k)$ for the decoupled dynamical system for $y$ with jump.
Figure 32.: The graph of mean level $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (with jump).

Figures 32: (a), (b) and (c) are the graphs of $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, and $u_3(\hat{m}_k, k)$ against time $t_k$ for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively. By plotting the real data sets (shown in Figure 38), it is easily seen that the graphs of $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ are similar to the graph of the Henry Hub Natural gas, Crude Oil, and Coal data set, respectively. We expect this to happen because $u_j$, $j \in I(1, 3)$ are the equilibrium spot price processes described in (9.3).

The graph of the interaction parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, and $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled dynamical system for $y$ with jump and estimates in Table 22 are given below:
The graph of interaction coefficients $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ (with jump).

Figures 33 (a) – (i) show the graph of the $\epsilon$-sub-optimal interaction coefficient parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$. The interaction coefficients $\kappa_{j,l}$, $j \neq l$ are negligible, because each estimate is $<< 10^{-15}$. Thus, this shows that the model describing the mean spot price, $y_j$, is mainly characterized by the market potential $\kappa_{j,j}^{-1} (u_{j}^{j-1} - y_j) y_j$, $j \in I(1, n), i \in I(1, K^*)$.

The table below shows the estimates of the diffusion coefficient’s parameters for $y$. 

Figure 33.: The graph of interaction coefficients $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ (with jump).
Table 23: Estimates $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$ (with jump).

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<td>0.8995</td>
</tr>
<tr>
<td>1108</td>
<td>2.0262</td>
<td>0</td>
<td>0.6325</td>
</tr>
<tr>
<td>1109</td>
<td>1.7828</td>
<td>0</td>
<td>0.6116</td>
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<tr>
<td>1110</td>
<td>1.2706</td>
<td>0</td>
<td>0.1001</td>
</tr>
</tbody>
</table>

The graph of the diffusion coefficient’s parameter for the decoupled dynamical system for $y$ with jump are given below:
Figure 34.: The graph of interaction coefficients $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$ (with jump).

Figures 34 (a) – (i) show the graph of the $\epsilon$-sub-optimal interaction measure of fluctuation coefficient parameters $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$.

The following table gives the drift coefficient’s parameter estimates $\beta_1(\hat{m}_k, k)$, $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\beta_3(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, and $\gamma_{3,3}(\hat{m}_k, k)$ for the dynamical system for $p$ with jump.
Table 24: Estimates $\beta_1(\hat{m}_k, k), \gamma_{1,1}(\hat{m}_k, k), \gamma_{1,2}(\hat{m}_k, k), \gamma_{1,3}(\hat{m}_k, k), \beta_2(\hat{m}_k, k), \gamma_{2,1}(\hat{m}_k, k), \gamma_{2,2}(\hat{m}_k, k), \beta_3(\hat{m}_k, k), \gamma_{3,1}(\hat{m}_k, k), \gamma_{3,2}(\hat{m}_k, k), \gamma_{3,3}(\hat{m}_k, k)$ (with jump).

| $t_k$ | Natural gas | | | | Crude oil | | | | Coal | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\beta_1$ | $\gamma_{1,1}$ | $\gamma_{1,2}$ | $\gamma_{1,3}$ | $\beta_2$ | $\gamma_{2,1}$ | $\gamma_{2,2}$ | $\gamma_{2,3}$ | $\beta_3$ | $\gamma_{3,1}$ | $\gamma_{3,2}$ | $\gamma_{3,3}$ |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0.1681 | 0.3497 | -0.0109 | 0.0248 | -0.4815 | 0.1626 | -0.4066 | -0.0123 | 0.7665 | -0.3259 | 0.0205 | -0.0198 |
| 13 | 0.1592 | 0.3755 | -0.0102 | 0.0228 | -0.7778 | -0.5752 | -0.0578 | 0.1870 | 1.0795 | -0.2904 | 0.0135 | -0.0217 |
| 14 | 0.3439 | 0.3755 | 0.0488 | -0.5478 | 1.7680 | -0.5555 | -0.2058 | 0.2911 | 0.8543 | -0.2056 | 0.0034 | -0.0064 |
| 15 | 0.3336 | 0.3652 | -0.0127 | 0.0213 | -0.8999 | 0.0601 | -0.1110 | 0.0405 | 0.5144 | -0.1264 | -0.0062 | 0.0127 |
| 16 | 0.4709 | 0.2780 | -0.0116 | 0.0104 | -0.8999 | 0.0601 | -0.1110 | -0.0129 | 0.0071 | -0.0002 | 0 | 0 |
| 17 | 0.3277 | 0.2780 | 0.0768 | -0.2633 | 1.3349 | 0.0027 | -0.0330 | -0.0750 | -0.6285 | -0.0569 | 0.0016 | 0.0262 |
| 18 | 0.3277 | 1.3156 | -0.1491 | -0.1646 | 1.5419 | -0.0088 | 0.1205 | -0.0892 | -1.3275 | -0.0703 | 0.0080 | 0.0386 |
| 19 | 0 | 0 | 0 | 0 | -0.1785 | -0.0368 | -0.0062 | 0.0189 | -0.8091 | -0.0001 | -0.0083 | 0.0466 |
| 20 | 0.6985 | 0.4990 | -0.0069 | -0.0187 | -0.1513 | -0.0778 | -0.0096 | 0.0255 | -0.0182 | 0.0080 | 0.0001 | 0.0003 |

Table 24 shows the estimates of the parameters $\hat{m}_k, \beta_1(\hat{m}_k, k), \gamma_{1,1}(\hat{m}_k, k), \gamma_{1,2}(\hat{m}_k, k), \gamma_{1,3}(\hat{m}_k, k), \hat{m}_k, \beta_2(\hat{m}_k, k), \gamma_{2,1}(\hat{m}_k, k), \gamma_{2,2}(\hat{m}_k, k), \beta_3(\hat{m}_k, k), \gamma_{3,1}(\hat{m}_k, k), \gamma_{3,2}(\hat{m}_k, k), \gamma_{3,3}(\hat{m}_k, k)$, for each of the energy commodity data sets. According to (9.4T), the estimate $\gamma_{j,l}(\hat{m}_k, k)$, $j \neq l$, is positive if commodity $p_j$ is cooperating with commodity $p_l$, and negative if commodity $p_j$ is competing with commodity $p_l$. There is no interaction between the
two commodities if $\gamma_{j,l}(\hat{m}_k, k) = 0$. It is apparent from the graph (from $\gamma_{1,3}(\hat{m}_k, k)$ in Column 6) that coal is competing with natural gas during this period because the estimates of $\gamma_{1,3}(\hat{m}_k, k)$ are mostly negative. It is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.

In the following, the graph of the drift coefficient’s parameters with estimates in Table 24 are given below:
Figure 35.: The graph of interaction coefficients $\gamma_{1,1}({\hat m}_k, k)$, $\gamma_{2,1}({\hat m}_k, k)$, $\gamma_{1,2}({\hat m}_k, k)$, $\gamma_{2,2}({\hat m}_k, k)$, $\gamma_{3,1}(\hat m_k, k)$, $\gamma_{3,2}(\hat m_k, k)$, $\gamma_{3,3}(\hat m_k, k)$ (with jump).

Figures 35 (a) – (i) show the graph of the $\epsilon$-sub-optimal interaction coefficient parameters $\gamma_{1,1}({\hat m}_k, k)$, $\gamma_{2,1}({\hat m}_k, k)$, $\gamma_{1,2}({\hat m}_k, k)$, $\gamma_{2,2}({\hat m}_k, k)$, $\gamma_{3,1}(\hat m_k, k)$, $\gamma_{3,2}(\hat m_k, k)$, $\gamma_{3,3}(\hat m_k, k)$. According to (9.47), the estimate $\gamma_{j,l}({\hat m}_k, k)$, $j \neq l$, is positive if commodity $p_l$ is cooperating with commodity $p_j$, and negative if commodity $p_l$ is competing with commodity $p_j$. There is no interaction between the two commodities if $\gamma_{j,l}({\hat m}_k, k) = 0$. It is apparent from the graph of $\gamma_{1,3}(\hat m_k, k)$ that coal is competing with natural gas because the estimates of $\gamma_{1,3}(\hat m_k, k)$ are mostly negative. Also, it is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.
Figure 36.: The graph of mean level $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ (with jump).

Figures 36: (a), (b) and (c) are the graphs of $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, and $\beta_3(\hat{m}_k, k)$ against time $t_k$ for the daily Henry Hub natural gas price, [27] daily crude oil price [28], and daily coal price data set, respectively.
Table 25: Estimates $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ (with jump).

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>Natural gas</th>
<th>Crude oil</th>
<th>Coal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{1,1}$</td>
<td>$\sigma_{1,2}$</td>
<td>$\sigma_{1,3}$</td>
</tr>
<tr>
<td>11</td>
<td>0 0 0</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>12</td>
<td>0.0485 0.0004 0.0032</td>
<td>0.2734 0.0166 0</td>
<td>0.0513 0 0.0000</td>
</tr>
<tr>
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<td>0.9445 0 0</td>
<td>0.2489 0 0</td>
</tr>
<tr>
<td>14</td>
<td>0.2120 0.1386 0.0133</td>
<td>0.3877 0.4773 0.1195</td>
<td>0.1365 0.0665 0.0086</td>
</tr>
<tr>
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<td>0.4246 0.1318 0.0021</td>
<td>0.03341 0.4894 0.1211</td>
<td>0.0112 0.6107 0.0696</td>
</tr>
<tr>
<td>16</td>
<td>0.5538 0.0778 0.1501</td>
<td>0.07751 0.2524 0.0811</td>
<td>0.0651 0.4251 0.0635</td>
</tr>
<tr>
<td>17</td>
<td>0.3907 0.0469 0.2230</td>
<td>0.08746 0.1848 0.2463</td>
<td>0 0.4458 0.0478</td>
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<tr>
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</tr>
<tr>
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<td>0.4999 0.0569 0.0127</td>
</tr>
<tr>
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<td>0.3789 0.3174 0.0046</td>
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<td>... ... ...</td>
</tr>
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</tr>
<tr>
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<td>0.08183 0.3163 0.0102</td>
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<td>0.1551 0.0009 0.0065</td>
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<td>0.7777 0 0.0033</td>
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<tr>
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</tr>
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<td>0.6875 0 0.1451</td>
</tr>
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<td>0.7298 0.2808 0.0147</td>
</tr>
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<td>0.6648 0.0915 0.0462</td>
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<td>0.6591 0.2874 0.0057</td>
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<td>0.1191 0.0011 0.0116</td>
<td>0.6260 0.1155 0.0393</td>
<td>0 0.0196 0.0060</td>
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<td>0.4992 0.0781 0.0382</td>
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<td>0.8431 0.2366 0.4511</td>
</tr>
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<td>0.2912 0.0021 0.0163</td>
<td>0.0385 0.0342 0.0037</td>
<td>0.2910 0.0489 0.0257</td>
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</table>

Table 25 gives the $\epsilon$-sub-optimal estimates of the parameters $u_1(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ for each of the energy commodity data sets.
Figure 37.: The graph of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (with jump).

Figures 37: (a), (b) and (c) are the graphs of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ against time $t_k$ for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively.
Table 26: Real and simulated estimates (with jump) for Natural gas, Crude oil, and Coal.

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>Natural gas</th>
<th>Crude oil</th>
<th>Coal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Real $p^R_j$</td>
<td>Simulated $p^S_j$</td>
<td>Real $p^R_j$</td>
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<td>3.3900</td>
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<tr>
<td>1110</td>
<td>3.4800</td>
<td>3.4604</td>
<td>102.3600</td>
</tr>
</tbody>
</table>

Table 26 shows the Real and simulated estimates for the spot price processes $p_j(t), j \in I(1, n)$.

The next figure shows the graph of the real and simulated prices (with jump) for Natural gas, Crude oil, and Coal data set.
Figures 38: (a), (b), and (c) show the graph of the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [27], daily crude oil data set [28], and daily coal data set [26], respectively. The red line represents the real data set $p$, while the blue line represent the simulated data set $\hat{p}_{m,k}$. The graph fits well. To reduce magnitude of error, we increase the magnitude of time delay. It is obvious that these curves fit better than the curves in Figure 31. It follows that the interconnected dynamical system with jump process incorporated into it performs better than the one without jump.
\textbf{Chapter 11}

\textbf{Forecasting}

11.1 Introduction

In this chapter, we shall sketch an outline about forecasting problem for the case where there is no jump. The sketch for the case where jump exist is similar. An \( \epsilon \)— sub-optimal simulated value \( (y^s(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \, p^s(\hat{m}_{i_k}^{i-1}, t_{k}^{i-1})) \) at time \( t_{i_k}^{i-1} \), \( i \in I(1, K^*) \), are used to define a forecast \( (y^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \, p^f(\hat{m}_{i_k}^{i-1}, t_{k}^{i-1})) \) for \( (y(t_{i_k}^{i-1}), p(t_{i_k}^{i-1})) \) at the time \( t_{i_k}^{i-1} \) for the system of energy commodity model.

11.2 Forecasting for Energy Commodity Model

In the context of Illustration 9.6.1, for \( i \in I(1, K^*) \), we begin forecasting from time \( t_{i_k}^{i-1} \). Using the data set up to time \( t_{i_k}^{i-1} \), we compute \( \hat{m}_{i_k}^{i-1}, \, m_a^{i-1}, \, u_j(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \, \beta_j(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \, \kappa_{j,l}(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \) \( \gamma_{j,l}(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \, \delta_{j,l}(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), \, \sigma_{j,l}(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}), j, l \in I(1, 3) \) for \( a \in I(0, k - 1) \). We assume that we have no information about the real data set \( \{y_j(t_{i_k}^{i-1})\}_{a=k}^{N_i} \). Under these considerations, imitating the computational procedure outlined in Section 10 and using solutions to (9.58)-(9.59), we find the estimate of the forecast \( y^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \) and \( p^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \) at time \( t_{i_k}^{i-1} \) as follows:

\[
\begin{align*}
\left\{\begin{array}{l}
y_j^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) = y_j^s(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) + \left( u_j(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) - y_j^s(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \right) \\
&\times \left[ \kappa_{j,l}(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) y_j^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \\
&+ \sum_{l \neq j} \kappa_{j,l} y_l^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \right] \Delta t \\
&+ \delta_{j,l}(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \left( u_j(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) - y_j^s(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \right) W_{j,l}(k) \\
&+ \left( u_j(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) - y_j^s(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) \right) \sum_{l \neq j} \delta_{j,l} y_l^f(\hat{m}_{i_k}^{i-1}, t_{i_k}^{i-1}) W_{j,l}(k)
\end{array}\right.
\end{align*}
\]

(11.1)
\[
\begin{align*}
p_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) &= p_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) + p_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \left[ \gamma_{j,j}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \left( y_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \right. \right. \\
&\left. \left. - p_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \right) + \beta_j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \right] \Delta t \\
&+ \sum_{l \neq j} \gamma_{j,l}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) p_f^l(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) Z_{j,l}(k) \\
&+ p_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \sum_{l \neq j} \sigma_{j,l}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) p_f^l(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) Z_{j,l}(k),
\end{align*}
\]

where the estimates \( u_j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}), \beta_j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}), \kappa_{j,l}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}), \gamma_{j,l}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}), \delta_{j,l}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}), \sigma_{j,l}(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \), \( j, l \in I(1, 3) \) are estimated with respect to the known past data set up to the time \( t_{k-1}^{i-1} \). We note that \( y_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \) is the \( \epsilon \)-sub-optimal estimate for \( y_j(t_{k-1}^{j}) \) at time \( t_{k-1}^{j} \).

To determine \( (y_f^j(\hat{m}_{k+1}^{j-1}, t_{k+1}^{j}), p_f^j(\hat{m}_{k+1}^{j-1}, t_{k+1}^{j})) \), we need \( u_j(\hat{m}_{k}^{j-1}, t_{k}^{j}), \beta_j(\hat{m}_{k}^{j-1}, t_{k}^{j}), \kappa_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}), \gamma_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}), \delta_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}), \sigma_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}) \), \( j, l \in I(1, 3) \). Since we only have information of real data up to time \( t_{k-1} \), we use the forecasted estimate \( y_f^j(\hat{m}_{k-1}^{j-1}, t_{k-1}^{j}) \) as the estimate of \( y_j(t_{k-1}^{j}) \) and to estimate \( u_j(\hat{m}_{k}^{j-1}, t_{k}^{j}), \beta_j(\hat{m}_{k}^{j-1}, t_{k}^{j}), \kappa_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}), \gamma_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}), \delta_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}), \sigma_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}) \), \( j, l \in I(1, 3) \).

Hence, we can write \( u_j(\hat{m}_{k}^{j-1}, t_{k}^{j}) \) as

\[
\begin{align*}
u_j^j(\hat{m}_{k}^{j-1}, t_{k}^{j}) &= u_j(\hat{m}_{k}^{j-1}, y_j(t_{k-1}^{j-1}), y_j(t_{k-1}^{j-1}+1), \ldots, y_j(t_{k}^{j-1})), \\
\kappa_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}) &= \kappa_{j,l}(\hat{m}_{k}^{j-1}, y_j(t_{k-1}^{j-1}), y_j(t_{k-1}^{j-1}+1), \ldots, y_j(t_{k}^{j-1})), \\
\delta_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}) &= \delta_{j,l}(\hat{m}_{k}^{j-1}, y_j(t_{k-1}^{j-1}), y_j(t_{k-1}^{j-1}+1), \ldots, y_j(t_{k}^{j-1})), \\
\beta_{j}(\hat{m}_{k}^{j-1}, t_{k}^{j}) &= \beta_{j}(\hat{m}_{k}^{j-1}, p_j(t_{k-1}^{j-1}), p_j(t_{k-1}^{j-1}+1), p_j(t_{k-1}^{j-1}+2), \ldots, p_j(t_{k}^{j-1})), \\
\gamma_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}) &= \gamma_{j,l}(\hat{m}_{k}^{j-1}, p_j(t_{k-1}^{j-1}), p_j(t_{k-1}^{j-1}+1), p_j(t_{k-1}^{j-1}+2), \ldots, p_j(t_{k}^{j-1})), \\
\sigma_{j,l}(\hat{m}_{k}^{j-1}, t_{k}^{j}) &= \sigma_{j,l}(\hat{m}_{k}^{j-1}, p_j(t_{k-1}^{j-1}), p_j(t_{k-1}^{j-1}+1), p_j(t_{k-1}^{j-1}+2), \ldots, p_j(t_{k}^{j-1})),
\end{align*}
\]

\( j, l \in I(1, n) \).

To find \( (y_f^j(\hat{m}_{k+2}^{j-1}, t_{k+2}^{j}), p_f^j(\hat{m}_{k+2}^{j-1}, t_{k+2}^{j})) \), we use the estimates

\[
\begin{align*}
u_j^j(\hat{m}_{k+1}^{j-1}, t_{k+1}^{j}) &= u_j(\hat{m}_{k+1}^{j-1}, y_j(t_{k}^{j-1}), y_j(t_{k}^{j-1}+1), \ldots, y_j(t_{k+1}^{j-1})), \\
\kappa_{j,l}(\hat{m}_{k+1}^{j-1}, t_{k+1}^{j}) &= \kappa_{j,l}(\hat{m}_{k+1}^{j-1}, y_j(t_{k}^{j-1}), y_j(t_{k}^{j-1}+1), \ldots, y_j(t_{k+1}^{j-1})),
\end{align*}
\]
11.2.1 Prediction/Confidence Interval for Energy Commodities

Continuing this process in this manner, to find \( \left( y_j^f(\hat{m}_{k+l}^{-1}, t_i^{-1}), p_j^f(\hat{m}_{k+l}^{-1}, t_i^{-1}) \right) \), we use the estimates

\[
\begin{align*}
\delta_{j,l}(\hat{m}_{k+1}^{-1}, t_i^{-1}) &= \delta_{j,l}(\hat{m}_{k+1}^{-1}, y_j(t_i^{-1}), y_j(t_i^{-1}+1), \ldots, y_j(t_i^{-1}+3), \ldots, y_j(t_i^{-1}+k)), \\
\beta_{j,l}(\hat{m}_{k+1}^{-1}, t_i^{-1}) &= \beta_{j,l}(\hat{m}_{k+1}^{-1}, p_j(t_i^{-1}), p_j(t_i^{-1}+1), \ldots, p_j(t_i^{-1}+k)), \\
\gamma_{j,l}(\hat{m}_{k+1}^{-1}, t_i^{-1}) &= \gamma_{j,l}(\hat{m}_{k+1}^{-1}, p_j(t_i^{-1}), p_j(t_i^{-1}+1), \ldots, p_j(t_i^{-1}+k)), \\
\sigma_{j,l}(\hat{m}_{k+1}^{-1}, t_i^{-1}) &= \sigma_{j,l}(\hat{m}_{k+1}^{-1}, p_j(t_i^{-1}), p_j(t_i^{-1}+1), \ldots, p_j(t_i^{-1}+k)).
\end{align*}
\]

11.2.1 Prediction/Confidence Interval for Energy Commodities

In order to be able to assess the future certainty, we also discuss about the prediction/confidence interval. We define the 100(1 − α)% confidence interval for the forecast of the state

\( (y_j^f(\hat{m}_{i_1}^{-1}, t_i^{-1}), p_j^f(\hat{m}_{i_1}^{-1}, t_i^{-1}) ) \) at time \( t_i^{-1}, l \geq k, j \in I(1, n) \) as

\[
\begin{align*}
&y_j^f(\hat{m}_{i_1}^{-1}, t_i^{-1}) \pm z_{1-\alpha/2}, \\
p_j^f(\hat{m}_{i_1}^{-1}, t_i^{-1}) \pm z_{1-\alpha/2}.
\end{align*}
\]
where \((s^j_y(m_l^{-1}, t_l^{-1}))^{1/2}\) is the estimate for the sample standard deviation for the forecasted state \(y_j\), and \((s^j_y(m_l^{-1}, t_l^{-1}))^{1/2}\) is the estimate for the sample standard deviation for the forecasted state \(p_j\) derived from the following iterative process

\[
\begin{align*}
\gamma_j, & = \frac{\sum_{l \neq j} \gamma_{j,l} (m_l^{i-1}, t_l^{i-1})}{\sum_{l \neq j} \gamma_{j,l}} \\
\delta_j, & = \frac{\sum_{l \neq j} \delta_{j,l} (m_l^{i-1}, t_l^{i-1})}{\sum_{l \neq j} \delta_{j,l}} \\
\delta_j, & = \frac{\sum_{l \neq j} \delta_{j,l} (m_l^{i-1}, t_l^{i-1})}{\sum_{l \neq j} \delta_{j,l}} \\
\end{align*}
\]

(11.3)

It is clear that the 95% confidence interval for the forecast at time \(t_k\) is

\[
\begin{align*}
\left( \gamma_j, \delta_j, \delta_j, \delta_j \right)
\end{align*}
\]

where the lower ends denote the lower bounds of the state estimate and the upper ends denote the upper bounds of the state estimate.

### 11.2.2 Illustration: Prediction/Confidence Interval for Energy Commodities with no jump

For the case of no jump, the following graphs show the simulated, forecasted and 95 percent confidence limit for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively.
Figures 39: (a), (b), and (c) show the graph of the forecast and 95 percent confidence limit for the case where there is no jump for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively. Figures 39: (a), (b), and (c) show two region: the simulation region $S$ and the forecast region $F$. For the simulation region $S$, we plot the real data set together with the simulated data set as described in Figure 38. For the forecast region $F$, we plot the estimate of the forecast as explained in Section 11. The upper and the lower simulated sketches in Figure 39 (a), (b), and (c) are corresponding to the upper and lower ends of the 95% confidence interval.

### 11.2.3 Illustration: Prediction/Confidence Interval for Energy Commodities with Jump

For the case of jump process, the following graphs show the forecast and 95 percent confidence limit for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively.
Figures 40: (a), (b), (c) show the graph of the forecast and 95 percent confidence limit for the case where there is jump for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively. Figure 40: (a), (b), and (c) show two region: the simulation region $S$ and the forecast region $F$. For the simulation region $S$, we plot the real data set together with the simulated data set as described in Figure 38.
Chapter 12

Conclusion and Future Work

It is easily seen from Figures 31 and 38 that the model with jump performs better than the model without jump. This is because the curve fits better at the jump times for the jump case than the case without jump.

For $j, l \in I(1, 3)$, the estimates for the drift interaction parameters $\gamma_{j,l}(\hat{m}_k, t_k^{i-1})$ for the case where jump is incorporated suggest that there is definitely interactions between the spot price of these three commodities. As discussed in (9.46) and (9.47), the sign of these parameters suggest if there is competition or cooperation between commodity $l$ and $j$. The estimate of the parameter $\gamma_{j,l}(\hat{m}_k, t_k^{i-1})$ in Tables 17 and 24 and Figures 28 and 35 suggest that these commodities either compete or cooperate with each other depending on the time period. We can also describe the relationship between any two commodity $j$ and $l$, $j \neq l \in I(1, 3)$ based on the overall average $\bar{\gamma}_{j,l} = \frac{1}{N} \sum_{k=1}^{N} \gamma_{j,l}(\hat{m}_k, t_k^{i-1})$. For example, for the case where jump is incorporated, $\bar{\gamma}_{1,3} = -0.0017$ and $\bar{\gamma}_{3,1} = -0.0095$. This suggests that on the average, there is competition between these two commodities. Also, $\bar{\gamma}_{1,2} = 0.0018$ and $\bar{\gamma}_{2,1} = 0.0083$. This indicates that on the average, there is cooperation between natural gas and crude oil. Finally, $\bar{\gamma}_{2,3} = -0.0146$ and $\bar{\gamma}_{3,2} = -0.0013$. Therefore, on the average, there is competition between crude oil and coal.

In the future, we plan to apply the Local Lagged Adapted Generalized Method of Moments to interconnected nonlinear stochastic dynamic model for log-spot price, expected log-spot price and volatility process. Also, we plan to incorporate delay in the multivariate interconnected nonlinear stochastic model. We plan to be able to apply the extended Local Lagged Adapted Generalized Method of Moments to other multivariate interconnected nonlinear dynamic model different from energy commodity model.
Appendix A

A.1 Existence and Positivity of delayed Volatility in Chapter 4

**Lemma A.1** Suppose \( u(t) \) is \( \mathcal{F}_t \) square-integrable, adapted, non-anticipative process. We have

\[
\mathbb{E} \left[ \left( \int_{t_0}^\infty u(t) dW(t) \right)^4 \right] = 0. \tag{A.1}
\]

**Proof.** We start by showing that (A.1) holds for simple predictable process using Definition 1.4.2. The extension of the stochastic integral to square-integrable adapted process follows from [91].

We denote \( W(t_i) \) by \( W_{t_i} \). Using (1.7), we have

\[
\mathbb{E} \left[ \left( \int_{t_0}^\infty u(t) dW(t) \right)^4 \right] = \mathbb{E} \left[ \left( \sum_{i=1}^n F_i(W_{t_i} - W_{t_{i-1}}) \right)^4 \right]
\]

\[
= 3 \sum_{i=1}^n \mathbb{E} [F_i^4] \Delta t_i^2 + \sum_{i,j=1}^n \mathbb{E} [F_i^2] [F_j^2] \Delta t_i \Delta t_j
\]

\[
= 0
\]

\( \square \)

Denote by \( C = C([-\tau, 0], \mathbb{R}) \) the Banach space of all continuous functions \([-\tau, 0] \rightarrow \mathbb{R}\) under the supremum norm

\[
||\alpha||_C = \sup_{\theta \in [-\tau, 0]} |\alpha(\theta)|, \quad \alpha \in C. \tag{A.2}
\]

Denote \( L^2(C, \mathbb{R}) \) to be the Banach space of all measurable maps \([t_0, T] \times \Omega \rightarrow \mathbb{R}\) which are \( L^2 \) with norm

\[
||\beta||_{L^2(C, \mathbb{R})}^2 = \mathbb{E}_\Omega [||\beta||_C^2]. \tag{A.3}
\]

Define \( u(t, \psi_t) \equiv \sigma^2(t, \psi_t) \). The differential equation (4.10) reduces to
\[ du(t, \psi_t) = f(t, u(t, \psi_t), w)dt, \quad u(t_0, \psi) = u_0(\psi) > 0. \]  
\hspace{1cm} (A.4)\\

where

\[ f(t, \psi_t, w) = \alpha + \beta \left[ \int_{t-\tau}^{t} \sqrt{u(s, \psi_s)dW_2(s)} \right]^2 + cu(t, \psi_t). \]  
\hspace{1cm} (A.5)\\

**Theorem A.1** Using Lemma A.1, \( f(t, \psi_t, w) \) defined in (A.5) satisfies

\[
\begin{cases} 
\|f(t, u_1, w) - f(t, u_2, w)\|_{L^2([t_0, T] \times L^2(\Omega, \mathbb{R}) \times \Omega, \mathbb{R})} & \leq L\|u_1 - u_2\|_{L^2([t_0, T] \times C, \mathbb{R})} \\
\|f(t, u, w)\| & \leq K(1 + \|u\|_{L^2([t_0, T] \times C, \mathbb{R})}) 
\end{cases} \]  
\hspace{1cm} (A.6)\\

for all \( u_1, u_2 \in L^2([t_0, T] \times C, \mathbb{R}) \)

**Proof.** For any \( u_1, u_2 \in L^2([t_0, T] \times C, \mathbb{R}) \)

\[
\begin{align*}
\|f(t, u_1, w) - f(t, u_2, w)\|_{L^2}^2 &= \\
&= \mathbb{E} \left[ \beta \left( \int_{t-\tau}^{t} \sqrt{u_1(s, \psi_s)dW(s)} \right)^2 - \left( \int_{t-\tau}^{t} \sqrt{u_2(s, \psi_s)dW(s)} \right)^2 \right]^2 \\
&\quad + c(u_1(t, \psi_t) - u_2(t, \psi_t))^2 \\
&\leq 2\mathbb{E} \left[ \beta^2 \left( \int_{t-\tau}^{t} \sqrt{u_1(s, \psi_s)dW(s)} \right)^2 - \left( \int_{t-\tau}^{t} \sqrt{u_2(s, \psi_s)dW(s)} \right)^2 \right]^2 \\
&\quad + c^2 |u_1(t, \psi_t) - u_2(t, \psi_t)|^2 \\
&\leq 4\beta^2 \mathbb{E} \left[ \left( \int_{t-\tau}^{t} \sqrt{u_1(s, \psi_s)dW(s)} \right)^4 \right] + 4\beta^2 \mathbb{E} \left[ \left( \int_{t-\tau}^{t} \sqrt{u_2(s, \psi_s)dW(s)} \right)^4 \right] \\
&\quad + 2c^2 \mathbb{E} |u_1(t, \psi_t) - u_2(t, \psi_t)|^2 \\
&\leq L\|u_1(t, \psi_t) - u_2(t, \psi_t)\|_{L^2}^2, \quad \text{where } L = 2c^2.
\end{align*}
\]

Likewise,

\[
\|f(t, u, w)\|_{L^2}^2 = \mathbb{E} \left[ \alpha + \beta \left( \int_{t-\tau}^{t} \sqrt{u(s, \psi_s)dW(s)} \right)^2 + cu(t, \psi_t) \right]^2 \\
&\leq 2\beta^2 \mathbb{E} \left[ \left( \int_{t-\tau}^{t} \sqrt{u(s, \psi_s)dW(s)} \right)^4 \right] + 2\mathbb{E} |\alpha + cu(t, \psi_t)|^2 \\
&\leq 4\mathbb{E} \left[ |\alpha|^2 + c^2 |u(t, \psi_t)|^2 \right] \\
&\leq K [1 + \|u\|_{L^2}], \quad \text{where } K = 4 \max\{||\alpha||^2, |c|^2\}.
\]
Next, we shall show that the solution $u(t, \psi)$ of the IVP (A.4) satisfies Lipschitz condition whenever the initial condition $y(t_0, \psi) = y_0(\psi)$ in (A.4) satisfies the following assumption:

$H_2$:

a). $u_0(\psi)$ satisfies Lipschitz condition, that is, for every $\psi_1, \psi_2 \in C$, there exist a constant $M_1 > 0$, such that

$$\|u_0(\psi_1) - u_0(\psi_2)\| \leq M_1 \|\psi_1 - \psi_2\|,$$

and

b). $\inf_{\psi} u_0(\psi) = M_2 > 0$.

**Theorem A.2** Every solution $u(t, \psi_t)$ satisfying (A.4) with initial condition satisfying $H_2(a)$ satisfies Lipschitz condition.

Furthermore, under condition $H_2(b)$, then $\sqrt{u(t, \psi_t)}$ satisfies Lipschitz condition.

**Proof.** For any solution $u_1(t, \psi_1), u_2(t, \psi_2)$ satisfying (A.4) and assumptions $H_2$, with $\psi_i \equiv \psi_{it}$, $i = 1, 2$, we have

$$\|u_1(t, \psi_1) - u_2(t, \psi_2)\|_2^2 =$$

$$E \left| \int_{t_0}^{t} \beta \left( \left( \int_{s-\tau}^{s} \sqrt{u_1(r, \psi_1)} dW(r) \right)^2 - \left( \int_{s-\tau}^{s} \sqrt{u_2(r, \psi_2)} dW(r) \right)^2 \right) ds \right|^2$$

$$+ c \left| u_1(s, \psi_1) - u_2(s, \psi_2) \right|^2 ds$$

$$\leq 2E \left| \int_{t_0}^{t} \beta \left[ \left( \int_{s-\tau}^{s} \sqrt{u_1(r, \psi_1)} dW(r) \right)^2 - \left( \int_{s-\tau}^{s} \sqrt{u_2(r, \psi_2)} dW(r) \right)^2 \right] ds \right|^2$$

$$+ 2E \left| \int_{t_0}^{t} (u_1(s, \psi_1) - u_2(s, \psi_2)) ds \right|^2$$

$$\leq 2E \left| \int_{t_0}^{t} \beta \left( \left( \int_{s-\tau}^{s} \sqrt{u_1(r, \psi_1)} dW(r) \right)^2 - \left( \int_{s-\tau}^{s} \sqrt{u_2(r, \psi_2)} dW(r) \right)^2 \right) ds \right|^2$$

$$+ 2E \left| c \int_{t_0}^{t} (u_1(s, \psi_1) - u_2(s, \psi_2)) ds \right|^2$$

$$\leq 2E \left| \int_{t_0}^{t} \beta \left( \left( \int_{s-\tau}^{s} \sqrt{u_1(r, \psi_1)} dW(r) \right)^2 - \left( \int_{s-\tau}^{s} \sqrt{u_2(r, \psi_2)} dW(r) \right)^2 \right) ds \right|^2.$$
\[ +2\mathbb{E} \int_{t_0}^{t} |c(u_1(s, \psi_1) - u_2(s, \psi_2))|^2 \, ds \]
\[ \leq 2b^2 T \int_{t_0}^{t} \mathbb{E} \left[ \left( \int_{s-\tau}^{s} \sqrt{u_1(r, \psi_1)} \, dW(r) \right)^2 - \left( \int_{s-\tau}^{s} \sqrt{u_2(r, \psi_2)} \, dW(r) \right)^2 \right] \, ds \]
\[ +2c^2 T \int_{t_0}^{t} \mathbb{E} |(u_1(s, \psi_1) - u_2(s, \psi_2))|^2 \, ds, \text{ using Holder’s inequality,} \]
\[ \leq 2c^2 T \int_{t_0}^{t} \| (u_1(s, \psi_1) - u_2(s, \psi_2)) \|^2 \, ds, \text{ using Lemma A.1}. \]

By Gronwall’s inequality, [70, 66], we have
\[ \| u_1(t, \psi_1) - u_2(t, \psi_2) \| \leq \| u_1(t_0, \psi_1) - u_2(t_0, \psi_2) \| e^{c^2 T^2} \]
\[ \leq M_3 \| u_0(\psi_1) - u_0(\psi_2) \|, \quad M_3 = e^{c^2 T^2}, \]
\[ \leq M \| \psi_1 - \psi_2 \|_0, \text{ where assumption } H_2(a) \text{ is used, and } M = M_1 M_3. \] (A.7)

Furthermore, using assumption \( H_2 \), there exist a positive constant \( M_4 \) such that \( M_4 \leq \| \sqrt{u_1(t, \psi_1)} + \sqrt{u_2(t, \psi_2)} \| \). Substituting this into equation (A.7), we have
\[ \| \sqrt{u_1(t, \psi_1) - u_2(t, \psi_2)} \|_{L^2([t_0, T] \times C, \mathbb{R})} \leq N \| \psi_1 - \psi_2 \|_0, \text{ where } N = M/M_4. \] (A.8)

In the next theorem, we show conditions for positivity of the solution \( u(t, \psi_t) \) of (A.4).

**Theorem A.3** Differential equation in (A.4) has a positive solution if \( \alpha > 0, \beta > 0 \).

**Proof.** Using the transformation
\[ z(t, \psi_t) = e^{-c(t-t_0)} u(t, \psi_t), \]
equation (A.4) reduces to
\[ dz = \left[ \alpha e^{-c(t-t_0)} + \beta \left( \int_{t-\tau}^{t} \sqrt{z(s, \psi_s)} \exp \left[ -\frac{c}{2} (t-s) \right] \, dW(s) \right)^2 \right] \, dt. \]

Hence, by hypothesis, \( z(t, \psi_t) \) is an increasing function of \( t \). Since \( u_0(t_0, \psi_t) > 0 \), then \( z(t, \psi_t) \) is positive. \( \square \)
Appendix B

B.1 Algorithm for Simulation

**Algorithm 1** Estimating parameters

Given initial parameters and initial predictions $\hat{x}(t_1|t_0)$ and $P(t_1|t_0)$,

for $k = 1$ to $n$

for $j = 0$ to 2

for $m = 1$ to 6

Compute $\hat{y}(t_k|t_{k-1})$ and $r_{j,m}(t_k|t_{k-1})$

Compute $\hat{x}(t_k|t_k)$ and $P(t_k|t_k)$ using (5.17),

Compute $\hat{x}(t_{k+1}|t_k)$ and $P(t_{k+1}|t_k)$ using (5.26),

Compute $e_k$ using (5.27),

end for

end for

end for

Return $e_k$.

Compute $L(\Theta)$ using (5.28),

$\hat{\Theta} = \arg\min L(\Theta)$

Return $L$. 

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### B.2 Expressions in Lemma 5.1

\[
D(t_k | t_{k-1})_{\mathbf{h}, \mathbf{h}^T} = \frac{\sigma_4 - \sigma_2}{2h^3} \sum_{p=1}^{n} e_p \mu_p \delta_p \mathbf{h} \delta_p^2 \mathbf{h}^T + e_p \delta_p^2 \mathbf{h} \mu_p \delta_p \mathbf{h}^T
\]

\[
+ \frac{\sigma_2}{2h^3} \sum_{p,q=1 \atop p \neq q}^{n} e_p \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T + e_p \mu_q \delta_q \mathbf{h} \mu_p \mu_q \delta_q \mathbf{h}^T
\]

\[
+ \frac{\sigma_2}{4h^4} \sum_{p,q=1 \atop p \neq q}^{n} e_q \delta_q^2 \mathbf{h} \mu_p \delta_p \mathbf{h} + e_p \mu_p \delta_p \mathbf{h} \delta_q \mathbf{h}^T
\]

\[
E(t_k | t_{k-1}) = \frac{\sigma_4 - \sigma_2}{2h^3} \sum_{p=1}^{n} e_p C^T \mu_p \delta_p \mathbf{h} \delta_p^2 \mathbf{h}^T + e_p C^T \delta_p^2 \mathbf{h} \mu_p \delta_p \mathbf{h}^T
\]

\[
+ \frac{\sigma_2}{2h^3} \sum_{p,q=1 \atop p \neq q}^{n} e_q C^T \mu_p \delta_p \mathbf{h} \mu_p \delta_q \mathbf{h}^T + e_p C^T \mu_q \delta_q \mathbf{h} \mu_p \mu_q \delta_q \mathbf{h}^T
\]

\[
+ \frac{\sigma_2}{4h^4} \sum_{p,q=1 \atop p \neq q}^{n} e_q C^T \delta_q^2 \mathbf{h} \mu_p \delta_p \mathbf{h} + e_p C^T \mu_p \delta_p \mathbf{h} \delta_q \mathbf{h}^T
\]

\[
J(t_k | t_{k-1}) = \frac{\sigma_4}{h^3} \sum_{p=1}^{n} e_p \mu_p \delta_p \mathbf{h}^T \mu_p \delta_p \mathbf{h} \delta_p \mathbf{h}^T
\]

\[
+ \frac{\sigma_2}{h^3} \sum_{p,q=1 \atop p \neq q}^{n} e_q \delta_q \mathbf{h}^T \delta_q \mu_p \delta_p \mathbf{h} + e_q \mu_p \delta_q \mathbf{h} \mu_p \mu_q \delta_q \mathbf{h}^T
\]

\[
+ \frac{\sigma_2}{4h^4} \sum_{p,q,r=1 \atop p \neq q \neq r}^{n} e_r \mu_p \delta_p \mathbf{h} \delta_q \mu_p \delta_p \mathbf{h} + e_r \mu_p \mu_p \delta_p \mathbf{h} \delta_q \mu_p \delta_p \mathbf{h}^T
\]

\[
+ e_q \mu_p \delta_p \mathbf{h} \delta_q \mu_p \mu_q \delta_q \mathbf{h}^T
\]

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\[ + \frac{\sigma_2 \sigma_4}{4h^5} \sum_{p,q \neq r}^{n} \left[ e_{q \mu \rho} \delta_{p}^{T} h_{p}^{T} + e_{p \mu \rho} \delta_{q}^{T} h_{q}^{T} + e_{p \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{p \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \rho} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\
+ \frac{\sigma^3}{4h^5} \sum_{p,q,r=1}^{n} \left[ e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} + e_{r \mu \delta} \delta_{q}^{T} h_{q}^{T} \right] \\]
\[ L_{i,j} = \frac{\sigma_4 - \sigma_2^2}{2h^4} \sum_{p=1}^{n} \left[ \mu_p \delta_p h_i \mu_p \delta_p h_j \delta_p^2 h_T + \mu_p \delta_p h_i \mu_p \delta_p h_j \delta_p^2 h_T + \delta_p^2 h_i \mu_p \delta_p h_j \mu_p \delta_p h_T \right] \\
+ \frac{\sigma_2^2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_j \mu_p \delta_p h_T + \frac{\sigma_2^2}{2h^2} \sum_{p=1}^{n} \delta_p^2 h_T \delta_{i,j} R_{i,j} \\
+ \frac{\sigma_2^2}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p h_j \mu_q \delta_q h_i \mu_p \delta_p h_T + \mu_q \delta_q h_i \mu_p \delta_p h_j \delta_q^2 h_T \\
+ \frac{\sigma_2^2}{2h^2} \sum_{p=1}^{n} \delta_p^2 h_i e_j R_{j,j} + \frac{\sigma_2^2}{2h^2} \sum_{p,q=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_j \delta_q^2 h_T \\
+ \frac{\sigma_4 \sigma_2 - \sigma_2^3}{8h^6} \sum_{p,q,r=1}^{n} \mu_p \delta_p h_i \mu_p \delta_p h_j \delta_p^2 h_T + \frac{\sigma_4 \sigma_2}{8h^6} \sum_{p,q=1}^{n} \delta_q^2 h_i \mu_p \delta_p h_j \mu_q \delta_q h_T \\
+ \frac{\sigma_4 \sigma_2 - \sigma_2^3}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_j \mu_p \delta_p h_T + \delta_q^2 h_i \mu_p \delta_p h_j \mu_q \delta_q h_T \\
+ \frac{\sigma_4 \sigma_2 - \sigma_2^3}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_j \mu_p \delta_p h_T + \frac{\sigma_2^2}{2h^2} \sum_{p=1}^{n} \delta_p^2 h_j e_i R_{j,i} \\
+ \frac{\sigma_4 \sigma_2 - \sigma_2^3}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_j \mu_p \delta_p h_T \\
+ \frac{\sigma_4 \sigma_2 - \sigma_2^3}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_j \mu_p \delta_p h_T \\
+ \frac{\sigma_2^2}{8h^6} \sum_{p,q,r=1}^{n} \mu_p \delta_p h_i \mu_q \delta_q h_r \delta_r^2 h_T + \mu_p \delta_p h_i \mu_r \delta_r \delta_p h_j \delta_p^2 h_T \\
+ \mu_p \delta_p h_i \mu_r \delta_r \delta_q h_j \delta_q^2 h_T + \frac{\sigma_2^2}{4h^2} \sum_{p=1}^{n} \delta_p^2 h_i e_j^T R_{j,j} + \delta_p^2 h_j e_i^T R_{i,i} \\
\]
\[
\begin{align*}
&+ \frac{\sigma^2_2}{8h^6} \sum_{p, q, r = 1}^{n} \sum_{p \neq q \neq r} \left( \mu_p \delta_{p \mu q} \delta_p \tilde{h}_i \mu_p \delta_{p \mu r} \delta_r \tilde{h}_j \mu_q \delta_q \mu_r \delta_r \tilde{h}^T \right) + \mu_p \delta_{p \mu q} \delta_p \tilde{h}_i \mu_p \delta_{p \mu r} \delta_r \tilde{h}_j \mu_q \delta_q \mu_r \delta_r \tilde{h}^T \\
&+ \frac{\sigma^2_2}{8h^6} \sum_{p, q = 1}^{n} \sum_{p \neq q} \left[ \delta^2_p \tilde{h}_i \mu_p \delta_{p \mu q} \delta_q \mu_q \delta_p \tilde{h}^T \right] + \mu_p \delta_{p \mu q} \delta_p \tilde{h}_i \mu_q \delta_q \mu_q \delta_p \tilde{h}^T \\
&+ \sigma^2_4 \sum_{p, q, r = 1}^{n} \sum_{p \neq q \neq r} \left( \delta^2_p \tilde{h}_i \mu_p \delta_{p \mu q} \delta_q \mu_q \delta_p \tilde{h}^T \right) + \mu_p \delta_{p \mu q} \delta_p \tilde{h}_i \mu_q \delta_q \mu_q \delta_p \tilde{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \tilde{h}_i \mu_q \delta_q \mu_q \delta_p \tilde{h}^T \\
\end{align*}
\]

\[
\mathbb{E} \left[ AA^T AA^T | Y_{t-1} \right] = \frac{1}{4h^6} \left( \sum_{p = 1}^{n} \sigma^2_6 \delta^2_p \tilde{h}_i \mu_p \delta_p \tilde{h}^T + \sum_{p, q = 1}^{n} \sum_{p \neq q} \sigma^2_2 \mu_p \delta_{p \mu q} \delta_p \tilde{h}_i \mu_p \delta_{p \mu q} \delta_q \mu_q \delta_p \tilde{h}^T \\
&+ \sum_{p, q = 1}^{n} \sum_{p \neq q} \sigma^2_4 \left( \delta^2_p \tilde{h}_i \mu_p \delta_q \tilde{h}^T + \delta^2_q \tilde{h}_i \mu_q \delta_q \tilde{h}^T + \mu_p \delta_{p \mu q} \delta_q \tilde{h}^T \right) \\
&+ \sum_{p, q = 1}^{n} \sum_{p \neq q} \sigma^2_4 \left( \delta^2_p \tilde{h}_i \mu_q \delta_q \tilde{h}^T + \delta^2_q \til{h}_i \mu_q \delta_q \til{h}^T \right) \right) + \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T \\
&+ \mu_p \delta_{p \mu q} \delta_p \til{h}_i \mu_q \delta_q \mu_q \delta_p \til{h}^T
\end{align*}
\]
\[
+ \sum_{p,q=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{p} \delta_{p} \mu_{q} \delta_{q} \tilde{h}_{p}^{2} \mu_{p} \delta_{p} \tilde{h}_{q}^{T} + \mu_{p} \delta_{p} \mu_{q} \delta_{q} \tilde{h}_{p}^{2} \mu_{q} \delta_{q} \tilde{h}_{q}^{T} + \mu_{p} \delta_{p} \mu_{q} \delta_{q} \tilde{h}_{p}^{2} \mu_{p} \delta_{p} \tilde{h}_{q}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{q} \delta_{q} \mu_{p} \delta_{p} \tilde{h}^{T} + \mu_{p} \delta_{p} \mu_{q} \delta_{q} \tilde{h}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2} \left[ \delta_{p}^{2} \mu_{p} \delta_{p} \mu_{q} \delta_{q}^{T} + \mu_{p} \delta_{p} \mu_{q} \delta_{q} \mu_{p} \delta_{p} \tilde{h}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2} \left[ \mu_{q} \delta_{q} \mu_{p} \delta_{p} \tilde{h}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2} \left[ \delta_{p}^{2} \mu_{p} \delta_{p} \mu_{q} \delta_{q}^{T} \right] \\
+ \sum_{p=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{q} \delta_{q} \tilde{h}_{q}^{2} \mu_{p} \delta_{p} \tilde{h}_{q}^{T} + \mu_{p} \delta_{p} \mu_{q} \delta_{q} \tilde{h}_{p}^{2} \mu_{q} \delta_{q} \tilde{h}_{q}^{T} + \mu_{p} \delta_{p} \mu_{q} \delta_{q} \tilde{h}_{p}^{2} \mu_{p} \delta_{p} \tilde{h}_{q}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{q} \delta_{q} \tilde{h}_{q}^{2} \delta_{p}^{2} \mu_{p} \delta_{p} \tilde{h}_{q}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{q} \delta_{q} \tilde{h}_{q}^{2} \delta_{p}^{2} \tilde{h}_{q}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{q} \delta_{q} \tilde{h}_{q}^{2} \delta_{p}^{2} \mu_{p} \delta_{p} \tilde{h}_{q}^{T} \right] \\
+ \sum_{p,q=1}^{n} \sigma_{2\sigma_{4}} \left[ \mu_{q} \delta_{q} \tilde{h}_{q}^{2} \delta_{p}^{2} \tilde{h}_{q}^{T} \right]
\]
\[
\begin{align*}
&+ \sum_{p,q=1 \atop p \neq q}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mu_q \delta_q \mathbf{h}^T \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_p \delta_p \mu_q \delta_q \mathbf{h}^T \delta_p^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2^3 \left[ \mu_q \delta_q \mathbf{h} \delta_q^2 \mathbf{h}^T \delta_r^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \mu_r \delta_r \mathbf{h} \mu_q \delta_q \mu_r \delta_r \mathbf{h}^T \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T \right] \\
&+ \sum_{p,q=1 \atop p \neq q}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \mu_q \delta_q \mu_p \delta_p \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_q \delta_q \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \delta_p^2 \mathbf{h}^T \mu_p \delta_p \mu_q \delta_q \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_q \delta_q \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \delta_p^2 \mathbf{h}^T \mu_r \delta_r \mathbf{h} \mu_p \delta_p \mathbf{h}^T + \mu_r \delta_r \mathbf{h} \mu_p \delta_p \mu_q \delta_q \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q=1 \atop p \neq q}^n \sigma_2 \sigma_4 \left[ \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_r \delta_r \mathbf{h} \mu_r \delta_r \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \delta_q^2 \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_r \delta_r \mathbf{h} \mu_r \delta_r \mathbf{h}^T \right] \\
&+ \sum_{p,q=1 \atop p \neq q}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q,r=1 \atop p \neq q \neq r}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T \right] \\
&+ \sum_{p,q=1 \atop p \neq q}^n \sigma_2 \sigma_4 \left[ \mu_p \delta_p \mathbf{h} \mu_q \delta_q \mathbf{h}^T + \mu_q \delta_q \mathbf{h} \mu_q \delta_q \mathbf{h}^T \mu_p \delta_p \mathbf{h} \mu_p \delta_p \mathbf{h}^T \right]
\end{align*}
\]
\[
\sum_{p,q,r=1}^{n} \sigma_2^3 \left[ \delta_p^2 \delta_q \delta_r \mu_p \delta_q \delta_r \delta_q + \mu_q \delta_q \mu_p \delta_q \hat{h} \mu_p \delta_q \hat{h}^T \right] \\
+ \sum_{p,q=1}^{n} \sigma_2 \sigma_4 \left[ \mu_q \delta_q \mu_p \delta_p \hat{h} \mu_p \delta_p \mu_q \delta_q \hat{h}^T \right] \\
+ \sum_{p,q=1}^{n} \sigma_2 \sigma_4 \left[ \delta_p^2 \delta_q (t_k, \hat{z}(t_k)) \mu_p \delta_p \hat{h} \mu_q \delta_q \delta_q \hat{h}^T + \delta_p^2 \delta_q \mu_q \delta_q \hat{h} \mu_q \delta_q \mu_q \delta_q \hat{h}^T \right] \\
+ \sum_{p,q=1}^{n} \sigma_2 \sigma_4 \left[ \delta_p^2 \delta_q \delta_q \delta_q \hat{h} \mu_q \delta_q \mu_q \delta_q \hat{h}^T \right] \\
+ \sum_{p,q=1}^{n} \sigma_2 \sigma_4 \left[ \mu_q \delta_q \delta_q \delta_q \delta_q \hat{h} \mu_q \delta_q \delta_q \delta_q \hat{h}^T \right]
\]
\[ \sum_{p.q,r=1}^{n} \sigma_2^3 \left[ \mu_q \delta_q \tilde{h}_p^2 \tilde{h}_p^T + \mu_{r} \delta_r \tilde{h}_{q} \delta_{q}^2 \right] + \mu_{q} \delta_{q} \tilde{h}_{p} \mu_{r} \delta_{r} \tilde{h}_{q} \delta_{q}^2 \tilde{h}_{p}^T + \mu_{q} \delta_{q} \tilde{h}_{p} \mu_{r} \delta_{r} \tilde{h}_{q} \delta_{q}^2 \tilde{h}_{p}^T \]
\[ + \sum_{p,q,r=1 \atop p \neq q \neq r}^{n} \sigma_{p}^{2} \left[ \mu_{q} \delta_{p} \bar{h} \mu_{q} \delta_{p} \bar{h}^{T} \delta_{p}^{2} \delta_{p}^{2} \bar{h} + \mu_{r} \delta_{p} \bar{h} \mu_{q} \delta_{r} \bar{h}^{T} \mu_{q} \delta_{p} \mu_{r} \delta_{p} \bar{h} \delta_{p}^{2} \bar{h} \right] \\
+ \sum_{p,q=1 \atop p \neq q}^{n} \sigma_{2} \sigma_{4} \left[ \mu_{q} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{p} \mu_{p} \mu_{q} \delta_{q} \bar{h} \\
+ \mu_{q} \delta_{q} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{p} \mu_{q} \delta_{q} \bar{h} + \mu_{q} \delta_{q} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{p} \mu_{q} \delta_{q} \bar{h} \delta_{p}^{2} \bar{h} \right] \\
+ \sum_{p,q=1 \atop p \neq q}^{n} \sigma_{2} \sigma_{4} \left[ \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{q} \delta_{q} \bar{h} \\
+ \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{q} \delta_{q} \bar{h} + \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{q} \delta_{q} \bar{h} \delta_{p}^{2} \bar{h} \right] \\
+ \sum_{p,q=1 \atop p \neq q}^{n} \sigma_{2} \sigma_{4} \left[ \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{q} \delta_{q} \bar{h} \\
+ \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{q} \delta_{q} \bar{h} + \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \mu_{q} \delta_{q} \mu_{q} \delta_{q} \bar{h} \delta_{p}^{2} \bar{h} \right] \\
+ \frac{\sigma_{2} R}{h^{2}} \sum_{p=1}^{n} \left[ 2 \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} + \mu_{p} \delta_{p} \bar{h}^{T} \mu_{p} \delta_{p} \bar{h} \right] \\
+ \frac{2 R}{4 h^{4}} \left[ \sum_{p=1}^{n} \sigma_{1} \delta_{p}^{2} \delta_{p}^{2} \bar{h}^{T} + \sigma_{2}^{2} \sum_{p,q=1 \atop p \neq q}^{n} \delta_{p}^{2} \delta_{q}^{2} \bar{h}^{T} \right] \\
+ \frac{\sigma_{2}^{2}}{h^{2}} \sum_{p=1}^{n} \mu_{p} \delta_{p} \bar{h} \mu_{p} \delta_{p} \bar{h}^{T} \left[ \sum_{i=1}^{n} \left( I_{n,n} R_{i,i} \right) + 2 R \right] \\
+ \frac{R}{4 h^{4}} \left[ \sum_{p=1}^{n} \sigma_{1} \delta_{p}^{2} \bar{h}^{T} \delta_{p}^{2} \bar{h} + \sigma_{2}^{2} \sum_{p,q=1 \atop p \neq q}^{n} \delta_{p}^{2} \bar{h} \delta_{q}^{2} \bar{h} \delta_{p} \delta_{q}^{2} \bar{h} \right] \\
+ \frac{2}{4 h^{4}} \left[ \sum_{p=1}^{n} \sigma_{1} \delta_{p}^{2} \delta_{p}^{2} \bar{h} \delta_{p}^{2} \bar{h}^{T} + \sigma_{2}^{2} \sum_{p,q=1 \atop p \neq q}^{n} \delta_{p}^{2} \delta_{q}^{2} \bar{h} \delta_{p} \delta_{q}^{2} \bar{h} \delta_{p} \delta_{q}^{2} \bar{h} \right] R \\
+ 3 RR^{T} \]
\[
\mathbb{E} \left[ AA^T AC^T | Y_{t-1} \right] =
\]
\[
S_x \left( \frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p \delta_q \mu_p \delta_q \delta_p h^T \delta_q h \right) + \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \delta_q \mu_p \delta_q \delta_p h^T \delta_q h 
\]
\[
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \delta_q \mu_q \delta_p \delta_q \delta_p h^T \delta_q h \right) + \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \delta_q \mu_q \delta_p \delta_q \delta_p h^T \delta_q h 
\]
\[
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \delta_q \mu_q \delta_q \delta_q \delta_p h^T \delta_q h \right) + \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \delta_q \mu_q \delta_q \delta_q \delta_p h^T \delta_q h 
\]
\[
\begin{align*}
\frac{\sigma^2}{8h^6} & \sum_{p,q,r=1}^{n} \mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h} + \frac{\sigma^2}{8h^6} \sum_{p,q,r=1}^{n} \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \\
&+ \sigma_4 \sigma_2 \frac{\mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h}}{p \neq q}
\end{align*}
\]

\[
\begin{align*}
&+ \frac{\sigma^2}{8h^6} \sum_{p,q,r=1}^{n} \left( \mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h} \right) + \\
&+ \frac{\sigma^2}{8h^6} \sum_{p,q,r=1}^{n} \left( \mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \delta_r \frac{\delta^2 h^T}{\delta p^2 h} \right)
\end{align*}
\]

\[
\begin{align*}
&+ \frac{\sigma^2}{4h^2} \sum_{p=1}^{n} \left( \delta^2 h_{i,j} R_{i,j} + \sigma_2 \delta^2 h_{i,j} R_{i,j} \right) \right) C^T + \frac{\sigma^2}{2h^2} \sum_{p=1}^{n} \delta^2 h_{i,j} R_{i,j}
\end{align*}
\]

\[
\textbf{E} \left[ AA^T C A^T | Y_{k-1} \right] =
\]

\[
S_x \left( \begin{array}{c}
\frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p \mu_q \delta_q C \frac{\delta^2 h^T}{\delta p^2 h} + \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q C \frac{\delta^2 h^T}{\delta p^2 h} \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q C \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \frac{\sigma_4}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q C \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \left( \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} \right) \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \left[ \delta^2 h_{i,j} R_{i,j} C \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} \right] \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \left[ \delta^2 h_{i,j} R_{i,j} C \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} \right] \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \left[ \delta^2 h_{i,j} R_{i,j} C \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} \right] \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \left[ \delta^2 h_{i,j} R_{i,j} C \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} + \mu_p \delta_p \mu_q \delta_q \mu_p \delta_p \mu_q \delta_q \frac{\delta^2 h^T}{\delta p^2 h} \right]
\end{array} \right)
\]

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\[
\begin{align*}
&+ \frac{\sigma_A \sigma^2}{8h^6} \sum_{p,q,r=1 \atop p \neq q \neq r}^n \delta_p^2 \delta_q^2 \delta_r^2 T C_{\delta_T^2} T + \frac{\sigma_A \sigma^2}{8h^6} \sum_{p,q=1 \atop p \neq q}^n \delta_p^2 \mu_p \delta_p \mu_q \delta_q T C_{\delta_T} T \\
&+ \frac{\sigma_A \sigma^2}{8h^6} \sum_{p,q,r=1 \atop p \neq q}^n \left[ \delta_p^2 \mu_p \delta_p \mu_q \delta_q T C_{\mu_p \delta_q T} + \mu_p \delta_p \mu_q \delta_q \delta_q T C_{\mu_p \delta_q T} \right] \\
&+ \frac{\sigma_A \sigma^2}{8h^6} \sum_{p,q,r=1 \atop p \neq q}^n \left[ \delta_p^2 \mu_p \delta_p \mu_q \delta_q T C_{\mu_p \delta_q T} + \mu_p \delta_p \mu_q \delta_q \delta_q T C_{\mu_p \delta_q T} \right] \\
&+ \frac{\sigma_A \sigma^2}{8h^6} \sum_{p,q,r=1 \atop p \neq q}^n \left( \mu_p \delta_p \mu_q \delta_q T C_{\mu_p \delta_q T} + \mu_p \delta_p \mu_q \delta_q \delta_q T C_{\mu_p \delta_q T} \right) \\
&+ \frac{\sigma_A \sigma^2}{8h^6} \sum_{p,q,r=1 \atop p \neq q}^n \left( \mu_p \delta_p \mu_q \delta_q T C_{\mu_p \delta_q T} + \mu_p \delta_p \mu_q \delta_q \delta_q T C_{\mu_p \delta_q T} \right) \\
&+ \frac{\sigma_A \sigma^2}{4h^2} \sum_{p=1}^n \left( \delta_p^2 h \sum_{i=1}^n R_{i,i} + 2 \delta_p^2 R h \right)
\end{align*}
\]

\[
\begin{align*}
E \left[ AC^T A^T | Y_{t_{k-1}} \right] &= E \left[ AA^T C A^T | Y_{t_{k-1}} \right] \\
E \left[ AC^T AC^T | Y_{t_{k-1}} \right] &= E \left[ AC^T C A^T | Y_{t_{k-1}} \right] \\
E \left[ AC^T C A^T | Y_{t_{k-1}} \right] &= \sum_{p=1}^n \frac{\sigma^2}{h^2} \mu_p \delta_p \hat{h}(\ddot{z}) C_{\mu_p \delta_p \hat{h}} T + \frac{\sigma_A \sigma^2}{4h^4} \delta_p^2 C_{\delta_p^2 \hat{h}} T + \delta_p^2 C_{\delta_p^2 \hat{h}} T + RC^T C
\end{align*}
\]
\[
\mathbb{E}[A^T C C^T | Y_{k-1}] = \left[ \sum_{p=1}^{n} \frac{\sigma_2}{h^2} \mu_p \delta_p \hat{h}(\hat{z}) \mu_p \delta_p \hat{h}^T + \frac{\sigma_4}{4h^4} \left( \delta_q \hat{h}(\hat{z}) \right) \left( \delta_q \hat{h}(\hat{z}) \right)^T \right] C C^T \\
+ \left[ \frac{\sigma_2}{4h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} \mu_p \delta_p \mu_q \delta_q \hat{h} + \delta_q \hat{h} \delta_q \hat{h} + R \right] C C^T
\]

\[
\mathbb{E}[A C^T C C^T | Y_{k-1}] = C C^T C C^T
\]

\[
\mathbb{E}[A C^T A A^T | Y_{k-1}] = S_x \left( \frac{\sigma_4}{2h^3} \sum_{p=1}^{n} \mu_p \delta_p \hat{h} \mu_p \delta_p \hat{h}^T \right) \\
+ \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \hat{h} C^T \mu_p \delta_p \hat{h} \delta_q h^T + \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \hat{h} C^T \mu_q \delta_q \hat{h} \mu_p \delta_p \mu_q \delta_q \hat{h}^T \\
+ \frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T + \frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p \hat{h} C^T \mu_q \delta_q \mu_p \delta_p \hat{h}^T \\
+ \frac{\sigma_4}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T + \frac{\sigma_2}{2h^4} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_q \delta_q \mu_p \delta_p \hat{h}^T \\
+ \frac{\sigma_4\sigma_2}{8h^6} \sum_{p=1}^{n} (\delta_p \hat{h} C^T \delta_p \hat{h} \delta_q h^T) + \frac{\sigma_4\sigma_2}{8h^6} \sum_{p=1}^{n} \delta_p \hat{h} C^T \delta_p \hat{h} \delta_q \hat{h}^T \\
+ \frac{\sigma_4\sigma_2}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T \\
+ \frac{\sigma_4\sigma_2}{8h^6} \sum_{p,q,r=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T \\
+ \frac{\sigma_4\sigma_2}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T \\
+ \frac{\sigma_3}{8h^6} \sum_{p,q,r=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T \\
+ \frac{\sigma_2}{8h^6} \sum_{p,q=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T + \frac{\sigma_3}{8h^6} \sum_{p,q,r=1}^{n} \mu_p \delta_p \mu_q \delta_q \hat{h} C^T \mu_p \delta_p \mu_q \delta_q \hat{h}^T
\]
B.3 Proof of Lemma 5.1

Proof.

\[ r_{0,2}(t_k|t_{k-1})_{\hat{h},\hat{h}} = \mathbb{E} \left[ (y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right] \]
\[ = \mathbb{E} \left[ (\tilde{D}_{\Delta_2} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta_2}^2 \hat{h} + v - C) \times \right. \]
\[ \left. (\tilde{D}_{\Delta_2} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta_2}^2 \hat{h} + v - C)^T | Y_{k-1} \right] \]
\[ = \mathbb{E} \left[ (\tilde{D}_{\Delta_2} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta_2}^2 \hat{h} + v) \left( \tilde{D}_{\Delta_2} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta_2}^2 \hat{h} + v \right)^T | Y_{k-1} \right] \]
\[ - \mathbb{E} \left( \tilde{D}_{\Delta_2} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta_2}^2 \hat{h} + v \right) C^T - C \mathbb{E} \left( \tilde{D}_{\Delta_2} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta_2}^2 \hat{h} + v \right)^T \]
\[ + CC^T \]
\[
\begin{align*}
r_{1,1}(t_k|t_{k-1}) &= \mathbb{E} \left[ (x(t_k) - \hat{x}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right] \\
&= \mathbb{E} \left[ \Delta x(t_k)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{k-1} \right] \\
&= \mathbb{E} \left[ S_x \Delta z(t_k) \left( \tilde{D}_{\Delta z} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta z}^2 \hat{h} + v - C \right) \right]
\end{align*}
\]

\[
\begin{align*}
r_{1,2}(t_k|t_{k-1}) &= \mathbb{E} \left[ S_x \Delta z(t_k)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T \left( \tilde{V}(t_k) - \hat{V}(t_k|t_{k-1}) \right)^T \right] \\
&= \mathbb{E} \left[ S_x \Delta z(t_k) \left\{ (y_1(t_k) - \hat{y}_1(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T, \ldots, \\
(y_n(t_k) - \hat{y}_n(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T \right\} \right] \\
&+ \tilde{x}(t_k|t_{k-1}) \mathbb{E} \left[ (y_1(t_k) - \hat{y}_1(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T, \ldots, \\
(y_n(t_k) - \hat{y}_n(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T \right]
\end{align*}
\]

\[
\begin{align*}
r_{0,3}(t_k|t_{k-1}) &= \left( \mathbb{E} \left( \Delta y_i(t_k) \Delta y_j(t_k) \Delta y(t_k)^T \right) \right)_{1 \leq i \leq n, 1 \leq j \leq n}
\end{align*}
\]

\[
\begin{align*}
r_{0,4}(t_k|t_{k-1}) &= \mathbb{E} \left[ \left( \tilde{D}_{\Delta z} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta z}^2 \hat{h} + v - C \right) \left( \tilde{D}_{\Delta z} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta z}^2 \hat{h} + v - C \right)^T \right] \\
&= \mathbb{E} \left[ AA^T AA^T | Y_{k-1} \right] - \mathbb{E} \left[ AA^T AC^T | Y_{k-1} \right] - \mathbb{E} \left[ AC^T CA^T | Y_{k-1} \right] \\
&+ \mathbb{E} \left[ AA^T CC^T \right] - \mathbb{E} \left[ AC^T AA^T \right] + \mathbb{E} \left[ AC^T AC^T \right] \\
&+ \mathbb{E} \left[ AC^T CA^T \right] - \mathbb{E} \left[ AC^T CC^T \right]
\end{align*}
\]

\[
\begin{align*}
M_{0,2}(t_k|t_{k-1}) &= \left( \mathbb{E} \left( \Delta y(t_k) \Delta y_i(t_k) \Delta y_j(t_k) \Delta y(t_k)^T \right) \right)_{1 \leq i \leq n, 1 \leq j \leq n}
\end{align*}
\]

\[
\begin{align*}
r_{2,2}(t_k|t_{k-1}) &= \mathbb{E} \left[ \Delta x(t_k) \Delta x(t_k)^T \Delta y(t_k) \Delta y(t_k)^T \right] \\
&= \mathbb{E} \left[ S_x \sum_{k=1}^{n} [\Delta z_i \Delta z_k S_{j,k}] \left( \tilde{D}_{\Delta z} \hat{h} + \frac{1}{2} \tilde{D}_{\Delta z}^2 \hat{h} + v - C \right) \right] \\
&= S_x Q_{i,j}, \quad \text{where } Q_{i,j} \text{ is defined below}
\end{align*}
\]

\[
\begin{align*}
r_{1,3}(t_k|t_{k-1}) &= \mathbb{E} \left[ \Delta x(t_k) \Delta y(t_k)^T \Delta y(t_k) \Delta y(t_k)^T \right] \\
|Q_{i,j}| &= \frac{1}{4h^4} \left[ \sigma_0 S_{j,i} \delta_i^2 \delta_j^2 h^2 + \sum_{p=1}^{n} \sigma_2 \sigma_4 S_{j,i} \delta_p^2 \delta_j^2 h^2 \right]
\end{align*}
\]

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\[
+ \sum_{p=1}^{n} \sigma_{2\sigma} S_{j,i} \left[ \delta^2_t \delta^2_p \tilde{h}^T + \delta^2_p \delta^2_t \tilde{h}^T \right] + \sum_{p,q=1 \atop p \neq q}^{n} \sigma_{2\sigma}^3 S_{j,i} \delta^2_p \delta^2_q \tilde{h}^T
\]
\[
+ \frac{\sigma_{2\sigma}}{4h^4} \sum_{p=1}^{n} S_{j,p} \left[ \delta^2_t \mu_i \delta_p \mu_p \delta^2_t \tilde{h}^T + \delta^2_p \delta^2_t \mu_i \delta_p \mu_p \tilde{h}^T + \delta^2_p \delta^2_t \mu_i \delta_p \mu_p \tilde{h}^T \right] + \delta^2_t \delta^2_p \mu_i \delta_p \mu_p \tilde{h}^T
\]
\[
+ \frac{\sigma_{2\sigma}}{4h^4} \sum_{p,r=1 \atop p \neq r}^{n} S_{j,p} \left[ \delta^2_t \mu_i \delta_p \mu_p \delta^2_t \tilde{h}^T + \delta^2_t \delta^2_p \mu_i \delta_p \mu_p \tilde{h}^T + \mu_i \delta_p \mu_p \delta^2_t \tilde{h}^T \right]
\]
\[
+ \sum_{p,q=1 \atop p \neq q}^{n} \sigma_{2\sigma} S_{j,p} \left[ \mu_i \delta_p \mu_q \delta^2_t \tilde{h}^T + \mu_i \delta_p \mu_q \mu_p \tilde{h}^T + \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T \right]
\]
\[
+ \frac{\sigma_{2\sigma}}{4h^4} \sum_{p,q=1 \atop p \neq q}^{n} S_{j,p} \left[ \delta^2_p \mu_i \delta_p \mu_p \delta^2_t \tilde{h}^T + \mu_i \delta_p \mu_q \delta^2_p \tilde{h}^T + \mu_p \delta_p \mu_q \delta^2_p \tilde{h}^T \right]
\]
\[
+ \frac{\sigma_{2\sigma}}{4h^4} \sum_{p,q=1 \atop p \neq q}^{n} S_{j,q} \left[ \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T + \mu_p \delta_p \mu_q \mu_p \tilde{h}^T + \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T \right]
\]
\[
+ \frac{\sigma_{2\sigma}}{4h^4} \sum_{p,q=1 \atop p \neq q}^{n} S_{j,i} \left[ \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T + \mu_p \delta_p \mu_q \mu_p \tilde{h}^T + \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T \right]
\]
\[
+ \frac{\sigma_{2\sigma}}{4h^4} \sum_{p,q=1 \atop p \neq q}^{n} S_{j,i} \left[ \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T + \mu_p \delta_p \mu_q \mu_p \tilde{h}^T + \mu_p \delta_p \mu_q \delta^2_t \tilde{h}^T \right]
\]
\[
+ \sum_{p,q=1 \atop p \neq q}^{n} S_{j,i} \left[ \mu_i \delta_p \mu_q \delta^2_t \tilde{h}^T + \mu_i \delta_p \mu_q \mu_p \tilde{h}^T + \mu_i \delta_p \mu_q \delta^2_t \tilde{h}^T \right]
\]
\[
+ \frac{\sigma_{2\sigma}}{2h^2} \sum_{p=1}^{n} S_{j,i} \left[ \delta^2_p \tilde{h} \tilde{C}^T + C \delta^2_p \tilde{h}^T \right] + \frac{\sigma_{2\sigma}}{2h^2} \sum_{p=1}^{n} S_{j,p} \left[ \mu_i \delta_p \mu_i \delta^2_p \tilde{h}^T + \mu_p \delta_p \mu_i \delta^2_t \tilde{h}^T \right]
\]

where \(\Delta x(t_k) = x(t_k) - \hat{x}(t_k|t_{k-1}), \Delta y(t_k) = y(t_k) - \hat{y}(t_k|t_{k-1})\) and \(M_{0,2}(t_k|t_{k-1})\) can be generated from \(r_{0,4}(t_k|t_{k-1})\).
B.4 Proof of Lemma 5.2

Proof. From (5.10) and applying Baye’s rule, we have

\[ P(x(t_k), y(t_k)|Y_{t_{k-1}}) = P(x(t_k)|Y_{t_k})P(y(t_k)|Y_{t_{k-1}}). \] (B.1)

Multiplying equation (B.1) by the product of two arbitrary functions \( s(x(t_k)) \) and \( u(y(t_k)) \), and taking the expectations, we have

\[
\int \int s(x)u(y)P(x,y|Y_{t_{k-1}})dxdy = \int \int s(x)u(y)P(x|Y_{t_k})P(y|Y_{t_{k-1}})dxdy = \mathbb{E} \left[ \mathbb{E} \left[ s(x(t_k))|Y_{t_k} \right] u(y(t_k))|Y_{t_{k-1}} \right].
\]

Hence,

\[
\mathbb{E} \left[ s(x(t_k))u(y(t_k))|Y_{t_{k-1}} \right] = \mathbb{E} \left[ \mathbb{E} \left[ s(x(t_k))|Y_{t_k} \right] u(y(t_k))|Y_{t_{k-1}} \right]. \] (B.2)

Equation (B.2) provides a systematic feasible procedure for solving for \( A_i, B_i, i = 0, 1 \), and \( A_2 \).

Substituting \( s = x(t_k) \) and \( u = 1 \), we have

\[
\mathbb{E} \left[ x(t_k)|Y_{t_{k-1}} \right] = \mathbb{E} \left[ \mathbb{E} \left[ x(t_k)|Y_{t_k} \right] |Y_{t_{k-1}} \right]. \] (B.3)

Hence

\[
\mathbb{E} \left[ x(t_k)|Y_{t_{k-1}} \right] = \mathbb{E} \left[ \mathbb{E} \left[ x(t_k)|Y_{t_k} \right] |Y_{t_{k-1}} \right].
\]

This implies that

\[
\hat{x}(t_k|t_{k-1}) = \mathbb{E} \left[ \hat{x}(t_k|t_k)|Y_{t_{k-1}} \right] = \mathbb{E} \left[ A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1})) + A_2(Y - \hat{Y})(y(t_k) - \hat{y}(t_k|t_{k-1}))|Y_{t_{k-1}} \right].
\]

Thus,

\[
r_{1,0}(t_k|t_{k-1}) = A_0(t_k|t_{k-1}) + A_2(t_k|t_{k-1})\sigma Y^2(t_k|t_{k-1}), \] (B.4)
where \( r_{1,0}(t_k|t_{k-1}) = \hat{x}(t_k|t_{k-1}) \). Substituting \( s = x(t_k) \) and \( u = (y(t_k) - \hat{y}(t_k|t_{k-1}))^T \), we have

\[
\begin{align*}
\mathbb{E} \left[ x(t_k)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{t_{k-1}} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ x(t_k)|Y_{t_k} \right] (y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{t_{k-1}} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ (A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1})) \\
+ A_2(Y - \hat{Y})(y(t_k) - \hat{y}(t_k|t_{k-1})) \right] (y(t_k) - \hat{y}(t_k|t_{k-1}))^T | Y_{t_{k-1}} \right],
\end{align*}
\]

Hence,

\[
r_{1,1}(t_k|t_{k-1}) = A_1(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1}) + A_2(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1}). \quad (B.5)
\]

Lastly, substituting \( s = x(t_k) \) and \( u = (y(t_k) - \hat{y}(t_k|t_{k-1}))^T (Y - \hat{Y})^T \), we have

\[
\begin{align*}
\mathbb{E} \left[ x(t_k)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T (Y - \hat{Y})^T | Y_{t_{k-1}} \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ x(t_k)|Y_{t_k} \right] (y(t_k) - \hat{y}(t_k|t_{k-1}))^T (Y - \hat{Y})^T | Y_{t_{k-1}} \right].
\end{align*}
\]

Hence,

\[
r_{1,2}(t_k|t_{k-1}) = A_0(t_k|t_{k-1})\sigma_{Y_2}^T(t_k|t_{k-1}) + A_1(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1}) \\
+ A_2(t_k|t_{k-1})M_{0,2}(t_k|t_{k-1}). \quad (B.7)
\]

The result follows by solving the systems of linear equations (B.4), (B.5), (B.7).

\[\square\]

**B.5 Proof of Lemma 5.3**

**Proof.**

First, we substitute \( s = (x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T \) and \( u = 1 \) into equation (B.2) and obtain

\[
\begin{align*}
\mathbb{E} \left[ (x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T | Y_{t_{k-1}} \right] \\
= \mathbb{E} \left[ \left( x(t_k) - \hat{x}(t_k|t_k) \right) \left( x(t_k) - \hat{x}(t_k|t_k) \right)^T | Y_{t_k} \right] | Y_{t_{k-1}} \\
= \mathbb{E} \left[ P(t_k|t_k) \right].
\end{align*}
\]

Hence,

\[ N_1 = B_0 + B_1r_{0,2}(t_k|t_{k-1}). \quad (B.8) \]

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Lastly, substituting

\[ s = (x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T \]
\[ u = (y(t_k) - \hat{y}(t_k|t_k-1))(y(t_k) - \hat{y}(t_k|t_k-1))^T \]

into equation \((B.2)\)

\[ N_2 = B_0 r_{0,2}(t_k|t_k-1) + B_1 r_{0,4}. \] \[(B.9)\]

The fifth and upper moments of \( \hat{y}(t_k) - \hat{y}(t_k|t_k-1) \) is neglected in \( N_2 \).

\[
\mathbb{E} \left[ (x(t_k) - \hat{x}(t_k|t_k-1))(y(t_k) - \hat{y}(t_k|t_k-1))^T A_1^T (y(t_k) - \hat{y}(t_k|t_k-1)) \times \\
(y(t_k) - \hat{y}(t_k|t_k-1))^T |Y_{k-1} \right]
\]

can be generated from \( r_{1,3} \),

\[
\mathbb{E} \left[ (x(t_k|t_k-1) - A_0)(y(t_k) - \hat{y}(t_k|t_k-1))^T A_1^T (y(t_k) - \hat{y}(t_k|t_k-1)) \times \\
(y(t_k) - \hat{y}(t_k|t_k-1))^T |Y_{k-1} \right]
\]

can be generated from \( r_{0,3} \),

\[
\mathbb{E} \left[ A_1(y(t_k) - \hat{y}(t_k|t_k-1))(x(t_k) - \hat{x}(t_k|t_k-1))^T (y(t_k) - \hat{y}(t_k|t_k-1)) \times \\
(y(t_k) - \hat{y}(t_k|t_k-1))^T |Y_{k-1} \right]
\]

can be generated from \( r_{1,3} \),

\[
\mathbb{E} \left[ A_1(y(t_k) - \hat{y}(t_k|t_k-1))(x(t_k|t_k-1) - A_0)^T (y(t_k) - \hat{y}(t_k|t_k-1)) \times \\
(y(t_k) - \hat{y}(t_k|t_k-1))^T |Y_{k-1} \right]
\]

can be generated from \( r_{0,3} \),

\[
\mathbb{E} \left[ A_1(y(t_k) - \hat{y}(t_k|t_k-1))(y(t_k) - \hat{y}(t_k|t_k-1))^T A_1^T (y(t_k) - \hat{y}(t_k|t_k-1)) \times \\
(y(t_k) - \hat{y}(t_k|t_k-1))^T |Y_{k-1} \right]
\]

can be generated from \( r_{0,4} \), where \( r_{1,3}, r_{0,3}, \) and \( r_{0,4} \) are defined in Appendix \( B.3 \). The conclusion of the Lemma follows by solving the systems of equation \((B.8)\) and \((B.9)\). \[\square\]
Appendix C

C.1 Proof of Lemma 6.1

Proof of Lemma 6.1 for small $m_k, m_{k-1} \leq m_k$, Proof.

\[
\bar{S}_{m_{k},k} = \frac{1}{m_k} \sum_{i=1-m_k}^{0} F^i x_{k-1} = \frac{1}{m_k} \left[ \sum_{i=1-m_k}^{-1-m_{k-1}} F^i x_{k-1} + \sum_{i=-m_{k-1}}^{-1} F^i x_{k-1} + F^0 x_{k-1} \right] 
\]

\[
= \frac{1}{m_k} \left[ \sum_{i=1-m_k}^{m_{k-1}} F^i x_{k-1} - F^{1-m_{k-1}} x_{k-1} - F^{-m_{k-1}} x_{k-1} 
\right. 
\]

\[
\left. + F^0 x_{k-1} \right] 
\]

\[
s_{m_{k},k}^2 = \frac{1}{m_k} \left[ \sum_{i=-m_k+1}^{0} (F^i x_{k-1})^2 - \frac{1}{m_k} \left( \sum_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2 \right] 
\]

\[
= \frac{1}{m_k} \left[ \sum_{i=-m_k+1}^{-m_{k-1}-1} (F^i x_{k-1})^2 + \sum_{i=-m_{k-1}}^{-1} (F^i x_{k-1})^2 + (F^0 x_{k-1})^2 
\right. 
\]

\[
\left. - \frac{1}{m_k} \left( \sum_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2 \right] 
\]

\[
= \frac{1}{m_k} \left[ \sum_{i=-m_{k-1}}^{-1} (F^i x_{k-1})^2 - \frac{1}{m_k} \left( \sum_{i=-m_{k-1}}^{-1} F^i x_{k-1} \right)^2 \right. 
\]

\[
+ \frac{1}{m_{k-1}} \left( \sum_{i=-m_{k-1}}^{-1} F^i x_{k-1} \right)^2 
\]

\[
+ \frac{1}{m_k} \left( (F^0 x_{k-1})^2 - (F^{-m_{k-1}} x_{k-1})^2 - (F^{-m_{k-1}+1} x_{k-1})^2 
\right. 
\]

\[
\left. - \frac{1}{m_k} \left( \sum_{i=-m_{k+1}}^{0} F^i x_{k-1} \right)^2 + \sum_{i=-m_k+1}^{-m_{k-1}+1} (F^i x_{k-1})^2 \right] 
\]
Substituting (C.3), we find an expression connecting $\bar{s}_{m,k}^2$, $\bar{s}_{m,k-1}^2$, and $s_{m,k-1}^2$. By definition and simplification,

$$m_k^2 \bar{s}_{m,k}^2 = \left[ \frac{\sum_{i=-m_k+1}^{0} F^i x_{k-1}^2}{m_k} \right]^2 = \frac{\sum_{i=-m_k+1}^{0} (F^i x_{k-1})^2}{m_k} + \frac{\sum_{l,s=-m_k+1}^{0} F^i x_{k-1} F^s x_{k-1}}{m_k}$$

Substituting (C.2) into (C.1), we have

$$s_{m,k}^2 = \frac{m_k-1}{m_k} \left[ \frac{m_k-1}{m_k} s_{m,k-1}^2 + \frac{m_k-1}{m_k} \bar{s}_{m,k-1}^2 \right] + \frac{(F^0 x_{k-1})^2}{m_k} - \frac{(F^{-m_k-1} x_{k-1})^2}{m_k} + \frac{(F^{-m_k-1} x_{k-1})^2}{m_k} + \frac{\sum_{i=-m_k+1}^{0} (F^i x_{k-1})^2}{m_k}$$

Likewise, using equation (C.2),

$$m_{k-1}^2 \bar{s}_{m,k-1}^2 = (m_k-2) s_{m,k-2}^2 + m_k-2 \bar{s}_{m,k-2}^2 + (F^{-1} x_{k-1})^2 - (F^{-m_k-2} x_{k-1})^2 + \sum_{i=-m_k-1}^{1} (F^i x_{k-1})^2 + \sum_{l,s=-m_k}^{l,s} F^l x_{k-1} F^s x_{k-1}.$$
Continuing in this sense and substituting $\bar{S}_{m_{i-1},i}^2$, $i = 2, \ldots, p - 1$ into $\bar{S}_{m_{k-1},k-1}^2$, we have

$$\begin{align*}
(m_{k-1})\bar{S}_{m_{k-1},k-1}^2 &= \sum_{i=2}^{p} \left[ \frac{m_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}} s_{m_{k-1},k-i}^2 + \frac{m_{k-p}}{\prod_{j=1}^{p-i} m_{k-j}} \bar{S}_{m_{k-p},k-p}^2 + \sum_{i=2}^{p} \frac{(F_{-i+1} x_{k-1})^2}{\prod_{j=1}^{i-1} m_{k-j}} \right] \\
&- \sum_{i=2}^{p} \left[ \frac{i-1}{\prod_{j=1}^{i-1} m_{k-j}} \sum_{l,i+2,m_{k-1}+1}^{m_{k-1}} (F_{l} x_{k-1})^2 \right] \\
&+ \sum_{i=2}^{p} \left[ \frac{i-1}{\prod_{j=1}^{i-1} m_{k-j}} \sum_{l,i+2,m_{k-1}+1}^{m_{k-1}} (F_{l} x_{k-1})^2 \right] + \sum_{i=2}^{p} \left[ \frac{i-1}{\prod_{j=1}^{i-1} m_{k-j}} \sum_{l,i+2,m_{k-1}+1}^{m_{k-1}} (F_{l} x_{k-1})^2 \right]
\end{align*}$$

(C.4)

Finally, the result follows by substituting (C.4) into (C.3).

\[ \square \]

C.2 Proof Lemma 6.1

Proof of Lemma 6.1 for small $m_k$, $m_k \leq m_{k-1}$, Proof. Following the same steps, if $m_k \leq m_{k-1}$,

$$\begin{align*}
\bar{S}_{m_{k},k}^2 &= \frac{m_{k-1}}{m_{k}} \left[ \sum_{i=1}^{p} \left( \frac{m_{k-i}}{\prod_{j=0}^{i-1} m_{k-j}} s_{m_{k-i},k-i}^2 + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}} \bar{S}_{m_{k-p},k-p}^2 \right) \\
&+ \omega_{m_{k-1},k-1}, m_k \leq m_{k-1} \\
\omega_{m_{k-1},k-1} &= \frac{m_{k-1}}{m_{k}} \left[ \sum_{i=1}^{p} \frac{(F_{-i+1} x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p} \frac{i+1}{\prod_{j=0}^{i-1} m_{k-j}} \sum_{l,i+2,m_{k-1}+1}^{m_{k-1}} (F_{l} x_{k-1})^2 \right] \\
&+ \sum_{i=1}^{p} \left[ \frac{i+1}{\prod_{j=0}^{i-1} m_{k-j}} \sum_{l,i+2,m_{k-1}+1}^{m_{k-1}} (F_{l} x_{k-1})^2 \right] \\
&- \frac{1}{m_{k}} \sum_{l,i,s=-m_{k}+1}^{0} \sum_{l \neq s} F_{l} x_{k-1} F_{s} x_{k-1} \\
\end{align*}$$

\[ \square \]

C.3 Proof Lemma 6.1

Proof of Lemma 6.1 for large $m_k$ Proof.
\[ s^2_{m,k} = \frac{1}{m_k-1} \left[ \sum_{i=-m_k+1}^0 \left( F^i x_{k-1} \right)^2 - \frac{1}{m_k} \left( \sum_{j=-m_k+1}^0 F^j x_{k-1} \right)^2 \right] \]

\[ = \frac{1}{m_k-1} \left[ \sum_{i=-m_k+1}^{-1} \left( F^i x_{k-1} \right)^2 - \frac{1}{m_k} \left( \sum_{i=-m_k+1}^{-1} F^i x_{k-1} \right)^2 \right] \]

\[ + \frac{1}{m_k-1} \left( \sum_{i=-m_k+1}^1 F^i x_{k-1} \right)^2 \]

\[ + \frac{1}{m_k-1} \left[ (F^0 x_{k-1})^2 - (F^{m_k-1} x_{k-1})^2 - (F^{-m_k-1} x_{k-1})^2 \right] \]

\[ = \frac{m_k-1}{m_k-1} s^2_{m_k,k-1} + \frac{m_k-1}{m_k-1} \bar{s}^2_{m_k-1,k-1} - \frac{m_k}{m_k-1} \bar{s}^2_{m_k,k} \]

\[ + \frac{m_k-1}{m_k-1} \bar{s}^2_{m_k,k-1} - (m_k-1) s^2_{m_k,k-1} + m_k \bar{s}^2_{m_k-1,k-1} + (F^0 x_{k-1})^2 - (F^{m_k-1} x_{k-1})^2 \]

\[ - (F^{-m_k-1} x_{k-1})^2 + \sum_{i=-m_k+1}^{0} \sum_{\substack{l,s \neq s \, \mid l}} F^l x_{k-1} \]

\[ + \frac{m_k}{m_k-1} \frac{\sum_{i=-m_k+1}^{0} \sum_{\substack{l,s \neq s \, \mid l}} F^l x_{k-1}}{m_k (m_k-1)} . \]

Hence,

\[ s^2_{m,k} = \frac{m_k-1}{m_k-1} s^2_{m_k,k-1} + \frac{m_k-1}{m_k-1} \bar{s}^2_{m_k-1,k-1} - \frac{m_k}{m_k-1} \bar{s}^2_{m_k,k} \]

\[ + \frac{m_k-1}{m_k-1} \bar{s}^2_{m_k,k-1} - (m_k-1) s^2_{m_k,k-1} + m_k \bar{s}^2_{m_k-1,k-1} + (F^0 x_{k-1})^2 - (F^{m_k-1} x_{k-1})^2 \]

\[ - (F^{-m_k-1} x_{k-1})^2 + \sum_{i=-m_k+1}^{0} \sum_{\substack{l,s \neq s \, \mid l}} F^l x_{k-1} \bar{s}^2_{m_k,k} \]

\[ + \frac{m_k}{m_k-1} \frac{\sum_{i=-m_k+1}^{0} \sum_{\substack{l,s \neq s \, \mid l}} F^l x_{k-1}}{m_k (m_k-1)} . \]  

(C.5)

Next, we find an expression connecting \( s^2_{m_k,k} \), \( \bar{s}^2_{m_k-1,k-1} \) and \( s^2_{m_k-1,k-1} \). By definition,

\[ m_k^2 \bar{s}^2_{m_k,k} = \left[ \sum_{i=-m_k+1}^{0} F^i x_{k-1} \right]^2 = \sum_{i=-m_k+1}^{0} \left( F^i x_{k-1} \right)^2 + \sum_{l,s=-m_k+1}^{0} \sum_{\substack{l \neq s \, \mid l}} F^l x_{k-1} F^s x_{k-1} \]

\[ = (m_k - 1) s^2_{m_k-1,k-1} + m_k \bar{s}^2_{m_k-1,k-1} + (F^0 x_{k-1})^2 - (F^{m_k-1} x_{k-1})^2 \]

\[ - (F^{-m_k-1} x_{k-1})^2 + \sum_{i=-m_k+1}^{0} \sum_{\substack{l,s \neq s \, \mid l}} F^l x_{k-1} \bar{s}^2_{m_k,k} \]

(C.6)

Substituting (C.6) into (C.5), we have

\[ s^2_{m_k} = \frac{m_k-1}{m_k} s^2_{m_k,k-1} + \frac{m_k}{m_k-1} \bar{s}^2_{m_k-1,k-1} + \left( \frac{m_k}{m_k-1} \bar{s}^2_{m_k,k-1} - (m_k-1) s^2_{m_k,k-1} + m_k \bar{s}^2_{m_k-1,k-1} + (F^0 x_{k-1})^2 - (F^{m_k-1} x_{k-1})^2 \right) \]

\[ + \frac{1}{m_k} \frac{\sum_{i=-m_k+1}^{0} \sum_{\substack{l,s \neq s \, \mid l}} F^l x_{k-1}}{m_k (m_k-1)} . \]  

(C.7)
Likewise,

\[ m_{k-1}^2 \bar{S}_{m_{k-1}, k-1}^2 = (m_{k-2} - 1)s_{m_{k-2}, k-2}^2 + m_{k-2} \bar{S}_{m_{k-2}, k-2}^2 + (F^{-1} x_{k-1})^2 \]

\[ - (F^{-m_{k-2}} x_{k-1})^2 - (F^{-m_{k-2}} x_{k-1})^2 + \sum_{i=-m_{k-1}}^{-1} (F^i x_{k-1})^2 \]

\[ + \sum_{l, s=-m_{k-1}}^{l \neq s} F^l x_{k-1} F^s x_{k-1}, \]

\[ m_{k-2}^2 \bar{S}_{m_{k-2}, k-2}^2 = (m_{k-3} - 1)s_{m_{k-3}, k-3}^2 + m_{k-3} \bar{S}_{m_{k-3}, k-3}^2 + (F^{-2} x_{k-1})^2 \]

\[ - (F^{-m_{k-3}} x_{k-1})^2 - (F^{-m_{k-3}} x_{k-1})^2 + \sum_{i=-m_{k-2}}^{-2} (F^i x_{k-1})^2 \]

\[ + \sum_{l, s=-m_{k-2}}^{l \neq s} F^l x_{k-1} F^s x_{k-1}. \]

Continuing in this sense and substituting \( \bar{S}_{m_{k-i}, k-i}, i = 2, \ldots, p - 1 \) into \( \bar{S}_{m_{k-1}, k-1} \), we have

\[ (m_{k-1}) \bar{S}_{m_{k-1}, k-1}^2 = \sum_{i=2}^{p} \left[ \sum_{j=1}^{m_{k-i} - 1} \frac{1}{\prod_{j=1}^{i-1} m_{k-j}} \right] s_{m_{k-i}, k-i}^2 + \frac{m_{k-p}}{\prod_{j=1}^{p} m_{k-j}} \bar{S}_{m_{k-p}, k-p}^2 + \sum_{i=2}^{p} \frac{(F^{-i+1} x_{k-1})^2}{\prod_{j=1}^{i-1} m_{k-j}} \]

\[ - \sum_{i=2}^{p} \frac{(F^{-i+2} m_{k-i} x_{k-1})^2}{\prod_{j=1}^{i-1} m_{k-j}} - \sum_{i=2}^{p} \frac{(F^{-i+2} m_{k-i} x_{k-1})^2}{\prod_{j=1}^{i-1} m_{k-j}} \]

\[ + \sum_{i=2}^{p} \left[ \sum_{l=-i+2}^{-i+1} \frac{F^{l} x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}} \right] \]

\[ + \sum_{i=2}^{p} \left[ \sum_{l=-i+1}^{-i+1} \frac{F^{l} x_{k-1} F^{s} x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}} \right] \]

\[ + \sum_{i=2}^{p} \left[ \sum_{l, s=1}^{l \neq s} \frac{F^{l} x_{k-1} F^{s} x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}} \right] \]

\[ (C.8) \]

Finally, the result follows by substituting (C.8) into (C.7).

\[ \square \]

C.4 Algorithm and Flowchart For Simulation

The simulated estimate \( y_{m_{k}, k}^s \) for the energy commodity model follows the Euler scheme

\[ y_{m_{k}, k}^s = y_{m_{k-1}, k-1}^s + \Delta t \bar{S}_{m_{k-1}, k-1} \Delta t + \bar{S}_{m_{k-1}, k-1} y_{m_{k-1}, k-1}^s \Delta W_{m, k}. \]

\[ (C.9) \]
Algorithm 2 Simulation scheme

Given initials $r$, $\epsilon$, $\{\tilde{s}_{m_0,0}^2\}_{m_0 \in OS_0}$, $\{s_{m-1,-1}^2\}_{m-1 \in OS_{-1}}$, $\{\tilde{s}_{m-1,-1}^2\}_{m-1 \in OS_{-1}}$, $\{y_{m_0,0}^s\}_{m_0 \in OS_0}$,

for $k = 1$ to $N$

for $m_{k-1} = 2$ to $r + k - 2$

Compute $a_{m_{k-1},k-1}$, $\tilde{\mu}_{m_{k-1},k-1}$

for $m_{k-2} = 2$ to $r + k - 3$

Compute $\tilde{S}_{m_{k-1},k-1}^2$, $s_{m_1,k}^2$, $y_{m_1,k}^s$, $\Xi_{m_1,k,y_1}$

end for

end for

if $\Xi_{m_1,k,y_1} < \epsilon$

Save $\hat{m}_k$, $\hat{m}_{k-1}$, $\hat{m}_{k-2}$

else

Find $\hat{m}_k$ that minimizes $\Xi_{m_1,k,y_1}$

end if

Compute $a_{\hat{m}_1,k}$, $\tilde{\mu}_{\hat{m}_1,k}$, $s_{\hat{m}_1,k}^2$, $y_{\hat{m}_1,k}^s$.

end for

Similar algorithm can be generated for the interest rate model.

REMARK 32 We give the first iterate for the energy commodity model.

Given initials $r$, $\epsilon$, $\{s_{m_0,0}^2\}_{m_0 \in OS_0}$, $\{s_{m-1,-1}^2\}_{m-1 \in OS_{-1}}$, $\{\tilde{s}_{m-1,-1}^2\}_{m-1 \in OS_{-1}}$, $\{y_{m_0,0}^s\}_{m_0 \in OS_0}$, $\{y_{m_0,0}^s\}_{m_0 \in OS_0}$.

Compute $a_{m_0,0}$, $\mu_{m_0,0}$.

For $k=1$:

Compute $y_{m_{1,1}}^s$ using (C.9). If $\Xi_{m_{1,1},y_1} < \epsilon$, save $\hat{m}_1$, $\hat{m}_0$, $\hat{m}_{-1}$, else, find values of $m_1$ that minimizes $\Xi_{m_{1,1},y_1}$.

Compute $a_{\hat{m}_0,0}$, $\mu_{\hat{m}_0,0}$, $s_{\hat{m}_1,1}^2$, $y_{\hat{m}_1,1}^s$.

Next, we give a flowchart similar to the algorithm above.
Flowchart 2: LLGMM Simulation Algorithm.
Appendix D

D.1 Proof of Lemma 9.3

Proof of Lemma 9.3 for small $m_k, m_{k-1} \leq m_k$, Proof.

\[ s_{m_k,k}^{i,j} = \frac{1}{m_k} \left[ \sum_{i=-m_k+1}^{0} (F^i x_i(k-1)) (F^j x_j(k-1)) \right. \]

\[ - \frac{1}{m_k} \left( \sum_{a=-m_k+1}^{0} F^a x_i(k-1) \right) \left( \sum_{a=-m_k+1}^{0} F^a x_j(k-1) \right) \]

\[ = \frac{1}{m_k} \left[ \sum_{i=-m_k+1}^{-m_k-1} (F^i x_i(k-1)) (F^j x_j(k-1)) + \sum_{i=-m_k+1}^{0} (F^i x_i(k-1)) (F^j x_j(k-1)) \right. \]

\[ + \left. (F^0 x_i(k-1)) (F^0 x_j(k-1)) \right] - \frac{1}{m_k^2} \sum_{a=-m_k+1}^{0} F^a x_i(k-1) \sum_{a=-m_k+1}^{0} F^a x_j(k-1) \]

\[ = \frac{m_k^{-1}}{m_k} s_{m_k-1,k-1}^{i,j} + \frac{m_k^{-1}}{m_k} \tilde{s}_{m_k-1,k-1}^{i,j} \tilde{S}_{m_k-1,k-1}^{j} - \tilde{s}_{m_k,k}^{i} \tilde{S}_{m_k,k}^{j} \]

\[ + \sum_{i=-m_k+1}^{-m_k-1} F^i x_i(k-1) F^j x_j(k-1) \]

\[ + \frac{F^0 x_i(k-1) F^0 x_j(k-1) - F^{-m_k-1} x_i(k-1) F^{-m_k-1} x_j(k-1)}{m_k} \]

Hence,

\[ s_{m_k,k}^{i,j} = \frac{m_k^{-1}}{m_k} s_{m_k-1,k-1}^{i,j} + \frac{m_k^{-1}}{m_k} \tilde{s}_{m_k-1,k-1}^{i,j} \tilde{S}_{m_k-1,k-1}^{j} - \tilde{s}_{m_k,k}^{i} \tilde{S}_{m_k,k}^{j} \]

\[ + \sum_{i=-m_k+1}^{-m_k-1} F^i x_i(k-1) F^j x_j(k-1) \]

\[ + \frac{F^0 x_i(k-1) F^0 x_j(k-1) - F^{-m_k-1} x_i(k-1) F^{-m_k-1} x_j(k-1)}{m_k} \]

\[ - \frac{F^{-m_k-1} x_i(k-1) F^{-m_k-1} x_j(k-1)}{m_k} \].

(D.1)
Next, we find an expression connecting $s_{m^2}^{ij}$, $s_{m^2,k^{-1}}^{ij}$, $s_{m^2,k^{-1}}^{ij}$, and $s_{m^2,k}^{ij}$. By definition and simplification,

$$m^2 s_{m,k}^{ij} = \sum_{i=-m_k+1}^{0} F_i x_i (k-1) \sum_{i=-m_k+1}^{0} F_i x_j (k-1)$$

$$= \sum_{i=-m_k+1}^{0} F_i x_i (k-1) F_i x_j (k-1) + \sum_{l,s=-m_k+1}^{0} F^l x_i (k-1) F^l x_j (k-1)$$

$$= (m_k^{-1}) s_{m^{-1},k^{-1}}^{ij} + m_k^{-1} s_{m^{-1},k^{-1}}^{ij} + F^0 x_i (k-1) F^0 x_j (k-1)$$

Substituting (D.2) into (D.1), we have

$$s_{m,k}^{ij} = \frac{m_k^{-1} s_{m^{-1},k^{-1}}^{ij}}{m_k^{-1}} + \frac{m_k^{-1} s_{m^{-1},k^{-1}}^{ij}}{m_k^{-1}} + \frac{F^0 x_i (k-1) F^0 x_j (k-1)}{m_k^{-1}}$$

Likewise, using (D.2),

$$m_k^{-1} s_{m^{-1},k^{-1}}^{ij} = (m_k^{-2}) s_{m^{-2},k^{-2}}^{ij} + m_k^{-2} s_{m^{-2},k^{-2}}^{ij} + F^{-1} x_i (k-1) F^{-1} x_j (k-1)$$

$$- F^{-1} x_i (k-1) F^{-1} x_j (k-1)$$

$$= \sum_{l,s=-m_k-1}^{0} F^l x_i (k-1) F^l x_j (k-1)$$

$$+ \sum_{l,s=-m_k-1}^{0} F^l x_i (k-1) F^l x_j (k-1)$$

$$+ \sum_{l,s=-m_k-1}^{0} F^l x_i (k-1) F^l x_j (k-1).$$

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Also,
\[
m^2_{k-2} \tilde{S}^{i}_{m_{k-2},k-2} \tilde{S}^{j}_{m_{k-2},k-2} = (m_{k-3})^{s^{i}_{m_{k-3},k-3}} + m_{k-3} \tilde{S}^{i}_{m_{k-3},k-3} \tilde{S}^{j}_{m_{k-3},k-3} + F^{-2} x_i(k-1) F^{-2} x_j(k-1) \]
\[
- F^{-m_{k-3}-2} x_i(k-1) F^{-m_{k-3}-2} x_j(k-1) \]
\[
- F^{-m_{k-3}-1} x_i(k-1) F^{-m_{k-3}-1} x_j(k-1) + \sum_{l=-m_{k-2}-1}^{-2} F^{l} x_i(k-1) F^{l} x_j(k-1) \]
\[
+ \sum_{l,s=-m_{k-2}-1}^{-2} F^{l} x_i(k-1) F^{s} x_j(k-1). \]

Continuing in this sense and substituting \( \tilde{S}^{i}_{m_{k-1},k-1} \tilde{S}^{j}_{m_{k-1},k-1} \), \( i = 2, \ldots, d - 1 \) into \( \tilde{S}^{i}_{m_{k-1},k-1} \tilde{S}^{j}_{m_{k-1},k-1} \), we have
\[
(m_{k-1})^{s^{i}_{m_{k-1},k-1}} \tilde{S}^{i}_{m_{k-1},k-1} + \sum_{l=1}^{d} \left[ \frac{m_{k-i}}{\prod_{a=1}^{d-1} m_{k-j}} \right] s^{i}_{m_{k-1},k-1} + \sum_{l=1}^{d} \frac{m_{k-d}}{\prod_{a=1}^{d-1} m_{k-j}} \tilde{S}^{i}_{m_{k-d},k-d} \tilde{S}^{j}_{m_{k-d},k-d} + \sum_{l=1}^{d} \frac{F^{l} x_i(k-1) F^{l} x_j(k-1)}{\prod_{a=1}^{d-1} m_{k-j}} \]
\[
- \sum_{l=1}^{d} \frac{F^{-m_{k-1}-2} x_i(k-1) F^{-m_{k-1}-2} x_j(k-1)}{\prod_{a=1}^{d-1} m_{k-j}} \]
\[
- \sum_{l=1}^{d} \frac{F^{-m_{k-1}-1} x_i(k-1) F^{-m_{k-1}-1} x_j(k-1)}{\prod_{a=1}^{d-1} m_{k-j}} + \sum_{l=1}^{d} \frac{F^{l} x_i(k-1) F^{l} x_j(k-1)}{\prod_{a=1}^{d-1} m_{k-j}} \]
\[
+ \sum_{l=1}^{d} \frac{F^{l} x_i(k-1) F^{s} x_j(k-1)}{\prod_{a=1}^{d-1} m_{k-j}} + \sum_{l=1}^{d} \frac{F^{l} x_i(k-1) F^{s} x_j(k-1)}{\prod_{a=1}^{d-1} m_{k-j}} \]

(D.4)

Finally, the result follows by substituting (D.4) into (D.3).

\[\square\]

D.2 Proof of Lemma 9.3 for small \( m_k \)

Proof of Lemma 9.3 for small \( m_k, m_k \leq m_{k-1} \), Proof. Following the same steps, if \( m_k \leq m_{k-1} \),
\[
\begin{align*}
\omega_{m_k-1,k-1}^{i,j} &= \frac{m_k-1}{m_k} \left[ \sum_{i=1}^{d} \frac{m_{k-1}}{m_{k-j}} \right] i,j \sum_{m_{k-1}}^{k-1} \frac{m_{k-d}}{m_{k-j}} \omega_{m_k,d-k}^{i,j} \sum_{m_{k-d}}^{k-1} \right] \\
\delta_{m_k-1,k-1}^{i,j} &= \frac{m_k-1}{m_k} \left[ \sum_{i=1}^{d} \frac{F^{i+i+1}x_i(k-1)F^{i+i}x_j(k-1)}{i+m_{k-j}} \right] - \sum_{i=1}^{d} \left[ \sum_{l=0}^{i} \frac{F^l x_i(k-1)F^l x_j(k-1)}{i+m_{k-j}} \right] \\
&\quad + \frac{1}{m_k} \left[ \sum_{l=0}^{i} \frac{F^l x_i(k-1)F^l x_j(k-1)}{l+m_{k-1}} \right] \\
&\quad - \frac{1}{m_k} \left[ \sum_{l=0}^{i} \frac{F^l x_i(k-1)F^l x_j(k-1)}{l+m_{k-1}} \right] \\
&\quad - \frac{1}{m_k} \left[ \sum_{l=0}^{i} \frac{F^l x_i(k-1)F^l x_j(k-1)}{l+m_{k-1}} \right] \\
&\quad + \frac{1}{m_k} \left[ F^0 x_i(k-1)F^0 x_j(k-1) - F^{-m_{k-1}} x_i(k-1)F^{-m_{k-1}} x_j(k-1) \\
&\quad - F^{-m_{k-1}} x_i(k-1)F^{-m_{k-1}} x_j(k-1) \right]
\end{align*}
\]

D.3 Proof of Lemma 9.3 for large \( m_k \)

Proof of Lemma 9.3 for large \( m_k \)

Proof.
\[\begin{align*}
&+ \frac{1}{m_k - 1} \left[ \sum_{l = -m_k + 1}^{-m_k - 1 + 1} F^s x_i(k - 1) \sum_{l = -m_k + 1}^{-m_k - 1 + 1} F^s x_j(k - 1) \right] \\
&- \frac{1}{m_k} \sum_{l = -m_k + 1}^{0} F^s x(k - 1) \sum_{l = -m_k + 1}^{0} F^s x(k - 1) \\
&= \frac{m_k - 1}{m_k - 1} s_{m_k - 1, k - 1}^{i,j} + \frac{m_k - 1}{m_k - 1} \tilde{s}_{m_k - 1, k - 1}^{i,j} - \frac{m_k - 1}{m_k - 1} \tilde{s}_{m_k, k - 1}^{i,j} \\
&+ \frac{F^0 x_i(k - 1) F^0 x_j(k - 1) - F^{-m_k - 1} x_i(k - 1) F^{-m_k - 1} x_j(k - 1)}{m_k - 1} \\
&- \frac{F^{-m_k - 1} x_i(k - 1) F^{-m_k - 1} x_j(k - 1)}{m_k - 1} + \frac{m_k - 1}{m_k - 1} \sum_{l = -m_k + 1}^{-m_k - 1 + 1} F^s x_i(k - 1) F^s x_j(k - 1) \\
&+ \frac{m_k - 1}{m_k - 1} \sum_{l = -m_k + 1}^{-m_k - 1 + 1} F^s x_i(k - 1) F^s x_j(k - 1).
\end{align*}\]

Hence,

\[s_{m_k, k}^{i,j} = \frac{m_k - 1}{m_k - 1} s_{m_k - 1, k - 1}^{i,j} + \frac{m_k - 1}{m_k - 1} \tilde{s}_{m_k - 1, k - 1}^{i,j} - \frac{m_k - 1}{m_k - 1} \tilde{s}_{m_k, k - 1}^{i,j} + \frac{F^0 x_i(k - 1) F^0 x_j(k - 1) - F^{-m_k - 1} x_i(k - 1) F^{-m_k - 1} x_j(k - 1)}{m_k - 1} \]

Next, we find an expression connecting \(\tilde{s}_{m_k, k}^{i,j}, \tilde{s}_{m_k - 1, k - 1}^{i,j}, \tilde{s}_{m_k - 1, k - 1}^{i,j}, \tilde{s}_{m_k, k - 1}^{i,j}\) and \(s_{m_k - 1, k - 1}^{i,j}\). By definition and simplification,

\[m_k^2 \tilde{s}_{m_k, k}^{i,j} \tilde{s}_{m_k, k}^{j,i} = \sum_{l = -m_k + 1}^{0} F^s x_i(k - 1) \sum_{l = -m_k + 1}^{0} F^s x_j(k - 1) \]

\[= \sum_{l = -m_k + 1}^{0} F^s x_i(k - 1) F^s x_j(k - 1) \]

\[+ \sum_{l, s = -m_k + 1}^{0} F^s x_i(k - 1) F^s x_j(k - 1) \]

\[= (m_k - 1) s_{m_k - 1, k - 1}^{i,j} + m_k - 1 \tilde{s}_{m_k - 1, k - 1}^{i,j} \tilde{s}_{m_k - 1, k - 1}^{j,i} \]

\[+ \frac{F^0 x_i(k - 1) F^0 x_j(k - 1) - F^{-m_k - 1} x_i(k - 1) F^{-m_k - 1} x_j(k - 1)}{m_k - 1} \]

\[+ \sum_{l = -m_k + 1}^{-m_k - 1 + 1} F^s x_i(k - 1) F^s x_j(k - 1) \]

\[+ \sum_{l, s = -m_k + 1}^{0} F^s x_i(k - 1) F^s x_j(k - 1) \]

\[(D.6)\]

Substituting \((D.6)\) into \((D.5)\), we have

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Continuing in this sense and substituting

\[
\bar{s}_{m,k} = \frac{m_{k-1} - 1}{m_k} s_{m,k-1,k} + \frac{m_{k-1}}{m_k} \bar{s}_{m,k-1,k-1} + \sum_{i=-m_k+1}^{-m_k} \left[ F^i x_i(k-1) F^i x_j(k-1) \right] \frac{m_{k-1} + 1}{m_k} \frac{m_{k-1} + 1}{m_k} F^i x_i(k-1) F^i x_j(k-1)
\]

Likewise,

\[
m_{k-1}^2 \bar{s}_{m,k-1,k-1} \bar{s}_{m,k-1,k-1} = (m_k - 2 - 1) s_{m_k-2,k-2} + m_{k-2} \bar{s}_{m_k-2,k-2} \bar{s}_{m_k-2,k-2} + F^{-1} x_i(k-1) F^{-1} x_j(k-1) - F^{-m_k-2-1} x_i(k-1) F^{-m_k-2-1} x_j(k-1) - F^{-m_k-2} x_i(k-1) F^{-m_k-2} x_j(k-1) + \sum_{i=-m_k}^{-m_k} F^i x_i(k-1) F^i x_j(k-1) + \sum_{l,s=-m_k}^{-m_k} F^l x_i(k-1) F^s x_j(k-1),
\]

\[
m_{k-2}^2 \bar{s}_{m,k-2,k-2} \bar{s}_{m,k-2,k-2} = (m_k - 3 - 1) s_{m_k-3,k-3} + m_{k-3} \bar{s}_{m_k-3,k-3} \bar{s}_{m_k-3,k-3} + F^{-2} x_i(k-1) F^{-2} x_j(k-1) - F^{-m_k-3-2} x_i(k-1) F^{-m_k-3-2} x_j(k-1) - F^{-m_k-3-1} x_i(k-1) F^{-m_k-3-1} x_j(k-1) + \sum_{i=-m_k}^{-m_k} F^i x_i(k-1) F^i x_j(k-1) + \sum_{l,s=-m_k}^{-m_k} F^l x_i(k-1) F^s x_j(k-1).
\]

Continuing in this sense and substituting \( \bar{s}_{m_k-i,k-i} \), \( i = 2, \ldots, d-1 \) into \( \bar{s}_{m_k-1,k-1} \), we have

\[
(m_{k-1}) \bar{s}_{m,k-1,k-1} = \sum_{l=2}^{d} \left[ \frac{m_{k-1} - 1}{m_k} \right] \bar{s}_{m_k-i,k-i} + \frac{m_{k-1}}{m_k} \bar{s}_{m_k-d,k-d} - \frac{m_{k-1}}{m_k} \bar{s}_{m_k-d,k-d} + \sum_{l=2}^{d} \frac{F^{-i+1} x_i(k-1) F^{-i+1} x_j(k-1)}{m_k} \]
\begin{align*}
- \sum_{i=2}^{d} \frac{F^{-i+2-m_{k-i}} x_i(k-1) F^{-i+2-m_{k-i}} x_j(k-1)}{\prod_{a=1}^{i-1} m_{k-j}} + \sum_{i=2}^{d} \frac{F^l x_i(k-1) F^l x_j(k-1)}{\prod_{a=1}^{i-1} m_{k-j}} \\
+ \sum_{i=2}^{d} \left[ \sum_{l,s=-i+2-m_{k-i}+1}^{-i+1} \frac{F^l x_i(k-1) F^s x_j(k-1)}{\prod_{a=1}^{i-1} m_{k-j}} \right]_{l \neq s}
\end{align*}

(D.9)

Finally, the result follows by substituting \((D.8)\) into \((D.7)\). \qed
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