Decomposition of Spectra from the Drum
With Applications to the Chesapeake Bay

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<td>b. ABSTRACT</td>
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Outline

• Review of Problem Statement / History
• Motivation / Applications
• Methods Applied – Toy Problems
• Analysis
• Conclusions
Drumhead Problem

• Given a sample of a drums sound, attempt to calculate the amplitudes of the modes – time-series at location of microphone

• Related to a famous problem posed by mathematician Mark Kac (1966) asking: “Can One Hear the Shape of a Drum?”

• Lead to idea of iso-spectral drums
Isospectral Drums

• Drums with differing boundaries that have identical $k_n$ values – so they sound alike!

Milnor, 1966  …  Driscoll, 1997 SIAM
Outline

• Review of Problem Statement / History
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• Methods Applied – Toy Problems
• Analysis
• Conclusions
Why Measure the Amplitudes?

• Being able to measure modes strengths suggests dynamics about the system.
• By measuring “windows” in time that overlap, the time-dependence of the amplitudes can be seen.
• Energy conservation – once a mode is excited, where does the energy go?
• Lead to prediction of amplitudes beyond the time window (forecasting).
Outline

• Review of Problem Statement / History
• Motivation / Applications
• Methods Applied – Toy Problems
• Analysis
• Conclusions
Methodology

• Three key issues:
  – Delta Function
  – Solution Architecture
  – Degeneracy of States

• Midn 1/C Grant Hundley
  – Developed a drum simulator
  – Working on numerical scheme to extract amplitudes from simulated time-series
Delta Function

\[ u(x, t) = \sum_{n=0}^{\infty} \left[ a_n(t) \sin(k_n x) + b_n(t) \cos(k_n x) \right] \]

\[ a_n(t) = \frac{1}{L} \int_{-L}^{+L} u(x, t)\text{data} \sin(k_m x) dx \]

\[ a_m(t) = \frac{1}{L} \int_{-L}^{+L} \sum_{n=0}^{\infty} \left[ a_n(t) \sin(k_n x) + b_n(t) \cos(k_n x) \right] \sin(k_m x) dx \]

\[ a_n(t) = \sum_{n=0}^{\infty} \left[ a_n(t) \frac{1}{L} \int_{-L}^{+L} \sin(k_n x) \sin(k_m x) dx + b_n(t) \frac{1}{L} \int_{-L}^{+L} \cos(k_n x) \sin(k_m x) dx \right] \]

\[ a_n(t) = \sum_{n=0}^{\infty} \left[ a_n(t) \delta_{nm} + b_n(t) \delta \right] \]

\[ a_m(t) = \Delta_{mn} a_n(t) \]

\[ a_n(t) = \int u(x, t)\text{data} \sin(k_n x) dx \]

\[ b_n(t) = \int u(x, t)\text{data} \cos(k_n x) dx \]
Delta Function Assumptions

\[ \omega_n = \frac{n \pi}{2T} \quad \text{and} \quad \omega_m = \frac{m \pi}{2T} \]

\( \omega_n \) and \( \omega_m \) have integer relationship

\[(T_1 = 4T')\]

Wavelength is set from the window and is symmetric

Define: \( \delta_{mn} \) and \( \varphi \)

\[ \delta_{mn} = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \sin(\omega_m t) \, dt \]

\[ \varphi = \frac{1}{T} \int_{-T}^{+T} \cos(\omega_n t) \sin(\omega_m t) \, dt \]

\[ \Delta_{mn} = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \sin(\omega_m t) \, dt \]

\[ \varepsilon_{mn} = \frac{1}{T} \int_{-T}^{+T} \cos(\omega_n t) \sin(\omega_m t) \, dt \]

Define: Delta and Epsilon

Should these conditions fail.
Delta Function

\[ \Delta_m = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \sin(\omega_m t) dt \]

\[ \Delta_m = \frac{1}{2T} \int_{-T}^{+T} \cos((\omega_n - \omega_m) t) - \cos((\omega_n + \omega_m) t) \, dt \]

\[ \Delta_m = \frac{1}{2T} \frac{1}{\omega_n - \omega_m} \sin((\omega_n - \omega_m) x) \left[ \frac{1}{T} \cos((\omega_n + \omega_m) x) \right] - \frac{1}{2T} \frac{1}{\omega_n + \omega_m} \sin((\omega_n + \omega_m) x) \left[ \frac{1}{T} \cos((\omega_n - \omega_m) x) \right] \]

\[ \Delta_m = \frac{1}{2T} \frac{1}{(n-m)\pi} \sin \left( \frac{(n-m)\pi}{2T} x \right) \left[ \frac{1}{T} \cos \left( \frac{(n+m)\pi}{2T} x \right) \right] - \frac{1}{2T} \frac{1}{(n+m)\pi} \sin \left( \frac{(n+m)\pi}{2T} x \right) \left[ \frac{1}{T} \cos \left( \frac{(n-m)\pi}{2T} x \right) \right] \]

\[ \Delta_m = \frac{2}{\pi(n-m)} \sin \left( \frac{\pi}{2} (n-m) \right) - \frac{2}{\pi(n+m)} \sin \left( \frac{\pi}{2} (n+m) \right), \quad \text{for even } n \to 2n \]

\[ \Delta_m = \text{sinc}(n-m) - \text{sinc}(n+m), \quad \text{assuming } m \text{ is free (real)} \]

\[ \Delta_{\pm m, 2n} = \delta_{\pm m, 2n}, \quad \text{assuming } n \text{ is even and } m \text{ is an integer} \]

\[ \varepsilon_m = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_n t) \cos(\omega_m t) dt \]

\[ \varepsilon_m = \frac{1}{2T} \int_{-T}^{+T} [\sin((\omega_n - \omega_m) t) - \sin((\omega_n + \omega_m) t)] \, dt \]

\[ \varepsilon_m = \phi \]
Delta Function \sim \text{Sinc}(\pi^*(n-m))

The $\Delta_{nm}$ function shown below shows its clear approximation to $\delta_{nm}$ when $(n, m)$ are integers. Also shown are the twin responses at $(-m, +m)$ for the $n = 10$ case.
Delta & Epsilon Function

- Delta function is a measure of the “mixing” between the eigenmodes.
- Slightly different Delta function for the cos*cos term (more on this later).
- Epsilon will be identically zero for symmetric domains.
Epsilon(\(\omega_m\))

Epsilon Timing Resolution Function

\[
\varepsilon(\omega_m = 10) = \cos(\pi(10-x))/(\pi(10-x)) - \cos(\pi(10+x))/(\pi(10+x))
\]
Solution Architecture:
Solutions to space-time problems

\[ u(x, t) = f(x) \cdot g(t) \]

\[ f(x) = \sum_{n=0}^{\infty} A_n f_D(k_n x) + B_n f_N(k_n x) \]

\[ g(t) = \sum_{n' = 0}^{\infty} C_{n'} g_D(\omega_{n'} t) + D_{n'} g_N(\omega_{n'} t) \]

\[ u(x, t) = \left[ \sum_{n=0}^{\infty} A_n f_D(k_n x) + B_n f_N(k_n x) \right] \left[ \sum_{n' = 0}^{\infty} C_{n'} g_D(\omega_{n'} t) + D_{n'} g_N(\omega_{n'} t) \right] \]

\[ u(x, t) = \sum_{n=0}^{\infty} AC_n f_{D,n} g_{D,n} + BC_n f_{N,n} g_{D,n} + AD_n f_{D,n} g_{N,n} + BD_n f_{N,n} g_{N,n} \]

\[ u(x, t) = \sum_{n=0}^{\infty} \sum_{n' = 0}^{\infty} AC_{n,n'} f_{D,n} g_{D,n'} + BC_{n,n'} f_{N,n} g_{D,n'} + AD_{n,n'} f_{D,n} g_{N,n'} + BD_{n,n'} f_{N,n} g_{N,n'} \]
Nature of Eigenmodes with both space and time in solution

• Two types:
  – Coupled - for each $k$ eigenvalue in space there exists a unique $\omega$ eigenvalue in time
    • Dispersion relationship, $\omega(k)$, for most differential equations in $(x,t)$.
    • Dispersion relationship is generally monotonic and increasing.
  – Decoupled – for systems not well motivated physically, yet can be described with a space-time basis set (river bank problem).
Solution Architecture

At one location, $x_1$, sample the data in a time-series.

Project out the sin and cosines.

\[
\alpha_m = \frac{1}{T} \int_{-T}^{+T} \sin(\omega_m t) \; u(x_1, t) \; dt
\]

\[
\beta_m = \frac{1}{T} \int_{-T}^{+T} \cos(\omega_m t) \; u(x_1, t) \; dt
\]

The outer product becomes a matrix of $\Delta_{mn}$ and $\varepsilon_{mn}$. 
Solution Architecture

\[ \begin{align*}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m
\end{pmatrix} &= 
\begin{pmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1n} \\
\Delta_{21} & \Delta_{22} & \Delta_{23} & \cdots & \Delta_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta_{m1} & \Delta_{m2} & \Delta_{m3} & \cdots & \Delta_{mn}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \cdots & \varepsilon_{1n} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \cdots & \varepsilon_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\varepsilon_{m1} & \varepsilon_{m2} & \varepsilon_{m3} & \cdots & \varepsilon_{mn}
\end{pmatrix}
\end{align*} \]

\[ \begin{pmatrix}
f_D(k_1x_1) \\
f_D(k_2x_1) \\
\vdots \\
f_D(k_mx_1)
\end{pmatrix} = 
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_m
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} & \cdots & \varepsilon_{n1} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} & \cdots & \varepsilon_{n2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\varepsilon_{1m} & \varepsilon_{2m} & \varepsilon_{3m} & \cdots & \varepsilon_{nm}
\end{pmatrix}
\begin{pmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1n} \\
\Delta_{21} & \Delta_{22} & \Delta_{23} & \cdots & \Delta_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta_{m1} & \Delta_{m2} & \Delta_{m3} & \cdots & \Delta_{mn}
\end{pmatrix}
\begin{pmatrix}
f_D(k_1x_1) \\
f_D(k_2x_1) \\
\vdots \\
f_D(k_mx_1)
\end{pmatrix} \]

- Mixing nature of Delta, Epsilon matrix can clearly be seen.
- No nodes can exist in the f(k*x) matrix.
- Solve for Amplitudes thru matrix inversion.
Solution Architecture

- Short-hand notation:

\[
\begin{align*}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} &= \begin{pmatrix}
\Delta_t & \varepsilon_t \\
\varepsilon_t^t & \Delta_t^c
\end{pmatrix} \begin{pmatrix}
f_D \\
f_D^c
\end{pmatrix} \begin{pmatrix}
AC \\
AD
\end{pmatrix} \\
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} &= \begin{pmatrix}
\Delta_t & \varepsilon_t \\
\varepsilon_t^t & \Delta_t^c
\end{pmatrix} \begin{pmatrix}
f_D & O \\
O & f_D
\end{pmatrix} \begin{pmatrix}
AC \\
AD
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} &= \frac{1}{T} \int_{-T}^{+T} \left( \begin{pmatrix} g_D & g_N \end{pmatrix} \otimes \begin{pmatrix} f_D & O \\
O & f_D \end{pmatrix} \begin{pmatrix} AC \\
AD \end{pmatrix} \right) dt = \frac{1}{T} \int_{-T}^{+T} \left( g_D \otimes \begin{pmatrix} AC \cdot f_D \\
AD \cdot f_D \end{pmatrix} \right) dt \\
\begin{pmatrix}
\alpha_m \\
\beta_m
\end{pmatrix} &= \frac{1}{T} \int_{-T}^{+T} \left( g_{D_m} \otimes \sum_{n=0}^{\infty} \left[ AC_n \cdot f_D(k_n \cdot x_1) \cdot g_{D_n} + AD_n \cdot f_D(k_n \cdot x_1) \cdot g_{N_n} \right] \right) dt = \frac{1}{T} \int_{-T}^{+T} \left( g_{N_m} \otimes u(x_1, t) \right) dt
\end{align*}
\]
Drum – Specific Solutions

• Drum problem: membrane stretched over a circular boundary (Dirichlet bc).
• Strike the drum.
• Use a microphone to record the time-series.
• Fourier analyze the time-series to obtain amplitudes for each $\omega_m : A_n$
Drum

\[ u(x, t) = \sum_{n=0}^{\infty} \left[ A_n J_n(k_n r) \sin(n \theta) + B_n J_n(k_n r) \cos(n \theta) \right] \left[ C_n \sin(\omega_n t) + D_n \cos(\omega_n t) \right] \]

\[ u(r, t) = \sum_{(n,n')=0}^{\infty} \left[ A_{nn'} J_n(k_{nn'} r) \sin(n \theta) + B_{nn'} J_n(k_{nn'} r) \cos(n \theta) \right] \left[ C_{nn'} \sin(\omega_{nn'} t) + D_{nn'} \cos(\omega_{nn'} t) \right] \]

- Spatial modes are a combination of Bessel function, \( J(k_n r) \) times \( \sin(n \theta) \) or \( \cos(n \theta) \)
- Temporal modes use \( \sin(\omega_n t) \) or \( \cos(\omega_n t) \)
- For each Bessel function, there exists multiple zero crossings, \( n' \)
- \( k_n \) values are non-integerlike, so \( \omega_n \) fail conditions for orthonormality
### Table 1: Bessel Function Zero Crossings

<table>
<thead>
<tr>
<th>n</th>
<th>$J_0(x)$</th>
<th>$J_1(x)$</th>
<th>$J_2(x)$</th>
<th>$J_3(x)$</th>
<th>$J_4(x)$</th>
<th>$J_5(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4048</td>
<td>3.8317</td>
<td>5.1356</td>
<td>6.3802</td>
<td>7.5883</td>
<td>8.7715</td>
</tr>
<tr>
<td>2</td>
<td>5.5201</td>
<td>7.0156</td>
<td>8.4172</td>
<td>9.7610</td>
<td>11.0647</td>
<td>12.3386</td>
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</tbody>
</table>

### Table 2: Bessel Function Zero Crossings

<table>
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<th>$k_{nn'}$</th>
<th>n</th>
<th>$n'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4048</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5.5201</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>8.6537</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>11.792</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>3.8317</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7.0156</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10.173</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>13.324</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>5.1356</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>8.4172</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>11.62</td>
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<td>3</td>
</tr>
<tr>
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<td>2</td>
<td>4</td>
</tr>
<tr>
<td>6.3802</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>9.761</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>13.015</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>16.223</td>
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<td>4</td>
</tr>
<tr>
<td>7.5883</td>
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<td>1</td>
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<tr>
<td>11.065</td>
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<td>2</td>
</tr>
<tr>
<td>14.373</td>
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<td>3</td>
</tr>
<tr>
<td>17.616</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

### Table 3: Bessel Function Zero Crossings

<table>
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<tr>
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<th>n</th>
<th>$n'$</th>
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<tbody>
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<td>2.4048</td>
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<tr>
<td>17.616</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
Degeneracy of States

\[ u(r, t) = \begin{pmatrix} g_D(\omega_1 t) & \cdots & g_D(\omega_m t) \end{pmatrix} \begin{pmatrix} g_N(\omega_1 t) & \cdots & g_N(\omega_m t) \end{pmatrix} \]

\[ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \]

the Degeneracy Matrix \( \equiv D \)

\[ u(r, t) = \begin{pmatrix} g_D(\omega_1 t) & \cdots & g_D(\omega_m t) \end{pmatrix} \begin{pmatrix} J_0(k_0 r_1) \sin(0 \theta_1) \cdot AC_1 + J_0(k_0 r_1) \cos(0 \theta_1) \cdot AC_3 \\ J_1(k_{11} r_1) \sin(1 \theta_1) \cdot AC_2 + J_1(k_{11} r_1) \cos(1 \theta_1) \cdot AC_4 \\ J_2(k_{21} r_1) \sin(2 \theta_1) \cdot AC_5 + J_2(k_{21} r_1) \cos(2 \theta_1) \cdot AC_6 \\ \vdots \\ J_n(k_{nn} r_1) \sin(n \theta_1) \cdot AC_n + J_n(k_{nn} r_1) \cos(n \theta_1) \cdot AC_n \end{pmatrix} \]

\[ \begin{pmatrix} J_0(k_0 r_1) \sin(0 \theta_1) \cdot AD_1 \\ J_0(k_0 r_1) \cos(0 \theta_1) \cdot AD_2 \\ J_1(k_{11} r_1) \sin(1 \theta_1) \cdot AD_3 + J_1(k_{11} r_1) \cos(1 \theta_1) \cdot AD_4 \\ J_2(k_{21} r_1) \sin(2 \theta_1) \cdot AD_5 + J_2(k_{21} r_1) \cos(2 \theta_1) \cdot AD_6 \\ \vdots \\ J_n(k_{nn} r_1) \sin(n \theta_1) \cdot AD_n + J_n(k_{nn} r_1) \cos(n \theta_1) \cdot AD_n \end{pmatrix} \]
Degeneracy and Sampling

- Due to symmetry in the solution, degenerate states are produced.
- Due to degeneracy, there is a 2-1 ratio of unknowns-knowns.
- Solution: add another sample location
The second sampled point →
Calculating the Amplitudes
Drum Solution in Short-hand

- Number of sample locations “squares-off” the degeneracy matrix, allowing the system to be solvable.

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\eta
\end{pmatrix} =
\begin{pmatrix}
\Delta_t & 0 \\
0 & \Delta_t
\end{pmatrix}
\begin{pmatrix}
\mathbb{D} \\
\mathbb{D}
\end{pmatrix}
\begin{pmatrix}
f_{D_1} & 0 \\
0 & f_{D_1} \\
f_{D_2} & 0 \\
0 & f_{D_2}
\end{pmatrix}
\begin{pmatrix}
AC \\
AD
\end{pmatrix}
\]

\[
\begin{pmatrix}
AC \\
AD
\end{pmatrix} =
\begin{pmatrix}
\mathbb{D} \\
\mathbb{D}
\end{pmatrix}
\begin{pmatrix}
f_{D_1} & 0 \\
0 & f_{D_1} \\
f_{D_2} & 0 \\
0 & f_{D_2}
\end{pmatrix}^{-1}
\begin{pmatrix}
\Delta_t & 0 \\
0 & \Delta_t
\end{pmatrix}^{-1}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\eta
\end{pmatrix}
\]
Drum Problems

- n=0 \sin(n\theta) term needs to be removed, breaking the 2-1 ratio to less than 2-1.
- Test method against drum simulation, with known inputs to the amplitudes, A_n.
- Further degeneracies exist due to closeness of k-eigenvalues.
Chesapeake Bay Problems

• No known analytic solution to the Bay (ie. No analytic hints as to any degeneracy)
• At a given samples location, calculate the projections, \((\alpha_m,\beta_m)\) for a range of \((\omega_m, T)\).
• Observe the patterns of \(\omega_m\) and compare the sequence to \(k_n\)’s.
• Guess the dispersion relationship (map from \(k_n\) to \(\omega_m\))
Degeneracy of States

• From observables at frequencies $\omega_m$, observe projection changes as the period, $T$, is changed.
• Select an appropriate period, $T$.
• Construct Degeneracy matrix based on best guess of dispersion relation as well as k-eigenvalues density.
Timing Resolution Nature of Delta Matrix

\[ \Delta \omega_{mn} = \frac{\pi}{T} \]

\[ dk = \frac{\pi}{\frac{\delta \omega}{\delta k} T} \]
$\Delta(\omega_m, T)$

Timing Resolution - Delta Matrix - $\Delta(T, \omega_m)$

$T$ - Period

$\omega_m$ - Frequency
Delta($\omega_m, T$)

Timing Resolution - Delta Matrix - $\Delta(T, \omega_m)$

$T$ - Period

$\omega_m$ - Frequency
Future Plans:

• Run thru the toy model (Drum).
• Add source terms to Chesapeake Bay model.
• Add Delta(spatial) matrix.
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Chesapeake Bay Problem

- Take data at stations around the Bay, collecting time-series of vector flows.
- How many stations are needed to provide enough data to fully calculate the modes?
Image Processing of the Chesapeake
Advantages

• Eigenmodes fill domain (space), suggest future behavior