Chesapeake Bay Analysis using

Time and Spatial Generalized Eigenfunctions

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Normal Mode Analysis of the Chesapeake Bay

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Time Series Analysis of the Chesapeake Bay

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http://web.usna.navy.mil/~rmm/
From 2004 - Present:

- USNA has begun studies specific to the Chesapeake Bay
- Current efforts span five departments:
  - Physics, Math, Chemistry, Oceanography and Naval Architecture
- Efforts have begun to join CBOS
- Instrumentation has begun in the Severn River and College Creek
- Study of small estuaries - “feeder” systems - to the Bay started
- Three Trident scholars: Gillary and Brasher (2004), Boe (2006)
- Differing approaches taken: Normal Mode Analysis,
  - Navier-Stokes integration, COMSOL model
- 100 modes calculated for Dirichlet and Neumann (2005)
- Feasibility study for Galerkin method to extract $\mathbf{u}(x, t)(2006)$
- Initial Value Problem/ Dual Time Problem / Dual Position Problem
- $\sim$ 10 monitoring stations needed for $\mathbf{u}(x_i, t)$
- Concerns about proper mesh for Bay
- How to handle varying wave speed in the Bay?
The basic unit of calculation used throughout this paper is the normal mode. Like the modes of a guitar string or an organ pipe, systems obeying the Helmholtz equation and Dirichlet or Neumann boundary conditions will resonate in states referred as “normal modes”. For the Chesapeake Bay, the modes calculated are energy potentials whose gradients and curls of gradients correspond to the vector current fields found in fluid mechanics ($\vec{u}$).

Briefly, the formulation leading to the calculation of fluid flow stems from the realization that the vector fields can be derived from two scalar fields, which are the solutions to the Helmholtz equation under Dirichlet and Neumann boundary conditions [4].

$$\overline{u} = \nabla \times [(\hat{n}\Psi) + \nabla \times (\hat{n}\Phi)].$$  \hspace{1cm} (1)

Here $\Psi$ is the stream potential where,

$$\overline{u}_D = (u,v)_D = \left( -\frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial x} \right),$$  \hspace{1cm} (2)

and $\Phi$ is the velocity potential where,

$$\overline{u}_N = (u,v)_N = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right),$$  \hspace{1cm} (3)

with $(u,v)$ representing the surface current velocities in the $x$ and $y$ directions respectively. The total velocity field is composed:
Monterey Bay

- Lipphardt et al. at Univ. of Delaware have continued work on Monterey Bay, developing near real-time “Nowcasts”.


- [http://newark.cms.udel.edu/~brucel/realtimemaps](http://newark.cms.udel.edu/~brucel/realtimemaps)
Image Processing of the Chesapeake
Approximated Boundaries
Galerkin Method:

\[ u(x, t) = \sum_{n=0}^{\infty} \left[ a(t)_n u(x)_{D,n} + b(t)_n u(x)_{N,n} \right] \]

\[ a(t)_m = \oint u(x, t)_{data} u(x)_{D,m} d\Omega \]

\[ a(t)_m = \oint \sum_{n=0}^{\infty} \left[ a(t)_n u(x)_{D,n} + b(t)_n u(x)_{N,n} \right] u(x)_{D,m} d\Omega \]

\[ a(t)_m = \sum_{n=0}^{\infty} \left[ a(t)_n \oint u_{D,n} u_{D,m} d\Omega + b(t)_n \oint u_{N,n} u_{D,m} d\Omega \right] \]

\[ a(t)_m = \sum_{n=0}^{\infty} \left[ a(t)_n \delta_{nm} + b(t)_n \delta \right] \]

\[ a(t)_m = \delta_{nm} a(t)_n. \]

\[ a(t)_n = \oint u(x, t)_{data} u(x)_{D,n} d\Omega \]
Partial Galerkin Method:

\[ \tilde{a}(t)_m = \int_{\Omega} u(\vec{x}', t) u(\vec{x}')_{D,n} d\Omega \]

\[ \tilde{a}(t)_m = \int_{\Omega} \sum_{n=0}^{\infty} \left[ a(t)_n u(\vec{x}')_{D,n} + b(t)_n u(\vec{x}')_{N,n} \right] u(\vec{x}')_{D,n} d\Omega \]

\[ \tilde{a}(t)_m = \sum_{n=0}^{\infty} \left[ a(t)_n \int_{\Omega} u_{D,n} u_{D,m} d\Omega + b(t)_n \int_{\Omega} u_{N,n} u_{D,m} d\Omega \right] \]

\[ \tilde{a}(t)_m = \sum_{n=0}^{\infty} \left[ \alpha_{D,nm} a(t)_n + \beta_{D,nm} b(t)_n \right] \]

(1)

Note that the \( \alpha_{D,nm} \) and \( \beta_{D,nm} \) exhibit wavelet-like responses in the spectral domain.
Dual Time Problem (Initial Value Problem)

\[ F(x, t)_{0,n} = f(x)_n g(t)_n \]

\[ F(x, t)_{\text{data}} = \sum_{n=0}^{\infty} \left[ A_n g(t)_{D,n} + B_n g(t)_{N,n} \right] \left[ C_n f(x)_{D,n} + D_n f(x)_{N,n} \right] \]

\[ F(x, t)_{\text{data}} = \sum_{n=0}^{\infty} \left[ AC_n g_{D,n} f_{D,n} + BC_n g_{N,n} f_{D,n} + AD_n g_{D,n} f_{N,n} + BD_n g_{N,n} f_{N,n} \right] \]

\[ \alpha(t)_{1,m} = \oint F(x, t)_{\text{data}} f(x)_{D,m} d\Omega \]

\[ \alpha(t)_{1,m} = \oint \sum_{n=0}^{\infty} \left[ AC_n g_{D,n} f_{D,n} + BC_n g_{N,n} f_{D,n} + AD_n g_{D,n} f_{N,n} + BD_n g_{N,n} f_{N,n} \right] f(x)_{D,m} d\Omega \]

\[ \alpha(t)_{1,n} = AC_n g(t_1)_{D,n} + BC_n g(t_1)_{N,n} \]

\[ \alpha(t)_{1,m} = \oint F(x, t)_{\text{data}} f(x)_{D,m} d\Omega \]

\[ \beta(t)_{1,m} = \oint F(x, t)_{\text{data}} f(x)_{N,m} d\Omega \]

\[ \gamma(t)_{2,m} = \oint F(x, t)_{\text{data}} f(x)_{D,m} d\Omega \]

\[ \Delta(t)_{2,m} = \oint F(x, t)_{\text{data}} f(x)_{N,m} d\Omega \]

\[ \alpha(t)_{1,n} = AC_n g(t_1)_{D,n} + BC_n g(t_1)_{N,n} \]

\[ \beta(t)_{1,n} = AD_n g(t_1)_{D,n} + BD_n g(t_1)_{N,n} \]

\[ \gamma(t)_{2,n} = AC_n g(t_2)_{D,n} + BC_n g(t_2)_{N,n} \]

\[ \Delta(t)_{2,n} = AD_n g(t_2)_{D,n} + BD_n g(t_2)_{N,n} \]
\[
\begin{pmatrix}
\alpha_n & \gamma_n \\
\beta_n & \Delta_n
\end{pmatrix} = \begin{pmatrix}
AC_n & BC_n \\
AD_n & BD_n
\end{pmatrix} \begin{pmatrix}
g(t_{1D,n} & g(t_{2D,n}) \\
g(t_{1N,n} & g(t_{2N,n})
\end{pmatrix}
\]

\[
\begin{pmatrix}
AC'_n & BC'_n \\
AD_n & BD_n
\end{pmatrix} = \begin{pmatrix}
\alpha_n & \gamma_n \\
\beta_n & \Delta_n
\end{pmatrix} \begin{pmatrix}
g(t_{1D,n} & g(t_{2D,n}) \\
g(t_{1N,n} & g(t_{2N,n})
\end{pmatrix}^{-1}
\]
Figure 1: Guitar String
Figure 1: Dual Time Problem - similar to the Initial Value Problem

Figure 2: Having found the amplitudes, the solution is projected forward in time.
Dual Position Problem (conjugate to time problem)

\[ F(x, t)_{0,n} = f(x)_n g(t)_n \]

\[ F(x, t)_{data} = \sum_{n=0}^{\infty} \left[ A_n g(t)_{D,n} + B_n g(t)_{N,n} \right] \left[ C_n f(x)_{D,n} + D_n f(x)_{N,n} \right] \]

\[ F(x, t)_{data} = \sum_{n=0}^{\infty} \left[ AC_n g_{D,n} f_{D,n} + BC_n g_{N,n} f_{D,n} + AD_n g_{D,n} f_{N,n} + BD_n g_{N,n} f_{N,n} \right] \]

\[ \alpha(x)_m = \int F(x, t)_{data} g(t)_{D,m} dt \]

\[ \alpha(x)_m = \int \sum_{n=0}^{\infty} \left[ AC_n g_{D,n} f_{D,n} + BC_n g_{N,n} f_{D,n} + AD_n g_{D,n} f_{N,n} + BD_n g_{N,n} f_{N,n} \right] g(t)_{D,m} \]

\[ \alpha(x)_n = AC_n f(x)_D + AD_n f(x)_{N,n} \]

\[ \alpha(x)_m = \int F(x, t)_{data} g(t)_{D,m} dt \]

\[ \beta(x)_m = \int F(x, t)_{data} g(t)_{N,m} dt \]

\[ \gamma(x)_m = \int F(x, t)_{data} g(t)_{D,m} dt \]

\[ \Delta(x)_m = \int F(x, t)_{data} g(t)_{N,m} dt \]

\[ \alpha(x)_n = AC_n g(t)_{D,n} + AD_n g(t)_{N,n} \]

\[ \beta(x)_n = BC_n g(t)_{D,n} + BD_n g(t)_{N,n} \]

\[ \gamma(x)_n = AC_n g(t)_{D,n} + AD_n g(t)_{N,n} \]

\[ \Delta(x)_n = BC_n g(t)_{D,n} + BD_n g(t)_{N,n} \]
\[
\begin{pmatrix}
\alpha_n & \gamma_n \\
\beta_n & \Delta_n
\end{pmatrix}
= 
\begin{pmatrix}
AC_n & AD_n \\
BC_n & BD_n
\end{pmatrix}
\begin{pmatrix}
f(x_1)_{D,n} & f(x_2)_{D,n} \\
f(x_1)_{N,n} & f(x_2)_{N,n}
\end{pmatrix}
\]

\[
\begin{pmatrix}
AC_n & AD_n \\
BC_n & BD_n
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_n & \gamma_n \\
\beta_n & \Delta_n
\end{pmatrix}
\begin{pmatrix}
f(x_1)_{D,n} & f(x_2)_{D,n} \\
f(x_1)_{N,n} & f(x_2)_{N,n}
\end{pmatrix}^{-1}
\]
Figure 3: Dual Position Problem - conjugate to the Dual Time Problem

Figure 4: Having found the amplitudes, the solution is projected across the spatial domain.
Multiple Position Problem

\[ F(x, t)_{0,n} = f(x)_n g(t)_n \]

\[ F(x, t)_{\text{data}} = \sum_{n=0}^{\infty} \left[ A_n g(t)_{D,n} + B_n g(t)_{N,n} + E_n t + F_n \right] \left[ C_n f(x)_{D,n} + D_n f(x)_{N,n} + G_n x + H_n \right] \]

\[ F(x, t)_{\text{data}} = \sum_{n=0}^{\infty} \left[ A_C n g_{D,n} f_{D,n} + B_C n g_{N,n} f_{D,n} + A_D n g_{D,n} f_{N,n} + B_D n g_{N,n} f_{N,n} \right] \]

\[ F(x, t)_{\text{data}} = \ldots \text{ 16 terms, needs 8 locations} \]

\[
\begin{pmatrix}
8x8 \\
\end{pmatrix} = \ldots
\]
Figure 5: Having found the amplitudes, the solution is projected across the spatial domain.
$\alpha_{nm}$ for mode $m=15$ out of 50 modes
fresh water

all the rest

\[ dx = c(x) \cdot dt \]

pick one element

\[ dx = 1 \]

salt water
Bathymmetry
Chesapeake Bay Analysis

• QUODDY Computer Model
  – Finite-Element Model
  – Fully 3-Dimensional
  – 9700 nodes
QUODDY

- Boussinesq Equations
  - Temperature
  - Salinity
- Sigma Coordinates
- No normal flow
- Winds, tides and river inflow included in model
Methods of solution

• Zel’dovich (1985): Velocity vectors fields can be extracted from two scalar potentials

\[ \vec{u} = \vec{\nabla} \times \left[ (\hat{n} \Psi) + \vec{\nabla} \times (\hat{n} \Phi) \right] \]

\[
\begin{align*}
\nabla^2 \Psi^D_n &= -\lambda_n \Psi^D_n \\
\nabla^2 \Phi^N_m &= -\mu_m \Phi^N_m
\end{align*}
\]

\[ \Psi^D \big|_{\text{boundary}} = 0 \]

\[ (\hat{n} \cdot \vec{\nabla} \Phi^N) \big|_{\text{boundary}} = 0 \]

• Lipphardt et al. (2000): Addition of forcing terms allows for non-conservation of mass through a boundary – ie. Water from rivers or the ocean is accounted.

\[
\begin{align*}
\nabla^2 \Theta (x, y, 0, t) &= S_\Theta (t) \\
(\hat{m} \cdot \vec{\nabla} \Theta) \big|_{\text{boundary}} &= (\hat{m} \cdot \vec{u}_{\text{model}}) \big|_{\text{boundary}}
\end{align*}
\]
Putting It All Together

- $\Psi$ is the stream potential (vorticity mode).
- $\Phi$ is the velocity potential (divergent mode).
- Situation analogous to $(\vec{E},\vec{B})$ fields from E&M.
- The vector field representation can be separated into two eigenvalue equations.
- Source term solved via Poisson’s equation.
- The total vector field is written as a sum over all states for each representation.

\[
(u, v) = \sum_{n=1}^{N} a_n (u_n, v_n)_D + \sum_{m=1}^{M} b_m (u_m, v_m)_N + (u(t), v(t))_S
\]
Time Series Analysis

- Chesapeake flow can be written as a Normal Mode expansion \( \Rightarrow u(r,t) \)

\[
\tilde{u}(\vec{r},t) = \sum_{n=1}^{N} a_n(t) \tilde{u}_{n,D}(\vec{r}) + b_n(t) \tilde{u}_{n,N}(\vec{r}) + \tilde{u}(\vec{r},t)_{src}
\]

- Use Galerkin method to extract \( a(t), b(t) \)

- Due to limitations in data collection:
  - use QUODDY (model based on data)
  - Partial domain Galerkin method
Galerkin Method

\[ f(\vec{r}, t) = \sum_{n=1}^{N} a_n(t) \psi_{n,D}(\vec{r}) + b_n(t) \phi_{n,N}(\vec{r}) \]

\[ a_m = \int \left\{ \sum_{n=1}^{N} a_n(t) \psi_n(\vec{r}) \cdot \psi_m(\vec{r}) + b_n(t) \phi_n(\vec{r}) \cdot \psi_m(\vec{r}) \right\} d\Omega_{\text{full}} \]

\[ a_m = \sum_{n=1}^{N} a_n(t) \left\{ \int \psi_n(\vec{r}) \cdot \psi_m(\vec{r}) d\Omega_{\text{full}} \right\} + \sum_{n=1}^{N} b_n(t) \left\{ \int \phi_n(\vec{r}) \cdot \psi_m(\vec{r}) d\Omega_{\text{full}} \right\} \]

\[ a_m = \sum_{n=1}^{N} a_n(t) \{ \delta_{nm} \} + \sum_{n=1}^{N} b_n(t) \{ 0 \} \]

\[ a_m = a_n(t) \]
Partial Domain Galerkin Method

\[ f(\vec{r}, t) = \sum_{n=1}^{N} a_n(t) \Psi_{n,D}(\vec{r}) + b_n(t) \Phi_{n,N}(\vec{r}) \]

\[ \tilde{a}_m = \int \left\{ \sum_{n=1}^{N} a_n(t) \Psi_n(\vec{r}) \cdot \Psi_m(\vec{r}) + b_n(t) \Phi_n(\vec{r}) \cdot \Psi_m(\vec{r}) \right\} d\Omega_{\text{partial}} \]

\[ \tilde{a}_m = \sum_{n=1}^{N} a_n(t) \left\{ \int \Psi_n(\vec{r}) \cdot \Psi_m(\vec{r}) d\Omega_{\text{partial}} \right\} + \sum_{n=1}^{N} b_n(t) \left\{ \int \Phi_n(\vec{r}) \cdot \Psi_m(\vec{r}) d\Omega_{\text{partial}} \right\} \]

let \( \alpha_{nm} = \int \Psi_n(\vec{r}) \cdot \Psi_m(\vec{r}) d\Omega_{\text{partial}} \)

let \( \beta_{nm} = \int \Phi_n(\vec{r}) \cdot \Psi_m(\vec{r}) d\Omega_{\text{partial}} \)

\[ \tilde{a}_m = \sum_{n=1}^{N} \{ \alpha_{nm} \} a_n(t) + \sum_{n=1}^{N} \{ \beta_{nm} \} b_n(t) \]
Fig. 6 Orthogonality between any two modes shows the degree
which information about the system overlaps. For a basis set, the
requirement is that all functions spanning the space have a zero in-
tegral when considering the product of any two modes over the do-
main. The Neumann modes for the Bay are decent as compared to
similar calculations on the unit square and circle.

Employing the same techniques for the Neumann modes,
agreement could not be reached within an acceptable range
(10-20%). While studying the unit square and circle in
order to establish baseline performance, the finite differ-
ence results typically varied more than FEMLAB, so most
likely the error is coming from the finite difference scheme
and not FEMLAB. Although not conclusive, the agree-
ment between the two methods indicates stability of the
solution set.

Although not shown in this report, the Dirichlet and
source terms were calculated for the Chesapeake Bay and
can be found in references [8] and [11]. Comparison with
the finite difference scheme will require further study to
validate either the FEMLAB result or the finite differences
result. Given that Neumann conditions are better suited for
finite element analysis leads one to trust the FEMLAB re-
results, however, improvements to the finite differences are
still possible, which will help validate the FEMLAB solu-
tion set. The solutions arising from the source term be-
behaved as expected. For a detailed inquiry into the complete
eigenmode set calculated to date for the Chesapeake Bay, consult reference [8]. These results can also be seen at the
website listed at USNA [9]. Future work on the velocity vector field for the Chesapeake Bay will include full three
dimensional analysis using FEMLAB and Quoddy in con-
junction.

Concurrent with this study, the residence time of par-
Fig. 7 The progression of eigenvalues for the Neumann modes of
ticulants was analyzed using the Navier-Stokes equations
and integrating the velocity fields [10]. By comparing trends
in velocity fields from an eigenmode calculation to a Navier-
Stokes calculation, one can analyze those parts of the ve-
locity domain that give rise to certain behaviors. This com-
parison represents the initial steps needed to perform a
full Normal Mode Analysis. Plans have begun to further instrument the Bay, leading to the question, how many
real-time measurements are required in order to predict
important behavior within the Chesapeake. Normal Mode
Analysis will be used to help answer this question.

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The boundary for the calculations for this paper were taken from the QUODDY model that the meshes will match. Clearly, the feature set of the boundary has been reduced by averaging the locations of the edges, effectively smoothing the Chesapeake. Even so, the minimal mesh for COMSOL is 3200 elements. By adding in features from the QUODDY mesh, the coarsest mesh will easily be over 5000 elements.

A quick review of oceanographic websites indicates that there are 10 data monitoring stations taking current data in the Chesapeake. This sparse amount of data suggests a data coverage of approximately 0.2%. One possibility for increasing the data coverage is to have many more data taking stations. This is costly and impractical as the Chesapeake is a major waterway for commerce and military use. An alternative approach is to take a truly poor mesh of the Chesapeake Bay. By severely lowering the mesh resolution, the data coverage will increase at the expense of numerical accuracy. By time averaging data, the longer time taken can also overcome poor spatial coverage. Each of these effects suggest an uncertainty principle \( \delta(\text{numeric}) \delta(\text{spatial}) \delta(\text{temporal}) = \text{constant} \). The standard checks for such an analysis will be limited to simple dimensional analysis or 1-D signal applications, as in section 4.

COMSOL is taking projects such as the Chesapeake Bay into new territory as the complexity rises. The bank of tests from simple systems may not easily apply to the richness of the systems capable of being calculated by COMSOL, yet difficult to analyze when combined with real-world data. This is an on-going project. Stay tuned.

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Conclusion

• Possibly found way to use ~10 monitoring stations to extract full Chesapeake Bay flow field.
• Time series has a good chance to work!
• Due to COMSOL and usage of fuller geometric solutions, challenge older precepts based on signal processing.
• If orthonormality is the strongest requirement, create post-processing options to massage results obtained from solvers. Given them tolerances to adjust results.
• Please allow for easier adjustment and creation of meshes!
• Visit us at: http://web.usna.navy.mil/~rmm/
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