Scaling and multiscaling in financial time series

Jean-François Muzy
CNRS UMR 6134, Université de Corse,
20250 Corté, France
muzy@univ-corse.fr

Emmanuel Bacry
CMAP, Ecole Polytechnique, Palaiseau, France

Alexey Kozhemyak
CMAP, Ecole Polytechnique, Palaiseau, France

Alain Arneodo
Lab. de Physique, ENS, Lyon, France

Jean Delour
BNP-Paribas, Paris, France

Didier Sornette
LPMC, Université de Nice, Nice, France
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**ABSTRACT:**
See also ADM001750, Wavelets and Multifractal Analysis (WAMA) Workshop held on 19-31 July 2004., The original document contains color images.
Outline

1/ A brief overview of financial markets
   - Basic definitions and problems related to finance
   - Scaling in finance

2/ Empirical properties of financial time series
   - Main “stylized facts”
   - Scaling properties

3/ Empirical models: From Bachelier to Mandelbrot
   - Fat tails: Truncated Levy models
   - Heteroskedasticity: Classical econometric models.
   - Multifractal Models

4/ The MRW model
   - Definition and scaling properties
   - Estimation issues

5/ Applications
   - Risk evaluation and forecasting
   - Portfolio theory and option pricing

6/ Conclusion and prospects
An overview of financial markets

- **Individuals**: Speculation, investment
- **Corporations, firms**: Raise funds (issue shares), investment
- **Banks, Financial institutions, Pension funds,...**: Hedging, arbitrage

- Markets: Financial securities (Stock, Bonds, options, futures,...), FX rates, Commodities,...
Some definitions: returns

- \( P(t) \): market asset price at time \( t \)
- \( X(t) = \ln P(t) \) return process
- Return at time \( t \) over a period \( \tau \):

\[
\begin{align*}
r(t, \tau) &= X(t + \tau) - X(t) = \ln P(t + \tau) - \ln P(t) \\
&\approx \frac{P(t + \tau) - P(t)}{P(t)} = R(t, \tau)
\end{align*}
\]

Dow-Jones Index

Annual net return: \( R \approx 11\% \)

Hold 1 usd over 25 years period \( \rightarrow (1.11)^{25} \approx 13.6 \text{ usd} \)
Some definitions: volatility

The *volatility* quantifies the size of return fluctuations.

\[ r(t, \tau) = \mu(\tau) + \sigma(\tau)\epsilon(t) \]

where \( \epsilon(t) \) is a normalized noise.

It is often identified to the *variance* \( \sigma^2 \) of return fluctuations.

![Graphs showing daily returns, Dow-Jones, and Gaussian white noise](image)
Problems of quantitative finance

- Rational investment and risk management
  - Price dynamics
  - Risk quantification and control
  - Financial instruments: derivatives

- Micro-economics
  - Market behavior under uncertainty
  - Agent based theory (Utility functions)

- Tools and fields
  - Probability and statistics, time series analysis, stochastic calculus,...
  - Econometrics, applied mathematics, statistical physics, physics of complex systems,...
Scaling in finance

• Supported by empirical observations

• Practical interests.
  - Stability over time scales (by aggregation)
  - The same model is valid over a wide range of scales.
  - Small number of parameters
  - Analytically tractable models

• Theoretical interest: universality and parcimony
  - No preferred scale ratio: scale invariance
  - Continuous time formulation.
  - Universality (small number of pertinent parameters)
Empirical properties of return time series

No return correlation

5 min return correlation function associated with 3 assets.

(from Bouchaud and Potters 1997)

Market efficiency: Return changes are unforecastable (martingale hypothesis)
⇒ Prices are *linearly* unforecastable
Empirical properties of return time series

- Return pdf have fat tails at small scales

Cumulative distribution and pdf of normalized 5-min returns of 1,000 largest US companies. (from Farmer 1999)

- Quasi-Gaussian at large scales
Empirical properties of return time series

Volatility clustering

Power-law volatility correlation of USD-DM rate.
(from Daracogna 2000)
Empirical properties of return time series

Log-normal volatility

S&P500 Volatility distribution.

(from Cizeau et al. 1997)
Empirical properties of return time series

Multi-scaling of returns

- Scaling of return absolute moments

\[ M(q, \tau) = \mathbb{E} [|r(\tau,t)|^q] \sim K_q \tau^{\zeta_q}. \]

- The return pdf varies strongly across scales
Empirical properties of return time series

Multi-scaling of returns


**Stock markets:** Brachet, Taflin, Tcheou (1997), Ausloos, Ivanova (2001), Bershaskii (2001),...

**Future markets:** Arneodo, Muzy, Sornette (1998), Muzy, Delour, Bacry (2000),...

\[ M(q, \tau) = \mathbb{E} \left[ |r(\tau, t)|^q \right] \sim K_q \tau^{\xi_q}. \]

Multiscaling of USD-DM FX rate (from Mandelbrot 2002)
Empirical models for return fluctuations

Bachelier model (1900)

The return process $X(t) = \ln P(t)$ is a Brownian motion:

$$dX(t) = \mu \, dt + \sigma \, dW(t)$$

- No correlation
- "Everything" can be computed (stochastic calculus,...)
- Universality

but

- Gaussian law at all scales: no fat tails
- Constant volatility: no volatility aggregation
- Simple scaling properties (self-similarity)

This model is still at the heart of most models used financial engineering.
Empirical models for return fluctuations
(Truncated) Levy models

• $\alpha$-stable process (Mandelbrot, Fama, 1963)
The return process $X(t)$ satisfies:

$$dX(t) = \mu \, dt + \sigma \, dL_\alpha(t)$$

where $L_\alpha(t)$ is an $\alpha$-stable Levy process. The returns $r(\tau, t)$ have therefore $\alpha$-stable laws.

- Fat tails
- Lot of possible computations
- Multi-scaling (“bi-scaling”)

but

- The variance is infinite
- Jumps
- No volatility clustering

• Truncated $\alpha$-stable process (Mantegna & Stanley, 1995):
The stable law is exponentially truncated in the tail.
Empirical models for return fluctuations

(G)ARCH models (Engle 1982, Bollerslev 1986)

The return at scale $\tau$, $r(n\tau, \tau)$, is conditionally Gaussian:

$$r(n\tau, \tau) \equiv r(n) = \sigma(n) \epsilon(n)$$

where $\epsilon(n)$ is a Gaussian white noise and the volatility $\sigma^2(n)$ is a regression from past squared returns and volatilities:

$$\sigma^2(n) = \alpha_0 + \sum_{i=1}^{p} \alpha_i r^2(n - i) + \sum_{j=1}^{q} \beta_j \sigma^2(n - i)$$

- Volatility clustering
- Easy to estimate (M.L.)
- Leptokurticity (heavy tail)

but

- Volatility correlations decrease rapidly
- No (multi-) scaling property
- Discrete time model (parameters change across scales)

GARCH(1,1) ($p = q = 1$) is a very popular model for volatility forecasting.
Empirical models for return fluctuations

Stochastic volatility models

(Taylor 1985, Hull & White 1987)

The return at scale $\tau$, $r(n\tau, \tau)$, is conditionally Gaussian:

$$r(n\tau, \tau) \equiv r(n) = \sigma(n) \epsilon(n)$$

where $\epsilon(n)$ is a Gaussian white noise and the volatility $\sigma^2(n)$ is itself a random process.

Usually $\omega(n) = \ln \sigma^2(n)$ is chosen to be AR(1):

$$\omega(n) = \phi \omega(n - 1) + \nu(n)$$

where $\nu(n)$ is a Gaussian white noise independent of $\epsilon(n)$

- Volatility clustering
- Easy to estimate (no exact M.L.)
- Leptokurticity

but

- Volatility correlations decrease rapidly
- Discrete time model (parameters change across scales)
- No multiscaling
Empirical models for return fluctuations

Multifractal models

\[ \mathbb{E}[|X(t)|^q] = K_q t^{\xi_q} \]

- MMAR model (Calvet, Fischer, Mandelbrot, 1999)
  The return \( X(t) = \ln P(t) \) is a Brownian motion compound with a multifractal “time” \( M(t) \):

\[
X(t) = B[M(t)] \\
M(t) \equiv \text{Multiplicative cascade}
\]

- MRW model (Bacry, Delour, Muzy, 2000)
  The return \( X(t) \) is obtained as the continuous limit of a stochastic variance model:

\[
X_l(t) = \sum_{k=1}^{t/l} \sigma_l(k) \epsilon_l(k) \\
M(t) = \int_0^t \sigma^2(u) \, du
\]
The MRW model
From Discrete cascades to stochastic variance models

\[ \{r_{\lambda_l}(\lambda t)\}_t = \lambda^H \{r_l(t)\}_t = W_\lambda \{r_l(t)\}_t \]

- Volatility Magnitude: \( \omega_l(t) = \frac{1}{2} \ln |r_l(t)|^2 \)
- Magnitude diffusion from coarse to fine scales:
  \[ \omega_{2^{-n-1}} = \omega_{2^{-n}} + \epsilon_{2^{-n-1}} \]
  with \( \epsilon = \ln(W) \) and \( \lambda^2 = \text{Var}(\epsilon) \)
- Ultrametric (tree) structure
  
  *Arneodo, Muzy, Sornette, 98*
  
  *Arneodo, Bacry, Muzy, Manneville, 98*

\[
\text{Cov} (\omega_l(t), \omega_l(t+\tau)) \simeq -\lambda^2 \ln(\tau/T), \quad l << \tau < T
\]
Magnitude correlation of the S&P 500 futures

Arneodo, Muzy, Sornette, 98

(a)

(b)
The MRW model

(Bacry, Delour, Muzy 2000)

\[ X(t) = \lim_{l \to 0} X_l(t) \]
\[ X_l(t) = X_l(t - l) + \sigma_l(t)\epsilon_l(t) \]

\( \epsilon_l(t) \) : Gaussian white noise
\( \sigma_l = e^{\omega_l(t)} \) : Stochastic volatility
\( \omega_l(t) \) : Gaussian (inf. div.) log-correlated magnitude:

\[ \text{Cov} (\omega_l(t), \omega_l(t + \tau)) \simeq -\lambda^2 \ln(\tau/T), \quad l < \tau \leq T \]

- Stationary (uncorrelated) increments
- Multifractal process
- Continuous scale invariance properties
- Fat tails
- Quasi-lognormal volatilities for \( \lambda^2 << 1 \)
- Volatility clustering

*Only 3 parameters:* noise variance \( \sigma^2 \), intermittency parameter \( \lambda^2 \) and volatility correlation time \( T \).
Multifractal scaling properties of MRW

(Bacry, Delour, Muzy, 2000)

\[ r(\lambda \tau, \lambda t) = \mathcal{L} e^{\Omega \lambda} r(\tau, t) \]

\[ \mathbb{E} [r(\tau, t)^q] = K_q \left( \frac{\tau}{T} \right)^{\zeta_q} \]

\[ \zeta_q = \frac{q}{2} - \lambda^2 q \left( \frac{q}{2} - 1 \right) \]

\[ \zeta_q < 1 \quad \leftrightarrow \quad K_q = +\infty \]

Analytical expression for the factors \( K_{2n} \):

\[ K_{2n} = T^n \sigma^{2n}(2n - 1)!! \prod_{k=0}^{n-1} \frac{\Gamma(1 - 2\lambda^2 k)^2 \Gamma(1 - 2\lambda^2 (k + 1))}{\Gamma(2 - 2\lambda^2 (q/2 + k - 1)) \Gamma(1 - 2\lambda^2)} \]
Fixed scale MRW returns
(Bacry, Khozemyak, Muzy, 2004)

• Fixed scale return \((\Delta < T)\):

\[
r_\Delta(k) \equiv r(\Delta, k\Delta) = X((k + 1)\Delta) - X(k\Delta)
\]

• Stochastic volatility

\[
r_{\Delta,k}(k) \overset{\text{Law}}{=} \varepsilon(k)e^{\Omega_\Delta[k]}
\]

where \(\varepsilon[k]\): gaussian white noise.

• In the small intermittency limit \(\lambda^2 << 1\):

\[
\Omega_\Delta(k) \approx \lambda \Gamma_\Delta(k)
\]

where \(\Gamma_\Delta(k)\) is a known Gaussian process ("renormalized magnitude")

• Moreover, if \(R_\Delta(k) = \varepsilon(k)e^{\lambda \Gamma_\Delta(k)}\)

\[
\mathbb{E}[(\ln |r_\Delta(k_1)||)^{p_1} \ldots] = \mathbb{E}[(\ln |R_\Delta(k_1)||)^{p_1} \ldots] \left(1 + o(\lambda^{2-\epsilon})\right)
\]

• If \(\mathbb{E}[|r_\Delta(k_1)|^{q_1} \ldots] < +\infty\),

\[
\mathbb{E}[|r_\Delta(k_1)|^{p_1} \ldots] = \mathbb{E}[|R_\Delta(k_1)|^{p_1} \ldots] \left(1 + o(\lambda^{2-\epsilon})\right)
\]
Magnitude correlation

\[
\ln |r_{\Delta}(k)| \overset{M}{\sim} \ln |\varepsilon_{\Delta}(k)| + \lambda \Gamma_{\Delta}(k)
\]

\[g(n) = n^2 \ln(n)\]

\[\text{Cov}(\Gamma_{\Delta}(k), \Gamma_{\Delta}(k+n)) = \ln\left(\frac{T \varepsilon^{3/2}}{\Delta}\right) + g(n) - \frac{1}{2} (g(n+1) + g(n-1))\]

\[\sim \ln\left(\frac{T}{n\Delta}\right) \text{ when } n \gg 1\]
Continuous deformation of increment pdf’s across scales

(Bacry, Delour, Muzy, 2000)

\[ r_\Delta(k) \overset{Law}{\simeq} \varepsilon_\Delta(k)e^{\lambda \Gamma \Delta(k)} \]

MRW

- Numerical estimation
- Castaing prediction

S&P Futures

- Numerical estimation
Multifractal estimation issues

- Small number of studies devoted to statistical estimation of random cascades.

- Finance:
  - Heuristical, Monte-Carlo estimates from “log-log”
  - GMM method from binomial cascade moments 
    *(Lux 2002)*

- MRW:
  - “Integral time” $T$
  - Decorrelation time
  - Variance $\sigma^2$
  - Return variance
  - Intermittency coefficient $\lambda^2$
  - Log-volatility correlation $\ln |r_\Delta(t)|$:
    $\text{Cov} (\ln r_\Delta(t), \ln r_\Delta(t + \tau)) \simeq -\lambda^2 \ln(\tau/T), \quad \Delta < \tau < T$
  - Absolute moments:
    $\mathbb{E} [ |r_\Delta|^{2q} ] = K_{2q} T^q \sigma^{2q} (2q - 1)!! \left( \frac{l}{T} \right)^{\zeta_{2q}}$

$\Rightarrow$ G.M.M. method
GMM estimation
Principle (Hansen 1982)

\{r(k)\}: data
\(\tilde{\theta}\): vector of \(p\) parameters to be estimated
\(\mu_j(\{r(k)\}, \tilde{\theta})\): “moments” satisfying the condition:

\[
\mathbb{E} \left[ \mu_j(\{r(k)\}, \tilde{\theta}) \right] = 0 \text{ for } 1 \leq j \leq m, \ m > p
\]

Let \(\bar{\mu}_j(\tilde{\theta})\) a sample estimator of \(\mathbb{E} \left[ \mu_j(\{r(k)\}, \tilde{\theta}) \right]\). Then the GMM estimator of \(\tilde{\theta}, \hat{\theta}\) is obtained as

\[
\hat{\theta} = \arg \min_{\tilde{\theta}} \left[ \bar{\mu}_i(\tilde{\theta}) W_{ij}^{-1} \bar{\mu}_j(\tilde{\theta}) \right]
\]

where \(W\) is a positive definite weighting matrix \(m \times m\) matrix. When \(W_{ij} = \mathbb{E} [\bar{\mu}_i \bar{\mu}_j]\) (or a consistent estimate of it), the GMM estimator is consistent and asymptotically efficient. Moreover,

\[
\sqrt{N}(\hat{\theta} - \tilde{\theta}) \xrightarrow{\text{Law}} \mathcal{N}(\tilde{0}, \Sigma(\tilde{\theta}))
\]

with

\[
\Sigma(\tilde{\theta})_{ij} = \frac{\partial \bar{\mu}_i}{\partial \theta_k} W_{kl}^{-1} \frac{\partial \bar{\mu}_k}{\partial \theta_j}
\]

→ Confidence intervals, tests,...
GMM estimation of MRW  
(Bacry, Muzy 2004)

\{r_\Delta(k)\}: return data  
\(\hat{\theta}\): vector \((\sigma^2, \lambda^2, T)\)

\(\mu_j(\{r_\Delta(k)\}, \hat{\theta})\): “moments”

\[
\mu_j = r_\Delta(k)^{2n} - M(\sigma^2, \lambda^2, T, n)
\]

\[
\mu_j = \ln |r_\Delta(k)| \ln |r_\Delta(k + n)| - C(\sigma^2, \lambda^2, T, n)
\]

satisfying the condition:

\[
\mathbb{E} \left[ \mu_j(\{r(k)\}, \hat{\theta}) \right] = 0 \text{ for } 1 \leq j \leq m.
\]

One then use an iterative procedure:

\[
W_0 = I \rightarrow (\sigma^2_0, \lambda^2_0, T_0) \rightarrow W_1 \rightarrow (\sigma^2_1, \lambda^2_1, T_1) \rightarrow \ldots
\]

MRW of 32.10³ samples

\(\lambda^2 = 0.03\) and \(T = 256\)

5% confidence intervals \(\lambda^2 \in [0.025, 0.032]\) and \(T \in [225, 503]\).
Parameter estimation for daily data

(Bacry, Muzy, 2002)

CAC 40 Index daily data
1/1/1973 to 31/12/97 (6239 points)
\( \lambda^2 = 0.03 \pm 0.01 \), \( T \in [0.5, 2] \) years

MRW sample
5000 points
\( \lambda^2 = 0.03 \), \( T = 250 \)
Parameter estimation intraday data  
(Bacry, Muzy, 2002)

S&P 500 intraday data (5mn ticks, 1996-1998)

: 5mn returns
: 30mn returns
: 1h returns
Parameter values for financial returns

\textit{(Bacry, Muzy, 2002)}

<table>
<thead>
<tr>
<th>Series</th>
<th>Size</th>
<th>$\lambda^2$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P500 index</td>
<td>$7 \times 10^4$</td>
<td>0.03</td>
<td>6 months</td>
</tr>
<tr>
<td>Future S&amp;P500</td>
<td>$7 \times 10^4$</td>
<td>0.025</td>
<td>2 years</td>
</tr>
<tr>
<td>FTSE100 index</td>
<td>$7 \times 10^4$</td>
<td>0.028</td>
<td>1.2 year</td>
</tr>
<tr>
<td>Future FTSE100</td>
<td>$7 \times 10^4$</td>
<td>0.029</td>
<td>1 year</td>
</tr>
<tr>
<td>Future JY/USD</td>
<td>$7 \times 10^4$</td>
<td>0.02</td>
<td>6 months</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>$7 \times 10^4$</td>
<td>0.030</td>
<td>1.5 year</td>
</tr>
<tr>
<td>Future Nikkei</td>
<td>$7 \times 10^4$</td>
<td>0.02</td>
<td>6 months</td>
</tr>
<tr>
<td>Japanese Yen</td>
<td>$7 \times 10^4$</td>
<td>0.022</td>
<td>1.0 year</td>
</tr>
<tr>
<td>French index</td>
<td>$6 \times 10^3$</td>
<td>0.029</td>
<td>1 year</td>
</tr>
<tr>
<td>Italian index</td>
<td>$6 \times 10^3$</td>
<td>0.029</td>
<td>2 years</td>
</tr>
<tr>
<td>Canadian index</td>
<td>$6 \times 10^3$</td>
<td>0.032</td>
<td>1.5 years</td>
</tr>
<tr>
<td>German index</td>
<td>$6 \times 10^3$</td>
<td>0.027</td>
<td>2 years</td>
</tr>
<tr>
<td>UK index</td>
<td>$6 \times 10^3$</td>
<td>0.023</td>
<td>2 years</td>
</tr>
<tr>
<td>hong-kong index</td>
<td>$6 \times 10^3$</td>
<td>0.037</td>
<td>3 years</td>
</tr>
</tbody>
</table>

\textit{Generic values:} \hspace{1em} $\lambda^2 = 0.03$, $T = 1$ year.
Applications
Risk evaluation and forecasting

• Volatility estimation and forecasting:
  - Model evaluation
  - Risk estimates
  - Fund manager comparison
  - Active trading
  - Option markets

• Classical econometric models
  - GARCH: Regression from past square returns and volat.
  - J.P. Morgan RiskMetrics: “Optimal” exponential smoothing of past square returns
  - Stochastic volatility: Wiener or Kalman filtering
MRW volatility prediction

(Bacry, Muzy, 2002)

\[ r_\Delta(k) = \varepsilon_\Delta[k]e^{\Omega_\Delta[k]} \]

• “Generic” estimates: \( \lambda^2 = 0.03, T = 1 \) year.

• Volatility prediction at scale \( \Delta_1 = l\Delta (l \geq 1) \) (Wiener filtering)
  
  – Method \( \text{MRWlin} \): Prediction of \( \sigma_{\Delta_1}[n] = e^{\Omega_{\Delta_1}[n]} \)
    \[ \hat{\sigma}_{\Delta_1}[n]^2 = [h_1 * r_\Delta^2](n) \quad (h_1 \text{ causal}) \]
  – Method \( \text{MRWlog} \) : Prediction of \( \Omega_{\Delta_1}[n] \)
    \[ \hat{\omega}_{\Delta_1}[n] = [h_2 * \ln(|r_\Delta|)](n) \quad (h_2 \text{ causal}) \]
  \( \rightarrow \) MLE of \( e^{\omega_{\Delta_1}[n]} \)

• Testing the prediction:
  
  • MSE (L2) Error:
    \[ e_{MSE}^2 = \mathbb{E} \left[ (\hat{\sigma}_{\Delta_1}[n]^2 - \sum_{i=1}^{l} |r_\Delta(i)|^2)^2 \right] \]
  
  • MAE (L1) Error:
    \[ e_{MAE} = \mathbb{E} \left[ |\hat{\sigma}_{\Delta_1}[n]^2 - \sum_{i=1}^{l} |r_\Delta(i)|^2| \right] \]
Volatility prediction
(Bacry, Muzy, 2002)

Comparisons on 10 daily Index series
\((h, s \subset [1 \text{ day}, 10 \text{ days}, 1 \text{ month}, 6 \text{ months}] \))

- **MSE (L2) Error**: Number of “hits”
  
  \[
  \begin{align*}
  \text{MRWlog} &= 66, \quad \text{GARCH} = 28, \quad \text{RM} = 6, \quad \text{Hist} = 0 \\
  \text{MRWlin} &= 81, \quad \text{GARCH} = 19, \quad \text{RM} = 0, \quad \text{Hist} = 0 \\
  \text{MRWlog} &= 57, \quad \text{MRWlin} = 43, \quad \text{RM} = 0, \quad \text{Hist} = 0
  \end{align*}
  \]

- **MAE (L1) Error**: Number of “hits”
  
  \[
  \begin{align*}
  \text{MRWlog} &= 80, \quad \text{GARCH} = 20, \quad \text{RM} = 0, \quad \text{Hist} = 0 \\
  \text{MRWlin} &= 88, \quad \text{GARCH} = 12, \quad \text{RM} = 0, \quad \text{Hist} = 0 \\
  \text{MRWlog} &= 72, \quad \text{MRWlin} = 28, \quad \text{RM} = 0, \quad \text{Hist} = 0
  \end{align*}
  \]

\[\implies \text{MAE} : \text{MRWlog} \text{ (or MRWlin)}\]

\[\implies \text{MSE} : \text{MRWlin}\]
Using intraday data for Volatility prediction

SP100 index : 04/08/97 – 17/12/01 (intraday 5mn)

<table>
<thead>
<tr>
<th>h=1 day, s = ...</th>
<th>MRWlog (daily)</th>
<th>MRWlog (intra)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE 1 day</td>
<td>+2.33</td>
<td>-4.30</td>
</tr>
<tr>
<td>10 days</td>
<td>+10.65</td>
<td>-15.79</td>
</tr>
<tr>
<td>1 month</td>
<td>+15.55</td>
<td>-16.93</td>
</tr>
<tr>
<td>MAE 1 day</td>
<td>-13.85</td>
<td>-10.88</td>
</tr>
<tr>
<td>10 days</td>
<td>-4.55</td>
<td>-24.88</td>
</tr>
<tr>
<td>1 month</td>
<td>-0.80</td>
<td>-28.00</td>
</tr>
</tbody>
</table>
Value at risk forecasting

- Definition of the Value at Risk $V(p)$ at level $p$:

$$\mathbb{P}(r_\Delta \leq -V(p)) = p$$

→ Intuitive interpretation: Most probable amplitude of the worst loss at scale $\Delta$ over an horizon $p^{-1}$ periods.

<table>
<thead>
<tr>
<th>Country</th>
<th>Gaussian VaR</th>
<th>Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>0.59</td>
<td>1.24</td>
</tr>
<tr>
<td>Japan</td>
<td>0.65</td>
<td>1.08</td>
</tr>
<tr>
<td>USA</td>
<td>1.85</td>
<td>2.26</td>
</tr>
<tr>
<td>GB</td>
<td>1.59</td>
<td>2.08</td>
</tr>
</tbody>
</table>

Gaussian most probable worst day versus observed worst day during the year 1994 for some international bond indices.

- Usage
  - Objective tool, easy interpretation
  - Widely used in performance evaluation
  - Optimization of risk allocation in a non Gaussian world

- Computation: Historical, Monte-Carlo, Analytical,...
  → Conditional Gaussian (Garch, RiskMetrics,...)
Value at risk forecasting using MRW
(Bacry, Khozemyak, Muzy, 2004)

- Normal law: \( n(x) \equiv \mathcal{N}(0, 1) \) and \( N(x) = \int_{-\infty}^{x} n(u)du \).

- Castaing formula for probability distribution:

\[
\mathbb{P}(r \leq x) = \int_{-\infty}^{+\infty} g_{\lambda}(u) N(e^{-u}x) \, du
\]

- “log-normal” volatility when \( \lambda^2 \to 0 \) (Hedgeworth expansion)

\[
\frac{\Omega_{\Delta}(k)}{\lambda} \xrightarrow{law} \mathcal{N}\left(0, \ln \frac{Te^{3/2}}{\Delta}\right)
\]

\[
\lambda g_{\lambda}(\lambda u) = n(u) + n'(u) (\lambda p_1(u) + \lambda^2 p_2(u) + \ldots)
\]

VaR prediction backtesting for BP-USD rate

- MRW, + Garch(1,1), − exact.
Volatility dynamics
(Sornette, Malavergne, Muzy, 2002)

- Conditional mean: \( \mathbb{E} \left[ \sigma^2(t + \tau) | \sigma^2(t) = \sigma_0^2 e^s \right] \)
- MRW theory (log-normal approx.):
  \[ \mathbb{E} [\tau, s] \sim E_0 \tau^{\alpha(s)} \]
  \[ \alpha(s, \tau) = C_\Delta s \]

Empirical estimates for S&P 100 index.
Other applications
Portfolio selection problem

- **Problem**: Optimal portfolio composition in order to minimize the risk and maximize the return.

- **Classical portfolio selection** (*Markowitz 1959*)
  - The **variance** fully defines the **risk** (Gaussian world or quadratic utility function)
  - The full information about the market is encoded in the mean returns $\mu_i$ and in the **covariance matrix** $\beta_{ij}$.
  - The solutions can be computed exactly and are located on the **efficient frontier** $\mu_p = \mu_0 + \sigma_p^2$.
  - No time scale (horizon) dependence (because $\mu$ and $\sigma^2$ are linear functions of the horizon)
  - Optimal portfolios are stable by linear superposition.
  - **CAPM**: Relates the mean return of some asset to its covariance with the “market portfolio”.

- **Non gaussian** return fluctuations
  - Risk dependent optimization
  - Non trivial time scale dependence (multi-period optimization)
  - Non linearities
**Other applications**

Portfolio selection for assets with heavy tails

- **α-stable models** *(Fama 1965, Walter et al. 1995)*
  - "Natural" extension of the gaussian framework (*α* = 2).
  - Most of Markowitz results can be generalized

- **Multivariate multifractal model**

  \[
  \{ r_i(k) = \varepsilon_i(k) e^{\Omega_i(k)} \}_{i=1,\ldots,N}
  \]

  Two “extreme” cases:
  - \( \Omega_i(k) = \Omega(k) \ \forall i \).
  - \( \text{Cov}(\Omega_i(k), \Omega_j(k')) = 0 \) for \( i \neq j \).

  - Higher order cumulant utility function

\[(\text{Muzy, Sornette, Delour, Arneodo, 2000})\]
Other applications
Options and other derivatives

- **Derivative security**: Financial instrument whose value depends on the value of more basic underlying variables (futures, options,...)

- **Option**: Contract that give the holder the *right* to buy/sell the underlying asset by a certain date (maturity) for a certain price (strike price)

- Usage: Hedging (reduce risk), speculation (take risk)

- Basic problem of mathematical finance: option pricing and associated hedging strategy.
  - Binomial random walk (*Cox & Rubinstein, 1979*)
  - Brownian motion (*Black & Scholes, 1971*)

Non Gaussian, heteroskedastic returns \(\Rightarrow\) Volatility smile
(Non constant implied volatility as a function of the strike and maturity)

- **Problem**: Extend Black & Scholes results to more general processes.
  - Stochastic volatility (*Hull & White 1988*)
  - \(\alpha\)-stable markets (*Rachev & al. 1994*)
  - Letptokurtic processes, MRW processes (*Bouchaud & al. 2000, 2002*)
Conclusion and prospects

Multifractal (cascade) models for financial time series:

- Parcimonious models that account for most observed “stylized facts”
- Relatively well known mathematical properties (Barral, Riedi, Bacry)
- Stable over “time-aggregation” (scaling)
- Versatility (log-infinitely divisible) (Barral, Riedi, Bacry)
- Econometry of multifractal processes (estimation, hypothesis testing,...)
- Financial engineering using multifractals (portfolio, stochastic calculus, options,...)

MRW model (log-normal): Stochastic volatility

- 3 parameters ( $\sigma^2$, $T$, $\lambda^2$)
- All features can be explained from volatility correlations
- Approx. log-normal “renormalized” volatility ($\lambda^2 \ll 1$)
- Estimation and forecasting
- Multivariate MRW
- Skewed MRW (Pochard, Bouchaud 2002)
Conclusion and prospects

Market microstructure and agent based models

- Order books dynamics (long-ranged)
- Order impact function and volume dynamics
- Agent based models: origin of the cascade (minority games, Lux-Marchesi model,...)
Return / Log-volatility correlations

Bouchaud, Matacz, Potters, 2001, Pochard, Bouchaud 2002

- Correlation \( \text{Cov} \left( -r(t), \omega(t + \tau) \right) \) for the S&P 100 (daily).

- Log-log representation: \( \text{Cov} \left( r(t), \omega(t + \tau) \right) \simeq -S\tau^{-1/2} \)