EFFICIENT GPS POSITION DETERMINATION ALGORITHMS

DISSERTATION

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AFIT/DS/ENG/07-09

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Abstract

This research is aimed at improving the state of the art of GPS algorithms, namely, the development of a closed-form positioning algorithm for a stand-alone user and the development of a novel differential GPS algorithm for a network of users.

The stand-alone user GPS algorithm is a direct, closed-form, and efficient new position determination algorithm that exploits the closed-form solution of the GPS trilateration equations and works in the presence of pseudorange measurement noise for an arbitrary number of satellites in view. A two-step GPS position determination algorithm is derived which entails the solution of a linear regression and updates the solution based on one nonlinear measurement equation. In this algorithm, only two or three iterations are required as opposed to five iterations that are normally required in the standard Iterative Least Squares (ILS) algorithm currently used. The mathematically derived stochastic model-based solution algorithm for the GPS pseudorange equations is also assessed and compared to the conventional ILS algorithm. Good estimation performance is achieved, even under high Geometric Dilution of Precision (GDOP) conditions.

The novel differential GPS algorithm for a network of users that has been developed in this research uses a Kinematic Differential Global Positioning System (KDGPS) approach. A network of mobile receivers is considered, one of which will be designated the 'reference station' which will have known position and velocity information at the beginning of the time interval being examined. The measurement situation on hand is properly modeled, and a centralized estimation algorithm processing several epochs of data is developed. The effect of uncertainty in the reference receiver's position and the

level of the receiver noise are investigated. Monte Carlo simulations are performed to examine the ability of the algorithm to correctly estimate the non-reference mobile users' position and velocity despite substantial satellite clock errors and receiver measurement noise.

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Thao Nguyen

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Nomenclature

N	Number of GPS pseudorange measurements available (T / ΔT)
n	Number of satellites in view
ΔT	Sampling time difference interval
T	Temporal horizon (time duration of interest)
(x_j, y_j, z_j)	User position at discrete time j
$x_{s_{i,j}}$	i^{th} Satellite position at discrete time j
$\rho_{i,j}$	Distance from the user's receiver to satellite i at time j
Z_i	i^{th} GPS pseudorange measurement
τ	User's receiver range equivalent clock bias
$\dot{\tau}$	User's receiver range equivalent clock drift
c	Speed of light
i	Satellite number index
j	Discrete time instance index
k	Users index
m	Number of reference stations
$\hat{x}$	Position estimate
θ	Parameter vector of interest (positions, velocities, clock biases)

EFFICIENT GPS POSITIONING DETERMINATION ALGORITHMS

I   Introduction

The NAVSTAR Global Positioning System (GPS) is a space-based satellite radio navigation system that provides three-dimensional (3-D) user positioning by solving a set of nonlinear trilateration equations using pseudorange measurements. The current method of solving the nonlinear equations is to linearize the pseudorange equations and calculate the user position iteratively, starting with a user-provided initial position guess. In this research, it is recognized up front that pseudorange measurements are noise corrupted. Hence, the stochastic nature of the measurements is reflected in the GPS pseudorange equations from the onset to develop a probabilistically sound GPS solution. By stochastically modeling the measurement situation at hand, solving for position or velocity becomes a stochastic estimation problem.

This research consists of two parts, both of which stochastically model the pseudorange measurement with random white noise and solve for position or velocity as a stochastic estimation problem. The first part is a direct, closed-form and efficient new position determination algorithm that exploits the closed-form solution of the GPS trilateration equations and works in the presence of pseudorange measurement noise for an arbitrary number of satellites in view. In some applications, a two-step GPS position determination algorithm, which entails the solution of a linear regression and updates the

solution based on one nonlinear measurement equation, is needed. In this algorithm, only two or three iterations are required, as opposed to five iterations that are required in the standard Iterative Least Squares (ILS) algorithm currently used. The mathematically derived stochastic model-based solution algorithm for the GPS pseudorange equations is also assessed and compared to the conventional ILS algorithm. Good estimation performance is achieved, even under high Geometric Dilution of Precision (GDOP) conditions.

The second part of this research investigates a Kinematic Differential Global Positioning System (KDGPS) algorithm. A number of mobile receivers is considered, one of which will be designated the 'reference station', which will have known position and velocity information at the beginning of the time interval examined. The measurement situation on hand is properly modeled, and a centralized estimation algorithm processing several epochs of data is developed. The effect of uncertainty in the reference receiver's position and the level of receiver noise are investigated. Monte Carlo simulations are performed to correctly estimate the non-reference mobile users' position and velocity despite substantial satellite clock errors and receiver measurement noise.

1.1 Global Positioning System

The Global Positioning System consists of a constellation of 24 satellite vehicles (SV) arranged in six orbital planes inclined at 55 degrees at an altitude of 20,200km. The constellation continually broadcasts signals that can be utilized by a receiver on the user's platform based on the concept of one-way time of arrival (TOA) ranging. The GPS

receiver determines the range from each visible satellite to the user's platform. This measured range is called a "pseudorange" since there are errors present on the GPS signal.

There are four unknown parameters involved with GPS positioning, which include 3 Cartesian position parameters x, y, and z, and GPS clock error. To determine a solution it is therefore required that at least four GPS SVs be within view of the receiver. The pseudoranges from the SVs are used to determine the user's position with respect to Earth. A typical GPS scenario is shown in Figure 1 below. SV geometry plays a critical part in determining GPS positioning. Poor SV geometry with respect to the receiver produces high Geometric Dilution of Precision (GDOP), which can adversely affect GPS position solutions [1].
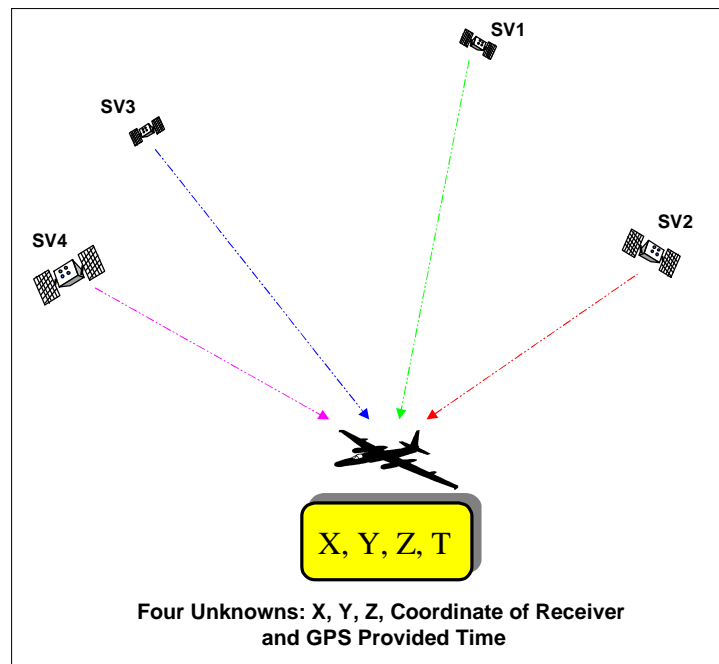
Figure 1. **A GPS Scenario [2]**

GPS provides two types of services. The Standard Positioning Service (SPS) is designated for the civilian users. The Precise Positioning Service (PPS) is intended for U.S. military and selected government agencies. Access to PPS signal is controlled through two cryptographic features denoted as Antispoofing (AS) and Selective Availability (SA). AS is a technique intended to defeat deception jamming, whereas SA is a method to intentionally inject additional error onto the GPS signal to deny full system accuracy.

SA decreased the positioning accuracy of stand-alone receivers to within 100-meters RMS. Military receivers utilized de-encryption techniques to remove SA and provide position accuracy of 10-meters root-mean-square (RMS) [1]. SA was effectively turned off in 2000.

1.1.1 Stand-Alone GPS Positioning

Single point, or stand alone, GPS techniques utilize signals broadcast from the GPS satellites as depicted in Figure 1. First, a nominal state (consisting of user position coordinates and receiver clock error) can be represented as

$$\hat{\mathbf{x}}_u = \begin{bmatrix} \hat{x}_u \\ \hat{y}_u \\ \hat{z}_u \\ c\hat{\delta t}_u \end{bmatrix} = approximate\ state \tag{1-1}$$

At a given measurement epoch, the GPS receiver generates a set of n pseudorange equations (where n is the number of satellites visible to the receiver). The pseudorange from the user receiver to the ith satellite is the sum of the true range plus the receiver range equivalent clock error (i.e., after SV clock, ionospheric, tropospheric, etc. errors have been corrected or deemed negligible)

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$$\rho_i = \sqrt{(x_i - x_u)^2 + (y_i - y_u)^2 + (z_i - z_u)^2} + c\delta t_u \qquad (1\text{-}2)$$

where

(x_u, y_u, z_u) $\qquad = $ ECEF position coordinates of the user (m)

(x_i, y_i, z_i) $\qquad = $ ECEF position coordinates of the ith satellite (m)

$c\delta t_u$ $\qquad\qquad = $ Range equivalent receiver clock error (m)

These equations are non-linear, and several techniques have been developed to solve for the user position. These include closed-form solutions, Kalman filtering, and ILS techniques based on linearization [1, 3]. Since the ILS algorithm is arguably the simplest approach, and most commonly used, it will be subsequently explained.

Since the position of the user is not known, an estimate of the user position $(\hat{x}, \hat{y}, \hat{z})$ is used to generate a set of estimated pseudoranges to each of the n satellites, i.e.,

$$\hat{\rho}_i = \sqrt{(x_i - \hat{x}_u)^2 + (y_i - \hat{y}_u)^2 + (z_i - \hat{z}_u)^2} + c\hat{\delta t}_u \qquad (1\text{-}3)$$

The relationship between the true and the estimated position with errors can be written as

$$\mathbf{x}_u = \hat{\mathbf{x}}_u + \Delta\mathbf{x}_u \qquad (1\text{-}4)$$

The approximated pseudorange equations (equation (1-3)) are then linearized using a first order Taylor series approximation to yield

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x} \qquad (1\text{-}5)$$

where

$$\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ : \\ \Delta\rho_n \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & -1 \end{bmatrix} \quad \Delta\mathbf{x} = \begin{bmatrix} \Delta x_u \\ : \\ \Delta c\hat{\delta t}_u \end{bmatrix} \qquad (1\text{-}6)$$

with the elements $\Delta\rho_i$ defined as

$$\Delta\rho_i = \hat{\rho}_i - \rho_i$$

$$a_{xi} = \frac{x_i - \hat{x}_u}{\hat{r}_i}, \quad a_{yi} = \frac{y_i - \hat{y}_u}{\hat{r}_i}, \quad a_{zi} = \frac{z_i - \hat{z}_u}{\hat{r}_i}$$

$$\hat{r}_i = \sqrt{(x_i - \hat{x}_u)^2 + (y_i - \hat{y}_u)^2 + (z_i - \hat{z}_u)^2}$$

Solving equation (1-5) has the solution

$$\Delta\mathbf{x} = \mathbf{H}^{-1}\Delta\boldsymbol{\rho} \tag{1-7}$$

The values obtained for $\Delta\mathbf{x}$ are used to update equation (1-4) for the user position.

There are three possible cases to be considered. If there are fewer than four satellite pseudoranges available, the position cannot be determined since Δx cannot be resolved. If there are exactly 4 distinct pseudoranges, there will be a unique solution. However, if there are more than four satellites visible, as is generally the situation, an overdetermined linear system is obtained, and no solution will be available that will perfectly solve the equation in $\Delta\mathbf{x}$. For this case, the least squares solution concept can be utilized.

Basic least-squares technique yields the solution

$$\Delta\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho} \tag{1-8}$$

Alternatively, a weighted least-squares solution

$$\Delta\mathbf{x} = (\mathbf{H}^T\mathbf{C}_{\boldsymbol{\rho}}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}_p^{-1}\Delta\boldsymbol{\rho} \tag{1-9}$$

can be used when the pseudorange measurements have different error statistics or when the pseudorange measurement errors are correlated. The matrix $\mathbf{C}_{\boldsymbol{\rho}}$ is the measurement error covariance matrix (diagonal terms are measurement error variances and off-diagonal terms are correlation between measurement errors). It is noted that this weighted solution is identical to the unweighted case if $\mathbf{C}_{\boldsymbol{\rho}} = \mathbf{I}$ (identity matrix).

For the over-determined case, there is generally no solution for $\Delta\mathbf{x}$ that exactly solves the measurement equation. However, measurement residuals, $\mathbf{v}$, can be applied to the measurements which would result in

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x} + \mathbf{v} \tag{1-10}$$

or

$$\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x} \tag{1-11}$$

Single point positioning estimates only receiver clock errors, and requires a correction for the satellite clock error. Satellite clock error corrections can be accomplished as described in ICD-GPS-200C [3]:

$$\rho_{corrected} = \rho + c\Delta t_{sv} \tag{1-12}$$

where

$\rho_{corrected}$	= pseudorange corrected for SV clock error
ρ	= original (raw) pseudorange measurement
Δt_{sv}	= SV clock correction = $a_{f0} + a_{f1}(t - t_0) + a_{f2}(t - t_0)^2 + \Delta t_r$
$a_{f0}, a_{f1}, a_{f2,} t_0$	= SV clock correction parameters from navigation message
Δt_r	= relativity correction = Fe $\sqrt{a}$ sin (E_k)
F	= constant = -4.442807633x10^{-10} sec/(meter)$^{1/2}$
e	= eccentricity from navigation message
$\sqrt{a}$	= square root of semi-major axis from navigation message
E_k	= Eccentric anomaly (from SV position calculation)

Stand-alone GPS positioning techniques are fast and reliable. However, the poor accuracy results (typically 30m-50m) are undesirable. In order to obtain the relative accuracy required for uses such as vehicle formation flying, kinematic and DGPS techniques must be employed.
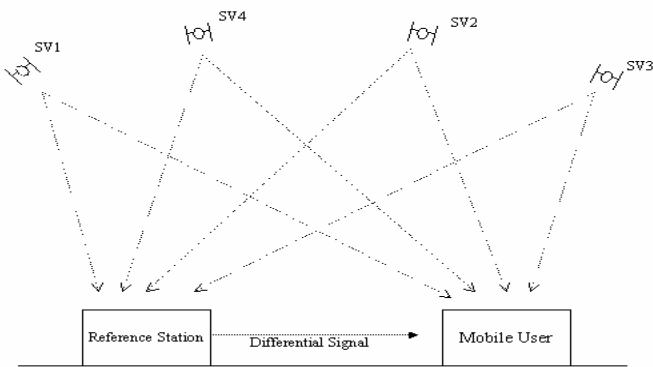
Figure 2. **A Typical DGPS Situation**

1.1.2 Differential GPS Positioning Techniques

The principal idea behind DGPS is that if a reference receiver is available with a known position, common errors between it and relatively close mobile receivers viewing the same satellites can be removed. Generally, there are two basic DGPS techniques, code and carrier-phase based [1].

1.1.3 Code-Based Algorithm

Let the reference receiver, *m,* have a known position represented as (x_m, y_m, z_m) and the reported i^{th} satellite position (via ephemeris data) be represented as (x_i, y_i, z_i). The geometric distance from the reference receiver to the ith satellite is

$$R_m^i = \sqrt{(x_i - x_m)^2 + (y_i - y_m)^2 + (z_i - z_m)^2} \qquad (1\text{-}13)$$

The reference receiver is then able to generate a pseudorange measurement to the ith satellite as

$$\rho_m^i = R_m^i + \varepsilon_{m,space} + \varepsilon_{m,user} + c\delta t_m \qquad (1\text{-}14)$$

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where $\varepsilon_{m,\text{space}}$, $\varepsilon_{m,\text{control}}$ and $\varepsilon_{m,\text{user}}$ are the space, control, and user segment induced pseudorange errors, respectively, and δt_m represents the reference receiver's clock offset from GPS system time. The errors are summarized in Table 1 [1]. The reference receiver simply resolves the difference between the generated pseudorange to the ith satellite, ρ_m^i and it's geometric range, R_m^i, to create the differential correction

$$\Delta\rho_m^i = \rho_m^i - R_m^j = \varepsilon_{m,space} + \varepsilon_{m,control} + \varepsilon_{m,user} + c\delta t_m \qquad (1\text{-}15)$$

This correction term is utilized by the user, or mobile, receiver, where it is differenced with the users' generated pseudorange measurement to the same satellite

$$\begin{aligned}\rho_u^i - \Delta\rho_m^i = R_u^j &+ \varepsilon_{u,space} + \varepsilon_{u,control} + \varepsilon_{u,user} + c\delta t_m \\ &- (\varepsilon_{m,space} + \varepsilon_{m,control} + \varepsilon_{m,user} + c\delta t_m)\end{aligned} \qquad (1\text{-}16)$$

If the user's receiver is located relatively nearby the reference receiver, the user's receiver pseudorange equation error components will be nearly identical to those of the reference receiver. Exceptions include errors that are not common to both receivers, i.e. multi-path and receiver noise. Therefore, the corrected user pseudorange is obtained

$$\rho_{u,corrected} = R_u^j + \varepsilon_u' + c\delta t_{combined} \qquad (1\text{-}17)$$

where ε_u' is the *residual* user segment error (multi-path, etc) and $\delta t_{combined}$ is the combined clock offset $(\delta t_u - \delta t_m)$. A typical comparison between stand-alone measurements and DGPS measurements is given in Table 1 [1].

Table 1.   Typical Satellite errors before/after DGPS correction [1]

Segment Source	Error Source	GPS 1σ Error (m)	Typical GPS 1σ errors after DGPS corrections
Space	Satellite clock stability	3.0	0
	Satellite perturbation	1.0	0
	Selective Availability	32.3	0
	Other (thermal, radiation, etc)	0.5	0
Control	Ephemeris prediction error	4.2	0
	Other (thruster performance, etc)	0.9	0
User	Ionospheric delay	5.0	0
	Tropospheric delay	1.5	0
	Receiver noise and resolution	1.5	2.1
	Multipath	2.5	2.5
	Other (inter-channel bias, etc)	0.5	0.5
System UERE	Total (rms)	33.3	3.3

1.1.4 Carrier Phase-Based Algorithm

Obviously, more precise position information can be obtained by measuring the amount of shift in the frequency (Doppler shift) of the received signal. Typically, this shift in carrier frequency arises from the relative motion of the GPS satellites to the user resulting in Doppler shift frequencies of $\Delta f = \pm 5000\, Hz$ with respect to the L1 and L2 carriers. Thus,

$$\Delta f = f_R - f_T \qquad (1\text{-}18)$$

where,

f_R = Frequency received at the receiver (Hz)
f_T = Known transmitted frequency (Hz)

Integration of the Doppler shift offset over time can result in extremely precise measurements (centimeter range for the L1 and L2 frequencies). Thus the carrier phase measurements, $\phi(t)$, can be calculated by integrating the Doppler measurements Δf_{meas} over the time epoch:

$$\phi(t) = \int_{t_0}^{t} \Delta f_{meas}(t)dt + \phi(t_0) \qquad (1\text{-}19)$$

The integer portion of the initial carrier-phase at the start of the integration, $\phi(t)$, is referred to as the "carrier phase integer ambiguity." This integer ambiguity exists because the receiver merely begins counting carrier cycles once the user tracks the satellite signal. Resolution of this integer ambiguity is paramount in determining the most precise range measurement possible. Several techniques have been utilized to resolve this problem, most popular of which are the least-squares iteration process or LAMBDA methods [4, 5, 6].

1.1.5 General Kinematic GPS Techniques

In some of the less advanced receivers, user velocity is calculated as the time derivative of the estimated position, i.e.,

$$\dot{u} = \frac{du}{dt} \approx \frac{u(t_2) - u(t_1)}{t_2 - t_1} \qquad (1\text{-}20)$$

In general, this approach yields poor results and is acceptable only if the user's velocity is constant over the selected time interval.

Many receivers process Doppler measurements which effectively estimate the Doppler frequency of the received satellite. The satellite velocity vector is computed using the ephemeris data and an orbital model that resides within the receiver [1]. At the receiver antenna, the received frequency, f_R, is given by the classical Doppler equation (neglecting relativistic effects) as follows

$$f_R = f_T(1 - \frac{(\mathbf{v}_r \cdot \mathbf{a})}{c}) \qquad (1\text{-}21)$$

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where,

f_T = transmitted satellite signal frequency (known)
$\mathbf{v}_r$ = satellite-to-user relative velocity vector
$\mathbf{a}$ = is the unit vector pointing along the line of sight from the user to the satellite
c = speed of light

The dot product represents the radial component of the satellite-user relative velocity vector along the instantaneous line of sight to the satellite vector, $\mathbf{a}_r$. The quantity $\mathbf{v}_r$ is given as the velocity difference

$$\mathbf{v}_r = \mathbf{v} - \dot{\mathbf{u}} \tag{1-22}$$

where $\mathbf{v}$ is the (known) velocity of the satellite and $\dot{\mathbf{u}}$ is the velocity of the user to be determined (both referenced to a common ECEF frame). Therefore, the Doppler offset due to the relative motion satisfies

$$\Delta f = f_R - f_T = -f_T \frac{(\mathbf{v} - \dot{\mathbf{u}}) \cdot \mathbf{a}}{c} \tag{1-23}$$

There are several techniques [1] to obtain user velocity, $\mathbf{u}$, from the measured Doppler frequency, Δf.

For the jth satellite, equation (1-23) yields

$$f_{R_j} = f_{T_j} \left\{ 1 - \frac{1}{c} [(\mathbf{v}_j - \dot{\mathbf{u}}) \cdot \mathbf{a}_j] \right\} \tag{1-24}$$

and the corrected satellite frequency is given by

$$f_{T_j} = f_o + \Delta f_{T_j} \tag{1-25}$$

where f_o is the nominal transmitted frequency and Δf_{T_j} is the correction determined from the navigation message update.

The measured value of the received frequency is in error due to the frequency bias offset. This offset is related to the drift rate of the user clock, relative to GPS time, by

$$f_{R_j} = f_j (1 + \dot{\delta t}_u) \tag{1-26}$$

where $\dot{\delta t}_u$ is considered positive if the user clock is running fast. Through algebraic manipulation, this can be rewritten as

$$\frac{c(f_j - f_{T_j})}{f_{T_j}} + v_{xj} a_{xj} + v_{yj} a_{yj} + v_{zj} a_{zj} = \dot{x}_u a_{xj} + \dot{y}_u a_{yj} + \dot{z}_u a_{zj} - \frac{c f_j \dot{\delta t}_u}{f_{T_j}} \tag{1-27}$$

where $\mathbf{v_j}$, $\mathbf{a_j}$ is the jth satellite velocity and acceleration component respectively and $\mathbf{v_u} = (\dot{x}_u, \dot{y}_u, \dot{z}_u)$ is the user velocity. To simplify this equation, we let

$$d_j = \frac{c(f_j - f_{Tj})}{f_{Tj}} + v_{xj} a_{xj} + v_{yj} a_{yj} + v_{zj} a_{zj} \tag{1-28}$$

Since the term f_j / f_{T_j} is $\cong 1$, equation (1-28) can be written as

$$d_j = \dot{x}_u a_{xj} + \dot{y}_u a_{yj} + \dot{z}_u a_{zj} - c \dot{\delta t}_u \tag{1-29}$$

We now have four unknowns that can be solved for by using measurements from four satellites and using the set of linear equations

$$d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \quad H = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ a_{x3} & a_{y3} & a_{z3} & 1 \\ a_{x4} & a_{y4} & a_{z4} & 1 \end{bmatrix} \quad g = \begin{bmatrix} \dot{x}_u \\ \dot{y}_u \\ \dot{z}_u \\ -c \dot{\delta t}_u \end{bmatrix} \tag{1-30}$$

with the general form

$$\mathbf{d} = \mathbf{Hg} \tag{1-31}$$

which can be solved as

$$\mathbf{g} = \mathbf{H^{-1} d} \tag{1-32}$$

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The previously stated technique for obtaining user velocity, equation (1-32), uses measurements that may be corrupted by measurement noise and/or multi-path errors. A Kalman Filter method may be used to compute a smoothed navigation solution.

The Kalman filter technique is a recursive algorithm that provides optimal estimates of user position, velocity, and clock drift (PVT) based on noise statistics and current measurements. The filter contains a dynamical model of the GPS receiver platform and outputs a set of user receiver position and velocity state estimates as well as the associated error variances. Kalman Filters entail an approach which simultaneously estimates eight states: 3 position states, 3 velocity states, 1 receiver clock bias and 1 receiver clock drift. In general, the velocity estimates are only valid for low dynamic situations.

Generally, the dynamical model can be derived from a Taylor series expansion of the receiver position $\mathbf{u}$ at time t shortly after time t_0:

$$
\begin{aligned}
\mathbf{u}(t) = \mathbf{u}(t_0) + \frac{d\mathbf{u}(t)}{dt}\big|_{t=t_0} (t - t_0) + \frac{1}{2!}\frac{d\mathbf{u}^2(t)}{dt^2}\big|_{t=t_0} (t - t_0)^2 + ... \\
...+ \frac{1}{3!}\frac{d\mathbf{u}^3(t)}{dt^3}\big|_{t=t_0} (t - t_0)^3 + h.o.t
\end{aligned}
\tag{1-33}
$$

In summary, the filter propagates the platform position from one time point to the next. Using these propagated states, the receiver calculates the anticipated pseudorange and delta pseudorange (the change in pseudorange per epoch for each satellite). Next, the pseudorange and delta pseudoranges are measured and the difference between the anticipated and the measured values (residuals or errors in the user position and velocity estimates) is taken. These errors are usually sent back through the algorithm to be utilized in future state estimates.

Utilizing a Kalman filter allows the use of fewer then 4 satellites and adjusts the state estimates to weight the effects of measurement noise. That is, when measurement noise is high, the filter places heavier weights on the state estimates while, on the other hand, the filter places heavier weights on the measurements when the noise is low.

1.2   Summary

This chapter described the conventional GPS techniques used to determine user position and velocity. The use of DGPS techniques allows the removal of nearly all common errors in each of the three segments (User, Control, and Space) as can be seen in Table 1. However, errors that are uncorrelated from receiver to receiver are not removed and, in particular, the receiver measurement noise is actually increased.

II   *A Direct, Closed-Form and Efficient New Position Determination Algorithm*

This Chapter presents the development of the closed-form GPS positioning algorithm for a stand-alone user. First, an overview of the closed-form solution is given as the basis for the underlying positioning concept. Second, the theoretical nature of using GPS pseudorange measurement equations in the presence of measurement noise for position determination is explored.   Third, the closed-form solution algorithm is developed, which is followed by the development of a Kalman-like update algorithm. The closed-form solution for a scenario with four satellites in view is then examined. The algorithm is then summarized step-by-step outlining how the Matlab simulation is developed. This chapter ends with the experiment setup and simulation results.

2.1   Overview

The current method of solving for GPS user's position is to linearize the pseudorange equations and calculate the user position iteratively, starting with a user-provided initial position guess [8]. For near-earth navigation, the center of the earth is a good initial estimate, and the currently used iterative least squares (ILS) algorithm converges to the GPS solution.   An area of potential improvement that has been investigated in recent years is the use of non-iterative closed-form solutions to the nonlinear pseudorange GPS equations. Closed-form solutions have been developed by

Bancroft [9], Leva [10], Krause [11], Abel and Chafee [12, 13], Hoshen [14], and by Nardi and Pachter [15, 16].

The goal of this research is to develop an efficient new position determination algorithm that uses the closed-form solution of the trilateration equations and works in the presence of pseudorange measurement noise for an arbitrary number of satellites. The new position determination algorithm will focus on the statistics of the position estimates and employ a Kalman-like filter.

2.1.1 Closed-Form Solutions

The closed-form solution developed herein is an improvement over [15] and [16] through the employment of a more rigorous mathematical formulation. In this research and in our previous work reported in [15] and [16], an over-determined system is treated, making use of all-in-view ($n \geq 5$) satellites as opposed to using just four satellites. Moreover, this work departs from a deterministic formulation of the problem ([9], [11], [13], and [14]) and specifically addresses the development of a reliable closed-form solution that works in the presence of measurement noise. Previous works, with the exception of [17], treated the pseudorange equations as a deterministic set of equations. In [18], the deterministic solution of [9] is adapted to account for measurement noise, and this approach is further developed in reference [19].

In this dissertation, pseudorange measurements are recognized to be noise corrupted. Hence, the stochastic nature of the measurements is reflected in the GPS pseudorange equations from the onset to develop a probabilistically sound GPS solution. By stochastically modeling the measurement situation at hand, solving for position becomes a stochastic estimation problem. The use of correct stochastic modeling and

estimation yields a GPS solution that, in addition to the position estimate, provides an estimate of the measurement noise intensity, provided that there are more than five satellites in view.

Thus, the estimation algorithm developed here provides a data-driven position (and user clock bias) estimation error covariance prediction. This prediction introduces a new confidence factor into GPS positioning that is critical for the downstream integration of GPS and Inertial Navigation Systems (INS) or Synthetic Aperture Radar (SAR) sensors. Moreover, an attractive feature of our solution is its good estimation performance, achieved even under poor GDOP conditions and in urban environments where the number of visible satellites may be reduced to four.

Moreover a direct, or autonomous, solution that does not require an initial position estimate is attractive for space navigation and for unusual planar array configurations using pseudolites, where the iterative process is sensitive to the initial position estimate (e.g., the application discussed in references [20] and [21]). Furthermore, fast solutions that require fewer iterations and Floating-Point Operations (FLOPS) are attractive for high-speed vehicles such as spacecraft, where the computational resources may be limited.

2.1.2 *Development of the Two-Step Close-Form Solutions*

The method of linear regression from statistics is used to obtain preliminary closed-form estimates of the position and user clock bias. The number of in-view satellites required is $n \geq 5$. In addition, a data-driven estimate of the pseudorange measurement noise intensity is derived. The data-driven estimation of the measurement noise intensity requires an additional satellite, thus, the two-step algorithm developed in

[22] requires at least six satellites in view (n ≥ 6). In section 2.4, the second step of the new algorithm is discussed in detail. In this second step, the closed-form solution is used in conjunction with one nonlinear measurement equation; thus, an update step, akin to a Kalman filter update technique is developed. This supplementary algorithm uses the solution of the closed-form algorithm as initialization. The two-step algorithm is validated in extensive simulations. Comparisons are drawn with results achieved using the conventional ILS algorithm currently used in GPS receivers. Good position and clock bias estimates are obtained using the two step algorithm with two to three iterations only, as opposed to five iterations in the ILS algorithm. Also, the FLOPS count is significantly reduced.

2.2   Background

This section is concerned with the theoretical nature of a position determination algorithm that uses the closed-form solution of the trilateration equations and works in the presence of pseudorange measurement noise for an arbitrary number of satellites. This section provides the basis theory that underlines the problem.

2.2.1 Pseudorange Corruptions

The GPS uses the radio timing principle to measure range between the satellites and the GPS receiver, making it a time-of-arrival system. If ranges were being measured directly, we would be dealing with a multilateration system, and obtaining a position fix would be easy. Under ideal error- and noise-free conditions, if both the satellite and the GPS receiver's clock were perfectly synchronized with GPS time with no error, the

measured range would be the true range [23]. However, the GPS receiver measures pseudoranges, which are corrupted by the receiver clock bias, measurement noise, and other error sources. The latter include atmospheric delays, satellite clock errors, ephemeris errors, and receiver-induced errors.

2.2.1.1 Receiver Clock Bias

The receiver clock bias caused by the difference between the receiver clock time and GPS time is by far the largest contributor to the difference between pseudorange and range. However, the receiver clock bias is common to the set of simultaneous pseudorange measurements, enabling it to be treated as an unknown parameter to be estimated along with the user position coordinates; hence the GPS solution consists of the user's three space coordinates and clock bias.

2.2.1.2 Ephemeris Corrections

Ephemeris corrections provided to the satellites from the control segment could be used to partially eliminate the satellite time error and the ephemeris errors. Known tropospheric and ionospheric error model corrections can be applied to partially compensate for tropospheric and ionospheric delay errors, and ionospheric errors can essentially be removed using dual-frequency measurements. Improved receiver designs are used to minimize the effects of the receiver-related errors, including receiver noise, code loop quantization errors, multipath effects, and interchannel errors.

2.2.2 Pseudorange Modeling in GPS

If the residual errors are grouped together under one random variable, v, the GPS pseudorange equation can be modeled as the true Euclidean range with an unknown clock

bias and measurement noise superimposed; thus, the stochastic nonlinear pseudorange measurement equation is

$$R_i = \sqrt{(u_x - x_i)^2 + (u_y - y_i)^2 + (u_z - z_i)^2} + b + v_i \qquad (2\text{-}1)$$

This equation represents the ith corrected pseudorange from satellite i, $i = 1, 2, 3, \ldots, n$, where n is the number of satellites in view; (u_x, u_y, u_z) are the unknown user position in earth-centered, earth-fixed (ECEF) coordinates; (x_i, y_i, z_i) are the known coordinates of the ith GPS satellite in ECEF coordinates; b is the unknown range-equivalent user clock bias; and v_i is zero-mean, Gaussian, pseudorange measurement noise. It is reasonable to assume that all receiver measurements are subject to the same receiver noise intensity; therefore, they will have the same variance, σ^2. However, the measurement noise terms v_i are not correlated between satellites.

Concerning the measurement noise, v: Given the number of contributing factors to pseudorange noise and the lack of knowledge of their characteristics, it is reasonable to assume that the residual pseudorange noise will have a zero-mean Gaussian distribution by invoking the Central Limit Theorem which states that the sum of many independent random variables, regardless of their contribution, will approach a Gaussian distribution [20]. The Gaussian pseudorange noise will not be white due to the correlated nature of the encompassed errors and noise. This concern is alleviated since there is no requirement for the pseudorange measurements to be uncorrelated in time, because the positioning problem will be treated as a static estimation problem, where a snap shot in time is treated as a new static estimation problem. On the other hand, it is desirable for the development of the stochastic estimation that pseudorange noise be uncorrelated

across satellites. This assumption will not hinder the development of the stochastic closed-form solution to the GPS pseudorange equation in this research, since the pseudorange measurements will be differenced, thereby eliminating some of the effects of correlated noise. Thus, the pseudorange measurement noise is modeled as

$$v_i \in N(0, \sigma^2), i = 1, 2, ..., n \tag{2-2}$$

2.3   *Summary of Results*

This section summarizes the theoretical development of the closed-form solution for user position estimation using GPS pseudorange measurements. The parameter vector consisting of user position and receiver clock bias can be defined as

$$\vec{u} = (u_x, u_y, u_z, b)^T$$

It is required to obtain an estimate of the parameter $\vec{u}$. It is also desirable to predict the estimation error covariance. The GPS position calculation algorithm described in Theorem 1 below provides the steps necessary to achieve these objectives. The derivation and proof of correction will be given in the sections that follow.

2.3.1 *Theorem 1*

Assume that the number of satellites in view is n ≥ 6. Given the ephemeredes $\{(x_i, y_i, z_i)\}_{i=1}^{n}$ of the n satellites and the n pseudoranges $\{R_i\}_{i=1}^{n}$ data, a position estimate and a prediction of its estimation error covariance can be obtained in two steps:

Step 1: Form the (*n*-1) x 4 regressor matrix

$$H = \begin{bmatrix} (x_n - x_1) & (y_n - y_1) & (z_n - z_1) & (R_1 - R_n) \\ (x_n - x_2) & (y_n - y_2) & (z_n - z_2) & (R_2 - R_n) \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ (x_n - x_{n-1}) & (y_n - y_{n-1}) & (z_n - z_{n-1}) & (R_{n-1} - R_n) \end{bmatrix},$$

the measurement vector $\vec{Z}$, whose elements are

$$Z_i = \frac{1}{2}\left(R_i^{\ 2} - R_n^{\ 2} + x_n^{\ 2} - x_i^{\ 2} + y_n^{\ 2} - y_i^{\ 2} + z_n^{\ 2} - z_i^{\ 2}\right), i = 1, 2, ..., n-1,$$

and the $(n-1)$ x $(n-1)$ covariance matrix $\tilde{R}^{-1}$, whose elements are

$$\left(\tilde{R}^{-1}\right)_{i,i} = r_i - \frac{r^2_{\ i}}{1 + \sum\limits_{k=1}^{n-1} r_k}, \quad i = 1, 2, ..., n-1$$

$$\left(\tilde{R}^{-1}\right)_{i,j} = r_i - \frac{r_i r_j}{1 + \sum\limits_{k=1}^{n-1} r_k}, \quad i \neq j, i = 1, 2, ..., n-1$$

where

$$r_i = -\left(\frac{R_n}{R_i}\right)^2$$

Then use this information to obtain the preliminary closed-form parameter estimate

$$\hat{\vec{u}}^- = \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1} \vec{Z}$$

Next calculate the return difference vector

$$\tilde{Z} = \vec{Z} - H\hat{\vec{u}}^-$$

The pseudorange measurement noise intensity (σ) estimate can be found from

$$\hat{\sigma}^2 \approx -\left(R_n - \hat{b}^-\right) + \sqrt{\left(R_n - \hat{b}^-\right)^2 + \frac{2(\tilde{Z}^T \tilde{R}^{-1} \tilde{Z})}{n-5}}$$

Finally calculate

$$c = \frac{\hat{\sigma}^2}{2} + \left(R_n - \hat{b}^-\right)^2$$

and compute the preliminary parameter estimation error covariance matrix

$$P_{\tilde{u}}^- = c\hat{\sigma}^2 \left(H^T \tilde{R}^{-1} H\right)^{-1}$$

Thus, at the conclusion of Step 1 we have

$$\vec{u} \in N\left(\hat{\vec{u}}^-, P_{\tilde{u}}^-\right)$$

Step 2: Use the estimate produced in Step 1, initialize

$$u_{x0} = \hat{\tilde{u}}_1^-, u_{y0} = \hat{\tilde{u}}_2^-, u_{z0} = \hat{\tilde{u}}_3^-$$

Form the (n-1) x 4 regressor vector

$$h \equiv \begin{bmatrix} \dfrac{(u_{x0} - x_n)}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}} \\[3ex] \dfrac{(u_{y0} - y_n)}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}} \\[3ex] \dfrac{(u_{z0} - z_n)}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}} \\[3ex] 1 \end{bmatrix}$$

Then form a scalar measurement

$$Z_n = R_n + \frac{(u_{x0} - x_n)x_n + (u_{y0} - y_n)y_n + (u_{z0} - z_n)z_n}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}},$$

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and the (correlation) vector p

$$p = \hat{\sigma}^2 \left(R_n - \hat{b}^-\right)\left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1} \begin{bmatrix} 1 \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{bmatrix}_{(n-1)x1}$$

Then the intermediate matrix variable Y is

$$Y = P_{\tilde{u}}^- + \frac{h^T P_{\tilde{u}}^- h - 1}{\left(1 - p^T h\right)^2} pp^T + \frac{1}{1 - p^T h}\left(P_{\tilde{u}}^- hp^T + ph^T P_{\tilde{u}}^-\right)$$

and K is the modified Kalman filter gain, given by

$$K = \frac{1}{1 - p^T h}\left(\frac{1}{1 + h^T Yh}Yh - p\right)$$

Update according to

$$\hat{\tilde{u}}^+ = \hat{\tilde{u}}^- + K\left(Z_n - h^T \hat{\tilde{u}}^-\right)$$

$$P_{\tilde{u}}^- = \left\{I - \left[\left(1 - p^T h\right)K + ph^T\right]\right\}Y$$

Iterate:

Akin to the iterated Kalman Filter algorithm used in Extended Kalman Filtering, the measurement Z_n and the vector h are updated about the improved position estimate. The algorithm iterates using the preliminary estimate and estimation error covariance available prior to the update and produced by the linear closed-form algorithm. Hence, in the h and Z_n formulae only, set

$$u_{x0} = \hat{\tilde{u}}_x^+, u_{y0} = \hat{\tilde{u}}_y^+, u_{z0} = \hat{\tilde{u}}_z^+$$

and repeat the calculation of $\hat{\vec{u}}^+$ and $P_{\vec{u}}^+$. The algorithm is hardwired to stop after 3 iterations as experiments have shown that the solution converges in 2 or 3 iterations. At the conclusion of Step 2 the information on the parameter extracted from the data is

$$\vec{u} \in N\left(\hat{\vec{u}}^+, P_{\vec{u}}^+\right)$$

The derivation and proof of correctness of the novel GPS positioning algorithm is outlined in the next sections, where the respective steps 1 and 2 are discussed, and is also adapted to the case where only four satellites are in view.

2.4 Closed-Form Solution

The derivation of the algorithm in step 1 is now presented in detail. Equation (2-1) can be written as

$$\left(u_x - x_i\right)^2 + \left(u_y - y_i\right)^2 + \left(u_z - z_i\right)^2 = \left(R_i - b - v_i\right)^2 \tag{2-3}$$

Expanding equation (3) results in the following

$$\begin{aligned} u_x{}^2 + u_y{}^2 + u_z{}^2 - b^2 - 2x_i u_x - 2y_i u_y - 2z_i u_z + 2R_i b \\ = R_i^2 - x_i^2 - y_i^2 - z_i^2 - 2R_i v_i + 2b v_i + v_i^2 \end{aligned} \tag{2-4}$$

It is noted that the first four terms in equation (2-4) are the unknown variables squared and that they are common to all n equations. This presents an opportunity for eliminating the nonlinear terms by differencing; hence, the nth equation is subtracted from the remaining $(n - 1)$ equations. The resulting $(n - 1)$ equations are linear in the unknown variables and can be expressed as

$$(x_n - x_i)u_x + (y_n - y_i)u_y + (z_n - z_i)u_z + (R_i - R_n)b$$

$$= \frac{1}{2}\left(R_i^2 - R_n^2 + x_n^2 - x_i^2 + y_n^2 - y_i^2 + z_n^2 - z_i^2\right) \qquad (2\text{-}5)$$

$$+ R_n v_n - R_i v_i - bv_n + bv_i + \frac{1}{2}\left(v_i^2 - v_n^2\right)$$

As a by-product of the preceding operation, the nonlinear n^{th} pseudorange equation remains

$$R_n = \sqrt{(u_x - x_n)^2 + (u_y - y_n)^2 + (u_z - z_n)^2} + b + v_n \qquad (2\text{-}6)$$

The nth equation will remain unused in this phase of the development but will subsequently be used as an additional "measurement" equation in the Kalman update in step 2. The linear regressions in equation (2-5) can be written compactly in matrix notation form as

$$\vec{Z} = H\vec{u} + \vec{V} \qquad (2\text{-}7)$$

where $\vec{Z}$ is the measurement vector, given by

$$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \bullet \\ \bullet \\ \bullet \\ Z_{n-1} \end{bmatrix}$$

and its element

$$Z_i = \frac{1}{2}\left(R_i^2 - R_n^2 + x_n^2 - x_i^2 + y_n^2 - y_i^2 + z_n^2 - z_i^2\right) \qquad (2\text{-}8)$$

H is the $(n-1)$ x 4 regressor matrix given by

$$H = \begin{bmatrix} (x_n - x_1) & (y_n - y_1) & (z_n - z_1) & (R_1 - R_n) \\ (x_n - x_2) & (y_n - y_2) & (z_n - z_2) & (R_2 - R_n) \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ (x_n - x_{n-1}) & (y_n - y_{n-1}) & (z_n - z_{n-1}) & (R_{n-1} - R_n) \end{bmatrix},$$

and $\vec{u}$ is the vector of unknowns, $\vec{u} = [u_x, u_y, u_z, b]^T$.

Finally, $\vec{V}$ is the $(n-1)$ error vector given by $\vec{V} = [V_1, V_2, \bullet \bullet \bullet, V_{n-1}]^T$, where

$$V_i = R_n v_n - R_i v_i - b(v_n - v_i) + \frac{1}{2}(v_i^2 - v_n^2), \ i = 1, 2, ..., n-1 \qquad (2\text{-}9)$$

To obtain an estimate of $\vec{u}$, the statistics of the equation error $\vec{V}$ must be derived from the known statistics of the pseudorange measurement noise v_i. According to equation (2-2), the following holds:

$$E\{v_i\} = 0$$
$$E\{v_i^2\} = \sigma^2 \qquad for\ each\ i = 1, 2, ..., n-1$$
$$E\{v_i v_j\} = 0\ (i \neq j)$$

Thus, the statistics of the error vector $\vec{V}$ are calculated as being

$$E\{V_i\} = 0$$
$$E\{V_i V_j\} = \begin{cases} \sigma^4 + \sigma_2[(R_n - b)^2 + (R_i - b)^2] & for\ i = j \\ \dfrac{\sigma^4}{2} + \sigma^2(R_n - b)^2 & for\ i \neq j \end{cases} \qquad (2\text{-}10)$$

The covariance matrix $P_{\vec{V}} = E(\vec{V}\vec{V}^T)$ can be expressed in tri-diagonal form as

$$P_{\vec{v}} = \sigma^2 \begin{bmatrix} d_1 & c & \bullet & \bullet & c \\ c & d_2 & c & \bullet & \bullet \\ \bullet & c & d_3 & \bullet & \bullet \\ \bullet & c & c & \bullet & c \\ c & \bullet & \bullet & \bullet & d_{n-1} \end{bmatrix}_{(n-1) \times (n-1)} \tag{2-11}$$

where

$$c = \frac{\sigma^2}{2} + \left(R_n - b\right)^2$$

and the diagonal elements

$$d_i = \sigma^2 + \left(R_n - b\right)^2 + \left(R_i - b\right)^2 \quad for\ i = 1, 2, ..., n-1$$

The linear regression in equation (2-7) is used to obtain an estimate of the unknown parameters $\vec{u}$. The aim is to obtain $\hat{\vec{u}}$ that minimizes the estimate error as weighted by the inverse covariance of the equation error vector. The minimum variance parameter estimate is

$$\hat{\vec{u}} = \left(H^T P_{\vec{v}}^{-1} H\right)^{-1} H^T P_{\vec{v}}^{-1} \vec{Z} \tag{2-12}$$

It is desirable to find a closed-form solution for $P_{\vec{v}}^{-1}$ to reduce the computation load of our GPS positioning algorithm. If $P_{\vec{v}}$ is expressed as $P_{\vec{v}} = c\sigma^2 \tilde{R}$, where

$$\tilde{R} = \begin{bmatrix} \dfrac{d_1}{c} & 1 & \bullet & \bullet & 1 \\ 1 & \dfrac{d_2}{c} & 1 & \bullet & 1 \\ 1 & 1 & \dfrac{d_3}{c} & \bullet & 1 \\ \bullet & & & \bullet & 1 \\ 1 & 1 & \bullet & \bullet & \dfrac{d_{n-1}}{c} \end{bmatrix} \tag{2-13}$$

then

$$P_{\tilde{V}}^{-1} = \frac{1}{c\sigma^2} \tilde{R}^{-1} \tag{2-14}$$

The elements of the diagonal of $\tilde{R}$ are a function of b, the clock bias error, and σ, the standard deviation of the measurement noise. For implementation purposes, it is desirable to remove this dependency before finding a solution for $\tilde{R}^{-1}$.

Since $\sigma^2 \ll (R_n - b)^2$, and for most positioning applications, $b \ll (R_i - b)^2$, the diagonal elements of $\tilde{R}$ can be simplified as shown in equation (2-15). To further strengthen the validity of the assumptions made to form equation (2-15), R_n can be selected as the largest of all available pseudoranges. Thus,

$$\frac{d_i}{c} \approx 1 + \frac{R_i^2}{R_n^2} \tag{2-15}$$

After several algebraic steps and applying the Matrix Inversion Lemma, it is found that

$$\left(\tilde{R}^{-1}\right)_{i,i} = \frac{1}{\sigma^2 c} \left[r_i - \frac{r_i^2}{1 + \displaystyle\sum_{k=1}^{n-1} r_k} \right], \quad i = 1, 2, ..., n-1$$

$$\left(\tilde{R}^{-1} \right)_{i,j} = \frac{1}{\sigma^2 c} \left[r_i - \frac{r_i r_j}{1 + \sum_{k=1}^{n-1} r_k} \right], \quad i \neq j, \ j = 1, 2, \ldots, n-1 \tag{2-16}$$

where

$$r_i = \left(\frac{R_n}{R_i} \right)^2$$

A remarkable property of the estimate in equation (2-12) is the fact that it is not dependent on σ, the pseudorange measurement noise variance. Equation (2-14) yields the equation error covariance $P_{\tilde{V}}$ as simply $\tilde{R}$ premultiplied by a scalar quantity. In equation (2-12), the scalar premultiplier of $P_{\tilde{V}}$ will cancel out; therefore, the minimum variance parameter estimate (position and clock bias) in equation (2-12) can be rewritten in an equivalent form as

$$\hat{\vec{u}} = \left(H^T \tilde{R}^{-1} H \right)^{-1} H^T \tilde{R}^{-1} \vec{Z} \tag{2-17}$$

Equation (2-17) is used for coding the MATLAB [24] algorithm. It must be noted that there are no large matrix inversions associated with this solution since $\tilde{R}^{-1}$ has been determined analytically and can be coded directly into the algorithm. The only inversion that needs to be performed is that of the small (4 x 4) matrix $\left(H^T \tilde{R}^{-1} H \right)$, which can easily be hardwired into the GPS receiver's algorithm.

Furthermore, it follows from equation (2-17) that the covariance of the estimate's error is given by

$$\begin{aligned} P_{\tilde{u}} &= E\left\{ \left(\vec{u} - \hat{\vec{u}} \right)\left(\vec{u} - \hat{\vec{u}} \right)^T \right\} \\ &= \sigma^2 c \left(H^T \tilde{R}^{-1} H \right)^{-1} \end{aligned} \tag{2-18}$$

Unlike the solution estimate, the covariance $P_{\hat{u}}$ is dependent on σ; hence, σ must be known or estimated to estimate the error covariance.

Substituting equation (2-7) into equation (2-17) yields

$$
\begin{aligned}
\hat{\vec{u}} &= \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1}\left(H\vec{u} + \vec{V}\right) \\
&= \vec{u} + \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1}\vec{V}
\end{aligned}
\tag{2-19}
$$

Now we substitute equation (19) into the return difference equation:

$$
\begin{aligned}
\tilde{Z} &= \vec{Z} - H\hat{\vec{u}} = H\vec{u} + \vec{V} - H\left(\vec{u} + \left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1}\vec{V}\right) \\
&= \left(I_{n-1} - H\left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1}\right)\vec{V}
\end{aligned}
$$

If we define the matrix $M \equiv I_{n-1} - H\left(H^T \tilde{R}^{-1} H\right)^{-1} H^T \tilde{R}^{-1}$, then the return difference can be expressed as

$$
\tilde{Z} = M\vec{V}
\tag{2-20}
$$

We now define the weighted return difference

$$
\tilde{\mathbf{Z}} \equiv R_e^{-1}\tilde{Z}
\tag{2-21}
$$

where R_e^{-1} is obtained from the Cholesky decomposition of $\tilde{R} : \tilde{R} = R_e R_e^{T}$. To avoid a large matrix inversion, as the size of R_e is $(n-1) \times (n-1)$, R_e^{-1} is expressed as a function of $\tilde{R}^{-1}$:

$$
R_e^{-1} = R_e^{T} \tilde{R}^{-1}
$$

We now calculate the scalar quantity

$$
\tilde{\mathbf{Z}}^{\mathrm{T}}\tilde{\mathbf{Z}} = \tilde{Z}^T\left(R_e^{-1}\right)^T R_e^{-1}\tilde{Z} = \tilde{Z}^T \tilde{R}^{-1}\tilde{Z} = \vec{V}^T M^T \tilde{R}^{-1} M\vec{V}
$$

It can be shown that

$$E\left\{\tilde{\mathbf{Z}}^{\mathrm{T}}\tilde{\mathbf{Z}}\right\}=c\sigma^{2}Tr(M)$$

and

$$Tr(M)=n-5 \tag{2-23}$$

Thus,

$$E\left\{\tilde{\mathbf{Z}}^{\mathrm{T}}\tilde{\mathbf{Z}}\right\}=c\sigma^{2}(n-5)=\left(\frac{\sigma^{4}}{2}+\sigma^{2}(R_{n}-b)\right)(n-5) \tag{2-24}$$

Rearranging equation (2-24) results in a quadratic equation in σ^{2}. Solving this quadratic equation and substituting

$$\tilde{\mathbf{Z}}^{\mathrm{T}}\tilde{\mathbf{Z}}=\tilde{Z}^{T}\tilde{R}^{-1}\tilde{Z}$$

yields the following data-driven estimate of the measurement noise intensity σ^{2}:

$$\hat{\sigma}^{2}\approx-\left(R_{n}-\hat{b}\right)+\sqrt{\left(R_{n}-\hat{b}\right)^{2}+\frac{2(\tilde{Z}^{T}\tilde{R}^{-1}\tilde{Z})}{n-5}} \tag{2-25}$$

In conclusion, the derived linear regression of equation (2-7), which consists of (n – 1) equations, requires that (n – 1) be at least 4 to provide an initial estimate of the four parameters in $\vec{u}$. This implies that a minimum satellite availability of 5 is required to produce the solution given in equation (2-17). At least 1 extra satellite is required for the data-driven prediction of the estimation error covariance $P_{\vec{u}}$. Thus, a minimum satellite availability of 6 is required to produce an initial estimate of the four parameters in $\vec{u}$ and the estimation error covariance. If the pseudorange measurement noise intensity is known, then 5 satellites in view suffice.

The solution, i.e., the preliminary parameter estimate $\hat{\vec{u}}$, is based on n - 1 equations only, although n measurements are available initially. This indicates that n

equations should be used to obtain the parameter estimate. The second step of the algorithm developed in this dissertation addresses this issue as it uses the remaining n^{th} pseudorange equation (see equation (2-6)) as a measurement update in a Kalman filtering-like update equation.

2.5   *Kalman Update Algorithm*

The proof of the correctness of the algorithm in step 2 is now given. The concept behind the Kalman update solution approach is similar to that of a conventional Kalman filter. The closed-form solution above provides a preliminary GPS solution estimate ($\hat{\hat{u}}$) and the associated error covariance matrix ($P_{\hat{u}}$). We now use the nth equation to update the previous estimate in the same way this would be accomplished during the update cycle of an extended Kalman filter.

The approach that is used begins with the linearization of equation (2-6) about a nominal position estimate. The linearized equation is then manipulated into the standard linear measurement form as described in [25] and used to update the estimate.

Since the measurement of equation (2-6) is nonlinear, it may be necessary for the process to continue in an iterative manner until convergence within a predefined tolerance is achieved. Simulation results show that the algorithm converges after two or three iterations.

It is noteworthy that the Kalman update algorithm presented in this section differs from the basic Kalman filter [26, 27] in that the "new" measurement used to update the preliminary estimate is correlated with the preliminary estimate. The conventional

Kalman filter update equation does not allow for correlation between the new measurement and the previous estimate; hence, a Kalman-like update equation that accommodates this correlation and that addresses the specific measurement situation on hand needs to be derived.

The first step in the mathematical development of the Kalman update algorithm is to linearize equation (2-6) about a nominal user position (u_{xo}, u_{yo}, u_{zo}) by performing a Taylor series expansion and neglecting second- and higher-order terms. Through equation manipulation and rearranging and redefining of terms, the following equation in the form of a linear scalar measurement model is obtained:

$$Z_n = h^T \vec{u} + v_n \tag{2-26}$$

where Z_n represents the scalar measurement and is defined as

$$Z_n = R_n + \frac{(u_{x0} - x_n)x_n + (u_{y0} - y_n)y_n + (u_{z0} - z_n)z_n}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}}$$

and h is a (4 x 1) regressor vector defined as

$$h \equiv \begin{bmatrix} \dfrac{(u_{x0} - x_n)}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}} \\ \dfrac{(u_{y0} - y_n)}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}} \\ \dfrac{(u_{z0} - z_n)}{\sqrt{(u_{x0} - x_n)^2 + (u_{y0} - y_n)^2 + (u_{z0} - z_n)^2}} \\ 1 \end{bmatrix} \tag{2-27}$$

Recall that $\vec{u}$ is the vector of unknowns $\begin{bmatrix} u_x, u_y, u_z, b \end{bmatrix}^T$, and v_n is the pseudorange measurement noise associated with the nth measurement, where $v_n \sim N(0, \sigma^2)$. The

initial nominal user position $\left(u_{x0}, u_{y0}, u_{z0}\right)$ is provided by the estimate of the closed-form algorithm in step 1:

$$u_{xo} = \hat{\hat{u}}_x, u_{y0} = \hat{\hat{u}}_y, u_{z0} = \hat{\hat{u}}_z$$

Equation (2-26) is in the desired linear measurement model form that can be used to update the solution obtained from the preliminary closed-form algorithm in a Kalman-like update step.

Now, Z_n is actually part of the measurements used to obtain the closed-form solution and not a new measurement, as would be the case in a conventional Kalman filter application. The noise in the new measurement and the previously derived position estimation error are therefore correlated. This is a violation of the basic assumptions used in the derivation of the conventional Kalman filter update equations. To derive the new Kalman-like update equation, it is necessary to know the relationship between the noise in the new measurement (v_n) and the preliminary estimate. This preliminary estimate is provided by the closed-form algorithm.

The linear regression used for the closed-form algorithm was defined in equation (2-7), and the statistics of the noise vector $\vec{V}$ were derived in equation (2-9). The closed-form algorithm produced an estimate of the GPS unknown parameters, $\hat{\hat{u}}$, defined in equation (2-17), and an estimate of its covariance matrix ($P_{\hat{u}}$), defined in equation (2-18). Using this knowledge of the estimated GPS solution, the true GPS parameter vector can be expressed as the random variable

$$\vec{u} = \hat{\hat{u}} + \vec{W} \tag{2-28}$$

where $\vec{W} \sim N\left(0, P_{\hat{u}}\right)$. The correlation of interest between v_n and $\vec{W}$ can be defined as

$$p \equiv E\{\vec{W}v_n\} = E\{v_n\vec{W}\} \tag{2-29}$$

To determine the relationship between $\vec{W}$ and v_n, the linear regression in equation (2-7) is multiplied from the left by $H^T\tilde{R}^{-1}$, yielding the expression

$$H^T\tilde{R}^{-1}\vec{Z} = H^T\tilde{R}^{-1}H\vec{u} + H^T\tilde{R}^{-1}\vec{V} \tag{2-30}$$

Equation (2-30) can be solved for $\vec{u}$ to obtain

$$\vec{u} = \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{Z} - \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V} \tag{2-31}$$

The first term on the right hand side of equation (2-31) is recognized from equation (2-17) as $\hat{\vec{u}}$; therefore, an expression for $\vec{W}$ in terms of $\vec{V}$ is obtained:

$$\vec{W} = \left(H^T\tilde{R}^{-1}H\right)^{-1}H^T\tilde{R}^{-1}\vec{V} \tag{2-32}$$

Furthermore, an expression that represents the variance between any single elements of $\vec{V}$ and v_n is determined by exploiting the noise statistics of $\vec{V}$ derived previously:

$$E\{V_iv_n\} = \sigma^2(R_n - b), i = 1, 2, ..., n-1 \tag{2-33}$$

Equation (2-33) yields the following covariance matrix:

$$E\left\{\vec{V}v_n\right\} = \sigma^2(R_n - b)\begin{bmatrix} 1 \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{bmatrix}_{(n-1)x1} \tag{2-34}$$

Using equations (2-32) and (2-34), an expression for the covariance between $\vec{W}$ and v_n is determined:

$$\mathrm{p} = \sigma^2 \left(R_n - b \right) \left(H^T \tilde{R}^{-1} H \right)^{-1} H^T \tilde{R}^{-1} \begin{bmatrix} 1 \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 1 \end{bmatrix}_{(n-1)x1} \tag{2-35}$$

Next, an augmented linear regression is formulated by combining equations (2-26) and (2-28). The augmented linear regression is expressed as

$$\vec{Z}_a = H_a \vec{u} + \vec{V}_a \tag{2-36}$$

where $\vec{Z}_a$ is the (5 x 1) augmented "measurement" vector defined as

$$\vec{Z}_a \equiv \begin{bmatrix} \hat{\vec{u}} \\ Z_n \end{bmatrix}$$

H_a is the (5 x 4) augmented regressor defined as

$$H_a \equiv \begin{bmatrix} I \\ h^T \end{bmatrix}$$

and $\vec{V}_a$ is the (5 x 1) augmented "measurement noise" vector defined as

$$\vec{V}_a \equiv \begin{bmatrix} \vec{W} \\ v_n \end{bmatrix}$$

In the derivation that follows, in order to distinguish the preliminary estimate $\hat{\vec{u}}$ and $P_{\vec{u}}$ as produced by the closed-form algorithm from the estimate that will be obtained through the Kalman update the following notation is used:

- $\hat{\vec{u}}^-$ and $P_{\vec{u}}^-$ represent the estimate and the estimation error covariance prior to the update, respectively.

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- $\hat{u}^+$ and $P_{\tilde{u}}^+$ represents the estimate and the estimation error covariance following the update, respectively.

To obtain the updated estimates from the augmented linear regression in equation (2-36) it is necessary to derive the covariance of the augmented noise vector $\vec{V}_a$. Since the statistics of the noise components in $\vec{V}_a$ have already being determined, the equation error covariance matrix, R_a, is given by

$$R_a = \begin{bmatrix} P_{\tilde{u}}^- & p \\ p^T & \sigma^2 \end{bmatrix} \tag{2-37}$$

The updated GPS minimum variance solution estimate and the associated covariance are then given by the expressions:

$$\hat{\bar{u}}^+ = P_{\tilde{u}}^+ H_a^T R_a^{-1} \vec{Z}_a \tag{2-38}$$

$$P_{\tilde{u}}^+ = \left(H_a^T R_a^{-1} H_a \right)^{-1} \tag{2-39}$$

The expressions in equations (2-38) and (2-39) are sufficient to obtain the required updates, but it is desirable to manipulate and reduce the equations into the more familiar and computationally efficient form of the classical Kalman filter update equations. After lengthy manipulations and applying the Matrix Inversion Lemma, the Kalman-like update equations in the desired form are obtained:

$$\hat{\bar{u}}^+ = \hat{\bar{u}}^- + K\left(Z_n - h^T \hat{\bar{u}}^- \right) \tag{2-40}$$

$$P_{\tilde{u}}^+ = \left\{ I - \left[\left(1 - p^T h\right) K + p \right] h^T \right\} Y \tag{2-41}$$

where the intermediate variable Y is the modified pre-update covariance matrix given by

$$Y = P_{\tilde{u}}^- + \frac{h^T P_{\tilde{u}}^- h - 1}{\left(1 - p^T h\right)^2} pp^T + \frac{1}{1 - p^T h} \left(P_{\tilde{u}}^- hp^T + ph^T P_{\tilde{u}}^- \right) \tag{2-42}$$

and K is the modified Kalman filter gain given by

$$K = \frac{1}{1 - p^T h}\left(\frac{1}{1 + h^T Yh}Yh - p\right)$$ (2-43)

The parameter estimate update in equation (2-38) appears to be identical to that of the classical Kalman filter update equations. However, this is not the case since the Kalman filter gain, equation (2-43), is not the same.

Remarks: Note that in the special case of the classical Kalman Filter with no correlation, $p = 0$ and $Y = P_{\tilde{u}}^-$. For this special case the classical Kalman Filter update formulae are indeed recovered:

$$K = \frac{1}{1 + h^T P_{\tilde{u}}^- h}P_{\tilde{u}}^- h$$
$$\hat{\tilde{u}}^+ = \hat{\tilde{u}}^- + K\left(Z_n - h^T \hat{\tilde{u}}^-\right)$$

and

$$P_{\tilde{u}}^+ = \left(I - Kh^T\right)P_{\tilde{u}}^-$$

Equations (2-40) to (2-43) are used in the MATLAB implementation of the Kalman update algorithm. The Kalman update algorithm is intended to refine the GPS closed-form solution estimate in a direct and non-recursive manner. However, since the measurement in equation (2-7) is nonlinear, it is necessary for the process to continue in an iterative manner until convergence within a predefined tolerance is achieved. Now recall that the new measurement used by the Kalman update algorithm is actually the nth pseudorange equation in equation (2-7) being linearized about the position estimate produced by the closed-form algorithm in step 1. This implies that how well the linearization fits the true unknown GPS parameters is dependent on how good the

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solution produced by the closed-form algorithm is to begin with. To alleviate this undesired dependency, after the Kalman update algorithm has been applied once and produces an improved solution estimate, equation (2-7) is once again linearized about the improved position estimate, producing a new and better linear measurement equation. This is akin to the iterated Kalman filter algorithm used in extended Kalman filtering.

The Kalman update algorithm is applied a second time using the preliminary estimate and estimation error covariance available prior to the update and produced by the linear closed-form algorithm, not the solution obtained as a result of the previous application of the Kalman update. Theoretically, this process can be continued recursively until convergence to the best possible solution is achieved; however, for the scenario that was used in the experiment, after two or three applications, the change in the solution estimate was insignificant. Hence, the algorithm was hardwired to perform three iterations.

2.6 Position Determination Algorithm for Four Satellites in View

In previous sections, it was required that $n \geq 5$. The derivation in step 1 of the algorithm for the case of four satellites in view is now investigated. Setting $n = 4$, equation (2-5) becomes:

$$
\begin{aligned}
(x_4 - x_i)u_x &+ (y_4 - y_i)u_y + (z_4 - z_i)u_z + (R_i - R_4)b \\
&= \frac{1}{2}\left(R_i^2 - R_4^2 + x_4^2 - x_i^2 + y_4^2 - y_i^2 + z_4^2 - z_i^2\right) \\
&+ R_4 v_4 - R_i v_i - b v_4 + b v_i + \frac{1}{2}\left(v_i^2 - v_4^2\right)
\end{aligned}
\tag{2-44}
$$

The linear regression in equation (2-44) can be compactly written in matrix notation form as

$$\vec{Z} = H\vec{u} + \vec{V} \tag{2-45}$$

where $\vec{Z}$ is the measurement vector, now given by

$$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

and the element

$$Z_i = \frac{1}{2}\left(R_i^2 - R_4^2 + x_4^2 - x_i^2 + y_4^2 - y_i^2 + z_4^2 - z_i^2\right) \tag{2-46}$$

H is the (3 x 4) regressor matrix given by

$$H = \begin{bmatrix} (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) & (R_1 - R_4) \\ (x_4 - x_2) & (y_4 - y_2) & (z_4 - z_2) & (R_2 - R_4) \\ (x_4 - x_3) & (y_4 - y_3) & (z_4 - z_3) & (R_3 - R_4) \end{bmatrix} \tag{2-47}$$

Finally, $\vec{V}$ is the (1 x 3) error vector given by $\vec{V} = [V_1, V_2, V_3]^T$, where

$$V_i = R_4 v_4 - R_i v_i + b(v_i - v_4) + \frac{1}{2}\left(v_i^2 - v_4^2\right), i = 1, 2, \text{ and } 3 \tag{2-48}$$

We now have three linear equations in four unknowns, and a stand-alone parameter estimate can not be obtained. The nonlinear fourth pseudorange equation is yet to be used:

$$R_4 = \sqrt{(u_x - x_4)^2 + (u_y - y_4)^2 + (u_z - z_4)^2} + b + v_4 \tag{2-49}$$

As in step 2 in the Kalman update algorithm, the fourth equation can be linearized about a nominal user position (u_{xo}, u_{yo}, u_{zo}) by performing a Taylor series expansion and

neglecting second- and higher-order terms. Through equation manipulation, the following equation in the form of a linear scalar measurement model is obtained:

$$z_4 = h^T \vec{u} + v_4 \tag{2-50}$$

where z_4 represents the scalar measurement and is defined as

$$z_4 = R_4 + \frac{(u_{x0} - x_4)x_4 + (u_{y0} - y_4)y_4 + (u_{z0} - z_4)z_4}{\sqrt{(u_{x0} - x_4)^2 + (u_{y0} - y_4)^2 + (u_{z0} - z_4)^2}} \tag{2-51}$$

and h is a 4 x 1 vector defined as

$$h \equiv \begin{bmatrix} \dfrac{(u_{x0} - x_4)}{\sqrt{(u_{x0} - x_4)^2 + (u_{y0} - y_4)^2 + (u_{z0} - z_4)^2}} \\ \dfrac{(u_{y0} - y_4)}{\sqrt{(u_{x0} - x_4)^2 + (u_{y0} - y_4)^2 + (u_{z0} - z_4)^2}} \\ \dfrac{(u_{z0} - z_4)}{\sqrt{(u_{x0} - x_4)^2 + (u_{y0} - y_4)^2 + (u_{z0} - z_4)^2}} \\ 1 \end{bmatrix} \tag{2-52}$$

Recall that $\vec{u}$ is the vector of unknowns, $[u_x, u_y, u_z, b]^T$, and v_4 is the pseudorange measurement noise associated with the fourth measurement, where $v_4 \sim N(0, \sigma^2)$. The initial nominal user position is assumed to be at the origin of the ECEF coordinates

$$u_{xo} = 0, u_{y0} = 0, u_{z0} = 0$$

Combining equations (2-45) and (2-51), we now have an augmented measurement model represented by

$$\begin{bmatrix} \vec{Z} \\ z_4 \end{bmatrix} = \begin{bmatrix} H \\ h^T \end{bmatrix} \vec{u} + \begin{bmatrix} \vec{V} \\ v_4 \end{bmatrix} \tag{2-53}$$

$$\vec{Z}_{aug} = H_{aug} \vec{u} + \vec{V}_{aug}$$

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The statistics of the equation error $\vec{V}$ and v_4 are the same as previously derived in the closed form for an arbitrary number of satellites in view. Thus, the expected value between these two errors can be calculated as

$$
\begin{aligned}
E\{V_i v_4\} &= E\left\{R_4 v_4{}^2 - R_i v_i v_4 - b(v_4 - v_i)\, v_4 + \frac{1}{2}\left(v_i^2 - v_4^2\right) v_4\right\} \\
&= R_4 E\{\, v_4{}^2\} - R_i E\{\, v_i v_4\} - b\left(E\{v_4{}^2\} - E\{\, v_i v_4\}\right) \\
&\quad + \frac{1}{2}\left(E\{v_i^2 v_4\} - E\{v_4^3\}\right) \\
&= R_4 \sigma^2 - b\sigma^2 + \frac{1}{2}\left(E\{v_i^2\}\, E\{\, v_4\} - E\{v_4^3\}\right) \\
&= \sigma^2 (R_4 - b),\ \ i = 1,\, 2,\, and\ 3
\end{aligned}
\tag{2-54}
$$

Using equations (2-10) and (2-54), the (4 x 4) covariance matrix $P_{\vec{V}_{aug}}$ can be expressed as

$$
P_{\vec{V}_{aug}} = \sigma^2 \begin{bmatrix} d_1 & c & c & c_1 \\ c & d_2 & c & c_1 \\ c & c & d_3 & c_1 \\ c_1 & c_1 & c_1 & 1 \end{bmatrix}_{4\,x\,4}
\tag{2-55}
$$

where

$$
c = \frac{\sigma^2}{2} + (R_4 - b)^2
$$
$$
c_1 = (R_4 - b)
$$

and the diagonal elements

$$
d_i = \sigma^2 + (R_4 - b)^2 + (R_i - b)^2 \quad for\ i = 1,\, 2,\, and\ 3
$$

Thus, the linear regression in equation (2-53) can be used to obtain an estimate of the unknown parameters $\bar{u}$. The aim is to obtain $\hat{\bar{u}}$ that minimizes the estimation error as weighted by the inverse covariance of the equation error vector.

The minimum variance parameter estimate is

$$\hat{\vec{u}} = \left(H_{aug}{}^T P_{\vec{V}_{aug}}^{-1} H_{aug}\right)^{-1} H_{aug}{}^T P_{\vec{V}_{aug}}^{-1} \vec{Z}_{aug} \tag{2-56}$$

To reduce the computation load of our GPS positioning algorithm, $P_{\vec{V}_{aug}}$ is expressed as

$P_{\vec{V}_{aug}} = c\sigma^2 \tilde{R}$, where

$$\tilde{R} = \begin{bmatrix} \dfrac{d_1}{c} & 1 & 1 & \dfrac{c_1}{c} \\ 1 & \dfrac{d_2}{c} & 1 & \dfrac{c_1}{c} \\ 1 & 1 & \dfrac{d_3}{c} & \dfrac{c_1}{c} \\ \dfrac{c_1}{c} & \dfrac{c_1}{c} & \dfrac{c_1}{c} & 1/c \end{bmatrix} \tag{2-57}$$

Then

$$P_{\vec{V}_{aug}}^{-1} = \frac{1}{c\sigma^2} \tilde{R}^{-1} \tag{2-58}$$

Again, since $\sigma^2 \langle\langle (R_4 - b)^2$, and for most positioning applications, $b \langle\langle (R_i - b)^2$, the diagonal elements of $\tilde{R}$ can be simplified as shown below in equation (2-59). To further strengthen the validity of the assumptions made to form equation (2-59), R_4 can be selected as the largest of all available pseudoranges. Thus,

$$\frac{d_i}{c} \approx 1 + \frac{R_i^2}{R_4^2} \equiv \tilde{d}_i \tag{2-59}$$

$$\frac{c_1}{c} \approx \frac{1}{R_4} \equiv \tilde{c}_1 \tag{2-60}$$

and 1/c can be approximated as

$$\frac{1}{c} \approx \frac{1}{R_4^2} \equiv \tilde{c}_1{}^2 \tag{2-61}$$

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Equation (2-61) shows that the error covariance $P_{\tilde{V}}$ is simply $\tilde{R}$ premultiplied by

a scalar quantity. In equation (2-56) the scalar premultiplier of $P_{\tilde{V}}$ will cancel out;

therefore, the minimum variance parameter estimate in equation (2-56) can be rewritten

as

$$\hat{\vec{u}} = \left(H_{aug}{}^{T} \tilde{R}^{-1} H_{aug} \right)^{-1} H_{aug}{}^{T} \tilde{R}^{-1} \vec{Z} \tag{2-62}$$

Equation (2-62) is used for coding the MATLAB [24] algorithm. Given that equation (2-49) is nonlinear, a few iterations will be required in order to convert the estimated

position $\hat{\vec{u}}$ to within a predefined tolerance.

2.7 *Experimental Setup*

The closed-form linear regression algorithm developed in this dissertation

requires at least six pseudoranges measurements to produce a stand-alone GPS solution

and a prediction of the position estimation error covariance. In terms of satellite

availability, the worst-case scenario occurs at latitudes in the range of 35˚ - 55˚, where

there are six or fewer GPS satellites available 20 percent of the time. However, more

than six satellites are in view most of the time. Therefore, for the case of $n = 4$, the four

pseudorange measurements are randomly selected from the available pseudorange

measurements. For $n \geq 5$, the two-step algorithm uses all n available pseudorange

measurements to produce the GPS solution. Satellite availability is not dependent on user

position longitude; hence, a fixed user position in the 35˚ - 55˚ latitude range over the

continental United States, 40˚ N latitude, 105˚ W longitude, at an altitude of 300 m, was

selected.

The geographic coordinates are converted to ECEF coordinates and used to generate the experimental datasets using GPSoft's Satellite Navigation Toolbox for MATLAB [28]. The Satellite Navigation Toolbox is used to generate realistic GPS satellite position data from which true ranges can be calculated between all in-view GPS satellites and the position of the receiver. After adding a range equivalent user clock bias of 1000 m to all the ranges, a zero mean random noise of preselected standard deviation σ = 100 m is superimposed to represent the Gaussian measurement noise.

The Satellite Navigation Toolbox has the capability of simulating realistic noise-corrupted pseudorange measurements which can be applied directly to the GPS position determination algorithm, as would be the case in a real-world scenario. In our experiments, the GPSoft toolbox was used for simulating just the GPS satellite ephemeris data, and the simulated pseudoranges were generated as described above. This was done for the following reasons:

- It provides a more structured dataset for analysis of the algorithm, since only the desired effects are being considered and the amount of noise corruption on the pseudorange measurements is exactly controlled; and

- Since the pseudoranges are produced starting from exactly known position coordinates, comparisons against the true position for determining the algorithm's accuracy are now possible.

In addition to using the novel two-step GPS positioning algorithm developed in this dissertation, calculations were also performed using the conventional ILS algorithm to provide a comparison baseline. Similar to our algorithm, in the simulation an enhanced ILS algorithm from [16] which uses all *n* available pseudorange measurements

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to obtain the GPS solution is exercised. Thus, the regressor, or **H** matrix, is the conventional matrix of direction cosines with a vector of 1's populating the last column. The (*n* x 4) **H** matrix is a tall matrix, so the generalized inverse is used, resulting in a least-squares solution.

2.8   Simulation Results

The novel algorithm developed in this research was tested against the conventional ILS algorithm. The results discussed in this dissertation are the cumulative representation of 5,000 Monte Carlo runs. It was found experimentally that 5,000 Monte Carlo runs are enough for the average miss distance and its standard deviation to converge for the algorithms discussed in this dissertation. To provide a fair comparison of the results from each approach, the Gaussian pseudorange noise realization for each satellite is kept the same between both algorithms for any given Monte Carlo run.

The estimation results as a function of satellite availability are plotted in Figure 2 and tabulated in Table 2. "Miss dist." is the experimentally determined 3-D distance between the true user position and the estimated position averaging over 5,000 Monte Carlo runs, and "std(miss)" is the experimentally determined standard deviation of "miss dist." over the 5,000 Monte Carlo runs. The predicted standard deviation of the miss distance is gauged according to

$$\Pr edicted\ std(miss) = \sqrt{P_{\tilde{u}_{11}}^{+} + P_{\tilde{u}_{22}}^{+} + P_{\tilde{u}_{33}}^{+}}$$

where $P_{\tilde{u}}^{+}$ is the estimation error covariance matrix provided by the estimation algorithm.

All "miss dist." results have been normalized with respect to the measurement noise standard deviation σ. $\hat{\sigma}$ is the average of the predicted values of σ, and std($\hat{\sigma}$) is the standard deviation of this average. Both $\hat{\sigma}$ and std($\hat{\sigma}$) have also been normalized with respect to σ. The number of iterations and FLOPS are the experimentally recorded number of iterations and FLOPS required to produce the solution, averaged over the 5,000 Monte Carlo runs.

2.8.1 Iterative Least Squares Algorithm Benchmark

The experimental average miss distance and its standard deviation produced by the ILS algorithm are used as a baseline for comparison to the algorithm presented in this dissertation. The average non-dimensional miss distance is a function of the number of satellites in view and it ranged from 1.44 to 2.95 for this algorithm. The experimentally obtained non-dimensional standard deviation of the miss distance is relatively small and it ranged from 0.72 to 1.65. The average non-dimensional miss distance for the new algorithm ranged from 1.43 to 2.95, which is comparable to the results obtained by using the ILS algorithm.

Irrespective of satellite availability, it took the ILS algorithm 5 iterations to converge to the required threshold for accuracy. During our experiment, FLOPS counts ranged between 3119 and 5503 for the ILS algorithm while the close-form algorithm produced FLOPS counts between 3061 and 5017. The variance in FLOPS is a function of the size of **H**, which changes as a function of satellite availability; of course the "miss dist." decreases as the availability of satellites in view increases.

Table 2.   Average Results from 5,000 Monte Carlos Runs

	$n=4$		$n=6$		$n=7$		$n=8$		$n=9$	
	ILS Alg.	New Alg	ILS Alg.	New Alg	ILS Alg.	New Alg	ILS Alg.	New Alg	ILS Alg.	New Alg
$\hat{\sigma}/\sigma$		N/A		0.81		0.89		0.93		0.94
std($\hat{\sigma}$)/σ		N/A		0.60		0.47		0.40		0.35
Experiment RMS(miss)/σ		N/A		2.24		1.92		1.61		1.63
Predicted std(miss)/σ		N/A		2.15		1.97		1.64		1.72
No. of Iterations	5	5	5	3	5	3	5	3	5	3
FLOPS	3139	3061	4115	3080	4535	3675	5013	4194	5503	5017

2.8.2 Novel Algorithm Results

From a performance point of view, the novel two-step algorithm produced results comparable to the baseline ILS results. As shown in Table 2, the experiment RMS miss distances resulted from the two-step algorithm are similar to those yielded by the predicted standard deviation of the miss distance. There is an apparent trend that as more satellites became available, the error got smaller, with the exception of an abnormally at *n* = 9.

Furthermore, the two-step algorithm takes only three iterations to produce a position estimate and a prediction of the estimation error covariance, while the conventional ILS algorithm takes five iterations to produce only the position estimate. As a result, the FLOPS count for the two-step algorithm is consistently lower than the

FLOPS count for the ILS algorithm. Concerning the n dependence of the FLOPS count, we see that the miss distance decreases as n increases. Very good results are obtained for $n \geq 7$. However, even though the FLOPS count of both algorithms are proportional to satellite availability (n), the FLOPS count of the two-step algorithm increases at a faster rate. This is due to the estimation of σ in the novel algorithm, which requires operations on $(n-1)$ x $(n-1)$ matrices.

Concerning the number of iterations in the novel algorithm, the novel algorithm could be hard wired to only two iterations instead of three to lower the FLOPS count even further. Experimental results show that if the number of iterations is reduced to two, the average miss distance and the estimation error covariance remain practically unchanged for $n \geq 7$ and slightly change for $n = 6$.

2.8.2.1 Unconventional Geometries

It is interesting to exercise the algorithm in unconventional high-GDOP scenarios where conventional iterative algorithms tend to have difficulties. This type of scenario can be expected in test range applications ([20] and [21]). In this dissertation, the experimental test environment consists of a simulated ground planar array of 36 pseudolites evenly spaced in a circular pattern with a 10,000 meters radius and one pseudolite at the center. This pattern was selected because it represents the best achievable ground array for the conventional ILS algorithm if the user is directly above the center pseudolite [21]. This number of pseudolites was selected to achieve satellite availability levels that allow for evaluation of the algorithm's estimation capability.

In the simulations, the user position was 10,000 m directly above the center of the circular pattern, and the center pseudolite was moved away from the center to vary the geometry. In this test environment, the conventional ILS algorithm produced fairly good estimates of the four GPS parameters with a pseudolite directly below the user; however, the estimates quickly degraded as the center pseudolite was moved away from directly below the user, and it failed to produce a solution when the center pseudolite was offset by more than 400 m.

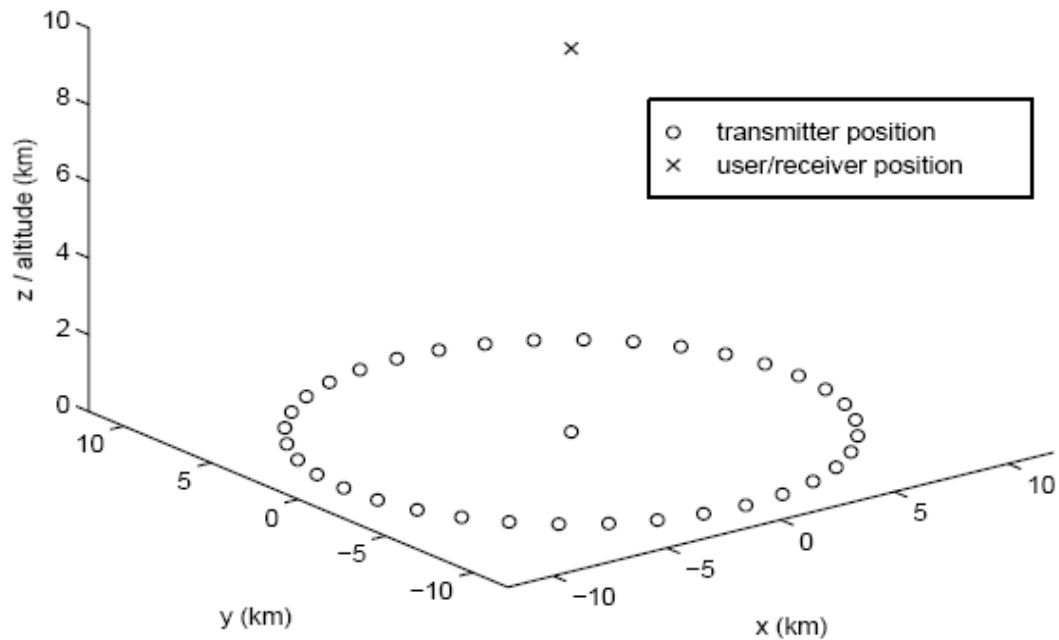**Figure 3.   Optimum Ground-Based Planar Array**

Unlike the results obtained for the typical near-earth GPS scenarios, the estimation of the 2-D u_x, and u_y user position coordinates was extremely good, with mean errors similar to those obtained with the conventional ILS algorithm; however, the user altitude (u_z) mean estimation error was very large, ranging from 2.9 x 10^5 meters to 1.1 x

10^6m. Given the extremely low estimation errors in the u_x and u_y user position coordinate estimates, it appears that the geometry produced by pseudolite ground planar arrays is bias. More research will be needed to fully characterize any potential usefulness of the closed-form algorithm for this application.

Table 3.   Ground Test Results from 5,000 Monte Carlos Runs

	Scenario	Centered Tx Offset = 0m	Centered Tx Offset = 400m
Close-Form	Estimated σ	95.56	95.22
	Error x	-0.53	0.12
	Error y	0.64	-0.14
	Error z	293416	1113025
	Miss distance	293416	1113025
Kalman –like Update	Error x	-0.52	0.17
	Error y	0.44	-0.90
	Error z	-10152	-9872
	Miss distance	-10152	-9872
Iterative Least Squared	Error x	-0.57	0.27
	Error y	0.68	0.19
	Error z	0.39	11.46
	Miss distance	0.97	11.47

Taking the estimates produced by step 1 and applying the Kalman update algorithm in step 2 improved the estimate of u_z. However, the error in u_z was still too large to render its estimate useful. Moreover, u_x and u_y were slightly affected, sometimes for the worse and other times for the better.

Based on the results, the Kalman update (step 2) does not prove very useful in this ground planar array test environment as it does not provide any significant improvement over the closed-form algorithm (step 1). It is noted that the user altitude (u_z) estimation error also increased from 0.39 to 11.46 for the ILS algorithm. Hence, both the ILS and

the two-step algorithm do not yield good estimates when our planar arrays of pseudolites

is used <u>and</u> the center pseudolite is offset by 400 m from the center of the array.

III   *Stochastic Model-Based DGPS Estimation Algorithm*

This chapter presents the development of the kinematic differential GPS algorithm for a network of users, the experiment setup, and simulation results from a 5,000 Monte Carlos runs.

3.1   Introduction

A novel kinematic differential GPS algorithm is presented to process GPS pseudorange measurements from multiple receivers obtained during multiple measurement epochs. Specifically, the accurate relative (and absolute) positioning of a team or formation of mobile vehicles is considered, and a general navigation web concept is advanced. The measurement situation is modeled under a stochastic framework, and, rather than differencing and double differencing as is currently practiced in conventional DGPS, the common errors are explicitly acknowledged and a centralized estimation algorithm is derived. In addition, the predicted covariance of the different receivers position estimation errors is obtained. Extensive simulations were performed to validate the novel kinematic DGPS algorithm. The results were compared to conventional DGPS scenarios, where the reference stations are stationary or there was only one available observation epoch.   The positioning accuracy improvements achieved were gauged against the performance of conventional DGPS.

3.1.1 Background

There are a large number of applications requiring the ability to obtain real-time kinematic positioning of high altitude formation-flying vehicles which include Low Earth Orbit (LEO) military satellite formations for earth observation/surveillance [29]. Therefore, formations of satellites might replace large complex satellites and allow lower overall cost and a higher degree of redundancy [30]. Research is being conducted to replace or augment the existing inertial navigation systems on board satellites with GPS receivers to allow meeting the stringent positioning requirements for the formation.

While the majority of current research is focused on utilizing carrier phase GPS techniques to acquire position and velocity information for each epoch [30, 31, 32, 33, 34], this study will examine the general principals and trends in utilizing pseudorange measurements only over a series of measurement epochs.

3.1.2 Problem Statement

GPS positioning accuracy is limited by measurement errors that can be classified as either common mode or non-common mode. Common mode errors have nearly identical effects on all receivers operating in a limited geographic area, as is the case in formation flight. The common mode errors are the satellite ephemerides errors, the satellite clock errors, and the effects of the ionosphere and troposphere. Non-common mode errors are distinct even for two receivers with small antennae separation and consist of receiver noise and resolution, multipath effects, and inter-channel bias, etc.

The DGPS navigation concept is based on the realization that close-by (< 50 km) GPS receivers are equally affected by the common errors. Conventional DGPS uses a reference station at a known ECEF position to determine corrections that other local, and

potentially mobile, GPS receivers within close range (≤ 50km) of the reference station can use to reduce the effects of GPS common mode errors. Standard DGPS navigation exploits the known position of a reference station and the existence of a communications channel to the moving vehicle (the "rover") to broadcast corrections to the GPS receiver on the rover, thus improve the latter's positioning accuracy. In its most basic form the DGPS methodology entails the application of reference station broadcast differential corrections to the user (rover) measured pseudoranges. Alternatively, the positioning error measured by the reference receiver is directly subtracted from the rover's position measurement. Thus DGPS yields a stand-alone and improved user (rover) position estimate [35].

3.1.3 Summary of Current Knowledge

Using GPS for relative and absolute positioning of formation flying vehicles is relatively new [30, 31, 32, 33]. The majority of the current research involves utilizing the Carrier Phase DGPS techniques for tracking and controlling vehicles. Simulated results indicate that ≤ 2cm rms position errors are possible [31].

3.1.4 Scope

The positioning error in DGPS is caused by receiver noise and resolution, multipath, and inter-channel bias, etc. Multipath error is addressed by carefully choosing the antenna's location, using choke ring antennae, and applying advanced signal processing techniques. Hence there is a strong incentive to develop methodologies for mitigating the effects of measurement noise and residual errors in DGPS. Obviously, an approach which relies on averaging out the random measurement noise and residual errors is called for. In Kinematic GPS (KGPS), the mitigation of the random residual

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errors and measurement noise induced effects can be addressed by several approaches. One approach is the application of a kinematic model to user motion and clock error during the (short) measurement interval. The other possible approach is the use of a measurement record obtained over multiple observation epochs. The third potential approach is the centralized processing of the GPS pseudoranges taking into account the underlying temporal dependence of the kinematic variables. Hence, improved user position, velocity, and possibly, acceleration estimates are obtained. This improvement in navigation performance is obtained irrespective of whether differential corrections, as provided by DGPS, are applied to the raw pseudorange measurements. Thus, in this research KGPS is used to mean not just a mobile user, but also the application of a dynamic model to rover motion.

Hence, provided that a communications channel from the reference station to the user is established and a measurements record obtained over multiple observation epochs is available, one is strongly motivated to combine KGPS and DGPS. In this way, improved user position, velocity and acceleration estimates are obtained by virtue of employing KGPS while, at the same time, the ill effects of measurement noise and residual errors are mitigated and the maximum benefit is extracted out of DGPS. In this dissertation the concept of synergistically employing DGPS and KGPS navigation in a monolithic computational algorithm is undertaken. The developed algorithm is referred to as Kinematic Differential GPS (KDGPS). Specifically, the DGPS scenario with one reference station at a surveyed position is referred to as conventional KDGPS. The DGPS scenario with multiple reference stations at known locations is referred to as augmented, or network, KDGPS. When the navigation of a team of mobile vehicles (rovers) is

considered, such that now the "reference stations" are mobile and their positions, along with the user's position, are not known and need to be determined, we then refer to relative, or generalized KDGPS. The latter is the object of this research.

In conventional DGPS, a "team" consisting of two members only is considered, viz., the user and the reference station. Also, a one-way communications channel between the reference station and the user is established. In conventional DGPS the reference station is stationary and it is located at a known (surveyed) position. When navigating near an array of stationary reference stations which surround the rover, the ionospheric corrections transmitted to the rover are calculated by interpolating the corrections computed by the GPS reference stations – one then refers to network DGPS [36]. The corrections to the pseudorange measurement errors are obtained using the conventional method of differencing and double differencing. All available a-priori information should be used. The ionospheric error can be reduced using modeling and interpolation. However, since the many GPS users use single frequency receivers, the residual ionospheric errors remain a concern. Furthermore, predicted satellite ephemerides and predicted satellite clock error corrections, as provided by NASA's Internet-based Global Differential GPS (IGDG) system which uses a world wide network of stationary reference stations, should be applied [37], [38]. At the same time, as it is desirable to move from meter level to centimeter level positioning in the relentless quest for higher accuracy, residual errors after DGPS correction cannot be neglected.

In this research we are mainly interested in a scenario where the GPS receivers are on vehicles, and multiple rovers in close proximity must be accurately positioned relative to each other, as in satellite or aircraft formation flight control. One then refers to

relative GPS [39, 40, 41].   However, rather than using the conventional differencing and double differencing method [39, 40, 41], in this research a stochastic modeling framework is used where the common errors are explicitly acknowledged and treated as parameters to be estimated. The use of multiple GPS receivers which are connected by a two-way data link and centralized processing of the measurements using statistical methods of parameter estimation has the potential to significantly improve the quality of the navigation solution. The correct treatment of the common GPS errors in a centralized computational algorithm is a very important concept. Thus, in this research ILS, linear regression, and the Singular Value Decomposition (SVD) method are used, and a more general DGPS paradigm is established where the concept of a reference station is done away with and a team of m users is considered. All the team members are treated as equals, and the designation of a reference station is not needed. Moreover, the requirement that one member of the team is stationary and at a known position is done away with. In our estimation algorithm, the inclusion of prior information on the reference stations' positions is allowed for, as in network adjustment in surveying [42], and the theory of conventional DGPS is recovered. In addition, the predicted covariance of the rovers' positions estimation error is obtained.

The pseudoranges to the n satellites in view measured by all the receivers are communicated to a central processor and are operated on by a centralized (optimal) estimation algorithm, where the common errors are properly accounted for, thus obtaining improved estimation performance. The central processor then communicates the correct position to each of the team members. Concerning communication needs: The central processor operates on all the m team members' (m x n) pseudoranges, and returns

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the estimated positions, velocities, and accelerations, of each of the m vehicles. The use of an architecture which employs a centralized processor reduces the number of required two-way communication channels from $m(m - 1)/2$ to m. Of course, the centralized processor could reside in one of the formation's vehicles, i.e., in the "leader", in which case only $(m – 1)$ two-way communication channels are required. Evidently,

$$m - 1 \; < \; 1/2 \; m(m - 1) \qquad\qquad \text{for} \;\; m > 2 \qquad\qquad (3\text{-}1)$$

Thus, a novel navigation web concept is advanced. Also, the navigation web development is motivated by self-calibration measurement methods from physics and surveying. For example, in [21] the self-calibration concept was exploited for GDOP reduction in a pseudolites-based measurements scenario. In geodesy, [42], the positioning accuracy of a network of surveyed landmarks is enhanced using additional angle and elevation measurements. However, the common errors in the pseudorange measurements are not explicitly acknowledged and are addressed using the conventional differencing and double differencing method. In this dissertation, the application of stochastic modeling to DGPS positioning is pursued, and the required mathematical development is presented. Indeed, improved relative and absolute position estimates for all the team members are obtained.

3.1.5 Main Contribution

For a special case of a conventional DGPS scenario and in a deterministic world, the results obtained by standard double differencing will be the same as those provided in this dissertation - if the effects of nonlinearity caused by linearization in the commonly used ILS algorithm are not considered, and without an estimate of the position error. After all, if it all comes down to a solution of a linear deterministic system, then whether

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one differences to eliminate the common variables using either Gauss elimination or uses matrix inversion, the result is the same - provided that numerical problems are avoided. The latter is guaranteed if the condition number of the measurement matrix is sufficiently low, which is the case in GPS when the GDOP is low.

In practice, the pseudorange measurements are corrupted by random measurement noise and residual errors. Hence, in this research the emphasis is on the stochastic modeling of the DGPS measurement situation at hand. By falling back on the classical statistical theory of linear regression, ILS and SVD method, the DGPS problem is correctly formulated and a state (e.g. position) estimate is rigorously obtained, including a prediction of the variance of the estimation error. Moreover the SVD method adapted from numerical analysis yields insight into the structure of the DGPS problem, in particular when there is no designated reference station and all users are mobile in which case good relative positioning is achieved. In addition, treating the common errors as parameters to be estimated also yields estimates of the clocks' relative errors. In relative GPS good estimates of the relative positions of the rovers are obtained, however, it is not the relative positions, but rather it's the clocks' relative errors that are directly estimated.

In summary, using stochastic modeling, a general Kinematic DGPS algorithm is developed with the following qualities:

- No stationary reference station is envisaged and a mobile users' formation is considered. Improved relative positioning is achieved. This is often all one is interested in, as in formation flight control.

- The standard and network DGPS scenario are special cases. Improved absolute positioning is then achieved.

- DGPS is applied to KGPS. Thus, in this research DGPS and KGPS are jointly considered in a unified mathematical framework.

- The simulation experiments are carefully calibrated. Obviously, the positioning accuracy is of interest, but in addition the predicted covariance of the state/position estimation error is also derived and is compared with the experimentally measured estimation error covariance. Having a correct predicted estimation error covariance is particularly important if GPS measurements are to be blended with measurements provided by other sensors, as in INS aiding. The mathematical result concerning the probability that a measurement will fall inside the multidimensional 1-σ error ellipsoid is not well known, even in statistical circles. In two dimensions it is always 39%, irrespective of the covariance. This result is provided in the Appendix and was used to calibrate the experiments for this research.

- Interesting applications of DGPS, such as when all the users are mobile and there is no stationary reference station as in relative DGPS, are listed in the conclusions.

3.1.6 DGPS Novel Algorithms

Algorithms for the following navigational scenarios are derived:

- Augmented conventional/network DGPS (ADGPS): n satellites, one user, m $(1 \leq m)$ stationary reference stations at surveyed locations, one $(N = 0)$ measurements epoch.

- Augmented Kinematic DGPS (AKDGPS): n satellites, one user, m $(1 \leq m)$ stationary reference stations at surveyed locations, $N + 1$ $(1 \leq N)$ measurements epochs.

- General Kinematic DGPS (GKDGPS): n satellites, one user, m $(1 \leq m)$ mobile reference stations, $N + 1$ $(1 \leq N)$ measurements epochs.

The GKDGPS scenario can also be viewed as entailing: n satellites, a team of m $(m \geq 2)$ mobile users, $N + 1$ $(N \geq 1)$ measurements epochs. Two approaches to KDGPS are developed:

- The indirect approach to KDGPS, where standard DGPS corrections are applied to the pseudoranges and snapshots of the vehicle's position are initially calculated. Subsequently the KGPS fitting methodology is applied to mitigate the deleterious effects of measurement noise; and,

- Direct KDGPS, where the reference receivers' provided pseudoranges data and the users' pseudorange measurements are jointly processed in a centralized algorithm.

Obviously, both in DGPS and in KGPS, the correct treatment of common errors is most advantageous. This, in turn, requires a centralized computational algorithm. Linear regression and ILS from statistics and the SVD method from numerical analysis are used with the payoff being improved estimation performance. This is the rationale for direct KDGPS. In this research the emphasis is on the direct approach.

KDGPS algorithms are presented where the ILS approach to the solution of the GPS trilateration equations is used. The alternative direct approach [22] to the solution of the GPS trilateration equations can also be used.

3.2 Theory

This section describes the theory behind the development of the stochastic *model-based* DGPS estimation algorithm tested in the Monte Carlo simulations.

3.2.1 Overview

Conventional DGPS implementations typically employ a GPS receiver at a precisely known location (reference, or base, station) to observe the common mode errors in the pseudorange measurements from each satellite. The differential corrections calculated from these errors are then transmitted to all users within the vicinity. The base station is usually an all-in-view receiver that computes and transmits corrections for all visible satellites. A DGPS remote receiver then performs acquisition and tracking using normal operations, but it applies the received corrections to each pseudorange measurement prior to computing its position solution. In this manner, absolute position accuracies much better than the GPS precise positioning service (PPS) level are achieved.

The Radio Technical Commission for Maritime Services (RTCM) standard for DGPS corrections [43] makes the following recommendations: 1) A reference station should not attempt to remove the effects of ionospheric and tropospheric errors from the broadcast corrections. 2) The effect of reference station receiver clock error should be removed from the broadcast corrections. 3) The effect of satellite clock error modeled by

the standard polynomial correction applied at the rover's receiver should be removed from the broadcast corrections. 4) The reference station antenna should be sited or augmented with additional hardware, e.g., a choke ring, and the corrections are processed to minimize the effects of multipath.

Also, it is assumed that all measurements are taken simultaneously. Latency issues are not addressed.

The pseudorange from satellite i measured by user k^{th} receiver is conventionally modeled as

$$
\begin{aligned}
R_i^{(k)} = &\sqrt{\left(x_{s_i} - x_k\right)^2 + \left(y_{s_i} - y_k\right)^2 + \left(z_{s_i} - z_k\right)^2} + \tau_k + v_i^{(k)} + \tau_{s_i} + E_i \\
&+ \Delta_{ion_i} + \Delta_{trop_i}
\end{aligned}
\tag{3-2}
$$

where τ_k is the kth receiver range-equivalent clock bias, τ_s_i is the ith satellite range-equivalent clock bias, E represents the ephemeris (or SA induced) error, Δ_{ion} represents the ionospheric error, Δ_{trop} represents the tropospheric error, and $v_i^{(k)}$ represents the receiver i^{th} channel measurement error. The last three terms in equation (3-2) represent the common mode errors. They are referred to as such because they affect all receivers, including the reference station(s), in a limited geographic area, in the same manner.

In conventional DGPS navigation the reference station broadcasts real-time (pseudorange) corrections that enable a GPS user to eliminate the effects of the common mode errors from his positioning solution. The extent to which this objective is achieved depends on the ability of the reference station to separate the common mode and non-common mode errors.

A reference receiver at an accurately surveyed location (x, y, z) can calculate the reference-to-i^{th} satellite range as

$$\hat{R}_i = \sqrt{\left(x_{s_i} - x\right)^2 + \left(y_{s_i} - y\right)^2 + \left(z_{s_i} - z\right)^2} \tag{3-3}$$

The range differential correction for the ith satellite is determined by differencing the calculated and measured reference-to-satellite ranges:

$$\Delta_{DGPS} = \hat{R}_i - R_i - (\tau_{s_i} + E_i + \Delta_{ion_i} + \Delta_{trop_i} + v_i) \tag{3-4}$$

The sign of Δ_{DGPS} is motivated by the RTCM DGPS standard [43] which states that the correction will be added by the remote user.

Based on the RTCM specifications, the broadcast corrections should be corrected to remove the reference receiver and modeled satellite clock errors. Equation (3-3) shows the actual reference station calculation in conventional DGPS and equation (3-4) shows the remaining error sources in the transmitted correction. Therefore, the broadcast correction will take the form of equation (3-4). Note that the Δ_{DGPS} signal contains the desired common mode error sources, which will cancel the corresponding errors in the user position calculation.

It is possible to do temporal filtering of the basic DGPS corrections before they are broadcast to the user. The purpose of the preliminary filtering is to reduce the non-common mode receiver measurement noise prior to broadcast. The basic correction for each in view satellite is then processed by its own Kalman filter. Reference station algorithms which entail the temporal filtering of the basic DGPS corrections before they are broadcast to the user are not addressed in this research. However, in this research, the KGPS concept is synergistically applied to DGPS in a centralized estimation algorithm.

3.3 *Novel Kinematic DGPS Algorithm*

This chapter provides a detailed description of the novel kinematic DGPS algorithm developed to estimate user position and velocity for a network of users.

3.3.1 *Overview*

The development of a GPS based relative position sensing system is undertaken, with application to aircraft or spacecraft formation flying. Accurate real-time solutions for the relative vehicle positions are required. Thus, this research addresses the general case where all sites are in motion with respect to the earth and to each other, a situation we refer to as general kinematic differential positioning (GKDGPS). There are also terrestrial applications for which precise relative positioning is desired between two sites, both in motion and both far enough from fixed land based sites that conventional DGPS, as described in the previous section, is not feasible. This application might involve the precise determination of the distance between two ships far out at sea. If both carry GPS receivers, the precise relative track between the two could be determined by processing the data with the GKDGPS algorithm. The same applies to formations of land, air, or space vehicles.

Centralized estimation is pursued: The GPS measurements from all vehicles are collected and one solves for the state of the entire formation. All the measurements must be combined to resolve the entire formation state. Thus, a novel web-based navigation concept is developed. This concept provides optimal estimates of the formation's state. In a large formation, high communication rates might be required in order to transfer the large amounts of raw GPS data as is required in DGPS.

3.3.2 Problem Statement

Two or more mobile users are considered over a pre-specified, presumably short, time interval. The number of users and the specific GPS satellites in view are held constant during the measurement interval. Only pseudorange measurements are used.

A specific case with two users is developed to demonstrate the applicability of the novel algorithm. At the beginning of the time interval, prior information on user 2's position (and velocity) is available. Pseudorange data is collected over several epochs during the measurement interval. A fixed sampling rate is used. The prior information and all pseudorange measurements are processed simultaneously at the end of the measurement interval and estimates of the position and velocity of both users 1 and 2 are obtained. The effect of having varying degrees of uncertainty in user 2's "reference station" prior position information is examined. In addition, the effect of receiver noise on the algorithm's ability to correctly estimate the users' positions and velocities in the presence of user and satellite clock errors - the latter, resulting from Selective Availability (SA) if it were turned back on - is investigated.

3.3.3 Theory

The developed novel algorithm integrates kinematic modeling and differential GPS. This is achieved by maximizing the use of the information in the pseudorange measurements from m users receiving n satellites that are available at $N + 1$, relatively close, time instants during the measurement interval. The correct treatment of the common (satellite clock) errors and proper stochastic modeling of the measurement situation on hand are responsible for achieving improved user positioning.

A sampling interval of ΔT seconds is used. The duration, T, of the measurement interval consists of $N + 1$ discrete measurement epochs. Thus, the time instants when measurements are taken are

$$t_j = \Delta T \cdot j, \quad j = 0, 1, \ldots, N \tag{3-5}$$

The number of epochs available over the duration of the measurement interval is $N + 1$, where

$$N = T / \Delta T \tag{3-6}$$

For the short measurement duration being stipulated, the users' kinematics are modeled as constant speed and rectilinear motion

$$\vec{x}_k(t) = \vec{x}_{ok} + \vec{V}_k \, t \tag{3-7}$$

where $k = 1, 2, \ldots, m$ and m is the number of users; $\vec{x}_k$, $\vec{x}_{ok}$, $\vec{V}_k \in \mathbf{R}^3$. The kinematic model could include higher order terms such as acceleration, jerk, or more importantly, could represent a more complex motion, e.g., a satellite trajectory parameterized by its orbital parameters. However, for the generic scenario examined in this research, equation (3-7) will be the kinematic model chosen for the users' motion.

The pseudorange from the kth user to the ith satellite at time t is modeled as

$$\rho_i^{(k)}(t) = \sqrt{\left[x_k(t) - x_{s_i}(t)\right]^2 + \left[y_k(t) - y_{s_i}(t)\right]^2 + \left[z_k(t) - z_{s_i}(t)\right]^2} + \tau_k + \tau_{s_i} \tag{3-8}$$

and the measurement equation is

$$Z_i^{(k)}(t) = \rho_i^{(k)}(t) + v_i^{(k)}(t) \tag{3-9}$$

where

- τ_k is the kth user range-equivalent clock bias with $k = 1, 2, \ldots, m$.

- τ_{s_i} is the ith satellite range-equivalent clock bias with $i = 1, 2, ..., n$.

- $(x_{s_i}, y_{s_i}, z_{s_i})$ is the ith satellite position at time t from the ephemeris data with

  $i = 1, 2, ..., n$.

- $(x_k(t), y_k(t), z_k(t))$ is the kth user position at time t with $k = 1, 2, ..., m$.

- i is the satellite number (the number of satellites in view is n).

- $v_i^{(k)}(t)$ is the measurement noise in the ith channel of user k^{th} receiver at time

  t.

The measurement noise at time j is modeled as

$$v_{i,j}^{(k)}(t) \approx N(0, \sigma^2) \tag{3-10}$$

and

$$E(v_{i,j}^{(k)} v_{i',j'}^{(k')}) = 0 \quad for \ i \neq i', \ j \neq j', \ k \neq k' \tag{3-11}$$

Incorporating the kinematic model of equation (3-7) for the users' positions for each time instant j, i.e., setting

$$x_k(t) \rightarrow x_{0_k} + (V_{k_x} \Delta T) j \tag{3-12}$$

$$y_k(t) \rightarrow y_{0_k} + (V_{k_y} \Delta T) j \tag{3-13}$$

$$z_k(t) \rightarrow z_{0_k} + (V_{k_z} \Delta T) j \tag{3-14}$$

where the subscript 0 denotes the initial position at time $t = 0$ ($j = 0$), allows equation (3-8) to be written as

$$\rho_{i,j}^{(k)} = \sqrt{\begin{aligned}&[x_{0_k} + (V_{x_k} \Delta T) j - x_{s_i}(t)]^2 + [y_{0_k} + (V_{y_k} \Delta T) j - y_{s_i}(t)]^2 \\ &+ [z_{0_k} + (V_{z_k} \Delta T) j - z_{s_i}(t)]^2\end{aligned}} + \tau_k(t) + \tau_{s_i} \tag{3-15}$$

Note that in equation (3-15), $t = j\Delta\text{T}$, $j = 0, 1, \ldots, N$. To apply a kinematic model to the users' clock errors, set $\tau_k(t) = \tau_k + \dot{\tau}_k \Delta T \; j$, $j = 0,1,\ldots,N$. In the special case where two users ($m = 2$) and five ($n = 5$) satellites are considered, the parameter of interest is

$$\theta = [x_{0_1}, y_{0_1}, z_{0_1}, V_{x_1}\Delta T, V_{y_1}\Delta T, V_{z_1}\Delta T, x_{0_2}, y_{0_2}, z_{0_2}, V_{x_2}\Delta T, V_{y_2}\Delta T, V_{z_2}\Delta T, \tau_1,$$
$$\tau_2, \tau_{s_1}, \tau_{s_2}, \tau_{s_3}, \tau_{s_4}, \tau_{s_5}, \dot{\tau}_1\Delta T, \dot{\tau}_2\Delta T]^T \tag{3-16}$$

where

- $(x_{0_k}, y_{0_k}, z_{0_k})$ represents the kth user initial (at time $t = 0$) ECEF coordinates, $k = 1, 2, \ldots, m$.

- $(V_{x_k}, V_{y_k}, V_{z_k})$ represents the kth user velocity, $k = 1, 2, \ldots, m$.

- τ_k represents the kth user range-equivalent receiver clock error, $k = 1, 2, \ldots, m$.

- $\dot{\tau}_k$ represents the kth user velocity-equivalent receiver clock drift, $k = 1, 2, \ldots, m$.

- τ_{s_i} represents the ith satellite range-equivalent clock error, $i = 1, 2, \ldots, n$.

The parameter vector shown in equation (3-16) is $\theta \in \mathbf{R}^{21}$. In the general case, the parameter

$$\theta \in \mathbf{R}^{8m+n} \tag{3-17}$$

In the algorithm, the pseudoranges received are composed as follows. First, the pseudoranges received by the kth user from n satellites at time instant j are used to form the n x 1 vector

$$Z_j^{(k)} = \begin{bmatrix} \rho_{1,j}^{(k)} \\ \bullet \\ \bullet \\ \bullet \\ \rho_{n,j}^{(k)} \end{bmatrix} \tag{3-18}$$

Now, composing the received pseudoranges over the $N+1$ time instants yields the $n\,(N+1)$ x 1 vector

$$Z^{(k)} = \begin{bmatrix} Z_0^{(k)} \\ \bullet \\ \bullet \\ \bullet \\ Z_N^{(k)} \end{bmatrix} \tag{3-19}$$

and finally, composing for the m users yields the $mn(N+1)$ x 1 "measurement vector"

$$Z = \begin{bmatrix} Z^{(1)} \\ \bullet \\ \bullet \\ \bullet \\ Z^{(m)} \end{bmatrix} \tag{3-20}$$

The pseudorange expression, $\rho(\theta)$, is composed similarly. Define

$$f_{i,j}^{(k)}(\theta) = \sqrt{\begin{array}{c} [x_{0_k} + (V_{x_k}\Delta T)j - x_{s_i}(t)]^2 + [y_{0_k} + (V_{y_k}\Delta T)j - y_{s_i}(t)]^2 \\ \overline{+[z_{0_k} + (V_{z_k}\Delta T)j - z_{s_i}(t)]^2} \end{array}} \tag{3-21}$$

and

$$\rho_{i,j}^{(k)}(\theta) = f_{i,j}^{(k)}(\theta) + \tau_k + \dot{\tau}_k \Delta T \, j + \tau_{s_i} \tag{3-22}$$

We compose

$$f_j^{(k)} = \begin{bmatrix} f_{1,j}^{(k)} \\ \bullet \\ \bullet \\ \bullet \\ f_{n,j}^{(k)} \end{bmatrix} \qquad (3\text{-}23)$$

Next, composing the $N + 1$ time epochs we obtain

$$f^{(k)} = \begin{bmatrix} f_0^{(k)} \\ \bullet \\ \bullet \\ \bullet \\ f_N^{(k)} \end{bmatrix} \qquad (3\text{-}24)$$

and finally, composing for the m users yields the function $f : \mathbf{R}^{6m} \rightarrow \mathbf{R}^{mn(N+1)}$,

$$f = \begin{bmatrix} f^{(1)} \\ \bullet \\ \bullet \\ \bullet \\ f^{(m)} \end{bmatrix} \qquad (3\text{-}25)$$

The vector ρ is similarly composed, thus obtaining the function $\rho(\theta) : \mathbf{R}^{8m+\ n} \rightarrow \mathbf{R}^{mn(N+1)}$. The nonlinear GPS equations are

$$Z = \rho(\theta) + W \qquad (3\text{-}26)$$

where the $mn(N + 1)$ equation error W represents the composed measurement noise vector

$$W = \begin{bmatrix} w_{1,0}^{(1)} \\ \bullet \\ \bullet \\ \bullet \\ w_{n,N}^{(m)} \end{bmatrix}_{mn(N+1)x1} \qquad (3\text{-}27)$$

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with covariance provided in equation (3-11)

$$R_1 = E(WW^T) = \sigma^2 I_{mn(N+1)} \tag{3-28}$$

Linearization of equation (3-26) with respect to the parameter θ at the lth iteration, about the current parameter estimate $\hat{\theta}^{(l)}$, yields

$$Z = \rho(\hat{\theta}^{(l)}) + \frac{\partial \rho}{\partial \theta}\bigg|_{\hat{\theta}^{(l)}} (\theta - \hat{\theta}^{(l)}) + W \tag{3-29}$$

Equation (3-29) is rearranged to yield the linear regression in θ

$$Z + \frac{\partial \rho}{\partial \theta}\bigg|_{\hat{\theta}^{(l)}} \hat{\theta}^{(l)} - \rho(\hat{\theta}^{(l)}) = \frac{\partial \rho}{\partial \theta}\bigg|_{\hat{\theta}^{(l)}} \theta + W \tag{3-30}$$

Note that as a result of our carefully performed linearization of equation (3-20), the LHS of equation (3-30) consists not only of the measurement vector Z, as is the case in conventional ILS, and two additional terms are included. Obviously, when $\rho(\theta)$ is linear, there is no difference between our improved ILS algorithm and the conventional ILS algorithm, however when the nonlinearity is strong, the new ILS algorithm will outperform the conventional ILS algorithm.

The calculation of the regressor matrix requires the composition of the partials of $\rho(\theta)$ with respect to the parameter vector θ, viz., let

$$H_{i,j}^{(k)}(\theta) = \frac{\partial \rho_{i,j}^{(k)}(\theta)}{\partial \theta} \tag{3-31}$$

where $H_{i,j}^{(k)}$ is a 1 x ($7m + n$) row vector. Its entries are explicitly given by

$$H_{i,j}^{(k)}(\theta) = \frac{1}{f_{i,j}^{(k)}(\theta)} [(\theta)^T E_{i,k}(j) - x_{s_{i,j}} e_{1_{i,k}}(j) - y_{s_{i,j}} e_{2_{i,k}}(j) - z_{s_{i,j}} e_{3_{i,k}}(j)] \\ + e_{i,k} \tag{3-32}$$

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where

- $(x_{s_{i,j}}, y_{s_{i,j}}, z_{s_{i,j}})$ is the ith satellite ephemeris at time j.

- $e_{i,k}, e_{1_{i,k}}(j), e_{2_{i,k}}(j), e_{3_{i,k}}(j)$ are $8m + 4n$ row vectors of zeroes, with 1's, j, and j^2

located at positions indicated in their subscripts, according to

$$e_{1_{i,k}}(j) = e_{7k-6,7k-3,7m+4i-3}$$
$$e_{2_{i,k}}(j) = e_{7k-5,7k-2,7m+4i-2}$$
$$e_{3_{i,k}}(j) = e_{7k-4,7k-1,7m+4i-1}$$
$$e_{i,k} = e_{7k,7m+4i}$$

- The matrix

$$E_{i,k}(j) = e_{1_{i,k}}^{T}(j)e_{1_{i,k}}(j) + e_{2_{i,k}}^{T}(j)e_{2_{i,k}}(j) + e_{3_{i,k}}^{T}(j)e_{3_{i,k}}(j) \qquad (3\text{-}33)$$

The composition of the regressor matrix is similar to the process employed for Z and

f. First, for the n satellites, the n x $(7m + n)$ matrix

$$H_j^{(k)}(\theta) = \begin{bmatrix} H_{1,j}^{(k)}(\theta) \\ \bullet \\ \bullet \\ \bullet \\ H_{n,j}^{(k)}(\theta) \end{bmatrix} \qquad (3\text{-}34)$$

is formed, followed by composition over the $N + 1$ time epochs yielding the $n(N + 1)$ x

$(7m + n)$ matrix

$$H^{(k)}(\theta) = \begin{bmatrix} H_0^{(k)}(\theta) \\ \bullet \\ \bullet \\ \bullet \\ H_N^{(k)}(\theta) \end{bmatrix} \qquad (3\text{-}35)$$

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Next, set

$$H^{(k)}(\theta) := [H^{(k)}(\theta) \mid H_1]$$

where all the entries of the $n(N + 1)$ x n matrix H_1 are 0, except its kth column which is (0,

1, 2, ..., N)^T, and finally, for the m users, the $mn(N + 1)$ x ($8m + n$) regressor matrix

$$H(\theta) = \begin{bmatrix} H^{(1)}(\theta) \\ \bullet \\ \bullet \\ \bullet \\ H^{(m)}(\theta) \end{bmatrix} \tag{3-36}$$

is obtained. Thus Equation (3-29) yields the linear regression

$$Z + \left.\frac{\partial f}{\partial \theta}\right|_{\hat{\theta}^{(l)}} \hat{\theta}^{(l)} - f(\hat{\theta}^{(l)}) = \left.\frac{\partial \rho}{\partial \theta}\right|_{\hat{\theta}^{(l)}} \theta + W \tag{3-37}$$

$$= H(\hat{\theta}^{(l)})\theta + W$$

The function $\rho(\theta): R^{8m+ n} \to R^{mn(N+1)}$ is linear in the users' and satellites' clock error

parameters and therefore the function $\rho(\theta)$ is replaced by the function $f(\theta)$ in the LHS

of equation (3-37).

The solution for the users' "positions", viz., the parameter vector θ, may be obtained

using least squares, whereupon the Iterated Least Squares (ILS) algorithm is established:

$$\hat{\theta}^{(l+1)} = [H^T(\hat{\theta}^{(l)}) H(\hat{\theta}^{(l)})]^{-1} H^T(\hat{\theta}^{(l)})[Z + \left.\frac{\partial f}{\partial \theta}\right|_{\hat{\theta}^{(l)}} \hat{\theta}^{(l)} - f(\hat{\theta}^{(l)})] \tag{3-38}$$

where

- Z is the received "stacked up" pseudorange measurements $mn(N + 1)$ x 1
  vector, according to equation (3-20).

- f is the $mn(N + 1)$ x 1 vector of ranges calculated according to equation (3-21).

- H is the $mn(N + 1)$ x $(7m + n)$ regressor matrix formed from the partials of ρ according to equation (3-36).

3.3.4 Reduced parameter vector

The regressor H is further examined. As stated in equation (3-16), the parameter vector contains $(8m + n)$ variables: 3 position components, 3 velocity components, a clock bias variable, and a clock drift variable for each of the two $(m = 2)$ users, as well as the five $(n = 5)$ satellite clock errors - 21 variables in total. The main objective is to estimate the (non-reference) users' position and velocity rather than the users' and satellites' clock errors. With this in mind, the regressor's matrix structure is examined and the algorithm is modified according to the following analysis. To simplify the exposition, the users clock drifts are not considered herein and the reduced parameter vector has the dimension $7m + n$

Define the $mn(N + 1)$ x $(m + n)$ matrix

$$B = [b_1, b_2, b_3, ..., c_1, c_2, c_3, ..., c_n] \qquad (3\text{-}39)$$

where

- b_k is the column of H operating on user k^{th} clock error variable, $k = 1, 2, ..., m$.

- c_k is the column of H operating on satellite i^{th} clock error variable, $i = 1, 2, ...,$ n.

Indeed, the structure of the regressor matrix $H(\hat{\theta}^{(l)})$ is

$$H = [Diag(\tilde{H}^{(k)}(\hat{\theta}^{(l)}))_{k=1}^{m} \,|B] \qquad (3\text{-}40)$$

where the $n(N + 1)$ x 6 matrices $\tilde{H}^{(k)}$ are the $H^{(k)}$ matrices with only the six columns $(6k - 5)$, $(6k - 4)$, $(6k - 3)$, $(6k - 2)$, $(6k - 1)$ and 6k, retained; e.g., in the case where $m = 2$, and therefore $k = 1, 2$

$$H = \left[\begin{pmatrix} \tilde{H}^{(1)} & 0 \\ 0 & \tilde{H}^{(2)} \end{pmatrix} \Big| \; B \right] \tag{3-41}$$

where the $\tilde{H}^{(1)}$ and $\tilde{H}^{(2)}$ matrices are explicitly given by the partials of f in $(x_{0_1}, y_{0_1}, z_{0_1}, V_{x_1}\Delta T, V_{y_1}\Delta T, V_{z_1}\Delta T)$ and $(x_{0_2}, y_{0_2}, z_{0_2}, V_{x_2}\Delta T, V_{y_2}\Delta T, V_{z_2}\Delta T)$, respectively, and the matrix B is given by

$$B = \begin{bmatrix} & | & I_n \\ & | & \bullet \\ Diag\,(e_{n(N+1)})_{k=1}^{m} & | & \bullet \\ & | & \bullet \\ & | & I_n \end{bmatrix} \tag{3-42}$$

where

- $e_{n(N+1)}$ is a vector of ones of size $(N + 1)$ x 1 .

- I_n is an identity matrix of size n.

For example, for $m = 2$,

$$B = \begin{bmatrix} & | & I_n \\ & | & \bullet \\ \begin{pmatrix} e_{n(N+1)} & 0 \\ 0 & e_{n(N+1)} \end{pmatrix} & | & \bullet \\ & | & \bullet \\ & | & I_n \end{bmatrix} \tag{3-43}$$

The following holds

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$$\sum_{k=1}^{m} b_k = \sum_{i=1}^{n} c_i \tag{3-44}$$

Obviously, the matrix B (and therefore, the regressor H) is rank deficient. The rank deficiency is exactly 1. Thus, perform the full rank factorization of B,

$$B = B_1 K \tag{3-45}$$

where B_1 is full rank $(m + n - 1)$ and is an $mn(N + 1)$ x $(m + n - 1)$ matrix of the form

$$B_1 = [b_1, b_2, ..., b_{m-1}, \sum_{k=1}^{m} b_k, c_1, c_2, ..., c_{n-1}] \tag{3-46}$$

For example, in the special case where $m = 2$

$$B_1 = [b_1, b_1 + b_2, c_1, c_2, ..., c_{n-1}] \tag{3-47}$$

This choice of basis is responsible for inserting a column of ones in the regressor matrix H. In the parlance of linear regression, an "intercept" variable is then included. The latter has the beneficial effect of absorbing truncation errors caused by the linearization of the RHS of equation (3-30). This basis choice also has the effect of yielding the estimates of clock error differences, as indicated in equation (3-37); however, we are mainly concerned with the estimation of user 1's position and velocity parameters.

Solving equation (3-45) for K yields the blocked $(m + n - 1)$ x $(m + n)$ matrix

$$K = \begin{bmatrix} \tilde{I}_n & 0_{mx(n-1)} & b \\ 0_{(n-1)xm} & I_{n-1} & -e_{n-1} \end{bmatrix} \tag{3-48}$$

where

$$\tilde{I}_n = \begin{bmatrix} I_{m-1} & -e_{m-1} \\ 0_{1x(m-1)} & 1 \end{bmatrix} \tag{3-49}$$

$$b = \begin{bmatrix} 0_{(m-1)x1} \\ 1 \end{bmatrix} \tag{3-50}$$

and

- 0 is a zeroes matrix.

- e_n is a vector of ones of length n.

Next, partition the regressor

$$H = [\tilde{H}|B] \tag{3-51}$$

where $\tilde{H} = Diag(\tilde{H}^{(k)}(\hat{\theta}^{(l)}))_{k=1}^m$ is an $mn(N+1)$ x 6m matrix consisting of the columns of H operating on the users' position and velocity parameters only. Also define the reduced, full rank, matrix

$$H_1 = [\tilde{H}|B_1] \tag{3-52}$$

Next, perform the full rank factorization of the regressor $H (= [\tilde{H}|B])$

$$H = H_1 K_1 \tag{3-53}$$

i.e., the following equation is solved for the $(7m + n - 1)$ x $(7m + n)$ matrix K_1 :

$$[\tilde{H}|B_1 K] = [\tilde{H}|B_1] K_1 \tag{3-54}$$

We calculate:

$$K_1 = \begin{bmatrix} I_{6m} & 0_{6mx(m+n)} \\ 0_{(m+n-1)x6m} & K_{(m+n-1)x(m+n)} \end{bmatrix} \tag{3-55}$$

Finally, the reduced parameter vector is defined

$$\theta_1 = K_1 \theta \tag{3-56}$$

The reduced parameter vector $\theta_1 \in 7^{m+n-1}$ consists of the users' position and velocity parameters we are interested in, as well as linear combinations of the user and satellite

clock errors. For the specific scenario examined in which we have two mobile users and five satellites visible, this yields the (18 x 1) parameter vector

$$\theta_1 = [x_{0_1}, y_{0_1}, z_{0_1}, V_{x_1}\Delta T, V_{y_1}\Delta T, V_{z_1}\Delta T, x_{0_2}, y_{0_2}, z_{0_2}, V_{x_2}\Delta T, V_{y_2}\Delta T, V_{z_2}\Delta T,$$
$$\tau_1 - \tau_2, \tau_2 + \tau_{s_5}, \tau_{s_1} - \tau_{s_5}, \tau_{s_2} - \tau_{s_5}, \tau_{s_3} - \tau_{s_5}, \tau_{s_4} - \tau_{s_5}]^T \tag{3-57}$$

Since

$$H\theta = H_1 K_1 \theta = H_1 \theta_1 \tag{3-58}$$

equation (3-37) is written with the reduced parameter θ_1 :

$$Z + H_2(\hat{\theta}_2^{(l)})\hat{\theta}_2^{(l)} - f(\hat{\theta}^{(l)}) = H_1(\hat{\theta}_2^{(l)})\hat{\theta}_1^{(l)} + W \tag{3-59}$$

where, in addition, the further reduced, (12 x 1), parameter vector is used

$$\theta_2 = [x_{0_1}, y_{0_1}, z_{0_1}, V_{x_1}\Delta T, V_{y_1}\Delta T, V_{z_1}\Delta T, x_{0_2}, y_{0_2}, z_{0_2}, V_{x_2}\Delta T, V_{y_2}\Delta T,$$
$$V_{z_2}\Delta T]^T \tag{3-60}$$

Thus, the further reduced parameter vector θ_2 is stripped of the clock error parameters of θ_1 , and θ_2 (and not θ_1) is used on the LHS of equation (3-59) because the function f is not dependent on the time parameters. Accordingly, the matrix H_2 is composed of the first 6m columns of H associated with the parameters featuring in θ_2 (positions and velocities, no clock errors). Thus, $H_2 = Diag(\tilde{H}^{(k)})_{k=1}^m$.

With reference to equation (3-57) it is noted that not using differencing and double differencing as is standard practice in DGPS and instead treating the common errors as parameters to be estimated yields estimates of the clocks' relative errors. In relative DGPS good relative position estimates are obtained but interestingly, it is not the relative positions, but rather, the clocks' relative errors that are directly estimated. Should

laser ranging be available, as in the European GIOVE-A experiment, clock and ephemeris errors could be separated.

3.3.5 Prior Information

The linear regression featuring in the ILS algorithm can be augmented to include prior information on user 2 (position and velocity), as in the conventional and network DGPS [36, 38, 42]. The prior information is provided in the form

$$
\begin{aligned}
x_{0_2} &\in N(\bar{x}_{0_2}, \sigma^2_{x_{0_2}}) \\
y_{0_2} &\in N(\bar{y}_{0_2}, \sigma^2_{y_{0_2}}) \\
z_{0_2} &\in N(\bar{z}_{0_2}, \sigma^2_{z_{0_2}}) \\
V_{x_2} &\in N(\bar{V}_{x_2}, \sigma^2_{V_{x_2}}) \\
V_{y_2} &\in N(\bar{V}_{y_2}, \sigma^2_{V_{y_2}}) \\
V_{z_2} &\in N(\bar{V}_{z_2}, \sigma^2_{V_{z_2}})
\end{aligned}
$$

(3-61)

Using very large σ parameters in equation (3-61) is tantamount to the stipulation that no prior information on user 2 initial state is available.

The linear regression in equation (3-59) is now augmented as follows:

$$
Z := \begin{pmatrix} Z \\ Z_1 \end{pmatrix}
$$

(3-62)

where

$$
Z_1 = \begin{pmatrix} \bar{x}_{0_2} \\ \bar{y}_{0_2} \\ \bar{z}_{0_2} \\ \bar{V}_{x_2} \\ \bar{V}_{y_2} \\ \bar{V}_{z_2} \end{pmatrix}
$$

(3-63)

In addition, the regressor H_1 is augmented

$$H_1 := \left[\frac{H_1}{M} \right] \tag{3-64}$$

where the 6 x $(7m + n - 1)$ selector matrix is, e.g., in the case where $m = 2$,

$$M = [0_{6x6}|I_6 \,|0_{6x(m+n-1)}] \tag{3-65}$$

and when appended to H_1 picks out the $x_{0_2}, y_{0_2}, z_{0_2}, V_{x_2}\Delta T, V_{y_2}\Delta T, V_{z_2}\Delta T$ elements of the

parameter vector θ_1. Moreover,

$$H_2 := \left[\frac{H_2}{0_{6x6m}} \right] \tag{3-66}$$

$$f := \left[\frac{f}{0_{6x1}} \right] \tag{3-67}$$

$$W := \begin{bmatrix} W \\ W_{mn(N+1)+1} \\ \bullet \\ \bullet \\ W_{mn(N+1)+6} \end{bmatrix} \tag{3-68}$$

Additionally, a weighting matrix R is included to correctly incorporate the

confidence level in the "reference " receiver's (user 2) prior information on position,

velocity, and possibly, range equivalent clock error:

$$R = Diag(R_1, R_2) \tag{3-69}$$

where

$$R_1 = \sigma^2 I_{mn(N+1)} \tag{3-70}$$

is determined by the measurement noise variance σ, and the diagonal matrix R_2 contains the prior information data, viz., the standard deviations of the reference station's initial position (and velocity)

$$R_2 = \begin{bmatrix} \sigma^2_{x_{0_2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2_{y_{0_2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{z_{0_2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{V_{y_2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{V_{z_2}} \end{bmatrix} \tag{3-71}$$

where

$$\begin{aligned} \sigma^2_{x_{0_2}} &= E(W^2_{mn(N+1)+1}) \\ \sigma^2_{y_{0_2}} &= E(W^2_{mn(N+1)+2}) \\ \sigma^2_{z_{0_2}} &= E(W^2_{mn(N+1)+3}) \\ \sigma^2_{V_{x_2}} &= E(W^2_{mn(N+1)+4}) \\ \sigma^2_{V_{y_2}} &= E(W^2_{mn(N+1)+5}) \\ \sigma^2_{V_{z_2}} &= E(W^2_{mn(N+1)+6}) \end{aligned} \tag{3-72}$$

This finally yields the enhanced ILS algorithm

$$\begin{aligned} \hat{\theta}_1^{(l+1)} &= [H_1^T(\hat{\theta}_2^{(l)}) R^{-1} H_1(\hat{\theta}_2^{(l)})]^{-1} H_1^T(\hat{\theta}_2^{(l)}) R^{-1} [Z \\ &\quad + H_2(\hat{\theta}^{(l)})\hat{\theta}^{(l)} - f(\hat{\theta}^{(l)})] \end{aligned} \tag{3-73}$$

and

$$l = 0, 1, \ldots, L.$$

Finally, if the satellite clock errors are known to high precision, as provided by NASA's Internet-Based Global Differential GPS, and the residual errors can be isolated to the ionospheric delay, then the additional prior information $\Delta_{ion_i} \approx N(25, 225)$ can be

used based on the ionospheric retardation. That is, the linear regression is augmented to include the equations

$$25 = \Delta_{\mathrm{i}on_i} + v_i, \quad v_i \approx N(0,\ 255), \quad \mathrm{i} = 1, \ldots, \mathrm{n}$$

3.4 Simulation Results

This section presents the simulation results and validates the algorithm's ability to correctly estimate the parameters of interest.

3. 4.1 Simulation Scenarios

In all the simulation scenarios two ($m = 2$) mobile users flying along parallel tracks, approximately 180 nautical miles (LEO) above the earth surface and separated by 10,000 meters, are considered. The number of satellites in view is $n = 5$ and $n = 8$ to illustrate the effect of satellite availability. The normalized user velocities V ΔT are $V_{x_1} = V_{x_2} = 100 m/\sec$ and $V_{y_1} = V_{y_2} = V_{z_1} = V_{z_2} = 0$. In the simulation experiments the users' range-equivalent clock bias are fixed at $\tau_1 = \tau_2 = 200m$, and all the satellites are subject to the same range-equivalent clock errors: $\tau_{s_i} = 100m, i = 1, 2, ..., n$ or $\tau_{s_i} = 30m, i = 1, 2, ..., n$. The number of measurement epochs considered varied between N = 1 and N = 11. In the case where N = 1, kinematic GPS is not possible and the users' velocities cannot be estimated. This scenario does however correspond to the conventional stand-alone DGPS paradigm and allows for a direct comparison to be made of the performance of the novel algorithm and the differencing-based conventional DGPS algorithm.

The random residual errors and receiver noise levels used in the simulation have σ of 0.75m, 1.50m, or 5.00m. Ten iterations (L=10) for the ILS algorithm were performed for all scenarios.

In the simulation experiments, the actual user 2 initial position is randomly chosen according to the specified normal distribution; the latter gauges our confidence in the user 2 prior position information. Both the nominal user 2's initial position and the σ information are provided to the DGPS algorithm, but not the realized (actual) user 2 position used to generate the pseudorange data given to the algorithm. The algorithm's estimation performance is gauged by measuring the distance of the computed position estimate from the actual true position which, of course, is known during the controlled simulation experiment. This applies to both user 1 and user 2. For each simulation scenario, 100 Monte Carlo (MC) runs were performed.

3.4.2 Numerical Results

The DGPS algorithm's ability to accurately estimate the position of user 1 in the face of large satellite clock errors is strongly dependent on the accuracy of the reference station's position information. This is in particular true when the number of satellites in view, n, is 5. When the number of satellites in view is increased to $n = 8$, the estimation accuracy improves considerably, even in the case of one measurement epoch ($N = 1$) only. This is to be expected: when $N = 1$, the number of unknowns is $4m + n\ 1 = 4 \times 2 + n\ 1 = 7 + n$. The number of equations is $mn = 2n$. Evidently, to obtain a navigation solution using the novel algorithm we need

$$7 + n \leq 2n \tag{3-74}$$

i.e.,

$$7 \leq n \tag{3-75}$$

At the completion of the MC experiment, the kth user position estimation error is:

$$e_k = \frac{1}{100} \sqrt{\left| \sum_{p=1}^{100} (x_{0_k} - \hat{x}_{0_k}^{(p)}) \right|^2 + \left| \sum_{p=1}^{100} (y_{0_k} - \hat{y}_{0_k}^{(p)}) \right|^2 + \left| \sum_{p=1}^{100} (z_{0_k} - \hat{z}_{0_k}^{(p)}) \right|^2} \tag{3-76}$$

where, unbeknown to the estimation algorithm, $(x_{0_k}, y_{0k}, z_{0_k})$ is the true initial position of

user k and is used to generate the data in the simulation, which the estimation algorithm

operates on, and $(\hat{x}_{0_k}^{(p)}, \hat{y}_{0_k}^{(p)}, \hat{z}_{0_k}^{(p)})$ is the algorithm-produced position estimate in the pth

MC run. The estimation error spread is gauged by the estimation error variance. The

latter is provided by the estimation algorithm and is referred to as the predicted

estimation error variance $\sigma_{P_k}, k = 1, 2$:

$$\sigma_{P_1} \equiv \sqrt{((H_1^T R^{-1} H_1)^{-1})_{1,1} + ((H_1^T R^{-1} H_1)^{-1})_{2,2} + ((H_1^T R^{-1} H_1)^{-1})_{3,3}} \tag{3-77}$$

and similarly

$$\sigma_{P_2} \equiv \sqrt{((H_1^T R^{-1} H_1)^{-1})_{7,7} + ((H_1^T R^{-1} H_1)^{-1})_{8,8} + ((H_1^T R^{-1} H_1)^{-1})_{9,9}} \tag{3-78}$$

Finally, the experimentally obtained estimation error variance is calculated as

$$\sigma_{E_k} = \frac{1}{100} \sqrt{\sum_{p=1}^{100} (x_{0_k} - \hat{x}_{0_k}^{(p)})^2 + \sum_{p=1}^{100} (y_{0_k} - \hat{y}_{0_k}^{(p)})^2 + \sum_{p=1}^{100} (z_{0_k} - \hat{z}_{0_k}^{(p)})^2} \tag{3-79}$$

The performance of the novel estimation algorithm is summarized in Tables 4 and 5 on

the next page. The user 1 position estimation error is less than 4 m even though the two

users' range equivalent clock errors are $\tau_1 = \tau_2 = 5m$, the eight satellites' large range

equivalent clock errors are $\tau_{s_i} = 100m$, $i = 1, 2, ..., 8$, the inaccuracy in the reference station

(user 2) position is 5m and the relatively large receiver noise variance is 5m.

Table 4.   Novel Algorithm Estimation Error & Accuracy [meters] with $m = 2$, $n = 8$, $N = 1$, $L = 10$ for 1 epoch

Referenced σ	Noise σ	e_1	σ_{P1}	σ_{E1}
0.0001	0.75	0.501	1.583	2.874
0.0001	1.50	0.875	2.945	3.459
0.0001	5.00	0.841	9.599	8.890
0.75	0.75	0.502	2.003	5.204
0.75	1.50	1.571	3.165	5.642
0.75	5.00	0.353	9.664	10.489
1.50	0.75	0.907	5.151	16.160
1.50	1.50	2.944	5.615	16.442
1.50	5.00	1.558	10.550	20.001
5.00	0.75	2.535	10.073	31.425
5.00	1.50	2.913	10.302	33.069
5.00	5.00	3.930	13.354	36.874

The estimation error is significantly reduced when five measurement epochs are used. In this case, and when the uncertainty in the user 2 position is small (Ref. $\sigma = 0{:}0001$), the following result is obtained in Table 5:

Table 5.   Novel Algorithm Estimation Error & Accuracy [meters] with $m = 2, n = 8, N = 5, L = 10$ for 5 epochs

Referenced σ	Noise σ	e_1	σ_{P1}	σ_{E1}
0.0001	0.75	0.225	0.523	0.714
0.0001	1.50	0.255	1.743	1.508
0.0001	5.00	0.285	3.486	5.129

Finally, the conventional DGPS algorithm's results are given in Table 6. As expected, the estimates in Table 5 are better than the estimates in Table 6. This applies to both the estimation error, and to the estimation error variance.

Table 6.   Conventional DGPS Estimation Error & Accuracy [meters] with $m = 2, n = 8, N = 1, L = 10$ for 5 epochs

Referenced σ	Noise σ	e_1	σ_{P1}	σ_{E1}
0.0001	0.75	0.290	1.355	0.983
0.0001	1.50	0.355	1.560	2.172
0.0001	5.00	0.528	2.486	7.100

Furthermore, with $n = 8$ satellites in view it is possible to obtain a good estimate of the users' relative positions even without prior information on the reference station's (user 2) position - as required in formation control.

IV Conclusions

This research presents the theories, models, and simulation results for a direct, closed-form and efficient new position determination algorithm developed for a stand-alone user and the stochastic model-based DGPS estimation algorithm constructed for a network of mobile users. The theoretical developments stochastically modeled the pseudorange measurement with random white noise and solved for position as a stochastic estimation problem. The following sections present the conclusions obtained from this research.

4.1 A Direct, Closed-Form and Efficient New Position Determination Algorithm

The performance of the novel two-step algorithm is comparable to that of the baseline ILS algorithm. Furthermore, it retains all the attractive features that motivated the development of the closed-form algorithm in the first place. Hence, considering the closed-form algorithm supplemented by the Kalman update algorithm as a single two-step GPS position determination algorithm, a novel algorithm with the following attributes has been developed:

- The performance under conventional navigation scenarios is comparable to that achieved by the conventional ILS algorithm.

- The algorithm is self-contained; hence it can be used under any geometrical conditions without the need for externally provided initialization, and a degree of autonomy is thus achieved.

- The algorithm is computationally efficient because it requires less iteration than the ILS, and its FLOPS count is lower.

- The algorithm has the capability to produce a data-driven estimate of the receiver measurement noise strength (σ) when the number of satellites in view n ≥ 6 and is able to predict its estimation error covariance.

- Good estimated performance under conditions of high GDOP is obtained. The horizontal positioning performance of the novel two-step algorithm under poor geometry conditions, e.g. when ground-based planar arrays of pseudolites are used, is similar to that of the conventional ILS algorithm. Moreover, there are no restrictions on the user position and an initial user position guess is not required as long as the user was within the confines of the outer radius of the circular pattern. This may prove beneficial in testing range applications where the conventional iterative algorithm is at risk of failure, imposing restrictions on the flight test trajectory and altitude.

In conclusion, the benefits of the two-step algorithms are computational efficiency, data-driven predictions of the noise strength of the pseudorange measurements and the estimation error covariance, no need for an initial position guess, and may provide better performance under poor geometry.

4.2   *Stochastic Model-Based DGPS Estimation Algorithm*

In this dissertation a network-based navigation concept is advanced and a novel algorithm for KDGPS data processing is presented. Specifically, the accurate relative (and absolute) positioning of a team (or formation) of mobile vehicles is considered. The measurement scenario is correctly modeled, that is, a stochastic framework is developed where rather than using double differencing the common errors are explicitly acknowledged and a novel centralized estimation algorithm is rigorously derived. The novel DGPS algorithm estimates the users' positions and velocities utilizing only pseudorange measurements. One of the benefits of the novel algorithm is its ability to account for the uncertainty in the reference receiver's position and allows for the inclusion of the information on the reference user's position accuracy. Indeed, DKGPS exploits the same self calibration concept as DGPS to provide comparable accuracies in a dynamic environment. DGPS can provide enhanced positioning accuracies (on the order of 1m) by using a stationary base station, a.k.a, reference station, at a surveyed location. Unlike DGPS, KDGPS does not require a designated base station and consequently cannot improve absolute accuracies. Instead, it provides a similar degree of relative position accuracy between the mobile GPS-equipped platforms as required in formation control. A data driven estimate of the predicted covariance of the position estimation error is provided. Last but not least, insight into the structure of the estimation problem on hand is gained. In relative DGPS good estimates of the relative positions of the rovers are obtained; however, it is not the relative positions, but rather, it's the clocks' relative errors that are directly estimated.

Simulations entailing MC experiment validate the algorithm's ability to estimate the variables of interest. The results are also compared to conventional DGPS scenarios, where the reference station(s) is (are) stationary and/or only one set of pseudorange measurements is available. The positioning accuracy improvements achieved with the novel algorithm are gauged against the performance of conventional DGPS.

The generic nature of the mathematical development presented in this dissertation which entails stochastic modeling, the use of the method of linear regression from statistics, and the adaptation of the SVD method from numerical analysis are conducive to a broad range of applications of the estimation algorithm. These entail

1) Augmented conventional DGPS.

a. Correctly treat measurement noise effects in conventional DGPS, where differencing causes the equation error variance to increase to $2\sigma^2$ and correlation is introduced. Furthermore, the equation error is no longer homoscedastic and a weighted ILS must be used.

b. Treat ephemeris errors (not just satellite clock errors) as unknown parameters to be estimated.

2) Formations of multiple spacecraft are envisioned for many planned space missions, and GPS will play an important role as a sensor. Tight control of the vehicles' positions relative to each other is very important. Formation technologies for spacecraft will enable multiple distributed spacecraft to act in a unified manner. This will enable new scientific missions involving distributed but coordinated measurements, leading to improved stellar interferometry, gravimetry, and synthetic aperture radars. GPS, and, in

particular generalized KDGPS, provides a promising relative navigation sensor. Similar success is foreseen in aircraft formation flying applications.

3) Relative GPS guidance, where a targeting system provides a moving target's coordinates using an on-board GPS receiver for aiding its inertial reference system. A GPS receiver is also on the weapon to track the same satellites as the GPS receiver on the targeting aircraft. Thus some of the GPS bias errors in the targeting data are removed and guidance accuracies can be improved to near "precision" level without the use of a terminal guidance seeker.

Appendix A: Acronym List

AFIT   Air Force Institute of Technology
AFRL   Air Force Research Laboratory
CPGPS Carrier Phase Global Positioning System
DCM   Direction Cosine Matrix
ECEF   Earth-Centered, Earth-Fixed
EKF   Extended Kalman Filter
GPS   Global Positioning System
INS   Inertial Navigation System
PR   Pseudorange
RMS   Root-Mean-Square
SA   Selective Availability
SV   Satellite Vehicle
GDOP Geometric Dilution Of Precision
UERE User Equivalent Range Error

Appendix B: Bivariate Gaussian Distribution

To evaluate the performance of the novel algorithm, the probability of the position estimate to be within an ellipsoid was computed. For a Bivariate Gaussian Distribution, the probability of the position estimates within the 1σ can be obtained as follows:

$$p = \frac{1}{2\pi|\det(P)|^{\frac{1}{2}}} \iint\limits_{X=\{\vec{x}|\vec{x}^T P^{-1}\vec{x}\le 1\}} e^{-\frac{1}{2}\vec{x}^T P \vec{x}} d\vec{x}$$

First, let the estimate error covariance matrix to be

$$P = T^T \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} T$$

where T is the orthonormal transformation matrix for P. The inverse of the error covariance matrix can be written as:

$$P^{-1} = T^T \begin{bmatrix} 1/\lambda_1^2 & 0 \\ 0 & 1/\lambda_2^2 \end{bmatrix} T$$

$$= \left(\begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix} T\right)^T \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix} T$$

Now, if we let

$$\vec{y} = \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix} T \vec{x}$$

then obviously

$$\vec{x} P^{-1}\vec{x} = \vec{y}^T \vec{y}$$

Thus, the transformation from y into x is

$$\Gamma = T^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where

$$\det(\Gamma) = \lambda_1 \lambda_2$$

Using the above relationships, the probability of the position estimate to be within the ellipsoid can be found as

$$
\begin{aligned}
p &= \frac{1}{2\pi |\det(P)|^{\frac{1}{2}}} \iint_{X=\{\vec{x}|\vec{x}^T P \vec{x} \le 1\}} e^{-\frac{1}{2}\vec{x}^T P \vec{x}}\, d\vec{x} \\
&= \frac{1}{2\pi |\det(P)|^{\frac{1}{2}}} \iint_{Y=\{\vec{y}|\vec{y}^T \vec{y} \le 1\}} e^{-\frac{1}{2}\vec{y}^T \vec{y}}\, d\vec{y} \\
&= \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} d\varphi \int_{\rho=0}^{1} \rho e^{-\frac{1}{2}\rho^2}\, d\rho \quad = \int_0^1 \rho e^{-\frac{1}{2}\rho^2}\, d\rho = -e^{-\frac{1}{2}\rho^2} \Big|_0^1 \\
&= 1 - 1/\sqrt{e} = 0.39
\end{aligned}
$$

irrespective of the covariance matrix P.

References

[1]   Kaplan, Elliott D. "Understanding GPS, Principles and Applications", Boston: Artech House, Inc., 1996.

[2]   Brian J. Young, "An Integrated Synthetic Aperture Radar/GPS/INS System For Target Geolocation Improvement", AFIT, 1999.

[3]   Raquet J. F., Captain, USAF, Class Notes, 2001.

[4]   Cox, D. B. and Brading, J. D. W., "Integration of LAMDA Ambiguity Resolution with Kalman Filter for Relative Navigation of Spacecraft", ION 55th Annual Meeting, June 1999.

[5]   Li, Z. and Gao, Y., "Construction of High Dimensional Ambiguity Transformations for the LAMDA Method", University of Calgary, 1999.

[6]   Teunissen, P. J. G., De Jonge, P. J., and Tiberius, C. C. J. M., "Performance of the LAMDA Method for Fast GPS Ambiguity Resolution", Navigation: Journal of The Institute of Navigation, Vol. 44, No. 3, Fall 1997.

[7]   Olsen, E. A., Park, C., and How, J., "3D Formation Flight using Differential Carrier-phase GPS Sensors", Stanford University, 1998.

[8]   Siouris, G. M., "Aerospace Avionics Systems: A Modern Synthesis". San Diego: Academic Press, Inc., 1993.

[9]   Bancroft, S., "An Algebraic Solution of the GPS equations, IEEE Transactions on Aerospace an Electronic Systems", 21 (1):56-59 (November 1985).

[10] Leva, J. L., "An Alternative Closed-Form Solution to the GPS Pseudo-Range Equations, IEEE Transaction on Aerospace and Electronic Systems", Vol. 32, No. 4, October 1996, pp. 1430-9.

[11] Krause, L. O., "A Direct Solution to GPS Type Navigation Equations, IEEE Transactions on Aerospace and Electronic Systems", 23(2):223-232 (March 1987).

[12] Abel, J. and Chaffee J., "Existence and Uniqueness of GPS Solutions, IEEE Transactions on Aerospace and Electronic Systems", 27(6):952-956 (November 1991).

[13] Chaffee, J. and Abel J., "On the Exact Solutions of Pseudorange Equations, IEEE Transactions on Aerospace and Electronic Systems", 30(4):1021-1030 (October 1994).

[14] Hoshen, J., "The GPS Equations and the Problem of Apollonius, IEEE Transactions on Aerospace and Electronic Systems", 32(3):1116-1124 (July 1996).

[15] Nardi, S. and Pachter, M., "GPS Estimation Algorithm Using Stochastic Modeling, Proceedings of the Conference on Decision and Control", Tampa, Fl, December 1998.

[16] Pachter, M. and Nardi S., "Stochastic modeling in GPS Estimation, Advances in Intelligent System: Concepts, Tools and Applications", Kluwer Academic, 35:389-400, 1999.

[17] Dailey, D. J. and Bell B. M., "A Method for GPS Positioning, IEEE Transactions on Aerospace and Electronic Systems", 32(3):1148-1154 (July 1996).

[18] Biton, I., Koifman, M., and Bar-Itzhack, I. Y., "Improved direct Solution of the Global Positioning System Equation", Journal of Guidance, Control, and Dynamics, Vol. 21, No. 1, January-February 1998, pp. 45-9.

[19] Wolfe, J. D., and Speyer, J. L., "Exact Statistical Solution of Pseudorange Equations", in ION GPS 2001, ION, Salt Lake City, UT, September 2001, pp. 3052-9.

[20] McKay, J. B., "Optimization of a GPS-Based Navigation Reference System". MS Thesis, School of Engineering, Air Force Institute of Technology, Wright-Patterson AFB, OH, December 1996.

[21] Pachter, M. and McKay J. B., "Geometry Optimization of a GPS-Based Navigation System", Navigation: Journal of the Institute of Navigation, Vol.44, No 4, Winter 1997-1998, pp 457-470.

[22] Pachter, M., and T. Q. Nguyen: "An Efficient GPS Position Determination Algorithm", Navigation: Journal of the Institute of Navigation, Vol. 50, No 2, Summer 2003, pp. 131-141.

[23] Spilker, J. J. Jr. and Parkinson B. W., "Overview of GPS Operation and Design, GPS: Theory and Applications Volume 1", edited by B. W. Parkinson. Washington, D.C.: American Institute of Aeronautics and Astronautics, 1996.

[24] The Math Works, Inc., Natick, MA. MATLAB (registered trademark) (Version 4.2c Edition), November 1994.

[25] Maybeck, P., "Stochastic Models, Estimation, and Control, I". New York: Academic Press, Inc., 1979. Republished, Arlington, VA: Navtech, 1994.

[26] Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems, ASME Transactions, Journal of Basic Engineering", 82:34-45 (1960).

[27] Kalman, R. E. and Bucy R. S., "New Results in Linear Filtering and Prediction Theory, Transactions of the ASME, Series D: Journal of Basic Engineering", 83:95-108 (1961).

[28] GPSoft, Athens, OH., "Satellite Navigation Toolbox for Matlab", January 1997.

[29] http://spacetechnology3.jpl.nasa.gov/proj/index.html.

[30] Olsen, E.A., Park, C., and How, J., "3D Formation Flight using Differential Carrier-phase GPS Sensors", Stanford University, 1998.

[31] Corazzini, T. and How, J., "Onboard GPS Signal Augmentation For Formation Flying", Stanford University, 1998.

[32] Corazzini, T., Robertson, A., Adams, J.C., Hassibi, A., and How, J., "GPS Sensing for Spacecraft Formation Flying", Stanford University, 1997.

[33] How, J., Twiggs, R., Weidow, D., Hartman, K., and Bauer, F., "Orion: A Low-Cost Demonstration of Formation Flying In Space Using GPS", American Institute of Aeronautics and Astronautics, 1998.

[34] Wu, S. and Yunck, T., "Precise Kinematic Positioning with Simultaneous GPS and Carrier Phase Measurements", Jet Propulsion Laboratory, California Institute of Technology.

[35] Parkinson, B. W., and J. J. Spilker: "Global Positioning System: Theory and Applications", Vols. 1 and 2, AIAA Press, 1996.

[36] Wanninger, L.: "Introduction to Network RTK", URL: http://www.network-rtk.info/intro/, December 2005.

[37] IGDG (2004). Internet-Based Global Differential GPS. Internet URL: http://gipsy.jpl.nasa.gov/igdg/.

[38] Kechine, M. O., C. C. J. M. Tiberius, and H. van der Marel: "Network Differential GPS – Kinematic Positioning with NASA's Internet-Based Global Differential GPS", Journal of Global Positioning Systems, Vol. 2, No 2, pp. 139-143, 2003.

[39] Williamson, W. R.: "Real Time, High Accuracy, Relative State Estimation for Multiple Vehicle Systems", Ph.D. dissertation, University of California, Los Angeles, 2000.

[40] Kawano, I., M. Mokuno, T. Kasai and T. Suzuki: "First Autonomous Rendezvous Using Relative GPS Navigation", NAVIGATION, Vol. 48, No 1, pp. 49-56, 2001.

[41] Montenbruck, O., T. Ebinuma, E. G. Lightsey and S. Leung: "A Real-Time Kinematic GPS Sensor for Spacecraft Relative Navigation", Aerospace Science and Technology, Vol. 6, No 6, pp. 435-449, October 2002.

[42] Leick, A.: "GPS Satellite Surveying ", Wiley, New York, 1995, pp. 178.

[43] RTCM Recommended Standards for Differential NAVSTAR GPS Service, Version 2.1, RTCM Special Committee No 104, January 3, 1994.

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14. ABSTRACT
      This research is aimed at improving the state of the art of GPS algorithms, namely, the development of a closed-form positioning algorithm for a stand-alone user and the development of a novel differential GPS algorithm for a network of users.
      The stand-alone user GPS algorithm is a direct, closed-form, and efficient new position determination algorithm that exploits the closed-form solution of the GPS trilateration equations and works in the presence of pseudorange measurement noise for an arbitrary number of satellites in view. A two-step GPS position determination algorithm is derived which entails the solution of a linear regression and updates the solution based on one nonlinear measurement equation. In this algorithm, only two or three iterations are required as opposed to five iterations that are normally required in the standard Iterative Least Squares (ILS) algorithm currently used. The mathematically derived stochastic model-based solution algorithm for the GPS pseudorange equations is also assessed and compared to the conventional ILS algorithm. Good estimation performance is achieved, even under high Geometric Dilution of Precision (GDOP) conditions.
The novel differential GPS algorithm for a network of users that has been developed in this research uses a Kinematic Differential Global Positioning System (KDGPS) approach. A network of mobile receivers is considered, one of which will be designated the 'reference station' which will have known position and velocity information at the beginning of the time interval being examined. The measurement situation on hand is properly modeled, and a centralized estimation algorithm processing several epochs of data is developed. The effect of uncertainty in the reference receiver's position and the level of the receiver noise are investigated. Monte Carlo simulations are performed to examine the ability of the algorithm to correctly estimate the non-reference mobile users' position and velocity despite substantial satellite clock errors and receiver measurement noise.

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