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   This final technical report presents a summary of a three-year project in which the mathematical basis of failure criteria for metals and composite materials was investigated. The investigation covered criteria associated with crack formation and structural instability. The main results are: (a) General procedures for the determination of mathematical functions that model the conditions for crack formation at critical points were developed and implemented into a generally available computer code. (b) Mathematical models for buckling failure were formulated, implemented and their properties clarified.

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RESEARCH ON THE p- AND hp-VERSIONS
OF THE FINITE ELEMENT METHOD

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1 Executive Summary

Multi-material interfaces are sites of failure initiation in composite materials. The development of reliable quantitative criteria for failure initiation in electronic components, adhesively bonded joints and laminated composites is obviously very important. At present there are no universally accepted procedures for the evaluation of fatigue and durability characteristics of structural, mechanical and electronic components made of composite materials.

This project was concerned with the development of mathematical methods and computational procedures for the determination of functionals which can be correlated with failure initiation events in composite materials subjected to thermal and mechanical loads. The approach is based on the assumption that failure initiation events are associated with the natural straining modes, analogously to the well established correlation between generalized stress intensity factors in linear elastic fracture mechanics and crack propagation events.

Failure criteria must be formulated in terms of functionals the exact values of which are finite. Stresses corresponding to the exact solution are usually infinity in singular points. The numerically computed stresses are, of course, finite but very sensitive to the discretization. Therefore stresses cannot be used for formulating failure criteria.

The specific objectives of the project were:
1. Develop procedures for the numerical determination of the eigenpairs $\lambda_i$ and $\tilde{\phi}_i$ that characterize the natural straining modes and natural flux states at singular points in heterogeneous bodies.

2. Develop a method for numerical determination of the generalized stress intensity and flux intensity factors in heterogeneous bodies subjected to thermal and mechanical loading.

In addition, development of methods for the estimation of limit loads for fiber-reinforced composites in compression was undertaken. The instability of fibers is an important consideration in the design of fiber-reinforced composites.

1.1 Summary of accomplishments

- A reliable numerical method for the determination of the flux and stress fields at multi-material interfaces in thermoelastic problems has been developed. The method involves numerical determination of the eigenpairs of the asymptotic expansion by a procedure called the modified Steklov method and determination of the coefficients by a procedure based on the complementary energy principle.

- A test implementation has been completed and the effectiveness of the method established through benchmark studies.

- Industrial application has been made possible by a Phase I STTR grant to Engineering Software Research and Development, Inc. (ESRD) and Washington University. The two-dimensional thermoelastic capabilities have been implemented in the commercial FEA code StressCheck. This is an unique capability which is now being used for investigation of the correlation of observed failure events in lap-shear test specimen with generalized stress intensity factors. This work, started on August 1, 1997 is being performed in collaboration with Raytheon TI Systems.

- The original scope of work was extended to include numerical simulation of failure of homogeneous and composite elastic materials through loss of stability. This work, performed in collaboration with Dr. Manil Suri of the University of Maryland and Dr. Ivo Babuška of The University of Texas, Austin, led to a clarification of some fundamental theorems related to the numerical simulation of problems of elastic stability. A doctoral dissertation has been completed [3].
1.2 Personnel supported

Faculty:
  Dr. Barna A. Szabo (PI)
Post-doctoral persons:
  Dr. Ricardo L. Actis (part time)
  Dr. Xian-Zhong Guo (part time)
  Dr. Gyorgy Kiralyfalvi
Graduate Students:
  Mr. Andre Tamagnini Noel (D.Sc. candidate, Graduated May, 1996)
  Mr. Gyorgy Kiralyfalvi (D.Sc. candidate, Graduated May, 1997)
  Ms. Li Zhang (D.Sc. candidate)

1.3 Consultative and advisory functions

The Principal Investigator presented briefings to eight Air Force contractors and one Navy laboratory:

1. McDonnell Douglas Aerospace St. Louis, MO (contact persons: Mr. Scott Fields, Mr. Daniel Dudley) April 25, 1996
2. Boeing Aerospace, Downey, CA. Dr. Saeed Paydarfar visited Washington University and was briefed on the scope and objectives of the project. May 19, 1997
5. Raytheon TI Systems, Dallas, TX (contact person: Dr. Terry Baughn) July 29, 1997
6. Lockheed-Martin, Fort Worth, TX (contact person: Mr. Michael Barnhart) July 30, 1997
8. Structures Division, NAWCAD, Patuxent River, MD (contact person: Dr. David Barrett 301-342-9360) September 8, 1997
9. Lockheed-Martin, Marietta, GA (contact person: Dr. Stephen P. Engelstad 770-494-9714) March 3, 1998. This presentation was made to members of a government/industry consortium known as the Composites Affordability Initiative. Members represent each of the major US aerospace companies and the Air Force and Navy. This particular meeting was hosted by Lockheed Martin.

1.4 Publications and presentations


1.5 Transitions

The new capabilities have been made available to Air Force laboratories and contractors through a professional quality software called StressCheck. StressCheck is being developed
and marketed by Engineering Software Research and Development, Inc., located in St. Louis, MO. The current users of Stress Check include Boeing Aircraft Company (on several locations); Piper Aircraft Co., Northrop Grumman Corporation; Cessna Aircraft Co. NASA Johnson Space Center and others.

A collaborative effort was started with Raytheon TI System for an experimental investigation of relationships between generalized stress intensity factors and failure initiation events at bonded interfaces.

McDonnell Douglas Aerospace St. Louis (now Boeing) funded a project with ESRD for a particular specialization of the material and geometric nonlinear analysis capabilities within the p-version of the finite element method for application to the analysis of cold-worked holes and attachment lugs. This technology was developed at Washington University under AFOSR sponsorship.

2 Technical description

Singular points are those points in a structural component where a reentrant corner occurs (like cracks and V-notches), material properties abruptly change along a free edge, interior points of three (or more) zones of different materials intersect, or an abrupt change in boundary conditions occurs, see Fig. 1.

![Figure 1: Typical singular points associated with multi-material interfaces.](image)

In the vicinity of these singular points the exact solution of the problem of elasticity is of the form:

\[ \tilde{u}_{EX} = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \tilde{\phi}_i(\theta) \]  \hspace{1cm} (1)

where \( \tilde{u}_{EX} \) is the exact displacement vector function, \( r \) and \( \theta \) are polar coordinates centered on the singular point, \( \tilde{\phi}_i \) \( \text{def} \{ \phi_{ijx}, \phi_{ijy} \} \) is a piecewise smooth vector function, and \( A_i \) are coefficients. Eq. (1) is an asymptotic expansion of the exact solution at the singular point.
Within a radius of convergence the exact solution of the problem of elasticity can be written in this form. The exponents \( \lambda_i \) (numbered such that \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \)) and the corresponding functions \( \phi_i(\theta) \) depend on the material properties and the geometric details at the singular point. These can be determined by solving an eigenvalue problem. Details are given in Section 2.3. A well known example is linear elastic fracture mechanics in two dimensions where \( \lambda_1 = 1/2 \) and

\[
\begin{align*}
\phi_{1|x} &= \frac{1}{2G} \left[ \left( \kappa - \frac{1}{2} \right) \cos \frac{\theta}{2} - \frac{1}{2} \cos \frac{3\theta}{2} \right], \\
\phi_{1|y} &= \frac{1}{2G} \left[ \left( \kappa + \frac{1}{2} \right) \sin \frac{\theta}{2} - \frac{1}{2} \sin \frac{3\theta}{2} \right].
\end{align*}
\]  

(2)  

(3)

\( G \) is the shear modulus, \( \kappa = 3 - 4\nu \) for plane strain, \( \kappa = (3 - \nu)/(1 + \nu) \) for plane stress where \( \nu \) is Poisson's ratio.

In linear elastic fracture mechanics \( \lambda_i \) and \( \phi_i \) have been determined by classical methods, and only the stress intensity factor, which is proportional to \( A_1 \), has to be determined by numerical means. For details see, for example, [6].

In the general case of multi-material singularities, such as those shown in Fig. 1, not only \( A_i \) but also \( \lambda_i \) and \( \phi_i(\theta) \) have to be determined by numerical means. The elastic stress is infinity in the singular point when \( 0 < \lambda_1 < 1 \) and \( A_1 \neq 0 \) and/or \( 0 < \lambda_2 < 1 \) and \( A_2 \neq 0 \), etc. A natural straining mode is the strain state associated with a particular term of the asymptotic expansion, eq. (1). As explained in Section 2.1, the natural straining modes provide a linkage between linear computations and observed failure initiation or failure propagation events.

In heat conduction the asymptotic expansion is analogous to eq. (1):

\[
T_{EX} = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \phi_i(\theta)
\]

(4)

where \( T_{EX} \) is the exact solution of the heat conduction problem and \( \phi_i \) are piecewise smooth scalar functions. The coefficients \( A_i \) are called flux intensity factors.

Numerical accuracy is essential because unless the accuracy of the computed data is known it would not be possible to tell whether the working hypothesis is wrong or the numerical errors are too large, or both. In some cases a large error in the working hypothesis is nearly canceled by a similarly large numerical error, leading to false conclusions.

### 2.1 Basic principles and assumptions

Consider the neighborhood of a singular point. It is assumed in the following that the principles of continuum mechanics remain valid everywhere within the body up to the fail-
ure initiation event. The possibility of strongly nonlinear behavior in the neighborhood of singular points is not excluded, however.

\[ u_{NL} = \{u_x, u_y\}_{NL} \]

Let \( u_{NL} \) be the solution of the general nonlinear continuum mechanics problem. It is expected that failure initiation will depend on \( u_{NL} \) (more precisely some functionals computable from \( u_{NL} \)) in the strongly nonlinear region of the singular point bounded by a boundary \( \Gamma_{NL} \), as shown in Fig. 2. This region is called the process zone. Let \( \Gamma_{EL} \) be a curve outside of \( \Gamma_{NL} \), and let \( G \) be an operator which associates the solution \( \tilde{u}_{NL} \) of the nonlinear problem inside \( \Gamma_{NL} \) with the boundary condition \( \bar{g}_{EL} \) specified on \( \Gamma_{EL} \), that is:

\[ G(\bar{g}_{EL}) = \tilde{u}_{EX}, \quad \bar{g}_{EL} = \tilde{u}_{NL}|_{\Gamma_{EL}} \]

where \( \bar{g}_{EL} = \tilde{u}_{NL}|_{\Gamma_{EL}} \) denotes the trace of \( \tilde{u}_{EX} \) on \( \Gamma_{EL} \). Denote the exact solution of the linear elastic problem by \( \bar{u}_{EL} \). The basic assumptions (which are valid in linear elastic fracture mechanics) are stated in the following:

**Assumption A:**

Inside of \( \Gamma_{EL} \) the error \( G(\tilde{u}_{NL}|_{\Gamma_{EL}}) - G(\bar{u}_{EL}|_{\Gamma_{EL}}) \) is so small that conclusions based on \( G(\tilde{u}_{EL}|_{\Gamma_{EL}}) \) are sufficiently close to the conclusions based on \( G(\tilde{u}_{NL}|_{\Gamma_{EL}}) \) for practical purposes. This assumption is expected to be valid whenever the nonlinear behavior is confined entirely to some small region inside \( \Gamma_{EL} \).

Assumption A leads to the important conclusion that failure initiation, which depends on the solution of the nonlinear problem inside of \( \Gamma_{NL} \), can be determined through the solution of the linear elastic problem, even though all basic assumptions of the linear theory may be
violated inside of $\Gamma_{NL}$. Consequently failure initiation in the neighborhood of the singular point can be predicted on the basis of the linear theory of elasticity. (This is because $\bar{u}_{EL}$ defines $\bar{u}_{NL}|_{\Gamma_{EL}}$.)

**Assumption B:**

There exists a physical principle which establishes the relationship between crack initiation and the stress field on the basis of information obtained from the linear solution $\bar{u}_{EL}$ only. Linear elastic fracture mechanics (LEFM) is a typical application of Assumption B.

In general, the linear solution $\bar{u}_{EL}$ is not known, only an approximation to $\bar{u}_{EL}$, which will be denoted by $\bar{u}_{FE}$, is known. Therefore the following assumption is necessary:

**Assumption C:**

There exist a norm $\| \cdot \|$ such that when $\|\bar{u}_{EL} - \bar{u}_{FE}\|$ is sufficiently small then the physical principle of Assumption B is not sensitive to the replacement of $\bar{u}_{EL}$ with $\bar{u}_{FE}$. Of course, the norm $\| \cdot \|$ is expected to depend on the physical principle of Assumption B, which is material-dependent. If conclusions are to be based on $\bar{u}_{FE}$ then $\bar{u}_{FE}$ has to be close to $\bar{u}_{EL}$ in this particular norm.

Based on these assumptions linear computations can be used for the prediction of failure initiation and failure propagation even though failure processes are highly nonlinear in nature. There are two key elements of failure initiation analysis:

1. A hypothesis concerning the relationship between certain parameters of the stress or strain field and observed failure initiation or crack propagation events.

2. Convincing experimental confirmation that the hypothesis holds independently of variations in geometric attributes, loading and constraints.

It would not sensible to perform experiments without a hypothesis based on the functionals that characterize the stress or strain fields in the neighborhood of critical points and computations cannot provide useful information about the conditions under which failure occurs without experimental data.

For details on the algorithms developed for the computation of the natural straining modes at multi-material interfaces and the generalized stress intensity factors we refer to [12], [7], [8], [10], [11].

**Remark 2.1** The assumption that the material is elastic on $\Gamma_{EL}$ is not essential. Similar considerations apply to nonlinear elasticity and the deformation theory of plasticity. In fact,
the methods of LEFM have been extended to the deformation theory of plasticity through the use of the J-integral [2], [5].

In the following the procedures developed for the computation of eigenpairs and their coefficients in heat conduction and elasticity are outlined and illustrated by examples. Additional information can be obtained from the references listed.

2.2 The problem of heat conduction

The index notation is used in the following. For two dimensional problems the range of the indices is 2, and for three dimensional problems the range is 3. The summation convention is used. The formulation of the mathematical is described for the problem of heat conduction.

The heat balance equation is analogous to the equilibrium equation in elasticity:

\[-q_{i,i} + Q = 0 \]  \hspace{1cm} (5)

where: \( q_i \) is the flux vector (in \( W/m^2 \) units) and \( Q \) is the rate of heat generation per unit volume (in \( W/m^3 \) units). Multiplying eq. (5) by the scalar function \( v \), integrating and applying Green’s lemma, we have:

\[- \int_{\Omega} q_i v_{,i} \, dV = \int_{\Omega} Q v \, dV - \int_{\Gamma} q_i n_i v \, dS. \]

Using Fourier’s law of heat conduction (which is analogous to Hooke’s law):

\[ q_i = -K_{ij} \mathcal{T}_j \]

where \( K_{ij} \) is assumed to be independent of \( \mathcal{T} \), we have for all \( v \in H^1(\Omega) \):

\[ \int_{\Omega} K_{ij} v_{,i} \mathcal{T}_j \, dV = \int_{\Omega} Q v \, dV - \int_{\Gamma} q_i n_i v \, dS. \] \hspace{1cm} (6)

This is analogous to the principle of virtual work in elasticity. Alternatively, the exact solution of the heat conduction problem is the minimizer of the functional \( \pi \):

\[ \pi(\mathcal{T}) \overset{\text{def}}{=} \frac{1}{2} \int_{\Omega} K_{ij} \mathcal{T}_i \mathcal{T}_j \, dV - \int_{\Omega} Q \mathcal{T} \, dV + \int_{\Gamma} q_i n_i \mathcal{T} \, dS \] \hspace{1cm} (7)

on the set of the admissible temperature fields. This is analogous to the principle of minimum potential energy in elasticity.
2.3 Computation of $\lambda_i$ and $\phi_i(\theta)$

The procedure for the numerical approximation of the eigenpairs $\lambda_i$ and $\phi_i(\theta)$ is briefly described for the problem of heat conduction. Additional details are available in [8].

Consider a small neighborhood of a singular point bounded by a circle of radius $R$ and denoted by $\Omega_R$, as shown in Fig. 3. Let $Q = 0$ on $\Omega_R \times t$ where $t$ is the thickness, assumed to be constant, and $u = 0$ on $\Gamma_1$ and $\Gamma_2$ and seek solutions of the form:

$$\mathcal{T} = r^\lambda \phi(\theta).$$

Noting that

$$\mathcal{T}_1 = \frac{\partial \mathcal{T}}{\partial r} + \frac{\partial \mathcal{T}}{\partial \theta} \frac{\partial \theta}{\partial x_1} = \frac{\partial \mathcal{T}}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial \mathcal{T}}{\partial \theta} \sin \theta$$

$$\mathcal{T}_2 = \frac{\partial \mathcal{T}}{\partial r} + \frac{\partial \mathcal{T}}{\partial \theta} \frac{\partial \theta}{\partial x_2} = \frac{\partial \mathcal{T}}{\partial r} \sin \theta + \frac{1}{r} \frac{\partial \mathcal{T}}{\partial \theta} \cos \theta$$

and $n_i = \{\cos \theta, \sin \theta\}$, we write:

$$- \int_{\Gamma_R} q_i n_i v \, dS = \lambda \int_{\Gamma_R} (K_{11} \cos^2 \theta + K_{12} \sin 2\theta + K_{22} \sin^2 \theta) \mathcal{T} v \, td\theta$$

$$\int_{\Gamma_R} ((K_{22} - K_{11}) \sin \theta \cos \theta + K_{12} \cos 2\theta) \frac{\partial \mathcal{T}}{\partial \theta} v \, td\theta.$$

We are now in a position to apply eq. (6) which results in:

$$(\mathcal{B}(\mathcal{T}, v) - \mathcal{N}(\mathcal{T}, v)) - \lambda \mathcal{M}(\mathcal{T}, v) = 0$$

(8)
where:

\[ B(\mathcal{T}, v) \overset{\text{def}}{=} \int_{\Omega_R} K_{ij}v_i \mathcal{T}_j \, dV \]

\[ M(\mathcal{T}, v) \overset{\text{def}}{=} \int_{\Gamma_R} \left( K_{11} \cos^2 \theta + K_{12} \sin 2\theta + K_{22} \sin^2 \theta \right) \mathcal{T} v \, t \, d\theta \]

\[ N(\mathcal{T}, v) \overset{\text{def}}{=} \int_{\Gamma_R} \left( (K_{22} - K_{11}) \sin \theta \cos \theta + K_{12} \cos 2\theta \right) \frac{\partial \mathcal{T}}{\partial \theta} \, v \, t \, d\theta \]

We seek \( \lambda > 0, \mathcal{T} \in H^1(\Omega_R), \partial \mathcal{T}/\partial \theta \in L^2(\Gamma_R) \) such that eq. (8) is satisfied for all \( v \in H^1(\Omega_R) \). This non-symmetric eigenvalue problem can be solved numerically.

**Remark 2.2** In the case of isotropic materials \( N(\mathcal{T}, v) = 0 \) hence the eigenvalue problem is symmetric. In the special case \( K_{ij} = K \delta_{ij} \), where \( K \) is constant, we have:

\[ \int_{\Omega_R} \mathcal{T}_i v_i \, dV - \lambda \int_{\Gamma_R} \mathcal{T} v t \, d\theta = 0. \]

The corresponding strong form is:

\[ \Delta \mathcal{T} = 0 \]

subject to the boundary conditions

\[ u = 0 \text{ on } \Gamma_1, \Gamma_2; \quad \frac{\partial \mathcal{T}}{\partial r} = \frac{\lambda}{R} \mathcal{T} \text{ on } \Gamma_R. \]

**Remark 2.3** \( N(\mathcal{T}, v) \) is non-symmetric, nevertheless all eigenvalues are real.

**Remark 2.4** \( \lambda_i \) do not converge monotonically. No minimum principle is involved.

**Remark 2.5** The formulation is analogous for any set of homogeneous boundary conditions on \( \Gamma_1, \Gamma_2 \).

The Steklov method on \( \Omega_R \) requires hp-meshing. This is because the rate of convergence of p-extensions is low due to the presence of the singular point. Therefore it is better to use a modified domain \( \Omega^*_R \) shown in Fig. 4. Using \( \Omega^*_R \) is called the *modified Steklov method*. Detailed discussions on the procedures are available in references [8], [12], [7].

**Remark 2.6** The modified Steklov method will find not only those eigenfunctions which lie in \( H^1(\Omega_R) \) but also eigenfunctions corresponding to negative values of \( \lambda \).
2.4 Extraction of the flux intensity factors

Analogously to eq. (1) the temperature field in the neighborhood of singular points is of the form:

\[ T = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \phi_i(\theta) \]  (9)

where: \( A_i \) represents the generalized flux intensity factors, \( \lambda_i \) and \( \phi_i \) are the eigenpairs characterized by the topological details at the singular point and the material properties.

Once the finite element solution is available, the flux intensity factors can be computed from the finite element solution by the contour integral method, the complementary energy method, and the \( L_2 \) projection method. These are briefly described in the following.

2.4.1 The contour integral method

The contour integral method is a procedure which utilizes the orthogonality of eigenfunctions and the fact that if \( \lambda_i \) is an eigenfunction then \(-\lambda_i\) is also an eigenfunction. For details we refer to [1], [6].

2.4.2 The complementary energy method

Define:

\[ \Pi_c(q_i) \overset{\text{def}}{=} \frac{1}{2} \int_{\Omega_R} C_{ij} q_i q_j \, dV - \int_{\Gamma_R} q_i n_i T_{FE} \, d\theta \]

where \( C_{ij} \) is the inverse of \( K_{ij} \); \( q_i \) satisfies the heat balance equation: \(-q_{i,i} + Q = 0\) (and prescribed homogeneous flux boundary conditions); \( T_{FE} \) is the temperature computed by the finite element method.
Letting \( q_i = -K_{ij} T_{ij} \) and minimizing the complementary energy functional \( \Pi_c(q_i) \) with respect to \( A_i \), yields approximations to the generalized flux intensity factors.

### 2.4.3 The \( L_2 \) projection method

This method involves the projection of \( u_{FE} \) onto the space spanned by the eigenfunctions. Specifically, \( A_i \) are determined from the condition:

\[
\int_{\Omega_R} \left( u_{FE} - \sum_{i=1}^{n} A_i r^{\lambda_i} \phi_i(\theta) \right) r^{\lambda_j} \phi_j(\theta) r dr d\theta = 0 \quad j = 1, 2, \ldots, n
\]

which yields \( n \) equations for \( A_i, i = 1, 2, \ldots, n \).

### 2.5 Example: The slit domain problem

Consider the problem \( \Delta \mathcal{T} = 0 \) on a unit circle slit along the axis \( x_2 \) with the boundary conditions

\[
\mathcal{T} = 0 \text{ on } \Gamma_1, \quad q_2 = 0 \text{ on } \Gamma_2, \quad q_n = q_i n_i = x_2 \text{ on } \Gamma_3.
\]

In this case: \( \lambda_1 = 0.25, \lambda_2 = 0.75, \lambda_3 = 1.25 \). This is a very challenging problem from the point of view of numerical approximation by the finite element method, owing to the fact that the lowest eigenvalue is 1/4, hence the theoretical rate of convergence\(^1\) of the p-version is 1/4 and the theoretical rate of convergence of the h-version is 1/8. The error is most effectively controlled by hp-extension, utilizing geometrically graded meshes [6].

The finite element mesh, consisting of 12 elements, and the temperature distribution at \( p = 8 \) (trunk space) is shown in Fig. 6. The p-convergence of the energy and \( A_1, A_2, A_3 \) are shown in Table 1. The extrapolated values are shown in the last row. It is seen that the rate of convergence is close to the theoretical rate of 0.25.

### 2.6 Example: Two-material internal interface problem

Consider the problem \( \Delta \mathcal{T} = 0 \) on a unit circle. On the first quadrant has the material properties are \( K_{11} = K_{22} = 10.0, K_{12} = 0 \) on the other three quadrants \( K_{11} = K_{22} = 1.0, K_{12} = 0 \). On the boundary \( q_n = f(\theta) \) where \( f(\theta) \) is a function given by Oh and Babuška in [4].

\(^1\)The theoretical rate of convergence is given by \( \beta \) in the a priori estimate \( \|u_{EX} - u_{FE}\|_{E(\Omega)} \leq k N^{-\beta} \) were \( k \) is a positive constant and \( N \) is the number of degrees of freedom. See [6].
2.7 The problem of thermo-elasticity

The equations of equilibrium are:

$$
\sigma_{ij,j} + F_i = 0 \tag{10}
$$

Multiplying (10) by a test function \( v_i \) and applying Green's lemma, the generic form of the principle of virtual work is obtained:

$$
\int_{\Omega} \sigma_{ij}^{(u)} \epsilon_{ij}^{(v)} \ dV = \int_{\Omega} F_i v_i \ dV + \int_{\Gamma} T_i v_i \ dS
$$

where

$$
\epsilon_{ij}^{(v)} \overset{\text{def}}{=} \frac{1}{2} (v_{i,j} + v_{j,i})
$$

is the small strain tensor corresponding to the virtual displacement \( v_i \) and

$$
\sigma_{ij}^{(u)} = E_{ijkl} (\epsilon_{kl}^{(u)} - \alpha_{kl} \mathcal{T})
$$

is the stress tensor corresponding to \( u_i \); \( \alpha_{kl} \) represents the coefficients of thermal expansion and \( \mathcal{T} \) is the temperature.

Remark 2.7 The temperature field \( \mathcal{T} \) is continuous but \( \epsilon_{kl}^{(u)} \) does not have to be continuous.

Remark 2.8 \( E_{ijkl} \) may be a function of \( \mathcal{T} \).

An alternative formulation is the principle of minimum potential energy:

$$
\Pi(u) \overset{\text{def}}{=} \frac{1}{2} \int_{\Omega} E_{ijkl} (\epsilon_{ij}^{(u)} - \alpha_{ij} \mathcal{T}) (\epsilon_{kl}^{(u)} - \alpha_{kl} \mathcal{T}) \ dV
- \int_{\Omega} F_i u_i \ dV - \int_{\Gamma} T_i u_i \ dS
$$
Figure 6: The slit domain problem: Mesh layout and contour plot of $T$, $p=8$, trunk space.

The exact solution of the problem of elasticity minimizes the potential energy:

$$\Pi(u_{EX}) = \min_{u \in \mathcal{E}(\Omega)} \Pi(u)$$

(11)

where $\Pi$ is the potential energy and $\mathcal{E}(\Omega)$ represents the space of admissible functions. In the case of isotropic materials the Euler equations are:

$$G \nabla^2 u_i + (\lambda + G)(u_{j,j})_{,i} = -F_i + \beta \mathcal{T}_i$$

where

$$\beta \overset{\text{def}}{=} (3\lambda + 2G)\alpha.$$

where $\lambda$ is the Lamé parameter, $G$ is the shear modulus and $\alpha$ is the coefficient of thermal expansion.
Table 1: p-Convergence of the energy and $A_1$, $A_2$, $A_3$. Twelve-element mesh. The extrapolated values are shown in the last row.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$N$</th>
<th>Potential Energy</th>
<th>Rate of Conv.</th>
<th>Est.'d Rel. Err.</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>-1.977649</td>
<td>0.00</td>
<td>35.62</td>
<td>0.794380</td>
<td>-0.901532</td>
<td>0.373088</td>
</tr>
<tr>
<td>2</td>
<td>36</td>
<td>-2.141858</td>
<td>0.39</td>
<td>23.31</td>
<td>1.004779</td>
<td>-0.953011</td>
<td>0.457475</td>
</tr>
<tr>
<td>3</td>
<td>66</td>
<td>-2.178293</td>
<td>0.29</td>
<td>19.56</td>
<td>1.110997</td>
<td>-0.964365</td>
<td>0.455113</td>
</tr>
<tr>
<td>4</td>
<td>108</td>
<td>-2.196471</td>
<td>0.24</td>
<td>17.39</td>
<td>1.168927</td>
<td>-0.968574</td>
<td>0.453430</td>
</tr>
<tr>
<td>5</td>
<td>162</td>
<td>-2.208072</td>
<td>0.23</td>
<td>15.85</td>
<td>1.203440</td>
<td>-0.969836</td>
<td>0.452681</td>
</tr>
<tr>
<td>6</td>
<td>228</td>
<td>-2.216175</td>
<td>0.22</td>
<td>14.68</td>
<td>1.226336</td>
<td>-0.970035</td>
<td>0.452696</td>
</tr>
<tr>
<td>7</td>
<td>306</td>
<td>-2.222206</td>
<td>0.22</td>
<td>13.74</td>
<td>1.243025</td>
<td>-0.970053</td>
<td>0.452696</td>
</tr>
<tr>
<td>8</td>
<td>396</td>
<td>-2.226874</td>
<td>0.22</td>
<td>12.97</td>
<td>1.255925</td>
<td>-0.970047</td>
<td>0.452703</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\infty$</td>
<td>-2.264978</td>
<td>0.25</td>
<td>0</td>
<td>1.359910</td>
<td>-0.970047</td>
<td>0.452696</td>
</tr>
</tbody>
</table>

2.8 Computation of thermal stress intensity factors

Outline of the solution algorithm.

1. Assuming that the displacement field at the singular point is of the form

   $$u_i = r^\mu \Phi_i(\theta)$$

   compute the eigenpairs $\mu_j$, $\Phi_i(\theta)$ $j = 1, 2, \ldots$ using the modified Steklov method where the second index on $\Phi_{ij}$ represents the ordinal number of the eigenfunction. This involves the solution of a non-symmetric eigenvalue problem of the form:

   $$(B(u_i, v_i) - N(u_i, v_i)) - \mu M(u_i, v_i) = 0$$

   The eigenvalues are usually complex. Both the real and imaginary parts must be considered.

2. Construct the homogeneous part of the statically admissible stress field $\sigma_{ij}^{(H)}$ from

   $$u_i = \sum_{j=1}^{N} C_j r^{\mu_j} \Phi_{ij}(\theta)$$

   where $C_j$ represents the generalized stress intensity factors. Specifically,

   $$\sigma_{ij}^{(H)} = \frac{1}{2} E_{ijkl}(u_{k,l} + u_{l,k}).$$
3. The particular solution $\sigma_{ij}^{(P)}$ satisfies

$$\sigma_{ij}^{(P)} = E_{ijkl} \alpha_{kl} T_j$$

on $\Omega_R$ and the homogeneous traction boundary conditions on $\Gamma_1, \Gamma_2$. In general it is difficult to construct $\sigma_{ij}^{(P)}$. Note, however, that $\sigma_{ij}^{(P)}$ is of the order $r^\lambda$ where $\lambda \overset{\text{def}}{=} \lambda_{\text{min}} > 0$ is the smallest eigenvalue of the thermal problem. On the other hand, $\sigma_{ij}^{(H)}$ is of order $r^{\mu-1}$ where $\mu \overset{\text{def}}{=} \mu_{\text{min}} > 0$ is the smallest eigenvalue of the elasticity problem.

4. Construct the complementary energy functional on $\Omega_R$:

$$\Pi_c(\sigma_{ij}) \overset{\text{def}}{=} \frac{1}{2} \int_{\Omega_R} C_{ijkl} \sigma_{ij} \sigma_{kl} \, dV - \int_{\Gamma_R} \sigma_{ij} n_j u_i^{(FE)} \, ds$$
\[
\begin{align*}
&= \frac{1}{2} \int_{\Omega_R} C_{ijkl}\sigma_{ij}^{(H)} \sigma_{kl}^{(H)} dV + \int_{\Omega_R} C_{ijkl}\sigma_{ij}^{(H)} \sigma_{kl}^{(P)} dV \\
&\quad + \frac{1}{2} \int_{\Omega_R} C_{ijkl}\sigma_{ij}^{(P)} \sigma_{kl}^{(P)} dV - \int_{\Gamma_R} \sigma_{ij} n_j u_{ij}^{(F)} t ds
\end{align*}
\]

This indicates that if \( R \) is sufficiently small then \( \sigma_{ij}^{(P)} \) may be neglected.

5. Compute the thermal stress intensity factors \( C_j^{(k)} \) by minimizing \( \Pi_c(\sigma_{ij}^{(H)}) \) on \( \Gamma_{R_k} \) for a sequence of decreasing radii \( R_k, k = 1, 2, \ldots, n \).

6. Use Richardson extrapolation to find

\[
C_j = \lim_{{R_k \rightarrow 0}} C_j^{(k)}.
\]

2.9 Example: Cracked panel subject to thermal load

A centrally cracked panel is subjected to \( T = 100 \) at the perimeter; \( T = 0 \) on the crack faces. \( K_{11} = K_{22} = 1.0; K_{12} = 0; E = 1.0, \nu = 0.3, \alpha = 0.01 \), plane strain. \( L = w = 50.0 \), \( a = 1 \) is shown in Fig. 9. The p-convergence of the first thermal stress intensity factor \( A_1 \) computed with Richardson extrapolation is shown in Fig. 11 and the results reported by Yosibash in [10] using direct computation are given in Table 2.

<table>
<thead>
<tr>
<th>( R/a )</th>
<th>0.5</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0006</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>0.4908</td>
<td>0.3781</td>
<td>0.3528</td>
<td>0.3491</td>
<td>0.3481</td>
</tr>
</tbody>
</table>

2.10 Sources of errors

The methods used for computing the finite element solution, the eigenpairs and the generalized flux intensity factors, are approximate methods, hence certain errors are incurred:

1. In numerical work the asymptotic expansion is truncated to a few terms.
2. The eigenpairs are only approximations. Therefore \( q_i \) do not satisfy the heat balance equation exactly.

3. The prescribed temperature on \( \Gamma_R \) is approximate, computed from the finite element solution.

Nevertheless, numerical experience has indicated that this method of extraction is superconvergent well beyond the range of accuracy required in engineering work [8].

### 2.11 Stability problems

This topic is of substantial interest in aerospace engineering because the sizes of compression members are determined primarily by stability considerations. It is also of great importance in micromechanics where the strength of the composite materials is often determined by the buckling of fibers. The formulation and investigation of stability problems in the fully three-dimensional setting was undertaken. Details are available in a doctoral dissertation [3]. A brief outline is presented in the following. Define:

\[
\sigma_{ij}^0 \overset{\text{def}}{=} \lambda \sigma_{ij}^*
\]

where \( \sigma_{ij}^* \) is the pre-buckling stress state. We are interested in finding \( \bar{u}_i \in \mathcal{E}(\Omega) \) such that:

\[
\int_{\Omega} C_{ijkl} u_{ij,k} v_{kl} \, dV + \lambda \int_{\Omega} \sigma_{ij}^* u_{\alpha,i} v_{\alpha,j} \, dV =
\int_{\Omega} F_i v_i \, dV + \int_{\Gamma_T} T_i v_i \, dS + \int_{\Omega} \mathcal{T} C_{ijkl} \alpha_{kl} \bar{u}_{ij} \, dV
\]
Figure 10: The mesh, temperature distribution and the resulting deformation.

for all \( v_i \in \tilde{E}(\Omega) \). Note: \( \sigma_{ij}^r \) must be such that:

\[
\left| \int_{\Omega} (C_{ijkl} + \lambda \sigma_{ij}^r \delta_{ij}) \ddot{u}_{i,j} \ddot{u}_{k,l} \, dV \right| < \infty
\]

for all \( \ddot{u}_i \in \tilde{E}(\Omega) \). The set of \( \lambda \) for which a solution exists is the resolvent set. The complement is the spectrum. The spectrum may be point, continuous or residual.

The work done by the initial stress \( \sigma_{ij}^0 \) due to the product terms of the Green-Lagrange strain tensor is incorporated in the strain energy:

\[
U(\ddot{u}_i) \overset{\text{def}}{=} \frac{1}{2} \int_{\Omega} C_{ijkl} \dddot{\varepsilon}_{ij} \dddot{\varepsilon}_{kl} \, dV + \frac{1}{2} \int_{\Omega} \sigma_{ij}^0 \dddot{u}_{\alpha,i} \dddot{u}_{\alpha,j} \, dV. \\
\text{work of } \sigma_{ij}^0
\]

where \( \ddot{u}_i \) is a small increment of displacement and \( \dddot{\varepsilon}_{ij} \) is the small strain.
Figure 11: p-Convergence of the first thermal stress intensity factor $C_1$ computed with Richardson extrapolation.

Biot (1938) and Prager (1947) proposed a different formulation. Their definition is:

$$U(\bar{u}_i) \overset{\text{def}}{=} \frac{1}{2} \int_\Omega C_{ijkl} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl} \, dV + \frac{1}{2} \int_\Omega \sigma_{ij}^0 (\bar{u}_{\alpha,i} \bar{u}_{\alpha,j} - \bar{\varepsilon}_{\alpha i} \bar{\varepsilon}_{\alpha j}) \, dV$$

The potential energy is:

$$\Pi(\bar{u}_i) \overset{\text{def}}{=} U(\bar{u}_i) - \int_\Omega \bar{F}_i \bar{u}_i \, dV - \int_{\partial \Omega_T} \bar{T}_i \bar{u}_i \, dS + \int_\Omega \bar{T} C_{ijkl} \alpha_{kl} \bar{u}_{i,j} \, dV$$

We seek $\bar{u}_i \in \bar{E}(\Omega)$ such that the potential energy is stationary:

$$\delta \Pi(\bar{u}_i) \overset{\text{def}}{=} \left( \frac{\partial \Pi(\bar{u}_i + \varepsilon v_i)}{\partial \varepsilon} \right)_{\varepsilon=0} = 0 \quad v \in \bar{E}(\Omega).$$

The principle of virtual work in the case of initial stress:

$$\int_\Omega C_{ijkl} \bar{u}_{i,j} v_{k,l} \, dV + \int_\Omega \sigma_{ij}^0 \bar{u}_{\alpha,i} v_{\alpha,j} \, dV =$$

$$\int_\Omega \bar{F}_i v_i \, dV + \int_{\partial \Omega_T} \bar{T}_i v_i \, dS + \int_\Omega \bar{T} C_{ijkl} \alpha_{kl} \bar{u}_{i,j} \, dV$$
Figure 12: Lockheed test problem 2: The function $u_n \overset{\text{def}}{=} u_x n_x + u_y n_y$

for all $v_i \in \hat{E}(\Omega)$.

Although two mathematical models of the general theory of elastic stability exist in the classical literature, some fundamental questions concerning the existence of a solution, the properties of the spectrum, and their relationship to loss of stability had not been investigated previously. Two working hypotheses were advanced:

1. The spectrum is a point spectrum, hence it is meaningful to consider the lowest nonzero eigenvalue as an indicator of the onset of instability;

2. The minimal eigenvalue of the finite dimensional problem converges to its infinite dimensional counterpart as the finite element space is enlarged (i.e., the degrees of freedom are increased).
Both classical formulations were implemented in fully three-dimensional setting so that numerical experiments could performed. For thin structures the results closely matched the classical results. It was found that the two classical models yield virtually identical results.

In a parallel investigation the first working hypothesis was proven by Professors Manil Suri and Ivo Babuška. At present it is not known whether the second hypothesis can be proven, but the available numerical results have not contradicted it.

The relationship between the limits of elastic stability estimated by the use of linear models and incremental models was investigated. It was found that for conservative loads a close relationship exists but the treatment of follower loads through the solution of linear eigenvalue problems does not appear possible, with the exception of very special cases, such as the buckling of circular rings. See, for example, [9].

The problem of modeling the elastic buckling of fibers in fiber-matrix composites was investigated. A model problem has been solved. For the investigation of the stability of a large number of fibers the use of periodic boundary conditions is necessary. Implementation of periodic boundary conditions and further investigation of the stability of fibers is being planned for 1998.

3 References


