ELECTROMAGNETIC SCATTERING OF AN ANISOTROPICALLY COATED TUBULAR CYLINDER

by

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Dissertation Supervisor: Hung-Mou Lee

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ELECTROMAGNETIC SCATTERING
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Abstract

The sum-difference surface current formulation is introduced to treat electromagnetic boundary value problems when anisotropic impedances are specified on both sides of a surface. It can also be applied to impedance coated bodies. This formulation preserves the duality nature of Maxwell equations and carries it over into the algebraic form of the integrodifferential operators in the equations for surface currents. Since a 90° rotation is equivalent to undergoing a duality transform for an incident plane wave, this particular symmetry in the algebraic form of the operators leads to sufficient conditions under which the on-axis backscattering of an anisotropic impedance coated scatterer having a 90° rotational symmetry is eliminated. The sum-difference formulation is utilized for solving the problem of electromagnetic scattering from an anisotropically impedance coated tubular cylinder of finite length. The solution has been coded in FORTRAN and tested. Some interesting results are presented and discussed.
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I. INTRODUCTION

Sometime ago the question was raised: "For electromagnetic boundary value problems with specified surface impedances, how can one go from a non-perfectly conducting surface on which both the electric and the magnetic equivalent surface currents are to be found, to a perfectly conducting surface on which the number of unknowns is halved [1]?" The answer to this question turns out to be one of algebra. It is well known that the impedance specified on the surface of a body separates its interior completely from its exterior. Therefore an impedance coated body can always be considered as a hollow volume enclosed by an infinitesimally thin shell with surface impedances specified both on the inside and the outside of the shell. The inside and the outside of the body can be considered as constituted of the same medium and the impressed electromagnetic excitation can be treated as continuous across the shell. On the outside surface, there are the equivalent total electric current \( \bar{K}^+ \) and total magnetic current \( \bar{L}^+ \); on the inside surface, there are \( \bar{K}^- \) and \( \bar{L}^- \). For an exterior problem, only \( \bar{K}^+ \) and \( \bar{L}^+ \) need to be found; for an interior problem, only \( \bar{K}^- \) and \( \bar{L}^- \) are necessary. A single formulation for solving both types of problems would appear to require finding all inside and outside currents therefore doubling the amount of work, but it turns out not to be the case because some of the currents are linear combinations of others. Furthermore, this formulation holds the key to answering the question posed above.

Since the shell is infinitesimally thin, from Maxwell equations the radiation to the outside and into the inside of the shell can both be given in terms of integrodifferential operators on the sum currents \( \bar{K} = \bar{K}^+ + \bar{K}^- \) and \( \bar{L} = \bar{L}^+ + \bar{L}^- \). Note that the outside currents will not contribute to the radiation in the interior while the inside currents will not contribute to the radiation to the exterior of the shell. For simplicity in the description, we consider the exterior problem of electromagnetic scattering. By definition, the radiation is the difference between the total field and the incident field. On the surfaces of the shell, this definition links the incident \( \bar{E} \) and \( \bar{H} \) fields and the difference currents \( \bar{K}^+ - \bar{K}^- \) and \( \bar{L}^+ - \bar{L}^- \) to the sum currents. It is therefore natural to treat the difference currents and the sum currents as the four
unknowns to be solved instead of the inside and the outside currents.

The surface impedance on the outside surface of the shell normalized to the medium is denoted by \( Z^+ \) and that on the inside surface is \( Z^- \). They can be tensors if the impedances are anisotropic and may vary from point to point. By forming the sum impedance \( Z = (Z^+ + Z^-)/2 \) and the difference impedance \( \Delta = (Z^+ - Z^-)/2 \), the impedance boundary conditions provide a set of relations between the difference and the sum currents. It turns out that the rank of \( Z \) determines how the unknown surface currents are solved. If \( Z \) is invertible, then the difference currents are linear combinations of the sum currents so that only the integrodifferential equations of the sum currents have to be solved. There are only two unknowns to be solved for both the exterior and the interior problems under this sum-difference current formulation. If \( Z \) is rank 0, then \( Z = 0 \); the impedance boundary condition requires that \( \bar{E} \) be proportional to a 90° rotation of \( \Delta \bar{K} \). The difference electric current is obtained from \( \bar{K} \) after the integrodifferential equation on \( \bar{K} \) is solved. This situation includes the case where \( Z^+ = Z^- = 0 \) when the surface is perfectly conducting, thus the result answers the question about the transition of the equations from a problem of two variables to one which has only a single variable.

Instead of dealing with an impedance coated body, this thesis presents the sum-difference currents formulation of electromagnetic boundary value problems for the scattering of an infinitesimally thin surface for which both the inside and the outside currents are true unknowns to be found. Extension of this formulation to impedance coated bodies is then discussed.

This formulation preserves the duality nature of Maxwell equations and carries it over into an explicit specific algebraic form of the integrodifferential operators in the equations for the sum currents. Since, for a plane wave, a 90° rotation is equivalent to undergoing a duality transform, this explicit symmetry in the algebraic form of the operators enables us to deduce sufficient conditions for eliminating the on-axis backscattering of an anisotropic impedance coated scatterer having a 90° rotational symmetry.

The sum-difference currents formulation is utilized for solving the problem of
electromagnetic scattering from an anisotropically impedance coated tubular cylinder of finite length. The solution has been coded in FORTRAN and tested. Some interesting results are presented and discussed.

In this thesis, the time dependence $e^{-i\omega t}$ is used. $\vec{E}$ represents the electric field intensity divided by the intrinsic impedance of the medium, $\sqrt{\mu/\varepsilon}$; therefore it has the same unit as $\vec{H}$ in amperes per meter. So are the electric and magnetic equivalent surface currents $\vec{K}$ and $\vec{L}$.
II. THE SUM-DIFFERENCE SURFACE CURRENT FORMULATION OF ELECTROMAGNETIC BOUNDARY-VALUE PROBLEMS

A. STRATTON-CHU FIELD FORMULATION AND RADIATION

On an orientable, piecewise regular surface, whether open or closed, having a surface electric current density \( \vec{K} \) and a surface magnetic current density \( \vec{L} \), we define the Stratton-Chu E-field formula as:

\[
\vec{E}_{s-c}(\vec{r}, S, \vec{K}, \vec{L}) = \frac{ik^2}{4\pi} \int_S \vec{K}(\vec{r}_o) G(\vec{r}-\vec{r}_o) da_o - \frac{i}{4\pi} \nabla \int_S \vec{K}(\vec{r}_o) \cdot \nabla_o G(\vec{r}-\vec{r}_o) da_o - \frac{k}{4\pi} \nabla \times \int_S \vec{L}(\vec{r}_o) G(\vec{r}-\vec{r}_o) da_o
\]  

(2-1)

where \( \vec{r} \) is a point which is not on \( S \), \( k = \omega \sqrt{\mu \varepsilon} \) and \( G(\vec{r}-\vec{r}_o) = \frac{e^{ik|\vec{r}-\vec{r}_o|}}{k|\vec{r}-\vec{r}_o|} \), then the Stratton-Chu H-field formula can be defined as:

\[
\vec{H}_{s-c}(\vec{r}, S, \vec{K}, \vec{L}) = \vec{E}_{s-c}(\vec{r}, S, \vec{L}, -\vec{K})
\]

\[
= \frac{k}{4\pi} \nabla \times \int_S \vec{K}(\vec{r}_o) G(\vec{r}-\vec{r}_o) da_o + \frac{ik^2}{4\pi} \int_S \vec{L}(\vec{r}_o) G(\vec{r}-\vec{r}_o) da_o - \frac{i}{4\pi} \nabla \int_S \vec{L}(\vec{r}_o) \cdot \nabla_o G(\vec{r}-\vec{r}_o) da_o
\]  

(2-2)

Note that if \( S \) is a closed surface and \( \vec{K} \) and \( \vec{L} \) are the actual total equivalent surface currents on \( S \), then \( \vec{E}_{s-c} \) and \( \vec{H}_{s-c} \) are respectively the \( \vec{E} \) and \( \vec{H} \) fields at \( \vec{r} \) due to all sources inside \( S \) if \( \vec{r} \) is located outside \( S \) and vice versa. This is a direct consequence of Maxwell’s equations [2] and under this circumstance, the Stratton-Chu formulae are equivalent to Maxwell’s equations. On the other hand, unlike the Poynting theorem, the Stratton-Chu field formulae over an open surface \( S \) are but integrodifferential operators on the tangential vector
fields $\vec{K}$ and $\vec{L}$ over $S$ without any special physical meaning attached.

To introduce equivalent surface currents on $S$, the direction of the unit normal vector $\hat{n}$ on the surface has to be chosen. Adopting the terminology for a closed surface, we can assign one side of any orientable surface $S$ as the "outside surface" $S^+$, albeit somewhat arbitrarily if $S$ is not closed. The outward normal $\hat{n}^+$ is the unit normal pointing out of this side of $S$ and for simplicity, we call $\hat{n}^+$ the normal of $S$ and denote it by $\hat{n}$. The other side of the surface $S$ is the "inside surface" $S^-$. At any point $\vec{r}$ on $S$, the inward normal $\hat{n}^-$ is $-\hat{n}$. As a convention, the fields and surface currents on $S^+$ and $S^-$ always carry the corresponding superscripts (Figure 2-1).

![Figure 2-1. Outside and Inside Surfaces, Normals and the Equivalent Currents.](image)

Each of the total surface currents $\vec{K}^\pm$ and $\vec{L}^\pm$ on $S^\pm$ consists of two parts: the incident
current (with the additional superscript "inc") and the scattering current (with the additional superscript "sc") corresponding to the incident field and the scattered field on the particular side of the surface \( S \):

\[
\vec{K}^\pm = \hat{n}^\pm \times \vec{H}^\pm = \hat{n}^\pm \times \vec{H}^{inc} + \hat{n}^\pm \times \vec{H}^{sc}
\]

\[
= \vec{K}^{inc} + \vec{K}^{sc}
\] (2-3)

\[
\vec{L}^\pm = \vec{E}^\pm \times \hat{n}^\pm = \vec{E}^{inc} \times \hat{n}^\pm + \vec{E}^{sc} \times \hat{n}^\pm
\]

\[
= \vec{L}^{inc} + \vec{L}^{sc}
\] (2-4)

Note that \( S \) is infinitesimally thin, hence \( \vec{H}^{+,inc} = \vec{H}^{-,inc} = \vec{H}^{inc} \) and \( \vec{E}^{+,inc} = \vec{E}^{-,inc} = \vec{E}^{inc} \) on \( S \) so that \( \vec{K}^{+,inc} = -\vec{K}^{-,inc} \) and \( \vec{L}^{+,inc} = -\vec{L}^{-,inc} \). Therefore the sum currents \( \vec{K} \) and \( \vec{L} \) on \( S \) defined below are also the corresponding sums of the scattering currents only:

\[
\vec{K} = \vec{K}^+ + \vec{K}^- = \vec{K}^{+,sc} + \vec{K}^{-,sc}
\]

\[
\vec{L} = \vec{L}^+ + \vec{L}^- = \vec{L}^{+,sc} + \vec{L}^{-,sc}
\] (2-5)

Since the Stratton-Chu field formulae are linear operators on the surface currents, the radiated fields from surface currents on \( S \) are determined by the sum currents only:

\[
\vec{E}^{sc}(\vec{r}) = \vec{E}_{s-c}(\vec{r}, \vec{S}^+, \vec{K}^+, \vec{L}^+) + \vec{E}_{s-c}(\vec{r}, \vec{S}^-, \vec{K}^-, \vec{L}^-) = \vec{E}_{s-c}(\vec{r}, \vec{S}, \vec{K}, \vec{L})
\]

\[
\vec{H}^{sc}(\vec{r}) = \vec{H}_{s-c}(\vec{r}, \vec{S}, \vec{K}, \vec{L}) = \vec{E}_{s-c}(\vec{r}, \vec{S}, \vec{L}, -\vec{K})
\] (2-6)

**B. CONDITION ON THE CURRENTS IMPOSED BY MAXWELL'S EQUATIONS**

As \( \vec{r} \) approaches \( \vec{r}^z \) on \( S^z \), the tangential components (denoted by the subscript
"tan") of Eq. (2-6) provide four equations relating the incident fields and the total currents on both sides of $S$ through the fact that the incident field is the difference between the total and the scattered field:

$$ \vec{E}^{inc}_{\tan} = \vec{E}^*_{\tan} - \vec{E}^{s-c}_{\tan}(\vec{r}, S, \vec{K}, \vec{L}) $$  \hspace{1cm} (2-7)

$$ \vec{E}^{-inc}_{\tan} = \vec{E}^{-}_{\tan} - \vec{E}^{-s-c}_{\tan}(\vec{r}, S, \vec{K}, \vec{L}) $$ \hspace{1cm} (2-8)

$$ \vec{H}^{inc}_{\tan} = \vec{H}^*_{\tan} - \vec{H}^{s-c}_{\tan}(\vec{r}, S, \vec{K}, \vec{L}) $$ \hspace{1cm} (2-9)

$$ \vec{H}^{-inc}_{\tan} = \vec{H}^{-}_{\tan} - \vec{H}^{-s-c}_{\tan}(\vec{r}, S, \vec{K}, \vec{L}) $$ \hspace{1cm} (2-10)

These four equations are not independent of each other. Because

$$ \hat{n}^\pm \times (\vec{E}(\vec{r}^\pm) \times \hat{n}^\pm) = \vec{E}^{\pm}_{\tan}(\vec{r}^\pm) = \hat{n}^\pm \times \vec{L}^\pm $$ \hspace{1cm} (2-11)

$$ \hat{n}^\pm \times (\vec{H}(\vec{r}^\pm) \times \hat{n}^\pm) = \vec{H}^{\pm}_{\tan}(\vec{r}^\pm) = -\hat{n}^\pm \times \vec{K}^\pm $$

the difference between Eqs. (2-7) and (2-8) trivially confirms the definition of the sum equivalent magnetic current while the difference between Eqs. (2-9) and (2-10) confirms the definition of the sum equivalent electric current as both can be deduced directly from Maxwell’s equations. We choose to use the sum of Eqs. (2-7) and (2-8) and that of Eqs. (2-9) and (2-10) as the two independent linear combinations out of Eqs. (2-7) through (2-10) to link the incident fields to the total surface currents on $S$ as dictated by Maxwell’s equations.
\[2\mathbf{E}_{\text{tan}}^{\text{inc}} = \mathbf{n} \times (\mathbf{L}^+ - \mathbf{L}^-) - \left\{ \mathbf{E}_{s-c}(\mathbf{r}^+, S, \mathbf{k}, \mathbf{L}) + \mathbf{E}_{s-c}(\mathbf{r}^-, S, \mathbf{k}, \mathbf{L}) \right\} = \mathbf{n} \times (\mathbf{L}^+ - \mathbf{L}^-) + M \mathbf{k} - N \mathbf{L}
\]
\[
2\mathbf{H}_{\text{tan}}^{\text{inc}} = -\mathbf{n} \times (\mathbf{k}^+ - \mathbf{k}^-) - \left\{ \mathbf{H}_{s-c}(\mathbf{r}^+, S, \mathbf{k}, \mathbf{L}) + \mathbf{H}_{s-c}(\mathbf{r}^-, S, \mathbf{k}, \mathbf{L}) \right\} = -\mathbf{n} \times (\mathbf{k}^+ - \mathbf{k}^-) + N \mathbf{k} + M \mathbf{L}
\] (2-12)

where \(M\) and \(N\) are linear integrodifferential operators on the tangential vector fields \(\mathbf{k}\) and \(\mathbf{L}\) over \(S\).

Under any orthonormal coordinates \((u, v)\) over \(S\) having \(\mathbf{u}, \mathbf{v}\) as the unit basis vectors and with \(\mathbf{n} = \mathbf{u} \times \mathbf{v}\), a tangential vector field \(\mathbf{A}\) over \(S\) can be written in matrix form as:

\[
\mathbf{A} = \begin{bmatrix} A_u \\ A_v \end{bmatrix}. \text{ Then } \mathbf{n} \times \mathbf{A} = \begin{bmatrix} -A_v \\ A_u \end{bmatrix} = -i \sigma_2 \mathbf{A}\] where \(\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\) is one of the Pauli spin matrices. Note that \(\sigma_2^2 = 1\). Using these matrix notations, we can rewrite Eq. (2-12) in the following form:

\[
\begin{bmatrix} 0 & i \sigma_2 \\ -i \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{k}^+ - \mathbf{k}^- \\ \mathbf{L}^+ - \mathbf{L}^- \end{bmatrix} = \begin{bmatrix} M & -N \\ N & M \end{bmatrix} \begin{bmatrix} \mathbf{k} \\ \mathbf{L} \end{bmatrix} - 2 \begin{bmatrix} \mathbf{E}_{\text{tan}}^{\text{inc}} \\ \mathbf{H}_{\text{tan}}^{\text{inc}} \end{bmatrix}
\] (2-13)

C. IMPEDEANCE BOUNDARY CONDITION

Maxwell’s equations alone cannot determine the electromagnetic fields completely. If \(S\) is an open surface, appropriate boundary condition which the fields satisfy on \(S\) must be specified. It is usually given in terms of the impedance boundary condition, a linear relation among the tangential components of the total \(\mathbf{E}\) and the total \(\mathbf{H}\) fields on \(S\). If \(S\) is a closed surface, there are two possibilities: One is to specify the electric and magnetic properties of
the volume within $S$ and require the fields to satisfy regularity conditions within $S$ and be linked to the fields outside through boundary conditions across $S$; another is to specify the impedance boundary condition on $S^+$ for an exterior problem or on $S^-$ for an interior problem. Note that an impedance boundary condition over a closed surface $S$ completely separates the exterior from the interior of $S$. Therefore, the surface impedance on $S^+$ can be arbitrary for an exterior problem while that on $S^-$ can be arbitrary for an interior problem. In this thesis, normalized surface impedances $Z^\pm$ are assumed to be specified on $S^\pm$ whether $S$ is an open or a closed surface. Note that an open surface $S$ can be considered as bounded within the closed surface formed by joining $S^+$ and $S^-$. 

The impedance boundary conditions on $S^\pm$ are defined by:

$$\hat{n}^\pm \times (\vec{E}^\pm \times \hat{n}^\pm) = Z^\pm (\hat{n}^\pm \times \vec{H}^\pm)$$ \hspace{1cm} (2-14)$$

or equivalently, in terms of the total surface currents:

$$\vec{n}^\pm \times \vec{L}^\pm = Z^\pm \vec{K}^\pm$$ \hspace{1cm} (2-15)$$

With the matrix notations for tangential vector fields over $S$ in the orthonormal coordinate system $(u, v)$ introduced before, we can consider $Z^\pm$ as 2$\times$2 matrices and rewrite Eq. (2-15) in the form:

$$\pm i \sigma_2 \vec{L}^\pm = Z^\pm \vec{K}^\pm = \frac{1}{2} Z^\pm [\vec{K}^\pm \pm (\vec{K}^+ - \vec{K}^-)]$$ \hspace{1cm} (2-16)$$

which can readily be recast into a relation among sum and difference currents:

$$\begin{bmatrix} -\Delta & -i \sigma_2 \\ Z & 0 \end{bmatrix} \begin{bmatrix} \vec{K}^+ - \vec{K}^- \\ \vec{L}^+ - \vec{L}^- \end{bmatrix} = \begin{bmatrix} Z & 0 \end{bmatrix} \begin{bmatrix} \vec{K} \\ \vec{L} \end{bmatrix}$$ \hspace{1cm} (2-17)$$
with

\[ Z = \frac{1}{2} \left( Z^+ + Z^- \right) \]  

(2-18)

and

\[ \Delta = \frac{1}{2} \left( Z^+ - Z^- \right) \]  

(2-19)

Eqs. (2-13) and (2-17) are a set of four two-dimensional vector equations to be solved for the sum and difference equivalent electric and magnetic surface current densities on \( S \).

D. ALGEBRA OF THE SUM-DIFFERENCE CURRENT EQUATIONS

The existence and uniqueness of a solution to either the exterior or the interior problem specified in terms of the impedance boundary condition have been well established [3]. Here we want to investigate how such a solution can be obtained from Eqs. (2-13) and (2-17). Clearly Eq. (2-17) defines uniquely the algebraic relationship between the difference and the sum currents if \( Z \) is invertible. For example, the difference currents can be expressed in terms of the sum currents:

\[
\begin{bmatrix}
\bar{K}^+ - \bar{K}^- \\
\bar{L}^+ - \bar{L}^-
\end{bmatrix} = - \begin{bmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix} R \begin{bmatrix} \bar{K} \\ \bar{L} \end{bmatrix}
\]  

(2-20)

where

\[
R = \begin{bmatrix} Z - \Delta Z^{-1} \Delta & -i\Delta Z^{-1} \sigma_2 \\ -i\sigma_2 Z^{-1} \Delta & \sigma_2 Z^{-1} \sigma_2 \end{bmatrix}
\]  

(2-21)
An equation for the sum currents is obtained by substituting Eq. (2-20) into Eq. (2-13):

\[
\begin{bmatrix}
M & -N \\
N & M
\end{bmatrix}
\begin{bmatrix}
\vec{K} \\
\vec{L}
\end{bmatrix}
+ R
\begin{bmatrix}
\vec{K} \\
\vec{L}
\end{bmatrix}
= 2
\begin{bmatrix}
\vec{E}_{\tan}^{\text{inc}} \\
\vec{H}_{\tan}^{\text{inc}}
\end{bmatrix}
\tag{2-22}
\]

which can be solved for \( \vec{K} \) and \( \vec{L} \). Eq. (2-20) in turn enables us to compute the difference currents algebraically then split the sum and the difference currents into total currents on \( S^z \).

If \( Z \) is not invertible, then the situation is more complicated. \( Z \) can either be of rank 0 when \( Z = 0 \) or rank 1 when \( \text{det } Z = 0 \) but \( Z \neq 0 \). Eqs. (2-13) and (2-17) can be combined into an equation for the sum currents \( \vec{K} \) and \( \vec{L} \):

\[
\begin{bmatrix}
1 & i \Delta \sigma_2 \\
0 & -iZ \sigma_2
\end{bmatrix}
\begin{bmatrix}
M & -N \\
N & M
\end{bmatrix}
\begin{bmatrix}
\vec{K} \\
\vec{L}
\end{bmatrix}
+ \begin{bmatrix}
Z & 0 \\
-\Delta & -i \sigma_2
\end{bmatrix}
\begin{bmatrix}
\vec{K} \\
\vec{L}
\end{bmatrix}
= 2 \begin{bmatrix}
1 & i \Delta \sigma_2 \\
0 & -iZ \sigma_2
\end{bmatrix}
\begin{bmatrix}
\vec{E}_{\tan}^{\text{inc}} \\
\vec{H}_{\tan}^{\text{inc}}
\end{bmatrix}
\tag{2-23}
\]

When \( Z = 0 \), Eq. (2-17) gives the null relations \( \vec{L}^- - \vec{L}^+ = i \sigma_2 \Delta (\vec{K}^+ - \vec{K}^-) \) and

\[
\vec{L} = i \sigma_2 \Delta \vec{K}
\]

hence \( \vec{L}^\pm = i \sigma_2 \Delta \vec{K}^\pm \). Eq. (2-23) becomes one for \( \vec{K} \) only:

\[
\begin{bmatrix}
1 & i \Delta \sigma_2 \\
0 & i \sigma_2 \Delta
\end{bmatrix}
\begin{bmatrix}
M & -N \\
N & M
\end{bmatrix}
\begin{bmatrix}
1 \\
i \sigma_2 \Delta
\end{bmatrix}
\vec{K}
= 2 \begin{bmatrix}
1 & i \Delta \sigma_2 \\
0 & i \sigma_2 \Delta
\end{bmatrix}
\begin{bmatrix}
\vec{E}_{\tan}^{\text{inc}} \\
\vec{H}_{\tan}^{\text{inc}}
\end{bmatrix}
\tag{2-24}
\]

Eq. (2-13) has to be used to find the difference electric current:

\[
\vec{K}^+ - \vec{K}^- = i \sigma_2 \left[ N + iM \sigma_2 \Delta \right] \vec{K} - 2i \sigma_2 \vec{H}_{\tan}^{\text{inc}}
\tag{2-25}
\]

Therefore,
\[
\vec{K}^\pm = \frac{1}{2} \left\{ 1 \pm i \sigma_2 \left[ N + iM \sigma_2 \Delta \right] \right\} \vec{K} = i \sigma_2 \vec{H}^{inc}_{tan}
\] (2-26)

Since the last term in Eq. (2-26) is \( \vec{K}^{\pm,inc} \),

\[
\vec{K}^{\pm,sc} = \frac{1}{2} \left\{ 1 \pm i \sigma_2 \left[ N + iM \sigma_2 \Delta \right] \right\} \vec{K}
\] (2-27)

\( \vec{L}^+ \) and \( \vec{L}^- \) can be obtained algebraically by multiplying \( i \sigma_2 \Delta \) to \( \vec{K}^+ \) and \( \vec{K}^- \) respectively. On the other hand, by Eq. (2-13),

\[
\vec{L}^{\pm,sc} = \frac{1}{2} i \sigma_2 \left\{ \Delta \mp \left[ M - iN \sigma_2 \Delta \right] \right\} \vec{K}
\] (2-28)

Note that the \( Z = 0 \) case includes the special situation \( Z^+ = Z^- = Z = \Delta = 0 \) when both sides of \( S \) are perfectly conducting. Under this circumstance \( \vec{L} = \vec{L}^{\pm} = 0 \) and the operator \( N \) is never involved.

When \( Z \) is rank 1, \( Z \neq 0 \) but \( \det Z = 0 \). The right hand side of Eq. (2-17) provides one linear relation between the components of \( \vec{L} - i \sigma_2 \Delta \vec{K} \) which can be used to reduce the four components of \( \vec{K} \) and \( \vec{L} \) as the unknown quantities in Eq. (2-23) to three so that the remaining three components can be solved. The left hand side of Eq. (2-17) assures that the same linear relationship between the components of \( \vec{L} - i \sigma_2 \Delta \vec{K} \) exists between corresponding components of the difference currents. Eq. (2-13) again has to be invoked to compute three other linearly independent combinations of the components of the difference currents from the sum currents.
E. CONSIDERATIONS FOR A CLOSED SURFACE

When \( S \) is a closed surface, the choice of \( Z^- \) can be arbitrary for an exterior problem such as scattering while the choice of \( Z^+ \) is arbitrary for an interior problem. It is desirable to choose \( Z^- = -Z^+ \) so that \( Z = 0 \) and \( \Delta = Z^+ = -Z^- \). Then we have \( \vec{L} = i \sigma_2 \Delta \vec{K} \) and only \( \vec{K} \) has to be computed. With such a choice, for an exterior problem,

\[
\vec{K}^{+,sc} = \frac{1}{2} \left( 1 - \sigma_2 M \sigma_2 Z^+ + i \sigma_2 N \right) \vec{K} \tag{2-29}
\]

\[
\vec{L}^{+,sc} = \frac{i \sigma_2}{2} \left( Z^+ + i N \sigma_2 Z^+ - M \right) \vec{K} \tag{2-30}
\]

and for an interior problem:

\[
\vec{K}^{-,sc} = \frac{1}{2} \left( 1 - \sigma_2 M \sigma_2 Z^- - i \sigma_2 N \right) \vec{K} \tag{2-31}
\]

\[
\vec{L}^{-,sc} = \frac{i \sigma_2}{2} \left( M - Z^- + i N \sigma_2 Z^- \right) \vec{K} \tag{2-32}
\]
III. A THEOREM OF ANISOTROPIC ABSORBERS

A. AXIAL RADIATION FROM A SURFACE OF 90° ROTATIONAL SYMMETRY

The xy-plane cross section of a surface $S$ having a 90° rotational symmetry around the z-axis is shown in Figure 3-1. Because of this symmetry, $S$ can be decomposed into four non-overlapping congruent pieces $S_1, S_2, S_3, S_4$ so that a 90° rotation will bring $S_i$ into $S_{i+1}$.

![Figure 3-1. Cross Section of a Surface of 90-Degree Rotational Symmetry.](image)

(These subscripts are considered as equal under modulo 4.) Therefore each piece $S_i$ of $S$ can
be parametrized in the same orthonormal coordinates \((u, v)\), with \(\dot{u} = \frac{\partial \vec{r}_i}{\partial u}, \dot{v} = \frac{\partial \vec{r}_i}{\partial v}\) the orthonormal basis vectors on \(S_i\), as follows:

\[
\vec{r}_1 = (x_o(u,v), y_o(u,v), z_o(u,v)) \\
\vec{r}_2 = (-y_o(u,v), x_o(u,v), z_o(u,v)) \\
\vec{r}_3 = (-x_o(u,v), -y_o(u,v), z_o(u,v)) \\
\vec{r}_4 = (y_o(u,v), -x_o(u,v), z_o(u,v))
\] (3-1)

where \(\vec{r}_i \in S_i\). As an example of a possible choice, \(u = \text{constant}\) and \(v = \text{constant}\) can be the lines of curvature of \(S_j\).

In terms of the coordinates \((u, v)\), the sum surface current distributions on \(S_i\) are:

\[
\vec{K}(\vec{r}_i) = \vec{K}_i(u,v) \\
\vec{L}(\vec{r}_i) = \vec{L}_i(u,v)
\] (3-2)

Since for \(r \gg r_o\),

\[
G(\vec{r}) = \frac{e^{ik|\vec{r}-\vec{r}_o|}}{kr}
\] (3-3)

\[
\nabla G(\vec{r}) = ik\vec{r}G = -\nabla_\omega G(\vec{r})
\] (3-4)

the radiation from such current distributions at a distance \(r \gg \max |\vec{r}_i|\) along the positive z-axis is, from Eq. (2-6):

16
\[ \mathbf{E}^sc(\mathbf{r}) = \mathbf{E}^sc(\mathbf{r}, S, \mathbf{K}, \mathbf{L}) \]

\[ = \frac{ik}{4\pi r} \int_S \left\{ \hat{x} \left[ K_x(\mathbf{r}) + L_y(\mathbf{r}) \right] + \hat{y} \left[ K_y(\mathbf{r}) - L_x(\mathbf{r}) \right] \right\} e^{ik\sqrt{(z-z_o)^2 + x_o^2 + y_o^2}} \, da_o \]

\[ = \frac{ik}{4\pi r} \int_S \left\{ \hat{x} \sum_{i=1}^{4} \left[ K_{ix}(u,v) + L_{iy}(u,v) \right] \right. \]

\[ + \left. \hat{y} \sum_{i=1}^{4} \left[ K_{iy}(u,v) - L_{ix}(u,v) \right] \right\} e^{ik\sqrt{(z-z_o)^2 + x^2 + y^2}} \, du \, dv \quad (3-5) \]

Note that this approximation is independent of the wavelength; it is applicable in regions closer to \( S \) than the usual Fresnel zone.

**B. CONDITION FOR VANISHING ON-AXIS BACKSCATTERING**

Consider two situations when a linearly polarized plane wave of unit strength is incident on \( S \) along the \( z \)-axis from the positive direction: Situation 1, identified with the superscript (1) has the wave polarized in the \( x \)-direction while Situation 2, identified with the superscript (2), has the wave polarized in the \( y \)-direction. The incident fields are respectively:

\[ \mathbf{E}^{inc.(1)} = \hat{x} \, e^{-ikz} \]
\[ \mathbf{H}^{inc.(1)} = -\hat{y} \, e^{-ikz} \quad (3-6) \]

\[ \mathbf{E}^{inc.(2)} = \hat{y} \, e^{-ikz} \]
\[ \mathbf{H}^{inc.(2)} = \hat{x} \, e^{-ikz} \quad (3-7) \]

Note that, as seen from the positive \( z \)-axis, the incident wave in Situation 2 is that of Situation 1 rotated by 90° counterclockwise. Furthermore, Situation 2 can be obtained from Situation 1 through the duality transformation \( \mathbf{E}^{inc} \rightarrow \mathbf{H}^{inc}, \mathbf{H}^{inc} \rightarrow -\mathbf{E}^{inc} \). Therefore, for a plane wave, a 90° rotation is equivalent to undergoing the duality transform.

Because of the rotational symmetry of \( S \), the currents excited on \( S_i \) under Situation 1 must appear on \( S_{i\perp} \) under Situation 2. Therefore:
\[ K_{i-1,x}^{(2)}(u,v) = -K_{i,y}^{(1)}(u,v) \]
\[ K_{i-1,y}^{(2)}(u,v) = K_{i,x}^{(1)}(u,v) \]  \hspace{1cm} (3-8)
\[ K_{i-1,z}^{(2)}(u,v) = K_{i,z}^{(1)}(u,v) \]

\[ L_{i-1,x}^{(2)}(u,v) = -L_{i,y}^{(1)}(u,v) \]
\[ L_{i-1,y}^{(2)}(u,v) = L_{i,x}^{(1)}(u,v) \]  \hspace{1cm} (3-9)
\[ L_{i-1,z}^{(2)}(u,v) = L_{i,z}^{(1)}(u,v) \]

Assume that \( Z \) on \( S \) is invertible, the sum currents on \( S \) are determined by Eq. (2-22). The tangential components of the incident fields which appear on the right-hand-side of that equation under these two situations are:

\[
\begin{bmatrix}
\vec{E}_{\text{inc},(1)}^{\text{tan}} \\
\vec{H}_{\text{inc},(1)}^{\text{tan}}
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\vec{u}} \cdot \hat{\vec{x}} \\
\hat{\vec{v}} \cdot \hat{\vec{y}} \\
-\hat{\vec{u}} \cdot \hat{\vec{y}} \\
-\hat{\vec{v}} \cdot \hat{\vec{x}}
\end{bmatrix}
\begin{bmatrix}
e^{-ikz}
\end{bmatrix}
\]  \hspace{1cm} (3-10)

\[
\begin{bmatrix}
\vec{E}_{\text{inc},(2)}^{\text{tan}} \\
\vec{H}_{\text{inc},(2)}^{\text{tan}}
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\vec{u}} \cdot \hat{\vec{y}} \\
\hat{\vec{v}} \cdot \hat{\vec{y}} \\
\hat{\vec{u}} \cdot \hat{\vec{x}} \\
\hat{\vec{v}} \cdot \hat{\vec{x}}
\end{bmatrix}
\begin{bmatrix}
e^{-ikz}
\end{bmatrix}
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\begin{bmatrix}
\vec{E}_{\text{inc},(1)}^{\text{tan}} \\
\vec{H}_{\text{inc},(1)}^{\text{tan}}
\end{bmatrix}
\]  \hspace{1cm} (3-11)

where \( I \) is the \( 2 \times 2 \) identity matrix. Therefore,
\[
\begin{bmatrix}
M & -N \\
N & M \\
\end{bmatrix}
\begin{bmatrix}
\vec{K}^{(1)} \\
\vec{L}^{(1)} \\
\end{bmatrix} + R
\begin{bmatrix}
\vec{K}^{(1)} \\
\vec{L}^{(1)} \\
\end{bmatrix} = 2
\begin{bmatrix}
\vec{E}^{inc,(1)}_{\tan} \\
\vec{H}^{inc,(1)}_{\tan} \\
\end{bmatrix}
\]
(3-12)

\[
\begin{bmatrix}
M & -N \\
N & M \\
\end{bmatrix}
\begin{bmatrix}
\vec{K}^{(2)} \\
\vec{L}^{(2)} \\
\end{bmatrix} + R
\begin{bmatrix}
\vec{K}^{(2)} \\
\vec{L}^{(2)} \\
\end{bmatrix} = 2
\begin{bmatrix}
\vec{E}^{inc,(2)}_{\tan} \\
\vec{H}^{inc,(2)}_{\tan} \\
\end{bmatrix} = 2
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
\vec{E}^{inc,(1)}_{\tan} \\
\vec{H}^{inc,(1)}_{\tan} \\
\end{bmatrix}
\]
(3-13)

Since
\[
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
M & -N \\
N & M \\
\end{bmatrix} = \begin{bmatrix}
M & -N \\
N & M \\
\end{bmatrix}
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\]
(3-14)

it follows that if
\[
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix} R = R
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\]
(3-15)

we can multiply \( \begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix} \) to Eq. (3-12) to get:

\[
\begin{bmatrix}
M & -N \\
N & M \\
\end{bmatrix}
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
\vec{K}^{(1)} \\
\vec{L}^{(1)} \\
\end{bmatrix} + R
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
\vec{K}^{(1)} \\
\vec{L}^{(1)} \\
\end{bmatrix} = 2
\begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
\vec{E}^{inc,(1)}_{\tan} \\
\vec{H}^{inc,(1)}_{\tan} \\
\end{bmatrix}
\]
(3-16)

Therefore the excited surface currents in these two situations are related by:

\[
\begin{bmatrix}
\vec{K}^{(2)} \\
\vec{L}^{(2)} \\
\end{bmatrix} = \begin{bmatrix}
0 & -I \\
I & 0 \\
\end{bmatrix}
\begin{bmatrix}
\vec{K}^{(1)} \\
\vec{L}^{(1)} \\
\end{bmatrix} = \begin{bmatrix}
-\vec{L}^{(1)} \\
\vec{K}^{(1)} \\
\end{bmatrix}
\]
(3-17)
Combining this result with Eqs. (3-8) and (3-9), we have:

\[
K_{i+1,x}^{(2)}(u,v) = -L_{i+1,x}^{(1)}(u,v) = -K_{i,y}^{(1)}(u,v)
\]
\[
K_{i+1,y}^{(2)}(u,v) = -L_{i+1,y}^{(1)}(u,v) = K_{i,x}^{(1)}(u,v)
\]
\[
L_{i+1,x}^{(2)}(u,v) = K_{i+1,x}^{(1)}(u,v) = -L_{i,y}^{(1)}(u,v)
\]
\[
L_{i+1,y}^{(2)}(u,v) = K_{i+1,y}^{(1)}(u,v) = L_{i,x}^{(1)}(u,v)
\]

so that

\[
\sum_{i=1}^{4} [K_{i,x}^{(1)}(u,v) + L_{i,y}^{(1)}(u,v)] = 0
\]

and

\[
\sum_{i=1}^{4} [K_{i,y}^{(1)}(u,v) - L_{i,x}^{(1)}(u,v)] = 0
\]

Hence, along the positive \( z \)-axis, from Eq (3-5),

\[
\tilde{E}^{zc}(\vec{r}) = \frac{ik}{4\pi r} \int_{S} \left\{ \hat{x} \sum_{i=1}^{4} [K_{i,x}^{(1)}(u,v) + L_{i,y}^{(1)}(u,v)]
\right.
\nonumber
\left. + \hat{y} \sum_{i=1}^{4} [K_{i,y}^{(1)}(u,v) - L_{i,x}^{(1)}(u,v)] \right\} e^{ik\sqrt{(z-z_{0})^{2}+x_{0}^{2}+y_{0}^{2}}} du dv
\]

\[
= 0
\]

and the backscattering from \( S \) along the positive \( z \)-direction must vanish if Eq. (3-15) is satisfied.
C. IMPEDANCE MATRICES FOR ZERO ON-AXIS BACKSCATTERING

It can be verified that the matrix
\[
\begin{bmatrix}
Z - \Delta Z^{-1} \Delta & -i \Delta Z^{-1} \sigma_2 \\
-i \sigma_2 Z^{-1} \Delta & \sigma_2 Z^{-1} \sigma_2
\end{bmatrix}
\]
commutes with \[
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\]
if and only if:

\[\sigma_2 Z^{-1} \Delta = -\Delta Z^{-1} \sigma_2 \] (3-22)

\[Z - \sigma_2 Z^{-1} \sigma_2 = \Delta Z^{-1} \Delta \] (3-23)

where both \(Z\) and \(\Delta\) are 2 \times 2 matrices. Under the assumption that the inverse of \(Z\) exists, we analyze Eqs. (3-22) and (3-23) as follows:

Because of the identity:

\[Z^{-1} = \frac{1}{\det Z} \sigma_2 Z^T \sigma_2 \] (3-24)

and, by multiplying \(\sigma_2\) to both sides of Eq. (3-22):

\[Z^{-1} \Delta = -\sigma_2 \Delta Z^{-1} \sigma_2 \] (3-25)

Eq. (3-23) can now be transformed to

\[Z = -\Delta (\sigma_2 \Delta Z^{-1} \sigma_2) + \sigma_2 Z^{-1} \sigma_2 = -\Delta \sigma_2 \Delta \sigma_2 (\sigma_2 Z^{-1} \sigma_2) + \sigma_2 Z^{-1} \sigma_2 \] (3-26)

\[= \frac{1}{\det Z} [1 - (\Delta \sigma_2)^2] Z^T \]
where

\[
(\Delta \sigma_2)^2 = \Delta \sigma_2 \Delta \sigma_2 = \begin{bmatrix}
\Delta_{11} \Delta_{22} - \Delta_{12}^2 & 0 \\
0 & \Delta_{11} \Delta_{22} - \Delta_{21}^2
\end{bmatrix}
\]  

(3-27)

It is observed that Eq. (3-27) is greatly simplified if \( \Delta \) is symmetric. Therefore, in this thesis, we consider only the case when \( \Delta = \Delta^T \). Then \( \Delta_{12} = \Delta_{21} \) so that:

\[
(\Delta \sigma_2)^2 = (\det \Delta) I
\]

(3-28)

From Eq. (3-26),

\[
Z = \left( \frac{1 - \det \Delta}{\det Z} \right) Z^T
\]

(3-29)

Eq. (3-29) can be satisfied only if \( \frac{1 - \det \Delta}{\det Z} = \pm 1 \). These two cases are:

Case I

\[
Z = -Z^T = \begin{bmatrix}
0 & z_{12} \\
-z_{12} & 0
\end{bmatrix}, \quad z_{12} \neq 0
\]

(3-30)

\[
\det \Delta = 1 + \det Z = 1 + z_{12}^2
\]

(3-31)

Case II

\[
Z = Z^T
\]

(3-32)

22
\[ \det Z + \det \Delta = 1 \quad (3-33) \]

On the other hand, substituting Eq. (3-24) into Eq. (3-22) yields:

\[ z_{11} (\Delta_{12} - \Delta_{21}) = (z_{12} - z_{21}) \Delta_{11} \quad (3-34) \]

\[ z_{22} (\Delta_{12} - \Delta_{21}) = (z_{12} - z_{21}) \Delta_{22} \quad (3-35) \]

\[ z_{11} \Delta_{22} + z_{22} \Delta_{11} = 2z_{21} \Delta_{12} = 2z_{12} \Delta_{21} \quad (3-36) \]

For Case I, Eqs (3-34) and (3-35) require that \( \Delta_{11} = \Delta_{22} = 0 \), Eq. (3-36) requires that \( \Delta_{12} = \Delta_{21} = 0 \). Therefore \( \Delta = 0 \). From Eq (3-31), \( 1 + z_{12}^2 = 0 \). Therefore \( z_{12} = \pm i \) so that \( Z^* = Z^- = Z = \pm \sigma_2 \).

For Case II, Eqs. (3-34) and (3-35) are trivially satisfied. Eq (3-36) becomes:

\[ z_{11} \Delta_{22} + z_{22} \Delta_{11} - 2z_{12} \Delta_{12} = 0 \quad (3-37) \]

Eq. (3-33) is explicitly:

\[ z_{11} z_{22} - z_{12}^2 + \Delta_{11} \Delta_{22} - \Delta_{12}^2 = 1 \quad -(3-38) \]

The sum of Eq. (3-37) with Eq.(3-38) is:

\[ (z_{11} + \Delta_{11})(z_{22} + \Delta_{22}) - (z_{12} + \Delta_{12})^2 = \det(Z + \Delta) = \det Z^* = 1 \quad (3-39) \]
Subtracting Eq. (3-37) from Eq. (3-38):

\[
(z_{11} - \Delta_{11})(z_{22} - \Delta_{22}) - (z_{12} - \Delta_{12})^2 = \det(Z - \Delta) = \det Z^- = 1 \tag{3-40}
\]

In summary, two sufficient conditions to satisfy Eqs. (3-22) and (3-23) have been deduced under the assumptions that \( \Delta \) is symmetric and \( Z \) is invertible. The first one is:

\[
Z^+ = Z^- = \pm \sigma_2 \tag{3-41}
\]

The other, with both \( Z^+ \) and \( Z^- \) symmetric, is:

\[
\det Z^+ = \det Z^- = 1 \tag{3-42}
\]

It should be noted that if \( S \) is a closed surface, the impedance boundary condition closes off the interior of the surface from its exterior. Therefore for the exterior problem, \( \det Z^+ = 1 \) is sufficient to eliminate the on-axis backscattering from a body of 90° rotational symmetry. Furthermore, \( Z^+ \) may vary with location. This is an extension of Weston's theory of isotropic absorbers [4].
IV. SCATTERING OF AN ANISOTROPICALLY COATED CYLINDER

In this chapter, the sum-difference surface current formulation developed in Chapter II is applied to the problem of scattering of an anisotropically coated tubular circular cylinder of finite length and negligible wall thickness. Due to the rotational symmetry, a Fourier series expansion can be utilized to reduce the variables of the problem to only the one along the axis of symmetry, chosen as the $z$-axis. The Fourier components $M_n$, $N_n$ of the operators $M$ and $N$ are deduced in terms of the Fourier components of $G(\vec{r} - \vec{r}_0)$ and its partial derivatives. To solve the set of integrodifferential equations so obtained, the equations and the surface currents are both weighted and expanded over the Chebyshev polynomials. The resultant infinite system of linear equations are then truncated and inverted numerically.

A. GEOMETRY AND COORDINATE SYSTEM

Figure 4-1 shows the geometry and coordinate system of a tubular circular cylinder of radius $a$ and length $2h$. The cylindrical coordinate system $(\rho, \phi, z)$ is scaled so that the surface of the cylinder $S$ is specified by $\rho = 1$ and $-1 \leq z \leq 1$. The radial vector is thus given by:

$$\vec{r} = a \rho \hat{\rho} + h z \hat{z}$$

$$= a \rho (\cos \phi \hat{x} + \sin \phi \hat{y}) + h z \hat{z}$$

(4-1)

To facilitate representing the tangent vectors over $S$ in matrix form, we chose $\hat{u} = \hat{\phi}$ and $\hat{v} = \hat{z}$ as the orthonormal basis vectors on $S$, so that $\hat{n} = \hat{\rho}$ is the outward normal to $S$.

To properly classify the polarization of the incident wave, the rectangular coordinate system $(x, y, z)$ is determined as follows: When a plane wave is not incident along the $z$-axis, the coordinates are chosen so that the wave is propagating in the $zx$-plane incident from the half-plane containing the positive $x$-axis. Axial incidences then are considered as the limiting cases as the direction of incidence approaches the positive or the negative $z$-axis from this half-plane. Therefore, even for axial incidence, a linearly polarized incident wave having its
electric field intensity vector $\vec{E}$ pointing in the $y$-direction is considered a TE wave while one having its $\vec{E}$ vector in the $zx$-plane is considered a TM wave. In the spherical coordinate system $(r, \theta, \phi)$, the direction of propagation of the incident wave $\hat{k}$ is given by $\hat{k} = -\hat{z}\cos \theta_i - \hat{x}\sin \theta_i$ where the incident angle $\theta_i$ varies over the range $0 \leq \theta_i \leq \pi$. For an incident plane wave of unit strength, the fields are:

**TE - polarization:**

$$
\vec{E}^{inc} = \hat{y} e^{i\vec{k} \cdot \vec{r}} \\
\vec{H}^{inc} = \hat{k} \times \vec{E}^{inc} = (\hat{x}\cos \theta_i - \hat{z}\sin \theta_i) e^{i\vec{k} \cdot \vec{r}}
$$

**TM - polarization:**
\[
\begin{align*}
\vec{H}^{\text{inc}} &= \hat{y} e^{i\vec{k} \cdot \vec{r}} \\
\vec{E}^{\text{inc}} &= \vec{H}^{\text{inc}} \times \vec{k} = - (\hat{x} \cos \theta_i - \hat{z} \sin \theta_i) e^{i\vec{k} \cdot \vec{r}}
\end{align*}
\]

where \( \vec{k} = k \hat{k} \). Note that \( \vec{k} \cdot \vec{r} = l_1 z \cos \theta_i + l_2 \rho \sin \theta_i \cos \phi \) where \( l_1 = kh \), and \( l_2 = ka \).

On the surfaces \( S \), the tangential components of TE-polarized plane wave are:

\[
\begin{align*}
E_{\phi}^{\text{inc}} &= \cos \phi e^{i\vec{k} \cdot \vec{r}} \\
H_{\phi}^{\text{inc}} &= - \cos \theta_i \sin \phi e^{i\vec{k} \cdot \vec{r}} \\
H_{z}^{\text{inc}} &= - \sin \theta_i e^{i\vec{k} \cdot \vec{r}}
\end{align*}
\]  \hspace{1cm} (4-1)

and the tangential components of TM-polarized plane wave are:

\[
\begin{align*}
\vec{E}_{\phi}^{\text{inc}} &= \cos \theta_i \sin \phi e^{i\vec{k} \cdot \vec{r}} \\
\vec{E}_z^{\text{inc}} &= \sin \theta_i e^{i\vec{k} \cdot \vec{r}} \\
H_{\phi}^{\text{inc}} &= \cos \phi e^{i\vec{k} \cdot \vec{r}}
\end{align*}
\]  \hspace{1cm} (4-2)

B. SCATTERED FIELDS

From Eq. (2-6), the scattered fields from the cylinder are given in terms of the sum equivalent electric current \( \vec{K} \) and the sum magnetic current \( \vec{L} \):

\[
\frac{4\pi}{l_1 l_2} \vec{E}^{\text{sc}}(\vec{r}) = - \frac{1}{k} \nabla \times \int_{-1}^{1} \int_{0}^{2\pi} \vec{L}(r_o) G(\vec{r} - \vec{r}_o) d\Phi_o d\zeta_o \\
+ i \int_{-1}^{1} \int_{0}^{2\pi} \vec{K}(r_o) G(\vec{r} - \vec{r}_o) d\Phi_o d\zeta_o \\
- \frac{i}{k^2} \nabla \int_{-1}^{1} \int_{0}^{2\pi} \vec{K}(r_o) \nabla_o G(\vec{r} - \vec{r}_o) d\Phi_o d\zeta_o
\]  \hspace{1cm} (4-3)

and
\[
\frac{4\pi}{l_1 l_2} \vec{H}^{sc}(\vec{r}) = \frac{1}{k} \nabla \times \oint_{-1}^{1} \int_{-1}^{1} 2\pi \vec{K}(r_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
+ i \oint_{-1}^{1} \int_{-1}^{1} 2\pi \vec{L}(r_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
- \frac{i}{k^2} \oint_{-1}^{1} \int_{-1}^{1} 2\pi \nabla_o G(\vec{r} - \vec{r}_o) d\Phi_o dz_o
\] (4-4)

Their components in cylindrical coordinates are:

\[
\frac{4\pi}{l_1 l_2} E^c(\rho, \phi, z) = \frac{1}{l_1} \frac{\partial}{\partial z} \oint_{-1}^{1} \int_{-1}^{1} 2\pi (\vec{\rho} \cdot \vec{\Phi}_o) L_\rho(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
- \frac{1}{l_2} \frac{\partial}{\partial \rho} \oint_{-1}^{1} \int_{-1}^{1} 2\pi L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
+ i \oint_{-1}^{1} \int_{-1}^{1} 2\pi (\vec{\rho} \cdot \vec{\Phi}_o) K_\rho(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
+ \frac{i}{l_2} \frac{\partial^2}{\partial \rho \partial \phi} \oint_{-1}^{1} \int_{-1}^{1} 2\pi K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o
\] (4-5)

\[
\frac{4\pi}{l_1 l_2} E^c(\phi, \rho, z) = \frac{1}{l_1} \frac{\partial}{\partial z} \oint_{-1}^{1} \int_{-1}^{1} 2\pi (\vec{\phi} \cdot \vec{\Phi}_o) L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
+ \frac{1}{l_2} \frac{\partial}{\partial \rho} \oint_{-1}^{1} \int_{-1}^{1} 2\pi L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
+ i \oint_{-1}^{1} \int_{-1}^{1} 2\pi (\vec{\phi} \cdot \vec{\Phi}_o) K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o \\
+ \frac{i}{l_2} \frac{\partial^2}{\partial \rho \partial \phi} \oint_{-1}^{1} \int_{-1}^{1} 2\pi K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\Phi_o dz_o
\] (4-6)
\[
\frac{4\pi}{l_2} E_z(x) = -\frac{1}{l_2 \rho} \frac{\partial}{\partial \rho} \int_{-1}^{1} \int_{0}^{2\pi} (\phi \cdot \phi_o) L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{1}{l_2} \frac{\partial}{\partial \phi} \int_{-1}^{1} \int_{0}^{2\pi} (\phi \cdot \phi_o) L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{i}{l_1 l_2} \frac{\partial^2}{\partial z^2} \int_{-1}^{1} \int_{0}^{2\pi} K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \left(1 + \frac{1}{l_1^2} \frac{\partial^2}{\partial z^2}\right) \int_{-1}^{1} \int_{0}^{2\pi} K_z(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o
\] (4-7)

\[
\frac{4\pi}{l_1 l_2} H_\rho(x) = -\frac{1}{l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} \int_{0}^{2\pi} (\phi \cdot \phi_o) K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{1}{l_2} \frac{\partial}{\partial \phi} \int_{-1}^{1} \int_{0}^{2\pi} K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{i}{l_2^2} \frac{\partial^2}{\partial \rho \partial z} \int_{-1}^{1} \int_{0}^{2\pi} L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{i}{l_1 l_2} \frac{\partial^2}{\partial z^2} \int_{-1}^{1} \int_{0}^{2\pi} L_z(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o
\] (4-8)

\[
\frac{4\pi}{l_1 l_2} H_\phi(x) = \frac{1}{l_1} \frac{\partial}{\partial z} \int_{-1}^{1} \int_{0}^{2\pi} (\phi \cdot \phi_o) K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
- \frac{1}{l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} \int_{0}^{2\pi} K_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ i \int_{-1}^{1} \int_{0}^{2\pi} (\phi \cdot \phi_o) L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{i}{l_2^2} \frac{\partial^2}{\partial \rho \partial z} \int_{-1}^{1} \int_{0}^{2\pi} L_\phi(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{i}{l_1 l_2} \frac{\partial^2}{\partial z^2} \int_{-1}^{1} \int_{0}^{2\pi} L_z(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o
\] (4-9)
\[ \frac{4\pi}{l_2} H_z^{se}(\vec{r}) = \frac{1}{l_2\rho} \frac{\partial}{\partial \rho} \int_{-1}^{1} \int_{0}^{2\pi} \rho (\dot{\phi} \dot{\phi}_o) K_{\phi}(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
- \frac{1}{l_2\rho} \frac{\partial}{\partial \phi} \int_{-1}^{1} \int_{0}^{2\pi} (\dot{\rho} \dot{\phi}_o) K_{\phi}(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ \frac{i}{l_2} \frac{\partial^2}{\partial \phi \partial \rho} \int_{-1}^{1} \int_{0}^{2\pi} L_{\phi}(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\
+ i \left( 1 + \frac{1}{l_2^2} \frac{\partial^2}{\partial z^2} \right) \int_{-1}^{1} \int_{0}^{2\pi} L_z(\vec{r}_o) G(\vec{r} - \vec{r}_o) d\phi_o dz_o \\ \] 

(4-10)

Note that in the above equations,
\[ \dot{\rho} = \dot{\rho}_o \cos(\phi - \phi_o) + \dot{\phi}_o \sin(\phi - \phi_o) \] 

\[ \dot{\phi} = -\dot{\rho}_o \sin(\phi - \phi_o) + \dot{\phi}_o \cos(\phi - \phi_o) \] 

(4-11)

(4-12)

Because of the rotational symmetry \( G(\vec{r} - \vec{r}_o) \) depends on \( \phi - \phi_o \) and Fourier series can be introduced to eliminate the variable \( \phi \). Define the Fourier series expansion of a function \( f(\phi, z) \) by:
\[ f(\phi, z) = \sum_{n=-\infty}^{\infty} e^{in\phi} f_n(z) \] 

(4-13)

then
\[ G(\vec{r} - \vec{r}_o) = \frac{e^{ik|\vec{r} - \vec{r}_o|}}{k |\vec{r} - \vec{r}_o|} = \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi_o)} G_n(l_1 |z - z_o|, l_2, \rho) \] 

(4-14)

and
\[ G_n(l_1|z-z_o|, l_2, \rho) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-i(n\phi - \phi_o)} G(\vec{r} - \vec{r}_o) \]  

(4-15)

Eqs. (4-5) to (4-10) become:

\[
\frac{2}{l_1^2} E_{\phi n}^{sc}(\rho, z) = \frac{1}{2l_1} \frac{\partial}{\partial z} \int_{-1}^{1} L_{\phi n}(z_o) [G_{n-1} + G_{n+1}] dz_o 
- \frac{i}{l_2 \rho} \int_{-1}^{1} L_{2n}(z_o) G_n dz_o
+ \frac{1}{2} \int_{-1}^{1} K_{\phi n}(z_o) [G_{n-1} - G_{n+1}] dz_o - \frac{n}{l_2^2} \frac{\partial}{\partial \rho} \int_{-1}^{1} K_{\phi n}(z_o) G_n dz_o
+ \frac{i}{l_1 l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} \left[ \frac{\partial}{\partial z_o} K_{2n}(z_o) \right] G_n dz_o
\]  

(4-16)

\[
\frac{2}{l_1^2} E_{\phi n}^{sc}(\rho, z) = \frac{i}{2l_1} \frac{\partial}{\partial z} \int_{-1}^{1} L_{\phi n}(z_o) [G_{n-1} - G_{n+1}] dz_o + \frac{1}{l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} L_{2n}(z_o) G_n dz_o
+ i \int_{-1}^{1} K_{\phi n}(z_o) \left[ \frac{1}{2} (G_{n-1} + G_{n+1}) - \frac{n^2}{l_2^2} G_n \right] dz_o 
- \frac{n}{l_1 l_2 \rho} \int_{-1}^{1} \left[ \frac{\partial}{\partial z_o} K_{2n}(z_o) \right] G_n dz_o
\]  

(4-17)

\[
\frac{2}{l_1^2} E_{zn}^{sc}(\rho, z) = \frac{1}{2l_2 \rho} \int_{-1}^{1} L_{\phi n}(z_o) [(n-1)G_{n-1} - (n+1)G_{n+1}] dz_o 
- \frac{1}{2l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} L_{\phi n}(z_o) [G_{n-1} + G_{n+1}] dz_o
- \frac{n}{l_1 l_2} \frac{\partial}{\partial z} \int_{-1}^{1} K_{\phi n}(z_o) G_n dz_o
+ \frac{i}{l_1 l_2} \frac{\partial}{\partial z} \int_{-1}^{1} \left[ \frac{\partial}{\partial z_o} K_{2n}(z_o) \right] G_n dz_o + i \int_{-1}^{1} K_{zn}(z_o) G_n dz_o
\]  

(4-18)

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$$\frac{2}{l_2 l_2^2} H_{\phi}^{sc}(\rho, z) = -\frac{1}{2l_1} \frac{\partial}{\partial \xi} \int_{-1}^{1} K_{\phi}(z_o)[G_{n-1} + G_{n+1}] dz_o$$

$$+ \frac{i n}{l_2 \rho} \int_{-1}^{1} K_{\phi}(z_o) G_n dz_o$$

$$+ \frac{1}{2} \int_{-1}^{1} L_{\phi}(z_o) [G_{n-1} - G_{n+1}] dz_o - \frac{n}{l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} L_{\phi}(z_o) G_n dz_o$$

$$+ \frac{i}{l_2^2 \rho} \int_{-1}^{1} \left[ \frac{\partial}{\partial z_o} L_{zn}(z_o) \right] G_n dz_o$$  \hspace{1cm} (4-19)$$

$$\frac{2}{l_2 l_2^2} H_{\phi}^{sc}(\rho, z) = -\frac{i}{2l_1} \frac{\partial}{\partial \xi} \int_{-1}^{1} K_{\phi}(z_o)[G_{n-1} - G_{n+1}] dz_o - \frac{1}{2} \frac{\partial}{\partial \rho} \int_{-1}^{1} K_{zn}(z_o) G_n dz_o$$

$$+ i \int_{-1}^{1} L_{\phi}(z_o) \left[ \frac{1}{2} G_{n-1} + G_{n+1} \right] dz_o - \frac{n^2}{l_2^2 \rho} G_n dz_o$$

$$- \frac{n}{l_2^2 \rho} \int_{-1}^{1} \left[ \frac{\partial}{\partial z_o} L_{zn}(z_o) \right] G_n dz_o$$  \hspace{1cm} (4-20)$$

$$\frac{2}{l_2 l_2^2} H_{\phi}^{sc}(\rho, z) = -\frac{1}{2l_2 \rho} \int_{-1}^{1} K_{\phi}(z_o)[(n-1)G_{n-1} - (n+1)G_{n+1}] dz_o$$

$$+ \frac{1}{2l_2} \frac{\partial}{\partial \rho} \int_{-1}^{1} K_{\phi}(z_o) [G_{n-1} + G_{n+1}] dz_o$$

$$- \frac{n}{l_1 l_2 \rho} \int_{-1}^{1} L_{\phi}(z_o) G_n dz_o$$

$$+ \frac{i}{l_2^2 \rho} \int_{-1}^{1} \left[ \frac{\partial}{\partial z_o} L_{zn}(z_o) \right] G_n dz_o + i \int_{-1}^{1} L_{zn}(z_o) G_n dz_o$$  \hspace{1cm} (4-21)$$

Note that in Eqs. (4-16) to (4-21), $G_n$ stands for $G_n({l_1 \vert z - z_o \vert, l_2 \rho})$. In shifting the $z$-derivative to the current densities $K_{zn}(z_o)$ and $L_{zn}(z_o)$, the fact that edge conditions [5] require that these components of the scattering currents vanish as $|z_o|$ approach 1 from below is used.
As \( \rho \to 1^+ \), the operators \( M_n \) and \( N_n \) can be deduced from Eqs. (4-16) to (4-21):

\[
M_{n,11}= -i l_1 l_2 \int_{l_1}^{l_2} dz_o \left[ \frac{1}{2} (G_{n-1} + G_{n+1}) - \frac{n^2}{l_2} G_n \right]_{\rho=1}
\]

\[
M_{n,12} = n \int_{l_1}^{l_2} dz_o \left[ G_n \right]_{\rho=1} \frac{\partial}{\partial z_o}
\]

\[
M_{n,21} = n \frac{\partial}{\partial z} \int_{l_1}^{l_2} dz_o \left[ G_n \right]_{\rho=1}
\]

\[
M_{n,22} = -i l_1 l_2 \int_{l_1}^{l_2} dz_o \left[ G_n \right]_{\rho=1} + \frac{1}{l_1^2} \frac{\partial}{\partial z} \left[ G_n \right]_{\rho=1} \frac{\partial}{\partial z_o}
\]

(4-22)

\[
N_{n,11} = \frac{i}{2} l_2 \frac{\partial}{\partial z} \int_{l_1}^{l_2} dz_o \left[ G_{n-1} - G_{n+1} \right]_{\rho=1}
\]

\[
N_{n,12} = \frac{i}{2} l_1 l_2 \int_{l_1}^{l_2} dz_o \frac{\partial}{\partial l_2} \left[ G_n \right]_{\rho=1}
\]

\[
N_{n,21} = \frac{l_1}{2} \int_{l_1}^{l_2} dz_o \left\{ \frac{n}{2} \frac{\partial}{\partial l_2} \left[ G_{n-1} + G_{n+1} \right]_{\rho=1} \right\}
\]

\[
N_{n,22} = 0
\]

(4-23)

Note that:

\[
\frac{i}{l_2} \left( \frac{\partial}{\partial \rho} \bigg|_{\rho=1^-} + \frac{\partial}{\partial \rho} \bigg|_{\rho=1^+} \right) G_n(l_1, |z - z_0|, l_2, \rho) = \frac{\partial}{\partial l_2} G_n(l_1, |z - z_0|, l_2, 1)
\]

(4-24)

Assuming that \( Z \) is invertible, from Eq. (2-23), the equations for the sum currents are therefore:

\[
\begin{bmatrix}
M_n & -N_n \\
N_n & M_n
\end{bmatrix}
\begin{bmatrix}
\vec{K}_n \\
\vec{L}_n
\end{bmatrix}
+ R
\begin{bmatrix}
\vec{K}_n \\
\vec{L}_n
\end{bmatrix}
= 2
\begin{bmatrix}
\vec{E}_{\text{tan},n}^{\text{inc}} \\
\vec{H}_{\text{tan},n}^{\text{inc}}
\end{bmatrix}
\]

(4-25)

where
\[ R = \begin{bmatrix} \Delta Z^{-1} & -i \Delta Z^{-1} \sigma_2 \\ -i \sigma_2 Z^{-1} \Delta & \sigma_2 Z^{-1} \sigma_2 \end{bmatrix} \] (4-26)

and the vectors \( \vec{K}_n, \vec{L}_n, \vec{E}_{\text{tan}, n}^{\text{inc}}, \vec{L}_{\text{tan}, n}^{\text{inc}} \) are two dimensional column matrix representations of the respective tangential vector fields on \( S \) over the orthonormal basis \( \hat{\phi}, \hat{\zeta} \). The incident fields are, for TE and TM polarizations respectively:

**TE:**

\[
E_{\phi, n}^{\text{inc}} = (-i)^{n-1} J'_n(l_2 \sin \theta_i) e^{-i l_1 z \cos \theta_i} \\
H_{\phi, n}^{\text{inc}} = -\frac{n(-i)^n \cos \theta_i}{l_2 \sin \theta_i} J_n(l_2 \sin \theta_i) e^{-i l_1 z \cos \theta_i} \\
H_{\zeta, n}^{\text{inc}} = -\sin \theta_i (-i)^{n-1} J_n(l_2 \sin \theta_i) e^{-i l_1 z \cos \theta_i} \] (4-27)

**TM:**

\[
E_{\phi, n}^{\text{inc}} = \frac{n(-i)^n \cos \theta_i}{l_2 \sin \theta_i} J_n(l_2 \sin \theta_i) e^{-i l_1 z \cos \theta_i} \\
E_{\zeta, n}^{\text{inc}} = \sin \theta_i (-i)^n J'_n(l_2 \sin \theta_i) e^{-i l_1 z \cos \theta_i} \\
H_{\phi, n}^{\text{inc}} = (-i)^{n-1} J'_n(l_2 \sin \theta_i) e^{-i l_1 z \cos \theta_i} \] (4-28)

**C. TRANSFORM TO SYSTEM OF LINEAR EQUATIONS**

Since the surface current components \( K_{\phi}(\phi, z_o) = O(1 - z_o^2)^{-1/2} \) and \( K_{\zeta}(\phi, z_o) = O(1 - z_o^2)^{1/2} \) as \( |z_o| \to 1^- \), representations of \( K_{\phi n}(z_o) \) and \( \frac{d}{dz_o} K_{\zeta n}(z_o) \) in Chebyshev polynomials of the first kind combined with the weighting factor conform to the proper edge behavior of the currents:

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\begin{align*}
K_{\phi,n}(z_0) &= \frac{1}{\pi \sin v} \sum_{q=0}^{\infty} K_{\phi,n}^q \ T_p(z_0) \\
&= \frac{1}{\pi \sin v} \sum_{q=0}^{\infty} K_{\phi,n}^q \ \cos qv \\
K_{z,n}(z_0) &= \frac{1}{\pi} \sum_{q=0}^{\infty} K_{z,n}^q \ \sin (q+1)v
\end{align*}

(4-29)

where \( T_p(z_0) \) is the Chebyshev polynomials of the first kind and \( z_0 = \cos v, \ -1 \leq z_0 \leq 1 \).

Similarly,

\begin{align*}
L_{\phi,n}(z_0) &= \frac{1}{\pi \sin v} \sum_{q=0}^{\infty} L_{\phi,n}^q \ \cos qv \\
L_{z,n}(z_0) &= \frac{1}{\pi} \sum_{q=0}^{\infty} L_{z,n}^q \ \sin (q+1)v
\end{align*}

(4-30)

For an invertible \( Z \), the coefficients in the above equations are to be determined from Eq. (4-23). To make use of the orthogonal property of the Chebyshev polynomials, the factor \( \sin v \sin (p + 1)v \) for \( p \geq 0 \) is multiplied to both sides of Eq. (4-23) before an integration over the range of \( v \) is carried out. The results are described term by term in the subsection to follow. This procedure creates an infinite system of linear equations to be solved numerically after it is truncated at an appropriate order determined by the electrical size of the cylinder.

1. **Incident Fields**

\[
\begin{bmatrix}
\bar{E}_{\text{inc},p}^\tan, n \\
\bar{H}_{\text{inc},p}^\tan, n
\end{bmatrix} = \frac{2}{p+1} \int_0^\pi \frac{d\bar{v}}{\pi} \sin \bar{v} \sin (p + 1)\bar{v} \begin{bmatrix}
\bar{E}_{\text{inc}}^\tan, n \\
\bar{H}_{\text{inc}}^\tan, n
\end{bmatrix}
\]

(4-31)
\[ E_{\Phi,n}^{inc,p} = (-i)^{n+p-1} \left( J_{p-1}(l_1 \cos \theta_i) + J_{p+1}(l_1 \cos \theta_i) \right) J_n'(l_2 \sin \theta_i) \]
\[ H_{\Phi,n}^{inc,p} = \frac{2n(-i)^{n+p}}{l_1 l_2 \sin \theta_i} J_{p+1}(l_1 \cos \theta_i) J_n(l_2 \sin \theta_i) \]
\[ H_{\tau,n}^{inc,p} = -(-i)^{n+p} \sin \theta_i \left( J_{p-1}(l_1 \cos \theta_i) + J_{p+1}(l_1 \cos \theta_i) \right) J_n'(l_2 \sin \theta_i) \]

\text{(4-32)}

\[ E_{\Phi,n}^{inc,p} = \frac{2n(-i)^{n+p}}{l_1 l_2 \sin \theta_i} J_{p+1}(l_1 \cos \theta_i) J_n(l_2 \sin \theta_i) \]
\[ E_{\tau,n}^{inc,p} = (-i)^{n+p} \sin \theta_i \left( J_{p-1}(l_1 \cos \theta_i) + J_{p+1}(l_1 \cos \theta_i) \right) J_n'(l_2 \sin \theta_i) \]
\[ H_{\Phi,n}^{inc,p} = (-i)^{n+p-1} \left( J_{p-1}(l_1 \cos \theta_i) + J_{p+1}(l_1 \cos \theta_i) \right) J_n'(l_2 \sin \theta_i) \]

\text{(4-33)}

where \( J'(\bullet) \) is a derivative with respect to the argument. Note that as \( \theta_i \) approaches 0 or \( \pi \),

only \( n = \pm 1 \) terms are nonzero. Hence only \( n = \pm 1 \) currents exist. For axial incident when

\( \theta_i = 0 \):

\[ E_{\Phi,1}^{inc,p} = \frac{(-i)^p}{l_1} J_{p+1}(l_1) = E_{\Phi,-1}^{inc,p} \]
\[ H_{\Phi,1}^{inc,p} = -\frac{(-i)^{p+1}}{l_1} J_{p+1}(l_1) = -H_{\Phi,-1}^{inc,p} \]

\text{(4-34)}

\[ E_{\Phi,1}^{inc,p} = \frac{(-i)^{p+1}}{l_1} J_{p+1}(l_1) = -E_{\Phi,-1}^{inc,p} \]
\[ H_{\Phi,1}^{inc,p} = \frac{(-i)^p}{l_1} J_{p+1}(l_1) = H_{\Phi,-1}^{inc,p} \]

\text{(4-35)}
2. The $R$ - Matrix Term

\[
\frac{2}{p+1}\int_0^\pi \frac{d\nu}{\pi} \sin \nu \sin(p+1)\nu K_{\phi n}(\cos \nu) \\
= \frac{2}{(p+1)\pi^2} \sum_{q=0}^\infty K_{\phi n}^q \int_0^\pi \! d\nu \sin(p+1)\nu \cos q\nu \tag{4-36}
\]

\[
= \sum_{q=0}^\infty A_{\phi}^{p,q} K_{\phi n}^q
\]

where

\[
A_{\phi}^{p,q} = \begin{cases} 
0 & \text{p+q odd} \\
\frac{4}{\pi^2(p+q+1)(p-q+1)} & \text{p+q even}
\end{cases} \tag{4-37}
\]

\[
\frac{2}{p+1}\int_0^\pi \frac{d\nu}{\pi} \sin \nu \sin(p+1)\nu K_{\nu n}(\cos \nu) \\
= \frac{2}{(p+1)\pi^2} \sum_{q=0}^\infty K_{\nu n}^q \int_0^\pi \! d\nu \sin \nu \sin(p+1)\nu \sin(q+1)\nu \tag{4-38}
\]

\[
= \sum_{q=0}^\infty A_{\nu}^{p,q} K_{\nu n}^q
\]

where

\[
A_{\nu}^{p,q} = \begin{cases} 
0 & \text{p+q odd} \\
-\frac{8(q+1)}{\pi^2(p+q+1)(p-q+1)(p+q+3)(p-q-1)} & \text{p+q even}
\end{cases} \tag{4-39}
\]

Therefore:
\[
\frac{2}{p+1}\int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v \begin{bmatrix} \vec{K}_n \\ \vec{L}_n \end{bmatrix} = \sum_{q=0}^{\infty} R \begin{bmatrix} A_{\phi}^{p,q} & 0 & 0 \\ 0 & A_{\theta}^{p,q} & 0 \\ 0 & 0 & A_{\phi}^{p,q} \end{bmatrix} \begin{bmatrix} \vec{K}_n \\ \vec{L}_n \end{bmatrix} 
\]

where \( R^{p,q} \) is a four-by-four matrix, given in terms of \( R \) by

\[
R_{i1}^{p,q} = R_{i1} A_{\phi}^{p,q} \\
R_{i2}^{p,q} = R_{i2} A_{\theta}^{p,q} \\
R_{i3}^{p,q} = R_{i3} A_{\phi}^{p,q} \\
R_{i4}^{p,q} = R_{i4} A_{\phi}^{p,q}
\]

for \( i = 1, 2, 3, 4 \) \hspace{1cm} (4-41)

Note that \( R_{i1}^{p,q} = 0 \) if \( p+q \) is odd.

3. The \( M_n, N_n \) Operators

Define the 4x4 matrix \( X^{p,q}_n \) by:

\[
\frac{2}{p+1}\int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v \begin{bmatrix} M_n & -N_n \\ N_n & M_n \end{bmatrix} \begin{bmatrix} \vec{K}_n \\ \vec{L}_n \end{bmatrix} = \sum_{q=0}^{\infty} X^{p,q}_n \begin{bmatrix} \vec{K}_n \\ \vec{L}_n \end{bmatrix} 
\]

\[
\frac{2}{p+1}\int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v M_{n,11} [K_{\phi,n}(s_o)] = \sum_{q=0}^{\infty} X^{p,q}_{n,11} K_{\phi,n}^q
\]

\[
= \sum_{q=0}^{\infty} X^{p,q}_{n,11} K_{\phi,n}^q
\]

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\[ X_{n,11}^p = -\frac{i l_1 l_2}{p+1} \left[ \frac{1}{2} (G_{n-1}^p - G_{n-1}^p + G_{n+1}^p - G_{n+1}^p + 2G_{n-1}^p - G_{n+1}^p + 2G_{n-1}^p - G_{n+1}^p) - \frac{n^2}{l_2^2} (G_{n-1}^p - G_{n+1}^p + 2G_{n-1}^p - G_{n+1}^p) \right] \quad (4-43) \]

\[ \frac{2}{p+1} \int_0^\pi \frac{d\nu}{\pi} \sin \nu \sin(p+1) \nu M_{n,21} [K_{\Phi,n}(z_\nu)] \]

\[ = \sum_{q=0}^\infty \frac{-2n}{p+1} \int_0^\pi \frac{d\nu}{\pi} \sin(p+1) \nu \frac{\partial}{\partial \nu} \int_0^\pi \frac{d\nu}{\pi} \cos \nu q v G_n K_{\Phi,n}^q \]

\[ = \sum_{q=0}^\infty \frac{2n}{p+1} \int_0^\pi \frac{d\nu}{\pi} \cos(p+1) \nu \int_0^\pi \frac{d\nu}{\pi} \cos \nu q v G_n K_{\Phi,n}^q \]

\[ = \sum_{q=0}^\infty X_{n,21}^p K_{\Phi,n}^q \]

and,

\[ X_{n,21}^p = 2n G_{n}^{p+1,q} \quad (4-44) \]

\[ \frac{2}{p+1} \int_0^\pi \frac{d\nu}{\pi} \sin \nu \sin(p+1) \nu N_{n,11} [K_{\Phi,n}(z_\nu)] \]

\[ = \sum_{q=0}^\infty \frac{-i l_1}{p+1} \int_0^\pi \frac{d\nu}{\pi} \sin(p+1) \nu \frac{\partial}{\partial \nu} \int_0^\pi \frac{d\nu}{\pi} \cos \nu q v [G_{n-1} - G_{n+1}] K_{\Phi,n}^q \]

\[ = \sum_{q=0}^\infty X_{n,31}^p K_{\Phi,n}^q \]

and,

\[ X_{n,31}^p = i l_2 [G_{n-1}^{p+1,q} - G_{n+1}^{p+1,q}] \quad (4-45) \]
\[
\frac{2}{p+1} \int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v \ N_{n,21} \left[ K_{\Phi,n}(c_\rho) \right] \\
= \sum_{q=0}^\infty \frac{l_1}{p+1} \int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v \int_0^\pi \frac{dv_\phi}{\pi} \cos qv_\phi \\
\left[ (n-1)G_{n-1} - (n+1)G_{n+1} - \frac{l_2}{2} \frac{\partial}{\partial l_2} (G_{n-1} + G_{n+1}) \right] K_{\Phi,n}^q \\
\sum_{q=0}^\infty X_{n,41}^{p,q} K_{\Phi,n}^q
\]

and,

\[
X_{n,41}^{p,q} = \frac{l_1}{2(p+1)} \left[ (n-1)(G_{n-1}^{p,q} - G_{n-1}^{p+2,q}) - (n+1)(G_{n+1}^{p,q} - G_{n+1}^{p+2,q}) \\
- \frac{l_2}{2} \frac{\partial}{\partial l_2} (G_{n-1}^{p,q} - G_{n-1}^{p+2,q} + G_{n+1}^{p,q} - G_{n+1}^{p+2,q}) \right]
\tag{4-46}
\]

\[
\frac{2}{p+1} \int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v \ M_{n,12} \left[ K_{\zeta,n}(z_\rho) \right] \\
= \sum_{q=0}^\infty \frac{-2n}{p+1} \int_0^\pi \frac{dv}{\pi} \sin v \sin (p+1)v \int_0^\pi \frac{dv_\phi}{\pi} \cos (q+1)v_\phi \ G_n K_{\zeta,n}^q \\
= \sum_{q=0}^\infty X_{n,12}^{p,q} K_{\zeta,n}^q
\]

and,

\[
X_{n,12}^{p,q} = -\frac{n(q+1)}{p+1} \left[ G_{n}^{p,q+1} - G_{n}^{p+2,q+1} \right]
\tag{4-47}
\]
\[
\frac{2}{p+1} \int_0^{\pi} \frac{d\nu}{\pi} \sin \nu \sin (p+1)\nu M_{n,22} [K_{z,n}(\tau_o)]
= \sum_{q=0}^{\infty} \frac{-2i l_1 l_2}{p+1} \left\{ \int_0^{\pi} \frac{d\nu}{\pi} \sin \nu \sin (p+1)\nu \int_0^{\pi} \frac{d\nu_o}{\pi} \sin \nu_o \sin (q+1)\nu_o G_n + \frac{1}{l_1^2} \int_0^{\pi} \frac{d\nu}{\pi} \sin (p+1)\nu \frac{\partial}{\partial \nu} \int_0^{\pi} \frac{d\nu_o}{\pi} \sin (q+1)\nu_o G_n \right\} K_{z,n}^q \]
= \sum_{q=0}^{\infty} X_{n,22}^{p,q} K_{z,n}^q
\]

and,

\[
X_{n,22}^{p,q} = -\frac{il_1 l_2}{2} \left[ \frac{1}{p+1} (G_n^{p,q} + G_n^{p+2,q+2} - G_n^{p,q+2} - G_n^{p+2,q}) - \frac{4(q+1)}{l_1^2} G_n^{p+1,q+1} \right] \tag{4-48}
\]

\[
\frac{2}{p+1} \int_0^{\pi} \frac{d\nu}{\pi} \sin \nu \sin (p+1)\nu N_{n,12} [K_{z,n}(\tau_o)]
= \sum_{q=0}^{\infty} \frac{l_1 l_2}{p+1} \int_0^{\pi} \frac{d\nu}{\pi} \sin \nu \sin (p+1)\nu \int_0^{\pi} \frac{d\nu_o}{\pi} \sin \nu_o \sin (q+1)\nu_o \left[ \frac{\partial}{\partial l_2} G_n \right] K_{z,n}^q \]
= \sum_{q=0}^{\infty} X_{n,32}^{p,q} K_{z,n}^q
\]

and,

\[
X_{n,32}^{p,q} = \frac{l_1 l_2}{4(p+1)} \frac{\partial}{\partial l_2} \left[ G_n^{p,q} + G_n^{p+2,q+2} - G_n^{p,q+2} - G_n^{p+2,q} \right] \tag{4-49}
\]

Note that, for \(n=0\), the expressions simplify to:
\[ X_{o,11}^{p,q} = -\frac{i l_1 l_2}{p+1}[G_{1}^{p,q} - G_{1}^{p+2,q}] \]

\[ X_{o,41}^{p,q} = -\frac{l_1}{p+1}[(G_{1}^{p,q} - G_{1}^{p+2,q}) + \frac{l_2}{2} \frac{\partial}{\partial l_2}(G_{1}^{p,q} - G_{1}^{p+2,q})] \]

\[ X_{o,22}^{p,q} = -\frac{i l_1 l_2}{2} \left[ \frac{1}{p+1}(G_{o}^{p,q} + G_{o}^{p+2,q} - G_{o}^{p,q+2} - G_{o}^{p+2,q}) - \frac{4(q+1)}{l_1^2} G_{o}^{p+1,q+1} \right] \quad (4-50) \]

\[ X_{o,32}^{p,q} = \frac{l_1 l_2}{4(p+1)} \frac{\partial}{\partial l_2}(G_{o}^{p,q} + G_{o}^{p+2,q} - G_{o}^{p,q+2} - G_{o}^{p+2,q}) \]

\[ X_{o,21}^{p,q} = X_{o,31}^{p,q} = X_{o,44}^{p,q} = X_{o,12}^{p,q} = 0 \]

Also note that \( X_{o}^{p,q} = 0 \) if \( p+q \) is odd.

From the symmetry of \( \begin{bmatrix} M_n & -N_n \\ N_n & M_n \end{bmatrix} \), we can deduce that:

\[ X_{n,13}^{p,q} = -X_{n,31}^{p,q} \quad X_{n,14}^{p,q} = -X_{n,32}^{p,q} \]

\[ X_{n,23}^{p,q} = -X_{n,41}^{p,q} \quad X_{n,24}^{p,q} = X_{n,42}^{p,q} = 0 \]

\[ X_{n,33}^{p,q} = X_{n,11}^{p,q} \quad X_{n,34}^{p,q} = X_{n,12}^{p,q} \]

\[ X_{n,43}^{p,q} = X_{n,21}^{p,q} \quad X_{n,44}^{p,q} = X_{n,22}^{p,q} \]

Note that for \( p+q \) odd,

\[ X_{n,11}^{p,q} = X_{n,41}^{p,q} = X_{n,22}^{p,q} = X_{n,32}^{p,q} = X_{n,23}^{p,q} = X_{n,33}^{p,q} = X_{n,14}^{p,q} = X_{n,44}^{p,q} = 0 \quad (4-52) \]

and for \( p+q \) even,

\[ X_{n,21}^{p,q} = X_{n,31}^{p,q} = X_{n,12}^{p,q} = X_{n,13}^{p,q} = X_{n,43}^{p,q} = X_{n,34}^{p,q} = 0 \quad (4-53) \]

The integrodifferential Eq. (4-25) is thus transformed into the infinite system of linear
equations of the unknown coefficients
\[
\begin{bmatrix}
\tilde{K}^q_n \\
\tilde{L}^q_n
\end{bmatrix}
\]

\[
\sum_{q=0}^{\infty} [X_n^{p,q} + R_n^{p,q}] \begin{bmatrix}
\tilde{K}^q_n \\
\tilde{L}^q_n
\end{bmatrix} = 2 \begin{bmatrix}
\tilde{E}^{inc,p}_{\tan,n} \\
\tilde{H}^{inc,p}_{\tan,n}
\end{bmatrix}
\] (4-54)

which is to be solved numerically.

D. RADIATION IN THE FAR FIELD

In the far field, we can write Eq. (4-3) as:

\[
\frac{4\pi}{l_1 l_2} E^{sc}(\vec{r}) = +i \int_{-1}^{1} d\zeta_o \int_{0}^{2\pi} d\phi_o \left[ \tilde{E}(\vec{r}_o) \times \hat{r} \right] G(\vec{r} - \vec{r}_o)
\]

\[
+ i \int_{-1}^{1} d\zeta_o \int_{0}^{2\pi} d\phi_o \left\{ \tilde{K}(\vec{r}_o) - i\tilde{r} \left[ K_0(\vec{r}_o) \sin \theta \sin (\phi - \phi_o) \right. \right.
\]

\[
\left. + K_2(\vec{r}_o) \cos \theta \right\} G(\vec{r} - \vec{r}_o)
\] (4-55)

or equivalently,
\[ \frac{\bar{E}^{sc}(\vec{r})}{G(\vec{r})} = \frac{il_1l_2}{2} \int_{-1}^{1} dz_o \int_{0}^{2\pi} d\phi_o \left\{ \hat{\theta} \left[ K_{\phi} \cos \theta \sin(\phi - \phi_o) - K_z \sin \theta + L_{\phi} \cos(\phi - \phi_o) \right] \\
+ \hat{\phi} \left[ K_{\phi} \cos(\phi - \phi_o) - L_{\phi} \cos \theta \sin(\phi - \phi_o) + L_z \sin \theta \right] \right\} e^{-il_1z_o \cos \theta + l_2z_o \sin \theta \cos(\phi - \phi_o)} \]
\]
\[ = \frac{l_1l_2}{2} \sum_{n=-\infty}^{\infty} (-i)^n e^{in\phi} \sum_{p=0}^{\infty} (-i)^{p+n} \]
\[ \cdot \left\{ \hat{\theta} \left[ \left( iJ_n(l_2 \sin \theta) \frac{ncot \theta}{l_2} K_{\phi,n} - J'_n(l_2 \sin \theta) L_{\phi,n} \right) J_p(l_1 \cos \theta) \\
- \frac{i}{2} J_n(l_2 \sin \theta) \sin \theta K_{\phi,n} \left( J_{p+2}(l_1 \cos \theta) + J_{p+1}(l_1 \cos \theta) \right) \right] \right\} \] (4.56)
\]
\[ \hat{\phi} \left[ \left( J'_n(l_2 \sin \theta) K_{\phi,n} + iJ_n(l_2 \sin \theta) \frac{ncot \theta}{l_2} L_{\phi,n} \right) J_p(l_1 \cos \theta) \\
- \frac{i}{2} J_n(l_2 \sin \theta) \sin \theta L_{\phi,n} \left( J_{p+2}(l_1 \cos \theta) + J_{p+1}(l_1 \cos \theta) \right) \right\} \}
\]

As \( \theta \to 0 \), only the \( n = \pm 1 \) terms are nonzero in this limit, and \( \hat{\theta} = \hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi \), \( \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi \). We have \( \hat{\theta} \pm i\hat{\phi} = (\hat{x} \pm \hat{y}) e^{\pm i\phi} \):

\[ \frac{\bar{E}^{sc}(\vec{r})}{G(\vec{r})} = \frac{l_1l_2}{4} \sum_{p=0}^{\infty} (-i)^p \left\{ \hat{x} \left[ \left( K_{\phi,1}^p - K_{\phi,-1}^p \right) + i \left( L_{\phi,1}^p + L_{\phi,-1}^p \right) \right] J_p(l_1) \right\} \] (4.57)
\]

Similarly, as \( \theta \to \pi \), only the \( n = \pm 1 \) terms are nonzero. \( \hat{\theta} = -\hat{\rho} = -\hat{x} \cos \phi - \hat{y} \sin \phi \), \( \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi \). We have \( \hat{\theta} \pm i\hat{\phi} = -(\hat{x} \pm \hat{y}) e^{\pm i\phi} \):

\[ \frac{\bar{E}^{sc}(\vec{r})}{G(\vec{r})} = \frac{l_1l_2}{4} \sum_{p=0}^{\infty} i^p \left\{ \hat{x} \left[ \left( K_{\phi,1}^p - K_{\phi,-1}^p \right) - i \left( L_{\phi,1}^p + L_{\phi,-1}^p \right) \right] J_p(l_1) \right\} \] (4.58)
\]
E. INSIDE AND OUTSIDE SURFACE CURRENTS

From Eq. (2-20) we have:

\[
\begin{bmatrix}
\bar{K}_n^{+p} - \bar{K}_n^{-p} \\
\bar{L}_n^{+p} - \bar{L}_n^{-p}
\end{bmatrix} = -\begin{bmatrix}
Z^{-1}\Delta & iZ^{-1}\sigma_2 \\
-i\sigma_2(\Delta - \Delta^{-1}\Delta) & -\sigma_2 \Delta Z^{-1}\sigma_2
\end{bmatrix}\begin{bmatrix}
\bar{K}_n^p \\
\bar{L}_n^p
\end{bmatrix}
\]  

(4-59)

Therefore the following matrix equation is obtained:

\[
\begin{bmatrix}
\bar{K}_n^{z_p} \\
\bar{L}_n^{z_p}
\end{bmatrix} = \frac{1}{2}\left\{ I + \begin{bmatrix}
Z^{-1}\Delta & iZ^{-1}\sigma_2 \\
-i\sigma_2(\Delta - \Delta^{-1}\Delta) & -\sigma_2 \Delta Z^{-1}\sigma_2
\end{bmatrix} \right\} \begin{bmatrix}
\bar{K}_n^p \\
\bar{L}_n^p
\end{bmatrix}
\]  

(4-60)
V. COMputation and results

The solution to the problem of the scattering of an anisotropically coated tubular cylinder of finite length as formulated in the previous chapter has been coded in FORTRAN and tested. The program listings are included in the Appendix. Computation has been carried out on the 32-bit Sun SPARC Station running under the Unix operating system in the Electrical and Computer Engineering Department and the Computer Center. The evaluation of the double series expansion coefficients of the Green's function and its derivatives for greater values of $ka$ and $kh$ have also been done on the 64-bit Cray Y-MP EL98 in the Visualization Lab so that the accuracy of the results can be accessed.

The program accepts the geometry of the cylinder, the surface impedance matrices, the incident wave frequency, direction and its polarization in terms of TE, TM or some linear combination of the two. It computes the sum surface currents and the far field radiation in the directions specified, including the strength and the phase of each field component. It also breaks down the sum surface currents into the outside and the inside currents.

In this chapter, some interesting results of computation for the scattering of a tubular cylinder having the length-to-diameter ratio $h/a$ of 4 or 6 are presented. All figures are attached at the end of this chapter. The wave is incident from the positive $z$-axis and is polarized in the $\hat{y}$ direction.

A. Comparison with Experimental Data

For backscattering from a perfectly conducting tubular cylinder of a $y$-polarized plane wave incident from the positive $z$-axis, experimental data are available [6] over a frequency band of well beyond three octaves, with the circumference-to-wavelength ratio $ka = 2\pi a/\lambda$ varied from 0.9448 to 3.3152. These data are measured with two sets of four cylinders each; one set having $h/a = 4$, the other with $h/a = 6$. Both data sets use the inner radii of the tubular cylinders as the parameter $a$ [7]. The cylinder length-to-wavelength ratio $2h/\lambda$ varies from 1.2030 to 4.2210 for the $h/a = 4$ case and from 1.8045 to 6.3315 for the $h/a$
= 6 case.

The experimental data are plotted against the results of theoretical computation. Figure 5-1 shows the backscattering from the set of cylinders having \( h/a = 4 \). Figure 5-2 shows those data from the set of cylinders with \( h/a = 6 \). The output of theoretical computation follows the measured data very closely. Note that the cutoff frequency of the dominant circular waveguide mode, TE\(_{11}\), occurs at \( 2h/\lambda = 2.344 \) for the cylinders with \( h/a = 4 \) and at \( 2h/\lambda = 3.516 \) for those with \( h/a = 6 \).

**B. NULL ON-AXIS BACKSCATTERING**

Three different ways of coating a surface impedance \( Z_s \) on a tubular cylinder of \( h/a = 6 \) are considered: Both on the outside surface and the inside surface; Only on the inside and leaving the outside surface perfectly conducting; Only on the outside and leaving the inside surface perfectly conducting. Here \( Z_s \) has the elements \( z_{s11} = 0.5, z_{s22} \) varies from 0.1 to 5, \( z_{s12} = z_{s21} = 0 \). At a fixed frequency for which \( 2h/\lambda = 3.194 \), slightly below the \( \text{TE}_{11} \) circular waveguide dominant mode cutoff of 3.516, the scattered fields are plotted in Figures 5-3 through 5-5. Figure 5-3 shows the results of computation for the case \( Z^* = Z^- = Z_s = Z \). As \( z_{s22} \) is varied through 2, the backscattering cross section vanishes as predicted in Chapter 3. Figure 5-4 shows the results for the case \( Z^* = 0 \) and \( Z^- = Z_s \). As the impedance on the inside surface is increased, the excited field inside the tubular cylinder is dissipated and the backscattered power decreases exponentially. Figure 5-5 shows the results for the case \( Z^* = Z_s \) and \( Z^- = 0 \). The backscattered power drops off rapidly at first as the impedance on the outside surface is increased. But the cross section quickly settles down to a fixed value presumably due to the current excited on the perfectly conducting inside surface of the cylinder.

Results of computation for the same configurations but at a higher frequency for which \( 2h/\lambda = 4.865 \), above the \( \text{TE}_{11} \) circular waveguide dominant mode cutoff of 3.516, are plotted in Figures 5-6 through 5-8. Now that the incident wave can propagate through the cylinder in the dominant waveguide mode, the backscattering cross sections are about an
order of magnitude smaller than in previous cases. Figure 5-6 shows the results of computation for the case \( Z^+ = Z^- = Z_r = Z \). Again the backscattering cross section vanishes as \( z_{22} \) is varied through 2. Figure 5-7 shows the results for the case \( Z^+ = 0 \) and \( Z^- = Z_r \) while Figure 5-8 shows the results for the case \( Z^+ = Z_r \) and \( Z^- = 0 \). It appears that, above cutoff, the contribution to the backscattering cross section from the inside of the tubular cylinder is minimal: once the outside current is reduced by the increase in surface impedance, the backscattering cross section is reduce by more than 10 dB as shown in Figure 5-8. When the impedance coating is applied in the inside surface, the maximal reduction in the backscattering cross section is only about 1.2 dB.

C. FREQUENCY DEPENDENCE

The axial backscattering of the two cases when the tubular cylinder is coated only on the inside or only on the outside with \( z_{21} = 0.5 \) and \( z_{22} = 2 \) are investigated for different frequencies with \( 2h/\lambda \) varying from 0.1 to 7.5. Figure 5-9 shows the results of the case when only the inside surface is coated so that the backscattering is mainly due to the current excited on the outside surface. The reflection from the ends of the cylinder causes the fluctuation in backscattering cross section. Being waves in free space on the outside of the cylinder, the maxima and minima are evenly spaced with the minima occurring when \( 2h/\lambda \) is a multiple of half integer. Figure 5-10 shows the results when only the outside surface is coated and the current on the inside surface dominates the contribution. The distinct feature in this case is that the backscattering cross section does not fluctuate with varying frequency below the waveguide mode cutoff. The incident wave is able to penetrate deeper into the cylinder with increasing frequency, resulting in a constantly rising strength of the backscattered field. Once beyond the circular waveguide mode cutoff, the wave can pass through the cylinder in the TE_{11} mode and the backscattering diminishes. The oscillation in the cross section at these higher frequencies represents the interference of reflected waves at the ends of the tube and the separation between maxima and minima should be determined by the guide wavelength at the particular frequency. These two situations should be
compared to Figure 5-11 which shows the results when both sides of the cylinder are perfectly conducting and the current can flow freely. The distinct notch in the cross section near the TE_{11} mode cutoff at 2h/\lambda = 3.516 and the subsequent faster variation in the cross section shows the combination of the two distinct features of Figures 5-9 and 5-10.

D. COMPUTATION ACCURACY

The main difficulty encountered in the computation is the evaluation of \( G_n^{p,q}(l_1, l_2) \) and its \( l_2 \)-derivative by double power series sum when \( l_1 \) becomes large. Computations for Figure 5-11, 5-13 and 5-15 use the \( G_n^{p,q} \) values evaluated with the Cray computer which has a 128-bit double precision number. Computations for Figures 5-12, 5-14 and 5-16 use the \( G_n^{p,q} \) values evaluated with the Sun SPARC Station which has a 64-bit double precision number. For \( 2h/\lambda \) greater than about 6.2, the SPARC Station fails to provide accurate results.

Figures 5-13 and 5-15 are axial backscattering from a cylinder of \( h/\lambda = 6 \) coated with the impedances having the elements \( z_{11}^+ = z_{22}^- = 0.5, z_{22}^+ = z_{11}^- = 0.4, z_{12}^+ = z_{21}^- = z_{12}^- = z_{21}^- = 0.3 \). Figure 5-13 shows the co-polarized backscattered field while Figure 5-15 shows the cross-polarized backscattering.
<table>
<thead>
<tr>
<th>Cylinder</th>
<th>$2h(cm)$</th>
<th>$2a^+$</th>
<th>$2a^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.566</td>
<td>0.9525</td>
<td>0.8915</td>
</tr>
<tr>
<td>2</td>
<td>4.796</td>
<td>1.27</td>
<td>1.199</td>
</tr>
<tr>
<td>3</td>
<td>6.064</td>
<td>1.588</td>
<td>1.516</td>
</tr>
<tr>
<td>4</td>
<td>7.396</td>
<td>1.908</td>
<td>1.849</td>
</tr>
</tbody>
</table>

Axial backscattering (perfectly conducting cylinder, $h/a = 4$)

Figure 5-1.
Axial backscattering (perfectly conducting cylinder, \( h/\lambda = 6 \))

<table>
<thead>
<tr>
<th>Cylinder</th>
<th>(2h (cm))</th>
<th>(2a^+)</th>
<th>(2a^-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.394</td>
<td>0.9525</td>
<td>0.8915</td>
</tr>
<tr>
<td>2</td>
<td>7.193</td>
<td>1.270</td>
<td>1.199</td>
</tr>
<tr>
<td>3</td>
<td>9.098</td>
<td>1.588</td>
<td>1.516</td>
</tr>
<tr>
<td>4</td>
<td>11.09</td>
<td>1.905</td>
<td>1.849</td>
</tr>
</tbody>
</table>

Figure 5-2.
$$Z^+ = Z^- = Z_s = \begin{bmatrix} 0.5 & 0 \\ 0 & z_{s22} \end{bmatrix}$$

Axial backscattering ($h/a = 6, \ 2h/\lambda = 3.194$)

Figure 5-3.
\[ Z^* = 0, \quad Z^* = Z_s = \begin{bmatrix} 0.5 & 0 \\ 0 & z_{s22} \end{bmatrix} \]

Axial backscattering \((h/a = 6, \ 2h/\lambda = 3.194)\)

Figure 5-4.
$Z^- = 0, \quad Z^+ = Z_s = \begin{bmatrix} 0.5 & 0 \\ 0 & z_{s22} \end{bmatrix}$

Axial backscattering ($h/a = 6, \ 2h/\lambda = 3.194$)

Figure 5-5.
\[ Z^+ = Z^- = Z_s = \begin{bmatrix}
0.5 & 0 \\
0 & z_{22}
\end{bmatrix} \]

**Axial backscattering** \((h/a = 6, \ 2h/\lambda = 4.865)\)

![Graph with two plots: Cross Section / \pi d^2 and Phase / \pi vs. Z_{422}](image-url)

**Figure 5-6.**
\( Z^+ = 0, \quad Z^- = Z_s = \begin{bmatrix} 0.5 & 0 \\ 0 & z_{s22} \end{bmatrix} \)

Axial backscattering \((h/a = 6, \quad 2h/\lambda = 4.865)\)

Figure 5-7.
\[ Z^* = 0, \quad Z^* = Z_s = \begin{bmatrix} 0.5 & 0 \\ 0 & z_{s22} \end{bmatrix} \]

Axial backscattering \( (h/a = 6, \ 2h/\lambda = 4.865) \)

Figure 5-8.
$Z^* = 0, \quad Z^- = Z_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 2.0 \end{bmatrix}$

Axial backscattering ($h/a = 6$)

Figure 5-9.
\[ Z^+ = 0, \quad Z^+ = Z_s = \begin{bmatrix} 0.5 & 0 \\ 0 & 2.0 \end{bmatrix} \]

Axial backscattering \((h/a = 6)\)

Figure 5-10.
Y-component of axial backscattering (perfectly conducting cylinder, $h/a = 6$, Cray G)

Figure 5-11
Y-component of axial backscattering (perfectly conducting cylinder, $h/a = 6$)

Figure 5-12.
\[
Z^+ = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \quad Z^- = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}
\]

Y-component of axial backscattering \((h/a = 6, \text{Cray G})\)

Figure 5-13.
\[ Z^+ = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \quad Z^- = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.5 \end{bmatrix} \]

Y-component of axial backscattering \((h/a = 6)\)

Figure 5-14.
\[
Z^+ = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \quad Z^- = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}
\]

X-component of axial backscattering \((h/a = 6, \text{ Cray G})\)

Figure 5-15.
\[ Z^+ = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.4 \end{bmatrix}, \quad Z^- = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.5 \end{bmatrix} \]

X-component of axial backscattering \((h/a = 6)\)

Figure 5-16.
VI. CONCLUSIONS

In this thesis, the sum-difference surface current formulation is introduced for solving electromagnetic boundary value problems when impedances are specified on both sides of a surface. For an impedance coated body, the body can be treated as being a surface separating the space into two regions of identical medium. For an exterior problem, the impedance normalized to the medium on the inside surface, $Z^-$, can be chosen arbitrarily; and for an interior problem, that on the outside surface, $Z^+$, can be arbitrary. The choice when $Z^- = -Z^+$ is of particular interest because the integrodifferential equation has only the sum of the equivalent electric surface currents on the outside and the inside surfaces as its unknown to be solved.

This formulation preserves the duality nature of Maxwell’s equations and carries it over into the algebraic form of the integrodifferential operators in the equations for the sum currents. Since a 90° rotation is equivalent to undergoing a duality transform for an incident plane wave, this particular symmetry in the algebraic form of the operators leads to the sufficient conditions that if $Z^+ = Z^- = \pm \sigma_2$, or if $Z^+$ and $Z^-$ are symmetric and $\text{det } Z^+ = \text{det } Z^- = 1$, the on-axis backscattering of an anisotropic impedance coated scatterer having a 90° rotational symmetry will be eliminated. Note that in the symmetric case, $Z^+$ and $Z^-$ may vary with location. This is an extension of Weston's result [4] for which the surface impedance is isotropic.

A FORTRAN program has been written which accepts the geometry of the cylinder, the surface impedance matrices, the incident wave frequency, direction and its polarization in terms of TE, TM or some linear combination of the two. It computes the sum surface currents and the far field radiation in the directions specified, including the strength and the phase of each field component. It also breaks down the sum surface currents into the outside and the inside currents.

The results of computation using this program agree with measured data of
backscattering from conducting tubular cylinders over a frequency band of more than three octaves. For a cylinder coated with surface impedance matrices satisfying the criteria for null on-axis backscattering, the numerical computation also validated the theoretical assertion.

Difficulties have been encountered about the computational accuracy in the evaluation of the double series Chebyshev expansion coefficients of the Green's function $G_n^{p,q}(l_1, l_2)$ and its $l_2$-derivative by double power series sum when the length of the cylinder is large compared to the wavelength: a compiler with a 64-bit double precision number can only handle a cylinder having a length up to about 6.2 wavelengths. Further work to explore the feasibility of asymptotic evaluation of these coefficients is recommended.
APPENDIX  PROGRAM LISTING

A. INCLUDE FILES

------------------------
REAL/TP.INC
C TYPE STATEMENTS FOR REAL AND INTEGERS AND DEFINITIONS OF CONSTANTS.
IMPLICIT DOUBLE PRECISION (A, B, D-H, O-Z)
IMPLICIT INTEGER*4 (I-N)
PARAMETER (PI=3.14159265358979323846264338327D0, PI2=PI+PI,
+ PI4=PI*PI)
PARAMETER (ONE=1.D0, TWO=2.D0, THR=3.D0, HXD=16.D0, ZERO=0.D0,
+ DEGP1=180.D0, EPS8=2.220446049250313D-16)
PARAMETER (ONEDE=ONE, HALF=ONE/TWO, THR=ONE/THR, QUAR=HALF*HALF)
------------------------
CMXPT.INC
C IMPLICIT TYPE STATEMENT AND CONSTANTS FOR COMPLEX NUMBERS.
IMPLICIT DOUBLE COMPLEX (C)
PARAMETER (CZERO=(0.D0, 0.D0), CONE=(1.D0, 0.D0))
PARAMETER (CONED=(-1.D0, 0.D0), CI1=(0.D0, 1.D0), CI2=(0.D0, -1.D0))
------------------------

B. INPUT DATA FILES

------------------------
CLYGEM.PRM
10 Maximum kh (integer)
5 Maximum ka (integer)
8 NREGNS (integer)
------------------------
CYLFLPT.PRM
1024 floating point zero IOBIT (1024 bits)
64 floating point precision IFFBIT (64 bits)
------------------------
INPUTDAT.PRM
.1D-1 DKA0, minimum ka (REAL) in the computation
.875D-2 DNLA, increment of ka (REAL)
40 NSTA, start point (INTEGER) in the computation
479 NEND, end point (INTEGER) in the computation
6.D0 RHA, ratio of h and a (REAL)
1 IE, if incident wave is TE-polarized it is 1, otherwise 0
0 IM, if incident wave is TM-polarized it is 1, otherwise 0
0.D0 THETA1, incident angle (REAL) is limited to 0 to 90 degree
6 NTH1AO, total number (INTEGER) of angle of THATA
3 NPHI, total number (INTEGER) of angle of PHI0 to be computed
0.D0 THETA0, initial theta angle (REAL)
1.D0 DELTHO, increment of theta angle (REAL)
0.DO PHI0, initial theta angle (REAL)
12.D0 DELPHI, increment of phi angle (REAL)
0 IK, set 1 to compute scattering currents on outer and inner surface
------------------------
IMPEDENC.PRM
0 IZ, if perfect conducting, IZ=0; otherwise, IZ=1.
(.5D0, 0.D0) Impedance Z+ (phi phi)
(.4D0, 0.D0) Impedance Z- (phi phi)
(.3D0, 0.D0) Impedance Z+ (phi z)
(.3D0, 0.D0) Impedance Z- (phi z)
(.3D0, 0.D0) Impedance Z+ (z phi)
------------------------
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C. SETUP PROGRAM AND CREATED FILES

PROGRAM SETUP

C******************************************************************************
C NOTES: The format statement 1001 need to be revised if other than
C double precision real numbers are used for DKHMAX and DKAMAX.
C Statement 1001 needs to be revised if KMAX or KAMAX exceeds
C 3 digits.
C The format statement 1002 need to be revised if other than
C double precision real numbers are used for FZERO AND PRECSN.
C Statement 1002 needs to be revised if I0BIT exceeds 6 digits
C or IPFBIT exceeds 4 digits.
C******************************************************************************

C INCLUDE 'REALTP.INC'
C INCLUDE 'CMFPRT.INC'
OPEN (20,FILE='CYLGEOM.PRM',IOSTAT=IOS,STATUS='OLD')
IF (IOS .NE. 0) THEN
WRITE(*,9000)
9000 FORMAT ('Cannot find the file CYLGEOM.PRM containing the ',/,
+ 'maximum values for ka and kh, and the parameter NREGS.' )
STOP
END IF
READ (20,*) KMAX
READ (20,*) KAMAX
READ (20,*) NREGS
CLOSE (20)
OPEN (20,FILE='CYFLPT.PRM',IOSTAT=IOS,STATUS='OLD')
IF (IOS .NE. 0) THEN
WRITE(*,9001)
9001 FORMAT ('Cannot find the file CYFLPT.PRM containing floating',/,
+ ' point zero bit I0BIT and precision IPFBIT.' )
STOP
END IF
READ(20,*) I0BIT
READ(20,*) IPFBIT
CLOSE (20)
OPEN (21,FILE='LIMITS.INC',STATUS='UNKNOWN')
WRITE (21,1001) KMAX, KAMAX
WRITE (21,1002) I0BIT, IPFBIT
CLOSE (21)
1001 FORMAT (6X,'PARAMETER (DKHMAX= ',I3,',D0, DKAMAX= ',I3,')
1002 FORMAT (6X,'PARAMETER (FZERO= ',I6, ',D0, PRECSN= ',I4, ',D0, ',
+ ' IPFBIT= ',I4, ',')
C
C Part 2

DKHMAX=KMAX
DKAMAX=KAMAX
ONEDEL=ONE-EPSS
KQDIM=INT((DKHMAX/PI)*NREGS+ONEDEL)
KNDIM=INT((DKAMAX/TWO)*NREGS+ONEDEL)
KQDIM=MAX(KQDIM,1)
KNDIM=MAX(KNDIM,1)
OPEN (21,FILE='MAINND.INC',STATUS='UNKNOWN')
WRITE (21,1003) NREGS, KNDIM, KQDIM
WRITE (21,1004)
WRITE (21,1005)
WRITE (21,1006)
CLOSE (21)
1003 FORMAT (6X,'PARAMETER (NREGS= ',I2, ', KNDIM= ',I3, ',
+ ', KQDIM= ',I3, ',')

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D. PROGRAM MAIN

PROGRAM MAIN

INCLUDE 'REALTP.INC'
INCLUDE 'CNPXTP.INC'
COMMON /GCONST/ DHR, DKA, HH, HA, HSQ, DASQ, HSNQ, DASQN, HHSQ, DHASQ,
+ RAH, RASHQ, DHA, DAN
COMMON /INPUT1/ DKA0, DINCA, NSTR, NEND, RHA
COMMON /INPUT2/ IE, IM, THETAI, THESINI, THECOSI, RTHEI
COMMON /INPUT4/ IZ, IX, IS, NYSM

CALL CHKINPUT
OPEN (21, FILE='rhzs41.dz', STATUS='UNKNOWN')
DO 8001 IS=NSTR, NEND
  DS=IS
  DKA=DKA0+DINCA*DS
  DHH=DKA*RHA
  CALL MAXDMM
  CALL RAPQMT
  CALL GNPQFN
  CALL BCRSCFL(IR, CESTH, CESPH)
  8001  CONTINUE

END
CALL RCSAREA(CESTH,CESPH)

CONTINUE
CLOSE(21)
STOP
END

E. SUBROUTINE CHKINPUT

SUBROUTINE CHKINPUT

INCLUDE 'REALTP.INC'
INCLUDE 'LIMITS.INC'
COMMON /INPUT1/ DKLO,DKRA,NEND,NEND,RHA
COMMON /INPUT2/ IE,IM,THETAI,THESIN,THECOS,RTHI
COMMON /INPUT3/ RTHI,RDELT,RPHI,RDELP,NHTAO,NPHI,THESIN,THECOS,
               + RHPI
COMMON /INPUT4/ IX,IX,IS,NSM

OPEN(20,FILE='INPUTDAT.PRM',IOMAT=IOST,STATUS='OLD')
IF (IOS.NE.0) THEN
WRITE(*,*) 'Fail to open input file INPUTDAT.PRM'
STOP
END IF

C Input kh and ka. These values are passed to other
C parts of this program through the common block /INPUT1/.
READ(20,* DKLO
READ(20,* DKRA
READ(20,* NEND
READ(20,* RHA

C Check against maximum kh and ka values.
NBTW=NEND-NSTR
DKH0=DKLO*RHA
DKH1=DKRA*NSTW-DKLO
IF (DKH1.GT. DKH0) THEN
   IF (DKH1.GT. DKH0) THEN
      WRITE(*,*) 'Both kh and ka values exceed the maximum allowed.'
      ELSE
      WRITE(*,*) 'The input kh value exceeds the maximum allowed.'
   END IF
   WRITE(*,*) 'The execution is terminated.'
   CLOSE(20)
   STOP
   ELSE IF (DKH1.GT. DKH0) THEN
      WRITE(*,*) 'The input ka value exceeds the maximum value allowed.'
      WRITE(*,*) 'The execution is terminated.'
      CLOSE(20)
      STOP
      END IF
      IF (DIHCH.LT. ZERO) OR (DIHCA.LT. ZERO) THEN
         WRITE(*,*) 'The increment DIHCH or DIHCA is less than zero.'
         WRITE(*,*) 'The execution is terminated.'
         CLOSE(20)
         STOP
      END IF

C Input the incident angle (THETAI) and polarization (TE or TM) of the
C incident wave. The incident angle is limited to 0 to 90
C degrees. SIN(THETAI) and COS(THETAI) are computed also. These
C parameters are passed to other parts of this program through the
C common block /INPUT2/.
READ(20,* IE
READ(20,* IM
READ(20,* THETAI

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C Check the input value of incident wave
  IF ((IE .NE. 1) .AND. (IE .NE. 0)) THEN
    WRITE ('*',') 'Improperly specified the polarization of incident ',
    + 'wave. Program is stopped.'
    STOP
  ELSE IF ((IM .NE. 1) .AND. (IM .NE. 0)) THEN
    WRITE ('*',') 'Improperly specified the polarization of incident ',
    + 'wave. Program is stopped.'
    CLOSE(20)
    STOP
  END IF
  IF ((THETA1 .LT. ZERO) .OR. (THETA1 .GT. 90)) THEN
    WRITE ('*',') 'Improperly specified incident angle. '
    + 'Program is stopped.'
    CLOSE(20)
    STOP
  END IF

C Calculate SIN(THETA1) and COS(THETA1)
  RTHEI=THETA1*PI/DEGPI
  THESIN=SIN(RTHEI)
  THECOSI=COS(RTHEI)

C*****************************************************************************
C Input the angles theta and phi at which the scattered fields are
C to be computed. They are specified in terms of the initial theta
C (THETAO) and phi (PHIO) angles, their respective increments DELTHO
C and DELPHI, and the total numbers of angles NTHTAO and NPHI to be
C computed. Thus NTHTAO and NPHI must be integers greater than 1. If
C either NTHTAO=0 or NPHI=0, no bistatically scattered fields will be
C computed. Note that the scattered electric field components are
C computed for all the phi-angles at a fixed theta, before the
C theta-angle is varied. All angles are specified in degrees. Theta is
C limited to 0 to 180 while phi is limited to 0 to 360 degrees.
C SIN(THETAO) and COS(THETAO) are computed also. These parameters are
C passed to other parts of this program through the common block /INPUT3/.
  READ(20,*), NTHTAO
  READ(20,*), NPHI
  IF ((NTHTAO .LT. 0) .OR. (NPHI .LT. 0)) THEN
    WRITE ('*',') 'Improperly specified number of output angles. ',
    + 'Program is stopped.'
    STOP
  ELSE IF ((NTHTAO .EQ. 0) .OR. (NPHI .EQ. 0)) THEN
    CLOSE(20)
    WRITE ('*',') 'Desired bistatic scattered field direction has not ',
    + 'been (properly) specified, they will not be computed.'
    NTHTAO=0
    NPHI=0
    THETAO=ZERO
    DELTHO=ZERO
    PHIO=ZERO
    DELPHI=ZERO
  ELSE
    READ(20,*), THETAO
    READ(20,*), DELTHO
    READ(20,*), PHIA
    READ(20,*), DELPHI
  END IF

C*****************************************************************************
C Check input values.
  IF ((DKH0 .LE. ZERO) .OR. (DKH1 .LE. ZERO)) THEN
    WRITE ('*',') 'Invalid kh, program is stopped.'
    STOP
  END IF
  IF ((DKA0 .LE. ZERO) .OR. (DKA1 .LE. ZERO)) THEN
    WRITE ('*',') 'Invalid ka, program is stopped.'
    STOP
  END IF

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IF ((DKA0 .GT. DKH0) .OR. (DKA1 .GT. DKH1)) THEN
WRITE(*,*) 'ka/kh > 1, program is stopped.'
END IF
C Output angle checking not required:
IF (NYHTAO .EQ. 0) GO TO 200
C Checking output angles:
IF ((THETAO .LT. ZERO) .OR. (THETAO .GT. DEGPI)) THEN
WRITE(*,*) 'The first output theta-angle lies outside the 0 to ', + '180 degrees range. Program is stopped.'
END IF
THTAI=THETAO-(NTHTAO-1)*DELTHO
IF ((THTAIF .LT. ZERO) .OR. (THTAIF .GT. DEGPI)) THEN
WRITE(*,*) 'Some of the specified output theta-angles lie', + 'outside the 0 to 180 degrees range. Program is stopped.'
END IF
PHIHX=TW0*DEGPI
IF ((PHIO .LT. ZERO) .OR. (PHIO .GT. PHIMX)) THEN
WRITE(*,*) 'The first output phi-angle lies outside the 0 to ', + '360 degrees range. Program is stopped.'
END IF
END IF
PHIF=PHIO+(NPHI-1)*DELPHI
IF ((PHIF .LT. ZERO) .OR. (PHIF .GT. TWO*DEGPI)) THEN
WRITE(*,*) 'Some of the output phi-angles lie outside', + 'the 0 to 360 degrees range. Program is stopped.'
END IF
CONTINUE
END IF

F. SUBROUTINE MAXODM

SUBROUTINE MAXODM

C******************************************************************************
C INCLUDE 'REALTP.INC'
C INCLUDE 'MAINDM.INC'
C COMMON /CRVTDM/ NMAX, MXNG, IQMAX, IQMA1, IQMAX2, IXCRNT, ICRNT, MXQBG, + MXQOG
C COMMON /CCONST/ DKH, DKHA, HA, HSQ, DA, DASQ, DSQ, HH, HDSQ, HDAQ,
+ RAH, RAASQ, DHA, DHA
C COMMON /RTTHETA/ DL1COSI, DL1SINI, DLL
C COMMON /INPUT1/ DKA0, DKNAG, NSTR, NEND, NHA, RHA
C COMMON /INPUT2/ IE, IM, THETAI, THESIN, THECOI, RTHEI
C COMMON /INPUT3/ RTHE, RDELT, RPHI, RDELP, NTHTAO, NPHI, THESES, THECOS,
+ RPHI
C ONEDEL=ONE-EPS8
IQMAX=INT((DKH/PI)*NREGNS+ONEDEL)
IQMAX=MAX(IQMAX,1)
IQMA1=IQMAX+1
IQMAX2=IQMA1+2
IXCRNT=2*IQMAX1
ICRNT=4*IQMAX1
MXQBG=IQMAX/2+1
MXQOG=IQMAX/2+1

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G. SUBROUTINE RAPQMT

SUBROUTINE RAPQMT

INCLUDE 'REALTP.INC'
INCLUDE 'CMPPXP.INC'
INCLUDE 'MAINMD.INC'
DIMENSION C0(2,2), C1R(2,2)
DIMENSION CZSUM(2,2), CZDIIF(2,2), CRI(4,4), CRC2(4,4), CRC3(4,4)
+ CRPQ1(KCRNT,KCRNT), CRPQ2(KCRNT,KCRNT)
+ CRPQ31(KCRNT,KCRNT), CRPQ32(KCRNT,KCRNT)
COMMON /INPUT2/ IE, IM, THETAI, THESINI, THECOSI, RTHEI
COMMON /INPUT4/ IZ, IZ1, IS, NYSM
COMMON /XPQTMP/ CRPQ1, CRPQ2
COMMON /XPQTMP2/ CRPQ31, CRPQ32
COMMON /CRNTM/ NMAX, MXNG, MAX, MAX1, MAX2, IXCRNT, ICRNT, MXQOG,
+ MXQOG

OPEN (20, FILE='IMPEDANCE.PRM', IOSTAT=IOS, STATUS='OLD')
IF (IOS .NE. 0) THEN
WRITE(*,9000)
9000 FORMAT ('Cannot find the file IMPEDANCE.PRM containing the '/,
+ 'impedances of the inner and outer surfaces.')
STOP
END IF
C
READ(20,*) IZ
IF (((IZ .NE. 0) .AND. (IZ .NE. 1)) THEN
WRITE (*,*) 'Improperly specified the impedance of cylinder ','
+ 'surface. program is stopped.'
CLOSE(20)
STOP
END IF
IF (IZ .EQ. 0) THEN
CLOSE(20)
RETURN
END IF
C Set up the matrix when Z and Delta are diagonal, or not diagonal but n <= 0.
C Initialize the matrix CRPQ1
DO 200 I=1,KCRNT

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DO 100 J=1,KCRNT
CRPQ1(I,J)=CZERO
CRPQ2(I,J)=CZERO
CRPQ3(I,J)=CZERO
CRPQ32(I,J)=CZERO
100 CONTINUE
200 CONTINUE
DO 400 I=1,2
DO 300 J=1,2
READ (20,*) CO(I,J)
READ (20,*) CIR(I,J)
CZSUM(I,J)=(CO(I,J)+CIR(I,J))*HALF
CZDIF(I,J)=(CO(I,J)-CIR(I,J))*HALF
300 CONTINUE
400 CONTINUE
CLOSE (20)
CZ11=CZSUM(1,1)
CZ12=CZSUM(1,2)
CZ21=CZSUM(2,1)
CZ22=CZSUM(2,2)
CRDET=CZ11*CZ22-CZ12*CZ21
CD11=CZDIF(1,1)
CD12=CZDIF(1,2)
CD21=CZDIF(2,1)
CD22=CZDIF(2,2)
IF (CRDET .EQ. CZERO) THEN
WRITE (*,9001)
STOP
END IF
C Check the diagonalization of the impedance matrixes
NSYM=1
IF (CZ12 .NE. CZERO) THEN
NSYM=0
ELSE IF (CZ21 .NE. CZERO) THEN
NSYM=0
ELSE IF (CD12 .NE. CZERO) THEN
NSYM=0
ELSE IF (CD21 .NE. CZERO) THEN
NSYM=0
END IF
CRDTZ=CRNE/CRDET
CR1(1,1)=CZ11-(CD11*CD11+CD12*CD21+CD21*CD12)*CRDTZ
+CD12*CD21*CRDTZ
CR1(1,2)=CZ12-(CD11+CD22)*CD12*CD21*CD12*CD22*CD21*CD12
+CD12*CD22*CR11*CRDTZ
CR1(2,1)=CZ21-(CD11*CD12+CD22)*CD22*CD21*CD12*CD22*CD21*CD12
+CD22*CD21*CRDTZ
CR1(2,2)=CZ22-(CD11*CD12+CD22)*CD22*CD21*CD12*CD22*CD21*CD12
+CD22*CD21*CRDTZ
CR1(1,3)=(CD12*CD11+CD12*CD12)*CRDTZ
CR1(1,4)=(CD12*CD21*CD12*CD22)*CRDTZ
CR1(2,3)=(CD12*CD21*CD12*CD22)*CRDTZ
CR1(2,4)=(CD12*CD11+CD12*CD12)*CRDTZ
CR1(3,1)=(CD11*CD22+CD11*CD21)*CRDTZ
CR1(3,2)=(CD12*CD22+CD12*CD21)*CRDTZ
CR1(4,1)=(CD11*CD22+CD12*CD21)*CRDTZ
CR1(4,2)=(CD12*CD11+CD22*CD21)*CRDTZ
CR1(3,3)=CZ11*CRDTZ
CR1(3,4)=CZ21*CRDTZ
CR1(4,3)=CZ12*CRDTZ
CR1(4,4)=CZ22*CRDTZ
C
DO 1300 IQE=0,MXQSG
IQE1=IQE+1
IQE=2*IQE
DQ=IQ
DQ1=IQ+1
IQK1=4*IQ+1


DO 1100 IPE=0, MXQEG
IPE1=IPE+1
IP=2.*IP
IPX1=4.*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP=IP
DP1=IP+1

DBPHI=PI*SQ*(DP+DQ1)*(DP1-DQ)
BPHI=4.*DO/DBPHI
DBZ=PI*SQ*(DP+DQ1)*(DP1-DQ)*(DP1+DQ1+ONE)*(DP-DQ1)
BZ=-8.*DO*DQ1/DBZ

CRPQ1(IPX1, IQX1)=CR1(1,1)*BPHI
CRPQ1(IPX2, IQX1)=CR1(2,1)*BPHI
CRPQ1(IPX3, IQX1)=CR1(3,1)*BPHI
CRPQ1(IPX4, IQX1)=CR1(4,1)*BPHI
CRPQ1(IPX1, IQX2)=CR1(1,2)*BZ
CRPQ1(IPX2, IQX2)=CR1(2,2)*BZ
CRPQ1(IPX3, IQX2)=CR1(3,2)*BZ
CRPQ1(IPX4, IQX2)=CR1(4,2)*BZ

CRPQ1(IPX1, IQX3)=CR1(1,3)*BPHI
CRPQ1(IPX2, IQX3)=CR1(2,3)*BPHI
CRPQ1(IPX3, IQX3)=CR1(3,3)*BPHI
CRPQ1(IPX4, IQX3)=CR1(4,3)*BPHI

CONTINUE
CONTINUE
DO 2300 IQO=0, MXQ0G
IQO1=IQO+1
IQ=2.*IQO+1
IQX1=4.*IQO+1
IQX2=IQX1+1
IQX3=IQX2+1
IQX4=IQX3+1
DO=IQ

DO 2200 IPO=0, MXQ0G
IPO1=IPO+1
IP=2.*IPO+1
IPX1=4.*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP=IP
DP1=IP+1

DBPHI=PI*SQ*(DP+DQ1)*(DP1-DQ)
BPHI=4.*DO/DBPHI
DBZ=PI*SQ*(DP+DQ1)*(DP1-DQ)*(DP1+DQ1+ONE)*(DP-DQ1)
BZ=-8.*DO*DQ1/DBZ

CRPQ1(IPX1, IQX1)=CR1(1,1)*BPHI
CRPQ1(IPX2, IQX1)=CR1(2,1)*BPHI
CRPQ1(IPX3, IQX1)=CR1(3,1)*BPHI
CRPQ1(IPX4, IQX1)=CR1(4,1)*BPHI
CRPQ1(IPX1, IQX2)=CR1(1,2)*BZ
CRPQ1(IPX2, IQX2)=CR1(2,2)*BZ
CRPQ1(IPX3, IQX2)=CR1(3,2)*BZ
CRPQ1(IPX4, IQX2)=CR1(4,2)*BZ

CRPQ1(IPX1, IQX3)=CR1(1,3)*BPHI
CRPQ1(IPX2, IQX3)=CR1(2,3)*BPHI
CRPQ1(IPX3, IQX3)=CR1(3,3)*BPHI
CRPQ1(IPX4, IQX3)=CR1(4,3)*BPHI

CONTINUE
CONTINUE
CRPQ1(IPX1, IQX4) = CR1(1, 4)*BZ
CRPQ1(IPX2, IQX4) = CR1(2, 4)*BZ
CRPQ1(IPX3, IQX4) = CR1(3, 4)*BZ
CRPQ1(IPX4, IQX4) = CR1(4, 4)*BZ

2200 CONTINUE
2300 CONTINUE

C Set up the matrix Rpq utilized when Z or Delta is nor diagonal, and n < 0.
CR2(1, 2) = -CR1(1, 2)
CR2(2, 1) = -CR1(2, 1)
CR2(1, 3) = -CR1(1, 3)
CR2(3, 1) = -CR1(3, 1)
CR2(2, 4) = -CR1(2, 4)
CR2(4, 2) = -CR1(4, 2)
CR2(3, 4) = -CR1(3, 4)
CR2(4, 3) = -CR1(4, 3)
CR2(1, 1) = CR1(1, 1)
CR2(2, 2) = CR1(2, 2)
CR2(1, 4) = CR1(1, 4)
CR2(4, 1) = CR1(4, 1)
CR2(3, 3) = CR1(3, 3)
CR2(4, 4) = CR1(4, 4)

DO 3300 IQE=0, MXQEG
IQE1=IQE+1
IQ=E=IQE
DQ=IQ
DO1=IQ+1
IQX1=4*IQ+1
IQX2=IQX1+1
IQX3=IQX2+1
IQX4=IQX3+1
DO 3100 IPE=0, MXQEG
IPE1=IPE+1
IPE=E=IPE
IP=2*IPE
IPX1=6*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP=IP
DPIP=IP-1
DBPHI=PSIQ*(DP-DQ1)*(DQ-DQ)
DBPHI=4.0/DBPHI
DBZ=PSIQ*(DP-DQ1)*(DQ-DQ)*((DQ1+ONE)*(DQ-DQ1))
BZ=-8.0*DBQ1/DBZ
CRPQ2(IPX1, IQX1) = CR2(1, 1)*BPHI
CRPQ2(IPX2, IQX1) = CR2(2, 1)*BPHI
CRPQ2(IPX3, IQX1) = CR2(3, 1)*BPHI
CRPQ2(IPX4, IQX1) = CR2(4, 1)*BPHI
CRPQ2(IPX1, IQX2) = CR2(1, 2)*BZ
CRPQ2(IPX2, IQX2) = CR2(2, 2)*BZ
CRPQ2(IPX3, IQX2) = CR2(3, 2)*BZ
CRPQ2(IPX4, IQX2) = CR2(4, 2)*BZ
CRPQ2(IPX1, IQX3) = CR2(1, 3)*BPHI
CRPQ2(IPX2, IQX3) = CR2(2, 3)*BPHI
CRPQ2(IPX3, IQX3) = CR2(3, 3)*BPHI
CRPQ2(IPX4, IQX3) = CR2(4, 3)*BPHI
CRPQ2(IPX1, IQX4) = CR2(1, 4)*BZ
CRPQ2(IPX2, IQX4) = CR2(2, 4)*BZ
CRPQ2(IPX3, IQX4) = CR2(3, 4)*BZ
CRPQ2(IPX4, IQX4) = CR2(4, 4)*BZ

3100 CONTINUE
3300 CONTINUE

DO 4300 IQ=0, MXQOG
IQQ=IQ+1
IQ=E=IQQ
IQX1=6*IQ+1
IQX2=IQX1+1

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IQX3=IQX2+1
IQX4=IQX3+1
DQ=IQ
DQ1=IQ+1
DO 4200 IP=0, NQ=QG
IP1=IP+1
IP=2*IP+1
IPX1=4*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP=IP
DP1=IP+1
DBPHI=PI*Q*(DP+DQ1)*(DP1-DQ)
DBPHI=4*DBPHI
DBZ=PI*Q*(DP+DQ1)*(DP1-DQ)*(DP1+DQ1+ON)*DQ1-DQ)
BZ=-8*Q/DQ1
CRPQ2(IPX1, IQX1)=CR2(1.1)*BPHI
CRPQ2(IPX2, IQX1)=CR2(2.1)*BPHI
CRPQ2(IPX3, IQX1)=CR2(3.1)*BPHI
CRPQ2(IPX4, IQX1)=CR2(4.1)*BPHI
CRPQ2(IPX1, IQX2)=CR2(1.2)*BZ
CRPQ2(IPX2, IQX2)=CR2(2.2)*BZ
CRPQ2(IPX3, IQX2)=CR2(3.2)*BZ
CRPQ2(IPX4, IQX2)=CR2(4.2)*BZ
CRPQ2(IPX1, IQX3)=CR2(1.3)*BPHI
CRPQ2(IPX2, IQX3)=CR2(2.3)*BPHI
CRPQ2(IPX3, IQX3)=CR2(3.3)*BPHI
CRPQ2(IPX4, IQX3)=CR2(4.3)*BPHI
CRPQ2(IPX1, IQX4)=CR2(1.4)*BZ
CRPQ2(IPX2, IQX4)=CR2(2.4)*BZ
CRPQ2(IPX3, IQX4)=CR2(3.4)*BZ
CRPQ2(IPX4, IQX4)=CR2(4.4)*BZ
4200 CONTINUE
4300 CONTINUE
C Prepare a matrix for computing the scattering currents on the outer
C and the inner surfaces
IF (IK .NE. 1) THEN
RETURN
END IF
CR3(1,1)=CR1(4,1)
CR3(1,2)=CR1(4,2)
CR3(1,3)=CR1(4,3)
CR3(1,4)=CR1(4,4)
CR3(2,1)=CR1(2,1)
CR3(2,2)=CR1(2,2)
CR3(2,3)=CR1(2,3)
CR3(2,4)=CR1(2,4)
CR3(3,1)=CR1(3,1)
CR3(3,2)=CR1(3,2)
CR3(3,3)=CR1(3,3)
CR3(3,4)=CR1(3,4)
CR3(4,1)=CR1(1,1)
CR3(4,2)=CR1(1,2)
CR3(4,3)=CR1(1,3)
CR3(4,4)=CR1(1,4)
C
DO 5200 IQE=0, MXQEG
IQE1=IQE+1
IQ=2*IQE
DQ=IQ
DQ1=IQ+1
IQX1=4*IQ+1
IQX2=IQX1+1
IQX3=IQX2+1
IQX4=IQX3+1
DO 5100 IPF=0, MXQEG
IPF1=IPF+1
C
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IP=2*IP
IPX1=4*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP=IP
DPI=IP+1
DBPHI=PI*Q*(DP+DQ1)*(DP1-DQ)
BPHI=4./DBPHI
DB2=PI*Q*(DP+DQ1)*(DP1-DQ)*(DP1+DQ1+ONE)*(DP-DQ)
BZ=-5.*DQ1/DB2
C
CRPQ31(IPX1, IQX1)=(CON+CR3(1,1))*BPHI
CRPQ31(IPX2, IQX1)=CR3(2,1)*BPHI
CRPQ31(IPX3, IQX1)=CR3(3,1)*BPHI
CRPQ31(IPX4, IQX1)=CR3(4,1)*BPHI
CRPQ31(IPX1, IQX2)=CR3(1,2)*BZ
CRPQ31(IPX2, IQX2)=(CON+CR3(2,2))*BZ
CRPQ31(IPX3, IQX2)=CR3(3,2)*BZ
CRPQ31(IPX4, IQX2)=CR3(4,2)*BZ
CRPQ31(IPX1, IQX3)=CR3(1,3)*BPHI
CRPQ31(IPX2, IQX3)=CR3(2,3)*BPHI
CRPQ31(IPX3, IQX3)=(CON+CR3(3,3))*BPHI
CRPQ31(IPX4, IQX3)=CR3(4,3)*BPHI
CRPQ31(IPX1, IQX4)=CR3(1,4)*BZ
CRPQ31(IPX2, IQX4)=CR3(2,4)*BZ
CRPQ31(IPX3, IQX4)=CR3(3,4)*BZ
CRPQ31(IPX4, IQX4)=(CON+CR3(4,4))*BZ

5100 CONTINUE
5200 CONTINUE

C
DO 6200 IQO=0, MXQOG
IQO=IQ+1
IQ=2*IQ+1
DQ=IQ
DQ1=IQ+1
IQX1=4*IQ+1
IQX2=IQX1+1
IQX3=IQX2+1
IQX4=IQX3+1
DO 6100 IPO=0, MXQOG
IPE1=IPO+1
IPO=2*IPO+1
IPX1=4*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP=IP
DPI=IP+1
DBPHI=PI*Q*(DP+DQ1)*(DP1-DQ)
BPHI=4./DBPHI
DB2=PI*Q*(DP+DQ1)*(DP1-DQ)*(DP1+DQ1+ONE)*(DP-DQ)

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H. SUBROUTINE GNPQFN

SUBROUTINE GNPQFN

C*******************************************************************************
INCLUDE 'REALTP.INC'
INCLUDE 'MAINDM.INC'
COMMON /GCONST/ DKH, DKA, HH, HA, HSQ, DASQ, HSQN, DASQN, HHSQ, HDAQ,
+ RAH, RASHQ, DHA, DAH
CHARACTER FILEEVEN1*12, FILEODD1*12
C Set up the file names
NC1=INT(DKA)
DC=NC1
NC=100000.*DKH+NC1
DC1=DKA-DC
NC2=INT(1000.*DC1)
C Set up the indices of file names of Green's function, and check whether
C these files exist. If these files exist, then use it directly. Otherwise,
C call subroutine XPQ2NL1.
FILEEVEN1='E'
FILEODD1='O'
IGE=8*2*(MAXPFG-1)*(MAXPFG+2)
IGO=8*2*(MAXPFG-1)*(MAXPFG+2)
WRITE (FILEEVEN1(2:8),'('I7.7')') NC
WRITE (FILEODD1(2:8),'('I7.7')') NC
WRITE (FILEEVEN1(10:12),'('I3.3')') NC2
WRITE (FILEODD1(10:12),'(I3.3)') NC2
OPEN (28,ACCESS='DIRECT',FILE=FILEEVEN1,RECL=1GE,IOSTAT=IOS, + STATUS='OLD')
   IF (IOS .NE. 0) THEN
     CLOSE (28)
     CALL XPQINI1(DKH,DKA)
   ELSE
     OPEN (29,ACCESS='DIRECT',FILE=FILEODD1,RECL=1GO,IOSTAT=IOS, + STATUS='OLD')
     IF (IOS .NE. 0) THEN
       CLOSE (29)
       CALL XPQINI1(DKH,DKA)
     ELSE
       CLOSE (28)
       CLOSE (29)
       CALL XPQINI(DKH,DKA)
     END IF
   END IF
RETURN
END

C***************************************************************
C SUBROUTINE XPQINI(DKHIN,DKAIN)
C***************************************************************
INCLUDE 'REALTP.INC'
INCLUDE 'CMPXTP.INC'
INCLUDE 'MAININD.INC'
DIMENSION GNE14,(MAXPEG+1)*(MAXPEG+2)/2),
     GNO(4,(MAXP0G+1)*(MAXPEG-2)/2),
     CNGNE(0:MAXPEG,0:MAXPEG,KNDIM1+1),
     CNGNO(0:MAXP0G,0:MAXPEG,KNDIM1+1),
     CDGNE(0:MAXPEG,0:MAXPEG,KNDIM1+1),
     CDGNO(0:MAXP0G,0:MAXPEG,KNDIM1+1)
COMMON /CRNDM/ NMAX,NMIN,IXMAX,IXMAX2,IXCRNT,ICRNT,MXSEQ,
                  MXQSG
COMMON /GPQTPMP/ CNGE,CDGNE,CNGO,CDGNO
SAVE /GPQTPMP/
C
CHARACTER FILEEVEN1*12, FILEODD1*12
DKH=DKHIN
DKA=DKAIN

C Initialize the matrix CNGE, CNGO, CDGNE, CDGNO
DO 105 IC=1,KNDIM1+1
   DO 102 IB=0,MXPEG
      DO 101 IA=0,MXPEG
         CGNE(IA,IB,IC)=ZERO
      CDGNE(IA,IB,IC)=ZERO
  101     CONTINUE
  102     CONTINUE
  104     CONTINUE
  105     CONTINUE
C
C Set up the file name
NC1=INT(DKAIN)
NC=100000.*DKH-NC1
DC1=DKA-DC
NC2=INT(1000.*DC1)
FILEEVEN1='E'
FILEODD1='O'
IGE=8*2*(MAXPEG+1)*(MAXPEG+2)
IGO=8*2*(MAXPEG+1)*(MAXPEG+2)
WRITE (FILEEVEN1(2:8),'(7I7)') NC
WRITE (FILEODD1(2, 8), '(I7.7)') NC
WRITE (FILEEVEN1(10, 12), '(I3.3)') NC2
WRITE (FILEODD1(10, 12), '(I3.3)') NC2
OPEN (28, ACCESS='DIRECT', FILE=FILEEVEN1, RECL=IGE, IOSTAT=IOS,
+ STATUS='OLD')
C
OPEN (29, ACCESS='DIRECT', FILE=FILEODD1, RECL=IGO, IOSTAT=IOS,
+ STATUS='OLD')

C
DO 900 NI=1, MXNG+1
C
C The following values N, DN and DNH are passed to the G-computation
C related subroutines through the common block /NCONST/.

READ (28, REC=NI) GNE
IRECE=0
DO 500 IQE=0, MXQE
DO 300 IPE=0, IQE-1
CGNE(IPE, IQE, NI)=CGNE(IQE, IPE, NI)
CDGNE(IPE, IQE, NI)=CDGNE(IQE, IPE, NI)
300 CONTINUE
DO 400 IPE=IQE, MXQE
IRECE=IRECE+1
GR=GNE(1, IRECE)
GI=GNE(2, IRECE)
GDR=GNE(3, IRECE)
GDI=GNE(4, IRECE)
CGNE(IPE, IQE, NI)=DCMPLX(GR, GI)
CDGNE(IPE, IQE, NI)=DCMPLX(GDR, GDI)
400 CONTINUE
500 CONTINUE
READ (29, REC=NI) GNO
IRECO=0
DO 800 IQO=0, MXQO
DO 600 IPO=0, IQO-1
CGNO(IPO, IQO, NI)=CGNO(IQO, IPO, NI)
CDGNO(IPO, IQO, NI)=CDGNO(IQO, IPO, NI)
600 CONTINUE
DO 700 IPO=IQO, MXQO
IRECO=IRECO+1
GR=GNO(1, IRECO)
GI=GNO(2, IRECO)
GDR=GNO(3, IRECO)
GDI=GNO(4, IRECO)
CGNO(IPO, IQO, NI)=DCMPLX(GR, GI)
CDGNO(IPO, IQO, NI)=DCMPLX(GDR, GDI)
700 CONTINUE
800 CONTINUE
900 CONTINUE
CLOSE (28)
CLOSE (29)
RETURN
END

C
SUBROUTINE XPQIN1(DKHN, DKAIR)
C
INCLUDE 'REALTP.INC'
INCLUDE 'CMPXTP.INC'
INCLUDE 'MAINMD.INC'
INCLUDE 'GPQMD.INC'
INCLUDE 'LIMITS.INC'
DIMENSION GNE(4, (MAXPEG+1)*(MAXPEG+2)/2),
+ GNO(4, (MAXPOG+1)*(MAXPOG+2)/2)
DIMENSION CGNE(0:MAXPEG, 0:MAXPEG, KNDIM1+1),
+ CGNO(0:MAXPOG, 0:MAXPOG, KNDIM1+1)
DIMENSION CDGNE(0:MAXPEG, 0:MAXPEG, KNDIM1+1),
+ CDGNO(0:MAXPOG, 0:MAXPOG, KNDIM1+1)
COMMON /CRNTDM/ NMX, MXNG, IQMAX, IQMAX1, IQMAX2, IXCRNT, ICNRT, MXQEG,
+ MXQOG

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COMMON /GCONST/ DKH, DKA, HH, HA, HSQ, DASQ, HSQN, DASQN, HSHS, HDASQ,
+RAH, RASHQ, DHA, DAH
COMMON /SUMLMT/ DHMAX, DAMAX, DSNQ, IHMAX, IAMAX, ISNG
COMMON /NCONST/ DN, DHH, N
COMMON /GPTMP/ CGNE, CDGNE, CGNO, CDGNO
CHARACTER FILEEVEN1*12, FILEODD1*12
SAVE /GPTMP/, /GCONST/, /SUMLMT/

C
DKH=DKHIN
DKA=DKAIN
C Initialize the matrix CGNE, CGNO, CDGNE, CDGNO
DO 105 IC= 1,XNDIM1+1
   DO 102 IB= 0,MXPEG
      DO 101 IA= 0.MXPPEG
         CGNE(IA,IB,IC)=CZERO
         CDGNE(IA,IB,IC)=CZERO
      101 CONTINUE
      CONTINUE
      DO 104 ID=0,MAXPOG
         DO 103 IB=0,MAXPOG
            CGNO(IE,ID,IC)=CZERO
            CDGNO(IE,ID,IC)=CZERO
         103 CONTINUE
      104 CONTINUE
   105 CONTINUE
C Determine the maximum number of terms for r- and m- series sums and
C pass them through /SUMLMT/:
C
   REFC=PO2*LOG(TWO)-LOG(PI2)-ONE
   REFH=HALF*(REFC-LOG(DKH))
   REFA=HALF*(REFC-LOG(DKA))
C
   IQMX2=IQMAX+2
   DHMAX=IQMX2
   DO 1100 WHILE ((DHMAX+HALF)*(LOG(DHMAX/DKH)+CNEN) .LT. REFH)
      DHMAX=DHMAX+ONE
   1100 CONTINUE
   IHMAX=INT(DHMAX)
C
   DAMAX=AINT(DKA)+ONE
   DO 1200 WHILE ((DAMAX+HALF)*(LOG(DAMAX/DKA)+CNEN) .LT. REFA)
      DAMAX=DAMAX+ONE
   1200 CONTINUE
C
   IAMAX=INT(DAMAX)
C
   ISNG=INT(TWO*DAMAX-ONE-EPS8)
   ISNG=MAX(IPFBT.ISNG.IQMX2)
C
C Checking dimensions.
   IF(IHMAX .GT. XMREG) THEN
      WRITE('(*,*) 'Warning: IHMAX = ',IHMAX,' > XMREG = ',XMREG
      WRITE('(*,*) 'IHMAX IS SET TO XMREG IN XPINI1'
      IHMAX=XMREG
   END IF
   ISN1=MXMSNG-N-1
   IF(ISNG .GT. ISN1) THEN
      WRITE('(*,*) 'ISNG = ',ISNG,' > MXMSNG-N-1 = ',ISN1
      WRITE('(*,*) 'ISNG IS REDUCED TO MXMSNG-N-1 IN XPINI1'
      ISNG=ISN1
   END IF
C
   DSNQ=ISNG
C
FILEEVEN1= 'E 
FILEODD1= 'O
NC1=AINT(DKA)
DC=NC1

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NC=DKX*100000.+NC1
DC1=DKA-DCC
NC2=IMT(1000.+DC1)
IGE=8*2*(MAXPFG-1)*(MAXPFG+2)
IGO=8*2*(MAXPFG-1)*(MAXPFG+2)
WRITE (FILEVEN1(2:8),'(17.7') NC
WRITE (FSIZED1(2:8),'(17.7') NC
WRITE (FILEVEN1(10:12),'(13.3') NC2
WRITE (FSIZED1(10:12),'(13.3') NC2
OPEN (28,ACCESS='DIRECT',FILE=FILEVEN1,RECL=IGE,STATUS='UNKNOWN')
OPEN (29,ACCESS='DIRECT',FILE=FSIZED1,RECL=IGO,STATUS='UNKNOWN')

C Initialize CGNE, CDGNEN, CGNO and CDNO
DO 1900 NI=1,NOGN+1
C The following values N, LN and DNN are passed to the G-computation
C related subroutines through the common block /NCONST/.
N=NI-1
LN=N
DNN=DN+HALF
IRECE=0
DO 1500 IQE=0,MXQBG
IQ=2*IQE
DO 1300 IPE=0,IQE-1
IP=2*IPE
CGNE(IPE,IQE,NI)=CGNE(IQE,IPE,NI)
CDGNEN(IPE,IQE,NI)=CDGNEN(IQE,IPE,NI)
1300 CONTINUE
DO 1400 IPE=IQE,MXQBG
IRECE=IRECE+1
IP=2*IPE
CALL GDN(IP, IQ, GI, GDI, GR, GDR)
GNE(1, IRECE)=GR
GNE(2, IRECE)=GI
GNE(3, IRECE)=GDR
GNE(4, IRECE)=GDI
CGNE(IPE, IQE, NI)=DCMPLX (GR, GI)
CDGNEN(IPE, IQE, NI)=DCMPLX (GDR, GDI)
1400 CONTINUE
1500 CONTINUE
WRITE (28,REC=NI) GNE
IRECO=0
DO 1800 IQO=0,MXQOO
IQ=2*IQO-1
DO 1600 IPO=0,IQO-1
IP=2*IPO-1
CGNO(IPO, IQO, NI)=CGNO(IQO, IPO, NI)
CDNO(IPO, IQO, NI)=CDNO(IQO, IPO, NI)
1600 CONTINUE
DO 1700 IPO=IQO,MXQOO
IRECO=IRECO+1
IP=2*IPO-1
CALL GON(IP, IQ, GI, GDI, GR, GDR)
GNO(1, IRECO)=GR
GNO(2, IRECO)=GI
GNO(3, IRECO)=GDR
GNO(4, IRECO)=GDI
CGNO(IPO, IQO, NI)=DCMPLX (GR, GI)
CDNO(IPO, IQO, NI)=DCMPLX (GDR, GDI)
1700 CONTINUE
1800 CONTINUE
WRITE (29,REC=NI) GNO
1900 CONTINUE
CLOSE(28)
CLOSE(29)
RETURN
END
C
SUBROUTINE GIN(IPIN, IQIN, GIOUT, GDOUT, GROUT, GDROUT)
C***************************************************************************
C This subroutine sets up P and Q dependent constants and passes along
C /PCONST/, then calls subroutines GDREG and GDSNG to compute G(P,Q,N)
C and its derivative.
C*******************************************************************************
C INCLUDE 'REALTP.INC'
COMMON /PCONST/ DP, DQ, DS, DD, DSH, DHH, DDSQ, DDSQ, IP, IQ, IS, ID
C Check and transform input variables.
   IP=IPIN
   IQ=IQIN
   ISPQ=IP*IQ
   IS=ISPQ/2
   IF ((ISPQ+1)/2 .GT. IS) THEN
      GIOUT=ZERO
      GDIOUT=ZERO
      GROUT=ZERO
      GDROUT=ZERO
      WRITE (*, 9001) ISPQ
   END IF
9001 FORMAT('The parameters P and Q have a sum of the ODD integer ',I4,
       +, ',', G(P, Q, N) and its derivative have been set to 0. ')
   RETURN
   END IF
   IF (IP .LT. IQ) THEN
      IP=IQIN
      IQ=IPIN
   END IF
C
   DP=IP
   DQ=IQ
C
   ID=IS-IQ
   DS=IS
   DD=ID
   DSH=DS+HALF
   DHH=DD+HALF
   DDSQ=DS*DS
   DDSQ=DD*DD
C
   CALL GDREG(GI,GDI,GRR,GDRR)
   CALL GDSNG(GRS,GRS)
   GIOUT=GI
   GDIOUT=GDI
   GROUT=GRS
   GDROUT=GDRR+GRS
   RETURN
END
CC
SUBROUTINE GDBEQ(GIOUT, GDIOUT, GROUT, GDROUT)
C*******************************************************************************
C This subroutine computes the regular part of G(P,Q,N) and its
c derivative.
C*******************************************************************************
C INCLUDE 'REALTP.INC'
   INCLUDE 'GQNDM.INC'
   COMMON /GCONST/ DKH, DKA, HH, HA, HSQ, DSQ, HSQN, DSQN, HHSQ, HDASQ, +
      RAIH, RAHSQ, DHI, DHA
   COMMON /GCONST/ DN, DHH, N
   COMMON /PCONST/ DP, DQ, DS, DD, DSH, DHH, DDSQ, DDSQ, IP, IQ, IS, ID
   COMMON /SUMLMT/ DHMAX, DMAX, DMIN, IHMAX, IAMAX, ISNG
   SAVE /GCONST/., /SUMLMT/
C
C Reserve working space to store r-independent numbers.
DIMENSION GIN(MAXREG), GRRR(MAXREG)
C
C Computation starts.
C
C Compute overall constant factors:
GRF=QUAR*DKH/PISQ/(DSSQ-QUAR)/((QUAR-DSQ)/(DN+ONE))
GIF=HALF/DSH
DJN=ZERO
DO 300 JN=1,N
DJN=DJN+ONE
HATN=HA/DJN
GRF=GRF*HATN*HATN
GIF=GIF*HATN*HA/(DJN+DSH)
300 CONTINUE
DJP=ZERO
DO 400 JP=1,IQ
DJP=DJP+ONE
GIF=(HHSQ/DJF/DJP)*GIF
400 CONTINUE
DO 500 JP=1,I,P
DJP=DJP+ONE
GIF=(HH/DJP)*GIF
500 CONTINUE
IF ((ID-1)/2 .GT. ID/2) GIF=-GIF
C
C Compute GI, GDI, GRREG, and GDRREG.
C Compute r-independent factors and store in GIM and GRRM:
SMH=DSH+ONE
SM1=DS
SM2=DS+DS
PM1=DP
QM1=DQ
THRH=HALF
DM1=ZERO
THRS=HALF+DS
THRD=HALF+DD
THRSN=HALF+DS
DO 600 JN=1,IHMAX
SMH=SMH+ONE
SM1=SM1+ONE
SM2=SM2+ONE
PM1=PM1+ONE
QM1=QM1+ONE
THRH=THRH+ONE
DM1=DM1+ONE
THRS=THRS+ONE
THRD=THRD+ONE
THRSN=THRSN+ONE
GIM(JM)=(SMH/SM2)*(SMH/PM1)*(SMH/QM1)*(HHSQ/DM1)
GRRM(JM)=(THRH/THRS)*(HHSQ/THRD)*(DM1/THRSN)
600 CONTINUE
C
C Compute r- and m-sum for GI and GRR.
C Setup initial r related values
DJR=DMAX
DNR=DN+DJR
DNHR=DN+HALF
DZNR=DN+DN
DNSHR=DN+DSH
DNR2=DN+ONE
C
m-sum for r=IAMAX
DNSHRM=DNSHR+DMAX
DN2RM=DN2R+DMAX
ANNEXTR=ONE+GIM(IHMAX)/DNSHRM
ANR0R=ONE+GRRM(IHMAX)/DN2RM
DO 700 JM=IAMAX-1,1,-1
DNSHRM=DNSHRM+ONE
DN2RM=DN2RM+ONE
ANNEXTR=ONE+ANNEXTR*GIM(JM)/DNSHRM
ANR0R=ONE+ANR0R*GRRM(JM)/DN2RM
700 CONTINUE
ANNEXT=ANNEXTR
DANEXT=DNR*ANNEXTR
ANR0=ANR0
DANR0=DNR*ANR0
DO 900 JR=1,AMAX,1,-1
C Compute factors for r=JR
ATMP=(DNHR/D2NR)*(DASQN/DJR)
ATMPI=ATMP/DNSHR
ATMPR=ATMP/DN2R
C Setup new r related values for r=JR-1
DJR=DJR+1ONE
DNR=DNR+ONE
DNHR=DNHR+ONE
D2NR=D2NR+ONE
DNSHR=DNSHR+ONE
DN2R=DN2R+ONE
C m-sum for r=JR-1
DNSHRM=DNSHR+DHMAX
DN2RM=DN2R+DHMAX
ANNEXTR=ONE+GIM(IHMAX)/DNSHRM
ANR0=ONE+GRRM(IHMAX)/DN2RM
DO 800 JM=IHMAX,-1,-1
DNSHRM=DNSHRM+ONE
DN2RM=DN2RM+ONE
ANNEXTR=ONE+ANNEXTR*GIM(JM)/DNSHRM
ANR0=ONE+ANR0*GRRM(JM)/DN2RM
800 CONTINUE
ANNEXT=ANNEXTR+ANNEXT*ATMPI
DANEXT=DNR*ANNEXT+DANEXT*ATMPI
ANR0=ANR0+ANR0*ATMPI
DANR0=DNR*ANR0+DANR0*ATMPI
900 CONTINUE
C GIOUT=ANNEXT*GIF
GDIOUT=DANEXT*GIF
GROUT=ANR0*GRRF
GDROUT=DANR0*GRRF
C RETURN
END
C*************************************************************************
C* SUBROUTINE GDSNG(GROUT, GDROUT)                                       *
C*************************************************************************
C This subroutine computes the 'singular' part of G(P,Q,N) and its         *
C derivative.                                                            *
C*************************************************************************
C INCLUDE 'REALTP.INC'
INCLUDE 'GPQDMD.INC'
COMMON /GCONST/ D8R,DKA,HH,HA,HSQ,DASQ,HASQ,DASQN,HASQN,HDSQ,         *
RAH,RHASQ,DHA,DAH
COMMON /NCONST/ DN,DRH,N
COMMON /PCONST/ DP,DPQ,DS,DDQ,DDH,DSQ,DDSQ,IP,IS,ID
COMMON /SUMLMT/ DHMAX,DAMAX,DNSG,IHMAX,IAMAX,ISNG
SAVE /GCONST/,/SUMLMT/
C Reserve working space to store r-independent numbers.
DIMENSION GSI(1:MXMSNG),GS2M(2:MXMSNG),
      GR2M(2:MXMSNG),GR2R(0:MXMSNG),GDR2R(0:MXMSNG)
C Computation starts.
C MXGR2M=ISNG+N
C Compute overall constant factors:
GRSF=QUAR/PSIQ/DKH
C Compute G1, GDR1, GR2, and GDR2:
C Compute initial constants:

S1LHA = TWO * LOG (HXX/RAH)
DX = HALF
S1SD0 = ZERO
DO 1000 JK = 1, ID
S1SD0 = S1SD0 + ONE / DK
DX = DK / ONE
1000 CONTINUE
S1SD0 = TWO * S1SD0
DO 1100 JK = ID + 1, IS
S1SD0 = S1SD0 + ONE / DK
DX = DK / ONE
1100 CONTINUE
S1SD0 = TWO * S1SD0
SNN0 = ZERO
SN10 = ZERO
SN20 = ZERO
IF ( N .GT. 0 ) THEN
DNJ = ONE
DNJH = HALF
DO 1200 JN = 1, N - 1
DNJI = ONE / DNJ
DNJHI = ONE / DNJH
SN10 = DNJ1 * DNJHI - SN10
SN20 = DNJH * DNJHI - SN20
DNJ1 = ONE
DNJH = DNJH + ONE
DNJ = ONE / DNJ
DNJH = ONE / DNJ
SN10 = S1LHA * HALF - SN10
SN10 = ( DNJ1 * DNJHI + SN10 ) * QUAR
SN20 = ( DNJH * DNJHI + SN20 ) * QUAR
SN20 = SN20 / HALF
SN10 = SN10 / QUAR
END IF
SN10 = S1LHA - SN10
SN20 = PI * SQRT ( THIR + SN20 )

C Compute r-independent terms and store in GR2M, GS1M, and GS2M:
GR2JM = ONE
S1SDM = S1SD0
S2SDM = ZERO
DM = ZERO
DMH = HALF
DO 1300 J = 1, MXGR2M
DM = DM + ONE
DMH = DMH + ONE
DM1 = ONE / DM
DMH1 = ONE / DMH
DMHSQ = DMH * DMH
DMHSS = DMHSS + DMHSQ - DSSQ
DMHSD = DMHSD + DSSQ - DMHSQ
GR2JM = ( DMHSS * DMH1 + DM1 ) * ( DMHSD * DMH1 + DM1 ) * GR2JM
GR2JM = GR2JM
TWSMHS = TWSMHS + DMHSQ - DSSQ
TWSMDH = TWSMDH + DSSQ - DMHSQ
S1SDM = S1SDM - ( TWSMHS + TWSMDH ) * DM1
S2SDM = S2SDM - ( TWSMHS + TWSMDH ) * DMH1
S1SDM = S1SDM - ( ONE * QUAR ) * DM1
S2SDM = S2SDM - ( ONE * QUAR ) * DMH1
GS1M(JM) = S1SDM
GS2M(JM) = S2SDM

1300 CONTINUE

C Compute m-sum for GR1, GDR1 and store in GR2R, GDR2R
IF ( N .GT. 0 ) THEN
DJR = DN + ONE
DNJRH = DJR - HALF
SNOR = SNN0
SGRM=SN0R+SN1SD0
SGRM=DRM*SGRM+ONEN
DJM=ONE
DJRM=DRM
DO 1400 J=1,N-1
SR1M=SN0R+GS1M(JM)
HSQ=HSQ/DJM/DJRM
SGRM=SGRM*HSQ+(SR1M*GR2M(JM))
SGRM=SGRM*HSQ+(DJR*SR1M+ONEN)*GR2M(JM)
DJM=DJM+ONE
DJRM=DJRM+ONE
1400 CONTINUE

GRM(N-1)=SGRM*TSWO
GDR(N-1)=SGRM*TSWO
DO 1600 J=1,N-2,0,-1
SN0R=SN0R+ONE/DJR+ONE/(DN-DJR)-ONE/(DN+DJR)
DJR=DJR+ONE
DJRM=DJR+ONE
SR1M=SN0R+SN1SD0
SGRM=DRM*SGRM+ONEN
DJM=ONE
DJRM=DRM
DO 1500 JR=1,N
SR1M=SN0R+GS1M(JM)
HSQ=HSQ/DJM/DJRM
SGRM=SGRM*HSQ+(SR1M*GR2M(JM))
SGRM=SGRM*HSQ+(DJR*SR1M+ONEN)*GR2M(JM)
DJM=DJM+ONE
DJRM=DJRM+ONE
1500 CONTINUE

GRM(JR)=SGRM*TSWO
GDRM(JR)=SGRM*TSWO
1600 CONTINUE

END IF

C
C Compute m-sum for GR2, GDR2 and store in GR2M, GDR2M
DRM=DN
SNR=SN10
SN2R=SN20
SRM=SN1R+SN1SD0
SR2R=SR2M*SR2M+SN2R
GDRM=DRM*SR2R-2M*SR2M
DJM=ONE
DJRM=DRM
DO 1700 J=1,N
SR2M=SN1R+GS1M(JM)
SR2R=SR2M*SR2M+SN2R+GS2M(JM)
GDRM=DRM*SR2R-2M+SR2M
HSQ=HSQ/DJM/DJRM
SR2R=SR2R*HSQ+(SR2R*GR2M(JM))
GDRM=DRM*SR2R+HSQ+GR2M(JM)
DJM=DJM+ONE
DJRM=DJR+ONE
1700 CONTINUE

GDRM(N)=SR2R
GDRM(N)=DR2R
DJR=ZERO
DO 1900 JR=1,N+1,MXGR2M
DJR=DJR+ONE
DRM=DN+ONE
DJRM=ONE/DJR
DRN=ONE/(DNR-HALF)
DR2N=ONE/(DRD+DN)
DNR=DN+DNR+ONE
DNR=DN+DNR+ONE
SN2R=SN2R/DJR=DNHRF+SN2R
SN2R=DNHRF+SN2R
SN2R=SR2M+SN1R
SN2R=SN2R+SN1SD0
SR2R=SR2M+SR2M+SN2R
90
DR2RM=DRN*SR2RM-TWO*SR2MM
DJM=ONE
DJR=DRN

DO 1800 JM=1, JR
SR2MM=SN1R+GS1M(JM)
SR2RM=SN2RM*SR2MM+SN2R+GS2M(JM)
DR2RM=DRN*SR2RM-TWO*SR2MM
HSQF=HSQ/DSW/DJEM
SR2MS=SR2RM*HSQF+SR2RM*GR2M(JM)
DR2RMS=DR2RMS*HSQF+DR2RMS*GR2M(JM)
DJM=DJM+ONE
DJRM=DJRM+ONE

1800 CONTINUE
GR2R(JR)=SR2RMS
GDR2R(JR)=DR2RMS

1900 CONTINUE
C
C Compute r-sum:
C
IF (RAHSQ .GT. HALF) THEN
C
Convergence acceleration via Euler-Abel Transformation:
C
DJR=ZERO
DJRH=DJR+HALF
DJRN=DJR+DN

DO 2000 JR=1, N-1
RFCTOR=ONE/DN
GR2R(0)=RFCTOR*GR2R(0)
GDR2R(0)=RFCTOR*GDR2R(0)

2000 CONTINUE
DJR=DJR+ONE
DJRH=DJR-ONE
DJRN=DJR+ONE

RFCTOR=(DJR*DZR/DJRN/DJRNN)*RAHSQ*RFCTOR
GR2R(JR)=RFCTOR*GR2R(JR)
GDR2R(JR)=RFCTOR*GDR2R(JR)

ELSE
RFCTOR=ONE
END IF

DO 2100 JR=N+1, MXGR2M
DJR=DJR+ONE
DJRH=DJR+ONE
DJRN=DJRNN+ONE

RFCTOR=(DJR*DZR/DJRN/DJRNN)*RAHSQ*RFCTOR
GR2R(JR)=RFCTOR*GR2R(JR)
GDR2R(JR)=RFCTOR*GDR2R(JR)

2100 CONTINUE
C
C Compute zeroth-order delta-r term:

DO 2300 JDDELTR=1, MXGR2M
GDR2R(JR)=GR2R(JR-1)-GR2R(JR)
GDR2R(JR)=NDR2R(JR-1)-GDR2R(JR)

2200 CONTINUE

2300 CONTINUE
SQR2R=HALF*GR2R(MXGR2M)
I. SUBROUTINE BCKSCFL

SUBROUTINE BCKSCFL(IR,CESTH,CESPH)

C ***********************************************
C INCLUDE 'REALTP.INC'
 INCLUDE 'CMPXTP.INC'
 INCLUDE 'MAINDM.INC'
C REAL*4 AKPHPR, AKPHPI, AKZPR, AKZPI, ALPHPR, ALPHPI, ALZPR, ALZPI,
+ AKPNR, AKPNH, AKZNR, AKZNH, ALPHNR, ALPHNH, ALNZNR, ALNZNH
DIMENSION CTH(KCRNT), CCLN(KCRNT)
DIMENSION CIW0(KKCRT), CKLN0(KKCRT), CXPQN0(KKCRT, KKCRT)
DIMENSION CRPQ1(KCRNT, KCRNT), CRPQ2(KCRNT, KCRNT),
+ CXPQ(KCRNT, KCRNT), CXRQ(KCRNT, KCRNT)
DIMENSION CRHP(61, 61), CKZP(61, 61), CLPHP(61, 61),
+ CLZP(61, 61), CKPH(61, 61), CKZM(61, 61),
+ CLPH(61, 61), CLZN(61, 61)
COMMON /INPUT2/ IE, IM, THRTAI, THESINI, THECOSI, RTHE1
COMMON /INPUT4/ IZ, IK, IS, NYSM
COMMON /KCONST/ DN, DNH, N
COMMON /XPOCXP/ CXPQ, CXPQ0, CXPQ00
COMMON /XQPOXM/ CRPQ1, CRPQ2
COMMON /CLKMTX/ CKPHP, CKZP, CLPHP, CLZP, CKPHN, CKZM, CLPHN, CLZN
COMMON /CRNNTM/ NMAX, MXNG, IOMAX, IQMAX, IQMAX1, IQMAX2, IXCRNT, ICRTNT, MXQEG,
+ MXQOG
COMMON /INPUT3/ RTHE, RDELT, RPHI, RDELP, NTHTAO, NPHI, THESIN, THECOS,
+ RPHI
CSTH=CZERO
CESPH=CZERO
DO 100 JQ=1, KKCRT
CKLN0(JQ)=CZERO
100 CONTINUE
DO 200 JQ=1, KCRNT
CKLN(JQ)=CZERO
200 CONTINUE
DO 400 IF=30, 30,
DO 300 JZ=-30, 30,
CKPHP(IF, JZ)=CZERO
CKZP(IF, JZ)=CZERO
CLPHP(IF, JZ)=CZERO
CLZP(IF, JZ)=CZERO
CKPHN(IF, JZ)=CZERO
CKZM(IF, JZ)=CZERO
CLPHN(IF, JZ)=CZERO
CLZN(IF, JZ)=CZERO
300 CONTINUE
400 CONTINUE
C IF (RTHE1 .EQ. 0.0) GO TO 2000
C If incident angle is not equal to 0, use this loop
C DO 1800 NA=0, NMAX
C CALL INCIDNT(NA, CW0, CW1)
C
C IF (NA .EQ. 0) THEN
CALL XPOC
ELSE
CALL XQPOC(NA)
ENDIF
C If the cylinder is made of perfect conductor and no coating on it
C IF (IZ .EQ. 0) THEN
DN=NA
C Use IMSL library to solve linear system
CALL DLSACG(IKCRNT, CXPQN0, KKCRT, CIW0, 1, CKLN0)
CALL ESCFAR(NA, CKLN0, CKLN, CETHN, CEPHN)
C
C If we calculate in X and Y components as theta approaches 0 or pi,
C then theta component is cos phi in X direction plus sine phi in Y
C direction, and phi component is negative sine phi in X direction plus
C cos phi in Y direction.
IF (RTHE .EQ. 0.0) THEN
CETHN=COX(RPHI)*CETHN+SIN(RPHI)*CEPHN
CEPHN=-SIN(RPHI)*CETHN+COX(RPHI)*CEPHN
ELSEIF (RTHE .EQ. PI) THEN
CETHN=-COX(RPHI)*CETHN+SIN(RPHI)*CEPHN
CEPHN=-SIN(RPHI)*CETHN+COX(RPHI)*CEPHN
ENDIF
C
C
CESTH=CESTH+CETHN
CESPH=CESPH+CEPHN
IF (NA .NE. 0) THEN
  IF (IE .EQ. 1) THEN
DO 600 I=2,KXCRT,2
  CKLN0(I)=-CKLN0(I)
600  CONTINUE
END IF
IF (IM .EQ. 1) THEN
DO 800 I=1,KXCRT,2
  CKLN0(I)=-CKLN0(I)
800  CONTINUE
END IF
NAI=-NAI
DN=NAI
CALL ESCFAR(NA1,CKLN0,CKLN,CETHN,CEPHN)
  IF (RTHE .EQ. 0.D0) THEN
    CETHN=COS(RPHI)*CETHN+SIN(RPHI)*CEPHN
    CEPHN=-SIN(RPHI)*CETHN+COS(RPHI)*CEPHN
  ELSEIF (RTHE .EQ. PI) THEN
    CETHN=-COS(RPHI)*CETHN-SIN(RPHI)*CEPHN
    CEPHN=-SIN(RPHI)*CETHN+COS(RPHI)*CEPHN
END IF
C
CESTH=CESTH+CETHN
CESPH=CESPH+CEPHN
END IF
END IF
C If the cylinder is coated with anisotropic material
IF (IZ .EQ. 1) THEN
DO 1100 I=1,KCRNT
  DO 1000 J=1,KCRNT
    CXRPQ(I,J)=CXQPQ(I,J)+CRPQ1(I,J)
1000  CONTINUE
1100  CONTINUE
DN=NA
CALL DSLACG(ICRNT,CXRPO,KCRNT,CIW,1,CKLN)
  IF ((IK .EQ. 1) .AND. (IR .EQ. 45)) THEN
    CALL KLCRNT(DN,CKLN,CIW)
  END IF
CALL ESCFAR(NA,CKLN0,CKLN,CETHN,CEPHN)
  IF (RTHE .EQ. 0.D0) THEN
    CETHN=COS(RPHI)*CETHN+SIN(RPHI)*CEPHN
    CEPHN=-SIN(RPHI)*CETHN+COS(RPHI)*CEPHN
  ELSEIF (RTHE .EQ. PI) THEN
    CETHN=-COS(RPHI)*CETHN-SIN(RPHI)*CEPHN
    CEPHN=-SIN(RPHI)*CETHN+COS(RPHI)*CEPHN
END IF
C
CESTH=CESTH+CETHN
CESPH=CESPH+CEPHN
  IF (NA .NE. 0) THEN
    IF (NYSM .EQ. 0) THEN
DO 1200 I=1,KCRNT
  DO 1200 J=1,KCRNT
    CXRPQ(I,J)=CXQPQ(I,J)+CRPQ2(I,J)
1200  CONTINUE
1300  CONTINUE
ELSE
DO 1500 I=1,KCRNT
  DO 1400 J=1,KCRNT
    CXRPQ(I,J)=CXQPQ(I,J)+CRPQ1(I,J)
1400  CONTINUE
1500  CONTINUE
END IF
CALL DSLACG(ICRNT,CXRPO,KCRNT,CIW,1,CKLN)
  IF (IE .EQ. 1) THEN
  END
DO 1600 I=2,KCRNT,4
   II=I+1
   CKLN(II)=CKLN(I)
   CKLN(I1)=CKLN(I1)
1600 CONTINUE
   END IF
   IF (II .EQ. 1) THEN
      DO 1700 I1=1,KCRNT,4
      I2=I+3
      CKLN(I)=CKLN(I)
      CKLN(I2)=CKLN(I2)
1700 CONTINUE
   END IF
   NA1=NA
   DN=NA1
   IF ((IX .EQ. 1) .AND. (IS .EQ. 199)) THEN
      CALL KCRNT(DN,KCRNT,CW)
      END IF
      CALL ESCFAR(NA1,CKLN0,CKLN,CETH,N,CEPHN)
      IF (RI .EQ. 0.0) THEN
        CETHN=OS(RPHI)*CETHN+SN(RPHI)*CEPHN
      CETHN=-SN(RPHI)*CETHN+OS(RPHI)*CEPHN
      ELSEIF (RI .EQ. PI) THEN
        CETHN=-OS(RPHI)*CETHN-SN(RPHI)*CEPHN
      CETHN=-SN(RPHI)*CETHN-COST(RPHI)*CEPHN
      END IF
      C
      CETHN=CETHN+CETHN
      CEPHN=CEPHN-CEPHN
      END IF
      END IF
5000 CONTINUE
      C
      If the cylinder is coated with anisotropic material and we want to calculate
      C equivalent currents K naL on inner and outer surfaces.
      IF (IX .EQ. 1) THEN
      IF (IS .EQ. 199) THEN
         OPEN(31,FILE='khzn1.dat',STATUS='UNKNOWN')
         OPEN(32,FILE='hz41.dat',STATUS='UNKNOWN')
      END IF
      DO 1900 IF=-30,30
      DO 1850 JZ=-30,30
      AKHPHP=REAL(CKPHP(IF,JZ))
      AKPHPI=IMAG(CKPHP(IF,JZ))
      AKZPR=REAL(CKZP(IF,JZ))
      AKZPI=IMAG(CKZP(IF,JZ))
      AKPHRO=REAL(CKPHN(IF,JZ))
      AKPHNI=IMAG(CKPHN(IF,JZ))
      AKZNR=REAL(CKZN(IF,JZ))
      AKZNII=IMAG(CKZN(IF,JZ))
      ALPHPR=REAL(CLPHP(IF,JZ))
      ALPHPI=IMAG(CLPHP(IF,JZ))
      ALZPR=REAL(CLZP(IF,JZ))
      ALZPI=IMAG(CLZP(IF,JZ))
      ALPHNR=REAL(CLPHN(IF,JZ))
      ALPHNI=IMAG(CLPHN(IF,JZ))
      ALZNII=REAL(CLZN(IF,JZ))
      ALZNII=IMAG(CLZN(IF,JZ))
      WRITE(31,'(A)'),AKHPHP,'AKPHPI'
      WRITE(31,'(A)'),AKZPR,'AKZPI'
      WRITE(31,'(A)'),ALPHPR,'ALPHPI'
      WRITE(31,'(A)'),ALZPR,'ALZPI'
      WRITE(31,'(A)'),AKPHNR,'AKPHNI'
      WRITE(31,'(A)'),AKZNII,'AKZNII'
      WRITE(31,'(A)'),ALPHNR,'ALPHNI'
      WRITE(31,'(A)'),ALZNII,'ALZNI'
1850 CONTINUE
1900 CONTINUE
CLOSE(31)
CIF incident angle is equal to zero, the program should go to this loop
Cbecause it need to compute when n=1 and n=-1 only.
2000  NA=1
   DN=NA
   CALL INCIDNT(NA,CIW0,CIW)
   CALL XPQN(NA)
C
   IF (IZ .EQ. 0) THEN
      CALL DLSAGC(IKCRNT,CRPQ0,KXCR0,CIW0,1,CKLN0)
      CALL ESCFAR(NA,CKLN0,CKLN,CETHN,CEPHN)
   C
   IF we calculate in X and Y components as theta approaches 0 or pi,
   then theta component is cosin phi in X direction plus sine phi in Y
   direction, and phi component is negative sine phi in X direction plus
   cosin phi in Y direction.
   IF (RTHE .EQ. 0.0D0) THEN
      CETHN=CETHN*CETHN+SFH(RPHI)*CEPHN
      CEPHN=-SIN(RPHI)*CETHN+CS(RPHI)*CEPHN
   ELSEIF (RTHE .EQ. PI) THEN
      CETHN=-CETHN-RPHI)*CEPHN
      CEPHN=-SIN(RPHI)*CETHN+CS(RPHI)*CEPHN
   END IF
   C
   CESTH=CESTH+CETHN
   CESPH=CESPH+CEPHN
   IF (NA .NE. 0) THEN
      IF (IE .EQ. 1) THEN
         DO 2400  I=2,KXCR0.2
            CKLN0(I)=CKLN0(I)
         2400  CONTINUE
      END IF
      IF (IM .EQ. 1) THEN
         DO 2500  I=1,KXCR0.2
            CKLN0(I)=CKLN0(I)
         2500  CONTINUE
      END IF
   NA1=NA
   DN=NA
   CALL ESCFAR(NA1,CKLN,CKLN,CETHN,CEPHN)
   IF (RTHE .EQ. 0.0D0) THEN
      CETHN=CETHN*CETHN+SFH(RPHI)*CEPHN
      CEPHN=-SIN(RPHI)*CETHN+CS(RPHI)*CEPHN
   ELSEIF (RTHE .EQ. PI) THEN
      CETHN=-CETHN-RPHI)*CEPHN
      CEPHN=-SIN(RPHI)*CETHN+CS(RPHI)*CEPHN
   END IF
   C
   CESTH=CESTH+CETHN
   CESPH=CESPH+CEPHN
   END IF
   END IF
   C
   IF (IZ .EQ. 1) THEN
      DO 3100  I=1,KCRNT
         DO 3000  J=1,KCRNT
            CXPQ(I,J)=CXPQ(I,J)+CRP(I,J)
         3000  CONTINUE
      3100  CONTINUE
   NA=NA
   CALL DLSAGC(IKCRNT,CXRPQ,KCRNT,CIW.1,CKLN)
   C
   IF ((IK .EQ. 1) .AND. (IR .EQ. 45)) THEN
      CALL KCRNT(NA,CKLN,CIW)
   END IF
   C
CALL ESCFAR(NA, CLN0, CLLN, CETHN, CEPHN)
IF (RTKE .EQ. 0.0D0) THEN
CETHN=COS(RPHI) * CETHN + SIN(RPHI) * CEPHN
CEPHN=-SIN(RPHI) * CETHN + COS(RPHI) * CEPHN
ELSEIF (RTKE .EQ. PI) THEN
CETHN=-COS(RPHI) * CETHN - SIN(RPHI) * CEPHN
CEPHN=-SIN(RPHI) * CETHN + COS(RPHI) * CEPHN
END IF
C
CESTH=CESTH + CETHN
CESPH=CESPH + CEPHN
IF (NA .NE. 0) THEN
IF (NYSM .EQ. 0) THEN
DO 3300 I=1, KCRNT
DO 3200 J=1, KCRNT
CXRFPQ(I, J)=CXRFPQ(I, J)+CRFPQ(I, J)
3200 CONTINUE
3300 CONTINUE
ELSE
DO 3400 I=1, KCRNT
DO 3400 J=1, KCRNT
CXRFPQ(I, J)=CXRFPQ(I, J)+CRFPQ(I, J)
3400 CONTINUE
3500 CONTINUE
END IF
CALL DLSCAC(1, KCRNT, CXRFPQ, KCRNT, CIW, 1, CLLN)
IF (IE .EQ. 1) THEN
DO 3600 I=2, KCRNT, 4
II=I-1
CLLN(II)=CLLN(I)
3600 CONTINUE
END IF
IF (IM .EQ. 1) THEN
DO 3700 I=1, KCRNT, 4
II=I-1
CLLN(I)=CLLN(II)
3700 CONTINUE
END IF
NA1=-NA
DN=NA1
C
IF ((IK .EQ. 1) .AND. (IS .EQ. 199)) THEN
CALL KLCRNT(DN, CLLN, CIW)
END IF
C
CALL ESCFAR(NA1, CLN0, CLLN, CETH, CEPH)
IF (RTKE .EQ. 0.0D0) THEN
CETH=COS(RPHI) * CETH + SIN(RPHI) * CEPH
CEPH=-SIN(RPHI) * CETH + COS(RPHI) * CEPH
ELSEIF (RTKE .EQ. PI) THEN
CETH=-COS(RPHI) * CETH - SIN(RPHI) * CEPH
CEPH=-SIN(RPHI) * CETH + COS(RPHI) * CEPH
END IF
C
CESTH=CESTH + CETH
CESPH=CESPH + CEPH
END IF
C
IF (IK .EQ. 1) THEN
IF (IS .EQ. 199) THEN
OPEN(31, FILE='khz41.dz', STATUS='UNKNOWN')
OPEN(32, FILE='lhz41.dz', STATUS='UNKNOWN')
END IF
DO 4200 IF=30, 30
DO 4100 JZ=-30, 30
97
AKPHPR=REAL(CKHPH(IP,JZ))
AKPHPI=IMAG(CKHPH(IP,JZ))
AKZPR=REAL(CKZP(IP,JZ))
AKZPI=IMAG(CKZP(IP,JZ))
AKPHNR=REAL(CKPHNR(IP,JZ))
AKPHNI=IMAG(CKPHNR(IP,JZ))
AKZNR=REAL(CKZNR(IP,JZ))
AKZNI=IMAG(CKZNR(IP,JZ))
ALPHPR=REAL(CLHPHI(IP,JZ))
ALPHPI=IMAG(CLHPHI(IP,JZ))
ALZPR=REAL(CLZP(IP,JZ))
ALZPI=IMAG(CLZP(IP,JZ))
ALPHNR=REAL(CLPHNR(IP,JZ))
ALPHNI=IMAG(CLPHNR(IP,JZ))
ALZNR=REAL(CLZNR(IP,JZ))
ALZNI=IMAG(CLZNR(IP,JZ))
WRITE(31,'(A,A)') AKPHPR,AKPHPI
WRITE(33,'(A)') AKZPR
WRITE(32,'(A)') ALPHPR
WRITE(32,'(A)') ALZPR
WRITE(31,'(A)') AKPHNR
WRITE(33,'(A)') AKZNR
WRITE(32,'(A)') ALPHNR
WRITE(32,'(A)') ALZNR
4100 CONTINUE
4200 CONTINUE
CLOSE(31)
CLOSE(32)
END IF
RETURN
END

SUBROUTINE INCIDENT(NA,CIW0,CIN)
C******************************************************************************
C This subroutine sets up the incident wave on an object which is coated
C with anisotropic material, or a perfect conductor.
C******************************************************************************
INCLUDE 'REALTP.INC'
INCLUDE 'CMXPRT.INC'
INCLUDE 'MAINDM.INC'

DIMENSION CIN(2),CIW0(2),CIW0(KRZAT),CIW0(KZAT),NJNS(KNND1+1),DJPS(KDZM1+2)
COMMON /INPUT2/ IE,IM,THETA,T,THESIN,TTHCOS,THTHEI
COMMON /INPUT3/ R,IE,REELTS,TRPHI,DELPHI,DELTHA,NPHI,THETSI,THECOS,
  + RHPI
COMMON /RTHPRT/ DLICOS,DLSIN,DL
COMMON /GOCONST/ DH,KHA,HH,HA,HSQ,DASQ,DSQN,DASQN,HSQ,DAQT,RHA,RHASQ,DHA,DHA
COMMON /CRNTDM/ NMAX,MXNG,IMAX,IMAX1,IMAX2,IXCNR,T,ICRRT,IXQEG,
  + MXQOG

C IF (IZ .EQ. 0) GO TO 4000
C This part computes the incident field on an anisotropic object
C Initialize the column matrix of sum current on the anisotropic
DO 100 IZ=1,IXCRRT
CIW0(IZ)=-0.0
100 CONTINUE
C If the incident angle is 90 degree or 0 degree
IF (KTHEI .EQ. ZERO) GO TO 3000
CALL DDSINS(DL2SIN,KNND1,NJNS)

98
DJN=DJNS(NP)
    IF (NA .EQ. 0) THEN
        DJNNM=DJNS(NP1)
    ELSE
        DJNNM=HALF*(DJNS(NA)-DJNS(NP1))
    ENDIF

C Use IMSL library to compute Bessel's function series.
CALL DBSUNS(DL1COS1,IQMAX2,DJPS)
DO 600 IP=0,KQDIM
  NIP=IP+1
  DNIP=NIP
  NIP1=NIP+1
  NIP2=NIP+2
  N11=NA+IP
  N12=NA+IP
  IE1=MOD(N11,N1)
  IE2=MOD(N12,N1)
C
  CF1=CI2**IE1
  CF2=CI2**IE2
  IPW1=4*IP+1
  IPW2=IPW1+1
  IPW3=IPW2+1
  IPW4=IPW3+1
C
  DJP=DJPS(NIP)
  DJP1=DJPS(NIP1)
  DJP2=DJPS(NIP2)
  IF (IE .EQ. 1) THEN
      CEEPHI=CF1*(DJP+DJP2)*DJNNM/NIP
      CBPHI=TWO*NA*CF2*(DJP1*DJN/DLL
      CBH1=CF2*THESIN1*(DJP+DJP2)*DJNNM/NIP
      ELSE
      CEEPHI=ZZERO
      CBPHI=CBPHI
      CBH1=ZZERO
      END IF
C
  IF (IM .EQ. 1) THEN
      CMEMPHI=TWO*NA*CF2*(DJP1*DJN/DLL
      CMEMPHI=CF2*THESIN1*(DJP+DJP2)*DJNNM/NIP
      ELSE
      CMEMPHI=ZZERO
      CMEMPHI=ZZERO
      END IF
C
  CIW(IPW1)=TWO*(CEEPHI+CMEMPHI)
  CIW(IPW2)=TWO*CMEMPHI
  CIW(IPW3)=TWO*(CEEPHI+CMEMPHI)
  CIW(IPW4)=TWO*CBPHI
600 CONTINUE
RETURN

C
3000 CALL DBSUNS(DKH,IQMAX2,DJPS)
DO 3200 IP=0,KQDIM
  JP=IP+2
  DJP=DJPS(JP)
  IE11=MOD(IP,4)
  IE12=MOD(IP+1,4)
  CF1=CI2**IE11
  CF2=CI2**IE12
  IPW1=4*IP+1
  IPW2=IPW1+2
C
  IF (IE .EQ. 1) THEN
      CEEPHI=CF1*DJP/DKH
      ZZERO

CEPHI=-CF2*DJP/DKH
ELSE
CEPHI=CZERO
CEPHI=CZERO
END IF
C
IF (IM .EQ. 1) THEN
CM2PHI=CF2*DJP/DKH
CM2PHI=CF1*DJP/DKH
ELSE
CM2PHI=CZERO
CM2PHI=CZERO
END IF
C
CIW(IPW1)=TWO*(CEPHI+CM2PHI)
CIW(IPW3)=TWO*(CEPHI+CM2PHI)
3200 CONTINUE
RETURN
C
C This part compute incident fields on a perfect conductor
C Initialize the column matrix of sum current on the conductor
4000 DO 4200 IX=1,KXCRT
CIW0(IX)=CZERO
4200 CONTINUE
C
C If the incident angle is 90 degree or 0 degree
IF (RTHETI .EQ. ZERO) GO TO 6000
C
CALL DBSJSN(DL2SINI, MXNG+1, DJNS)
DJN=DJNS(NP)
IF (NA .EQ. 0) THEN
DJN=DJNS(NP1)
ELSE
DJN=HALF*(DJNS(NA)-DJNS(NP1))
ENDIF
C
CALL DBSJSN(DL1COSI, IQMAX2, DJPS)
DO 4400 IF=0,KDIM
NIF=IF+1
DNIF=NIF
NIP1=NIF+1
NIP2=NIF+2
N11=NA+IP-1
N12=NA+IP
IE1=MOD(N11,4)
IE2=MOD(N12,4)
C
CF1=CI2**IE1
CF2=CI2**IE2
IPW1=2*IP+1
IPW2=IPW1+1
C
DJP=DJPS(NIP)
DJP1=DJPS(NIP1)
DJP2=DJPS(NIP2)
IF (IE .EQ. 1) THEN
CEPHI=CF1*(DJP+DJP2)*DJN/DNIF
ELSE
CEPHI=CZERO
ENDIF
C
IF (IM .EQ. 1) THEN
CM2PHI=TWO*NA*CF2*DJP1*DYN/DLL
CM2PHI=CF2*THESINE*(DJP+DJP2)*DJN/DNIF
ELSE
CM2PHI=CZERO
CM2PHI=CZERO
ENDIF
C
CIW0(IPW1)=TWO*(CCEPHI+CMEPHI)
CIW0(IPW2)=TWO*CM2E
4400 CONTINUE
RETURN
C
6000 CALL DBSINS(DKH,IQMAX2,DJPS)
DO 5200 IF=0,KQDIM
JP=IP+2
DJP=DJPS(JP)
IE11=MOD(IP,4)
IE12=MOD(IP+1,4)
CP1=CIZ**IE11
CP2=CIZ**IE12
IPW1=2*IP+1
C
IF (IE .EQ. 1) THEN
CCEPHI=CP1*DJP/DKH
ELSE
CCEPHI=CZERO
END IF
C
IF (IM .EQ. 1) THEN
CMEPHI=CP2*DJP/DKH
ELSE
CMEPHI=CZERO
END IF
C
CIW0(IPW1)=TWO*(CCEPHI-CMEPHI)
5200 CONTINUE
RETURN
END
C
SUBROUTINE XPQ0
C*****************************************************************************
C This subroutine computes the matrix XNF(P,Q) for N = 0 following a
C call to XPQ1NL. This matrix is kept in the common block:
C COMMON /XPQTMP/ CXFQ0
C*****************************************************************************
C INCLUDE 'REALTF.INC'
INCLUDE 'CMKTPF.INC'
INCLUDE 'MAINDM.INC'
C
DIMENSION CXFQ0(KCRNT,KCRNT),CXFQ0(KCRNT,KCRNT)
DIMENSION CXFQ0(KXCRNT,KXCRNT)
DIMENSION CQNFRE(0:MAXPQ,0:MAXPQ,KNDIML-1),
+ CQNFRE(0:MAXPQ,0:MAXPQ,KNDIML-1),
DIMENSION CQNFRE(0:MAXPQ,0:MAXPQ,KNDIML-1),
+ CQNFRE(0:MAXPQ,0:MAXPQ,KNDIML-1),
COMMON /GPQTMP/, GQNF,CGQNF,CGQNF,CGQNF
COMMON /XPQTMP/ CXFQ0,CXQNF,CXQNF
COMMON /XPQ1NL/ NMAX,NQMAX1,NIQMAX1,NQMAX2,NIQMAX2,IXCRNT,ICRNT,NXQEG,
+ MXQEG
COMMON /GCONST/ DQH,DK,HA,HQ,DSQ,HSQ,DASQ,HSQN,HSQH,HDASQ,
+ RA,HASA,HDA,HDA
COMMON /INPUT1/ IZ,IK,IS,NYSM
SAVE /XPQTMP/, /GPQTMP/, /GCONST/
C
IF (N .NE. 0) THEN
WRITE(*,*) 'Input N is not equal to 0 in XPQ0.'
WRITE(*,*) 'Execution is stopped.'
STOP
END IF
C
IF (IZ .EQ. 0) GO TO 4000
C
Initialize the XNF(P,Q) matrix.
C Null terms will be skipped later.

101
DO 200 IQ=1,KCRNT
DO 100 IP=1,KCRNT
CXPQN(IP, IQ)=CZERO
100 CONTINUE
200 CONTINUE
C
Form the XN(P,Q) matrix for n = 0 and using on a perfect conductor
CP31=DKA*CTI
C
DO 1300 IQE=0, MXQEG-1
IQB1=IQE-1
IQ=2*IQE
IQX1=4*IQ+1
IQX2=IQX1+1
IQX3=IQX2+1
IQX4=IQX3+1
DO1=IQ+1
DO2=DO1/HHSQ
DO 1100 IP=0, MXQEG-1
IPE1=IP-1
IF=2*IP
IPX1=4*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP1=IP+1
F41=DHK/DPL
CP11=F41*CP31
F51=QUAR*F41*DKA
CXPQN(IPX1, IQX1)=(CGNE(IPE1, IQE, 2)-CGNE(IPE, IQE, 2))*CP11
CXPQN(IPX4, IQX1)=(CGNE(IPE1, IQE, 2)-CGNE(IPE, IQE, 2)+HA*{(CDGNE(IPE1, IQE, 2)-CDGNE(IPE, IQE, 2)))*F51
CXPQN(IPX2, IQX2)=(TWO*FQ22*DP1*CGNE(IPE, IQE, 1)+HALF*{(CGNE(IPE, IQE, 1)+CGNE(IPE1, IQE, 1)-CDGNE(IPE, IQE, 1)))*CP11
CXPQN(IPX3, IQX2)=(CDGNE(IPE, IQE, 1)+CGNE(IPE1, IQE, 1)-CDGNE(IPE, IQE, 1)))*F51
CXPQN(IPX2, IQX3)=-CXPQN(IPX4, IQX1)
CXPQN(IPX3, IQX3)=CXPQN(IPX1, IQX1)
CXPQN(IPX1, IQX4)=-CXPQN(IPX3, IQX2)
CXPQN(IPX4, IQX4)=CXPQN(IPX2, IQX2)
1100 CONTINUE
1300 CONTINUE
DO 2300 IQO=0, MXQOG-1
IQO1=IQO+1
IQO=2*IQO+1
IQOL1=4*IQO-1
IQOL2=IQOL1+1
IQOL3=IQOL2+1
IQOL4=IQOL3+1
DO2=IQO-1
DO3=DO2/HHSQ
DO 2200 IPO=0, MXQOG-1
IPO1=IPO+1
IP=2*IPO+1
IPX1=4*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP1=IP+1
F41=DHK/DPL
CP11=F41*CP31
F51=QUAR*F41*DKA
CXPQN(IPX1, IQOL1)=(CGNO(IPO1, IQO, 2)-CGNO(IPO, IQO, 2)))*CP11
CXPQN(IPX4, IQOL1)=(CGNO(IPO1, IQO, 2)-CGNO(IPO, IQO, 2)+HA*{(CDGNO(IPO1, IQO, 2)-CDGNO(IPO, IQO, 2)))*F51
CXPQN(IPX2, IQOL2)=(TWO*FQ22*DP1*CGNO(IPO1, IQO1, 1)+HALF*{(CGNO(IPO1, IQO1, 1)+CGNO(IPO, IQO1, 1)-CDGNO(IPO, IQO1, 1)))*F51
+ CGNO(IPQ1.IQ01.1) * CGNO(IPQ1.IQ0.1) * CF11
+ CGNO(IPQ1.IQ01.1) * CGNO(IPQ1.IQ0.1) * F51
CXPNQ(IPQ3.IQX2) = (CXPNQ(IPQ3.IQX2) - CGNO(IPQ1.IQ01.1) * CF11)
CXPNQ(IPQ3.IQX2) = (CXPNQ(IPQ3.IQX2) - CGNO(IPQ1.IQ01.1) * F51)
CXPNQ(IPQ2.IQX1) = CXPNQ(IPQ2.IQX1)
CXPNQ(IPQ3.IQX2) = CXPNQ(IPQ3.IQX2)
CXPNQ(IPQ4.IQX4) = CXPNQ(IPQ4.IQX4)

2200 CONTINUE
2300 CONTINUE
RETURN
C Initialize the X(N,P,Q) matrix.
C Null terms will be skipped later.
C
4000 DO 4200 IQ=1,KXCRT
DO 4100 IP=1,KXCRT
CXPNQ(IPQ1.IQ01) = ZERO
4100 CONTINUE
4200 CONTINUE
C Form the X(N,P,Q) matrix for n = 0 and using on an anisotropic coat.
CF31 = DKA * CI1
C
DO 4500 IQE=0, MXQEG-1
IQE1=IQE-1
IQ=2*IQE
IQX1=2*IQ+1
IQX2=IQX1+1
DQ1=IQ-1
FQ22=DQ1/HHSQ
DO 4400 IPU=0, MXQEG-1
IFP1=IPU-1
IP=2*IP
IPX1=IPX1+1
DPI=IFP-1
FP41=DH1/DPI
CF11=FP41*CF31
F51=QUAR*F41*DKA
CXPNQ(IPQ1.IQX1) = (CGNE(IPQ1.IQ01, IQE, 2) - CGNE(IPQ1.IQ01, IQE, 1)) * CF11
CXPNQ(IPQ2.IQX2) = (TWO*FQ22*DPI*CGNO(IPQ1.IQ01, IQE, 1) - HALF* (C
+ CGNE(IPQ1.IQ01, IQE, 1)) * CF11
4400 CONTINUE
4500 CONTINUE
DO 5300 IQO=0, MXQOG-1
IQO1=IQO-1
IQ=2*IQO-1
IQX1=2*IQO+1
IQX2=IQX1+1
DQ1=IQ-1
FQ22=DQ1/HHSQ
DO 5200 IPO=0, MXQOG-1
IPO1=IPO-1
IP=2*IP-1
IPX1=2*IP+1
IPX2=IPX1+1
DPI=IP-1
FP41=DH1/DPI
CF11=FP41*CF31
F51=QUAR*F41*DKA
CXPNQ(IPQ1.IQX1) = (CGNO(IPQ1.IQ01, IQO, 2) - CGNO(IPQ1.IQ0, IQO, 2)) * CF11
CXPNQ(IPQ2.IQX2) = (TWO*FQ22*DPI*CGNO(IPQ1.IQ01, IQO, 1) + HALF* (C
+ CGNO(IPQ1.IQ01, IQO, 1)) * CF11
5200 CONTINUE
5300 CONTINUE
RETURN
SUBROUTINE XPNQ(NIN)
C*******************************************************************************
C This subroutine calls the subroutine GPN to update G(P,Q,N) for
C N = NIN+1, then forms the matrix XN(P,Q) for N = NIN > 0. This matrix
C is kept in the common block:
C COMMON /XPNQMP/ XPNQ
C*******************************************************************************
C WARNING:
C IT IS ASSUMED THAT XPNQ1 AND XPNQ FOR N FROM 1 TO NIN-1 HAVE BEEN
C CALLED SO THAT G(P,Q,N) AND ITS DERIVATIVE FOR N=NIN-1 AND N=NIN
C HAVE BEEN STORED IN THE COMMON BLOCK /XPNQMP/ WITH PROPER N-INDICES.
C*******************************************************************************
INCLUDE 'REALTP.INC'
INCLUDE 'CNPKTP.INC'
C DIMENSION CXPQN(KCNRN, KCRNT), CXPQ0(KCNRN, KCRNT)
DIMENSION CXPQ00(KCNRN, KCRNT)
DIMENSION CGNE(0:MAXPQG, 0:MAXPQG, KNIDM1+1),
+ CNGN(0:MAXPQG, 0:MAXPQG, KNIDM1+1)
DIMENSION CDGNE(0:MAXPQG, 0:MAXPQG, KNIDM1+1),
+ CDGN(0:MAXPQG, 0:MAXPQG, KNIDM1+1)
COMMON /GFQTMP/ CGNE, CNGN, CDGNE, CDGN
COMMON /XPNQMP/ CXPQN, CXPQ0, CXPQ00
COMMON /CRNPR/ NMAX, MXQG, IQMAX1, IQMAX2, ICNRN, ICRNT, IMXQG,
+ IMXQG
COMMON /GCONST/ DKH, DKA, HH, HA, HSQ, DASQ, HSQN, DSNQ, HHSQ, DDAQ,
+ RAR, RAHSQ, DRA, DAH
COMMON /INPGEN/ EN, DNH, N
COMMON /INPUT/ IX, IXR, IRS, NYSM
SAVE /XPNQMP/, /GFQTMP/, /GCONST/
C N=NIN
NA=ABS(N)
IF (NA .LT. 1) THEN
WRITE(*,*) 'Input ABS(N) is less than 1 in XPNQ.'
WRITE(*,*) 'Execution is stopped.'
STOP
END IF
C NUI=NA+1
NP1=NOI+1
NMI=NA
C DNO=NA
DNP=NOI
DNM=NA-1
C IF (IZ .EQ. 0) GO TO 4000
C Initialize the XN(P,Q) matrix.
C Null terms will be skipped later.
C DO 800 IQ=1,KCRNT
    DO 700 IP=1,KCRNT
        CXPQN(IP,IQ)=ZERO
    700    CONTINUE
800    CONTINUE
C Form the XN(P,Q) matrix.
    CF31=DKA*CTI
    F21=TW0*DNO
    FN11=F21*DNO/DASQ
C DO 1300 IQE=0,MXQEG-1

104
IQE1=IQE+1
IQ2=IQE
IQX1=4*IQ+1
IQX2=IQX1+1
IQX3=IQX2+1
IQX4=IQX3+1
DQ1=IQ+1
FQ12=DNQ0*DQ1
FQQ2=DQ1/HHQ0
DO 1100 IPB=0,MXQEG-1
IPE1=IPB
IP=2*IPB
IPX1=4*IP+1
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP1=IP+1
F41=HH/DP1
CF11=F41/CF31
F51=HALF*DKA*F41
CXPQN(IPX1, IQX1) = (CGNE(IPE, IQE, NOI) - CGNE(IPE1, IQE, NOI)) * FN11 +
- CGNE(IPE1, IQE, NOI) - CGNE(IPE, IQE, NOI) +
+ CGNE(IPE1, IQE, NOI) - CGNE(IPE, IQE, NOI) * CNM1 +
CXPQN(IPX4, IQX1) = (CGNE(IPE, IQE, NOI) - CGNE(IPE1, IQE, NOI) * DN1 +
- CGNE(IPE1, IQE, NOI) - CGNE(IPE, IQE, NOI)) * CF11 +
- HA*(CDGNE(IPE1, IQE, NOI) - CDGNE(IPE, IQE, NOI)) * F41
CXPQN(IPX2, IQX2) = (FQ22*DP1*CDNO(IPB, IQE, NOI) -
+ CGNE(IPE, IQE1, NOI) - CGNE(IPE1, IQE, NOI) -
+ CDGNE(IPE1, IQE, NOI) - CDGNE(IPE, IQE, NOI) * CF11 +
- CDGNE(IPE1, IQE1, NOI) - CDGNE(IPE, IQE, NOI) * F51
CXPQN(IPX3, IQX3) = CXPQN(IPX1, IQX1) +
CXPQN(IPX4, IQX4) = CXPQN(IPX2, IQX2) +
CXPQN(IPX4, IQX4) = CXPQN(IPX2, IQX2)
CXPQN(IPX4, IQX4) = CXPQN(IPX2, IQX2)

1100 CONTINUE
DO 1200 IPO=0,MXQOG-1
IP1= IPO+1
IP2= IPO+1
IPX1=4*IP+1
IPX2= IPX1+1
IPX3= IPX2+1
IPX4= IPX3+1
DP1= IP+1
F12=FQ12/DP1
CXPQN(IPX2, IQX1) = F21*CGNE(IPO1, IQE, NOI)
CXPQN(IPX3, IQX1) = (CGNE(IPO1, IQE, NOI) - CGNE(IPO1, IQE, NOI)) * CF31
CXPQN(IPX1, IQX2) = (CGNO(IPO1, IQE, NOI) - CGNO(IPO, IQE, NOI)) * F12
CXPQN(IPX3, IQX3) = CXPQN(IPX3, IQX1)
CXPQN(IPX4, IQX3) = CXPQN(IPX2, IQX1)
CXPQN(IPX3, IQX4) = CXPQN(IPX1, IQX2)
CXPQN(IPX4, IQX4) = CXPQN(IPX2, IQX2)

1200 CONTINUE
1300 CONTINUE
DO 2300 IQO=0,MXQOG-1
IQO1=IQO+1
IQ=2*IQO+1
IQX1=4*IQ+1
IQX2= IQX1+1
IQX3= IQX2+1
IQX4= IQX3+1
DQ1= IQ+1
FQ12= DNQ0*DQ1
FQQ2=DQ1/HHQ0
DO 2100 IPB=0,MXQBG-1
IPE1=IPB+1
IP=2*IPB
IPX1=4*IP+1
105
IPX2=IPX1+1
IPX3=IPX2+1
IPX4=IPX3+1
DP1=IP+1
F12=FQ12/DP1
CXPQN(IPX2, IQX1)=P21*CNO(IPE, IQO, NOI)
CXPQN(IPX3, IQX1)=(CGNO(IPE, IQO, NMI)+CGNO(IPE, IQO, NPI)) *CF11
CXPQN(IPX1, IQX2)=(CGNE(IPE1, IQO1, NOI)+CGNE(IPE1, IQO1, NOI)) *F12
CXPQN(IPX1, IQX3)=-CXPQN(IPX3, IQX1)
CXPQN(IPX4, IQX3)=CXPQN(IPX2, IQX1)
CXPQN(IPX3, IQX4)=CXPQN(IPX1, IQX2)

2100 CONTINUE
   DO 2200 IPO=0, MXPQG-1
   IP01=IPO+1
   IP=2*IP01+1
   IPX1=4*IP+1
   IPX2=IPX1+1
   IPX3=IPX2+1
   IPX4=IPX3+1
   DP1=IP+1
   F41=HH/DP1
   CF11=F41*CF31
   F51=HALP*DKA*F41
   CXPQN(IPX1, IQX1)=((CGNO(IPO1, IQO, NOI)+CGNO(IPO1, IQO, NOI))+CGNO(IPO1, IQO, NMI)+CGNO(IPO1, IQO, NPI)) *CF11
   CXPQN(IPX4, IQX1)=(CGNO(IPO1, IQO, NMI)+CGNO(IPO1, IQO, NPI)) *DNF+CN*(CDGNO(IPO1, IQO, NMI)+CDGNO(IPO1, IQO, NPI)) *F41
   CXPQN(IPX2, IQX2)=(FQ21*DP1*CGNE(IPO1, IQO1, NOI)+CGNO(IPO1, IQO1, NOI)+CGNO(IPO1, IQO1, NOI)+CDGNO(IPO1, IQO1, NOI)) *CF11
   CXPQN(IPX3, IQX2)=(CDGNO(IPO1, IQO, NOI)+CDGNO(IPO1, IQO, NOI)+CDGNO(IPO1, IQO, NOI)+CDGNO(IPO1, IQO, NOI)) *F51
   CXPQN(IPX2, IQX3)=-CXPQN(IPX4, IQX1)
   CXPQN(IPX3, IQX3)=CXPQN(IPX1, IQX1)
   CXPQN(IPX1, IQX4)=-CXPQN(IPX3, IQX2)
   CXPQN(IPX4, IQX4)=CXPQN(IPX2, IQX2)

2200 CONTINUE
2300 CONTINUE
RETURN
C Initialize the KN(P,Q) matrix.
C Null terms will be skipped later.
C
4000 DO 4800 IQ=1, KXCRK
   DO 4700 IP=1, KXCRK
   CXPQN(IP, IQ)=CZERO
4700 CONTINUE
4800 CONTINUE
C
C Form the KN(P,Q) matrix.
CP11=DKA*CT1
F21=TWO/DNO
FN11=F21*DNO/DASQ

C
   DO 5300 IQE=0, MXPQEG-1
   IQE1=IQE+1
   IQ=2*IQE
   IQX1=2*IQ+1
   IQX2=IQX1+1
   DQ1=IQ+1
   FQ12=DQ1/DASQ
   FQ22=DQ1/HASQ
   DO 5100 IPE=0, MXPQEG-1
   IPE1=IPE+1
   IP=2*IPE
   IPX1=2*IP+1
IPX2=IPX1+1
DPL=IP-1
F41=HH/DPI
CF11=F41*CF31
FS1=HALF*DKA*F41
CXPQNO(IPX1, IQX1)={(CGNE(IPN, IQE, NOI)-CGNE(IPN1, IQE, NOI))-FN1*+
+ CGNE(IPN, IQE, NOI)-CGNE(IPN1, IQE, NOI)+
+ CGNE(IPN, IQE, NOI)-CGNE(IPN1, IQE, NOI)+
+ CGNE(IPN, IQE, NOI)-CGNE(IPN1, IQE, NOI)}-CF11
CXPQNO(IPX2, IQX2)=(FQ22*DP1*CGNO(IPN, IQE, NOI)+
+ CGNE(IPN, IQE, NOI)+CGNE(IPN1, IQE, NOI)+
+ CGNE(IPN, IQE, NOI)-CGNE(IPN1, IQE, NOI)+
+ CGNE(IPN, IQE, NOI)-CGNE(IPN1, IQE, NOI)}*CF11

5100 CONTINUE
   DO 5200 IPO=0, MXQQG-1
   IP01=IPO+1
   IP=2*IPO+1
   IPX1=2*IP+1
   IPX2=IPX1+1
   DPL=IP-1
   F12=FQ12/DP1
   CXPQNO(IPX2, IQX1)=F21*CGNE(IP01, IQE, NOI)
   CXPQNO(IPX1, IQX2)=(CGNO(IP01, IQE, NOI)-CGNO(IP0, IQE, NOI))\*F12

5200 CONTINUE
5300 CONTINUE
   DO 6300 IQ0=0, MXQQG-1
   IQ01=IQ0+1
   IQ=2*IQ0-1
   IQX1=2*IQ+1
   IQX2=IQX1+1
   DQ1=IQ-1
   FQ12=DN0*DP1
   FQ22=DQ1/HSQ
   DO 6100 IPE=0, MXQS6-1
   IP1=IPE+1
   IP=2*IPE
   IPX1=2*IP+1
   IPX2=IPX1+1
   DPL=IP-1
   F12=FQ12/DP1
   CXPQNO(IPX2, IQX1)=F21*CGNO(IP1, IQE, NOI)
   CXPQNO(IPX1, IQX2)=(CGNE(IPN1, IQ01, NOI)-CGNE(IPN, IQ01, NOI))\*F12

6100 CONTINUE
   DO 6200 IPO=0, MXQQG-1
   IP01=IPO+1
   IP=2*IPO+1
   IPX1=2*IP+1
   IPX2=IPX1+1
   DPL=IP-1
   F41=HH/DPI
   CF11=F41*CF31
   FS1=HALF*DKA*F41
   CXPQNO(IPX1, IQX1)={(CGNO(IPO1, IQE, NOI)-CGNO(IP01, IQE, NOI))-FN1*+
+ CGNO(IPN1, IQE, NOI)-CGNO(IP1, IQE, NOI)-
+ CGNO(IPN1, IQE, NOI)-CGNO(IP1, IQE, NOI)-\*CF11
CXPQNO(IPX2, IQX2)=(FQ22*DP1*CGNE(IPN1, IQ01, NOI)+
+ CGNE(IPN, IQ01, NOI)+CGNE(IP1, IQ01, NOI)+
+ CGNE(IPN, IQ01, NOI)-CGNE(IP1, IQ01, NOI)+
+ CGNE(IPN, IQ01, NOI)-CGNE(IP1, IQ01, NOI)+\*CF11
6200 CONTINUE
6300 CONTINUE
RETURN
END

C
SUBROUTINE ESCFAR(NIN, CLN0, CLNL, CETRN, CEPHN)
C********************************************************************************
C This subroutine computes the scattered fields in far zone
C
C INCLUDE 'REALTFP, INC'
C INCLUDE 'CMPPFP, INC'
C INCLUDE 'MAINDFM, INC'

107
DIMENSION CKLN0(KXCRT)
DIMENSION CKLN(KCRN), DJNS(KNDIM1), DJPS(KQDIM1+2)
COMMON /CRNTDIM/ NMAX, MXNG, IQMAX, IQMAX1, IQMAX2, IXCNRNT, ICRNT, MXQEG,
+ MKQOG
COMMON /GCONST/ RKH, RKA, HH, NA, HSQ, DASQ, HSQN, DASQN, HHSQ, HHASQ,
+ RAM, RASHQ, DHA, DAH
COMMON /INPUT3/ RTHE, RDELT, RPHI, ROELP, NTHIAO, NPHI, THESIN, THECOS,
+ RPHI
COMMON /INPUT4/ IZ, IJ, IS, NYSM

N=NIN
DN1=N
NA=ABS(NIN)
NP=NA+1
NP1=NP+1
PHI1=RPHI*N
C12N=C11**N
CENP=CONE*COS(PHI1)+C11*SIN(PHI1)
CETHN=CZERO
CEPHN=CZERO

IF ((RTHE .EQ. 0.) .OR. (RTHE .EQ. PI)) GO TO 2000

TCOT=THECOS/THESIN
DL2SIN=DKA*THESIN
DL1COS=DKH*THECOS
DNL2=DN1*TCOT/DR

CALL DBSNS(DL2SIN, KNDIM1, DJNS)

DJN=DJNS(NP)
IF (N .EQ. 0) THEN
DDJN=-DJNS(NP1)
ELSE
DDJN=HALF*(DJNS(NA)-DJNS(NP1))
ENDIF

KN=NA/2
KNP=NP/2
IF ((N .LT. 0) .AND. (KNP .GT. KN)) THEN
DJN=-DJN
DDJN=-DDJN
ENDIF

CALL DBSNS(DL1COS, KQDIM1+2, DJPS)

IF (IZ .EQ. 0) THEN
DO 100 IP=0, IQMAX
INP=IP+1
NP2=INP+2
N11=IP
N12=IP+2
IE1=MOD(N11,4)
CF1=C12**IE1
IPW1=2*IP+1
IPW2=IPW1+1
DJP=DJPS(IP)
DJPS=DJPS(IP2)
DJPH=HALF*(DJP+DJPS)

CETHN=CETHN+C12N*DHA*CENP*C11*CF1*DJN*(DNL2*CKLN0(IPW1)*DJP
+ THECOS*CKLN0(IPW2)*DJPH)

CEPHN=CEPHN-C12N*DHA*CENP*DDJN*CKLN0(IPW1)*CF1

CONTINUE
ELSE
DO 200 IP=0, IQMAX
INP=IP+1
NP2=INP+2
N11=IP
CONTINUE
ENDIF

100 CONTINUE
ELSE
DO 200 IP=0, IQMAX
INP=IP+1
NP2=INP+2
N11=IP
CONTINUE
ENDIF

108
N12=IP+2
IE1=MOD(N11,4)
CF1=C12**IE1
IPW1=4*IP+1
IPW2=IPW1+1
IPW3=IPW2+1
IPW4=IPW3+1
DJP=DJPS(INP)
DJPH=DJPS(NP2)
DJPH2=HALF*(DJP+DJPH2)
CETHN=CETHN+CI12*DHA*CF1*(((CI1*DJN*DNL2*CKLN(IPW1)-DDJN
+*CKLN(IPW3))*DJP-CI1*DJN*THESIN*CKLN(IPW2)*DJPH)*CF1
+*CKLN(IPW3))*DJP-CI1*DJN*THESIN*CKLN(IPW4)*DJPH)*CF1
+ *CF1
200 CONTINUE
END IF
RETURN
C
2000 IF (NA .NE. 1) THEN
GO TO 3000
END IF
CALL DBSINS(DKH,KQDIM1,DJPS)
C
IF (IZ .EQ. 0) THEN
DO 2100 IP=0,IQMAX
INF=IP+1
IPW1=2*IP+1
JF=MOD(IP,4)
CF1=CI1**JP
CF2=C12**JP
C
DJP=DJPS(INP)
IF (RTHE .EQ. 0.) THEN
IF (N .LT. 0) THEN
CETHN=CETHN+DAH*CF2*CKLN0(IPW1)
CEPHN=CEPHN+DAH*CKLN0(IPW1)
ELSE
CETHN=CETHN+DAH*CF2*CKLN0(IPW1)
CEPHN=CEPHN+DAH*CF2*CI1*CKLN0(IPW1)
END IF
END IF
IF (RTHE .EQ. PI) THEN
IF (N .LT. 0) THEN
CETHN=CETHN+DAH*CF1*CKLN0(IPW1)
CEPHN=CEPHN+DAH*CKLN0(IPW1)
ELSE
CETHN=CETHN+DAH*CF1*CKLN0(IPW1)
CEPHN=CEPHN+DAH*CF1*CI1*CKLN0(IPW1)
END IF
2100 CONTINUE
END IF
ELSE
DO 2200 IP=0,IQMAX
INF=IP+1
IPW1=4*IP+1
IPW3=IPW1+2
JF=MOD(IP,4)
CF1=CI1**JP
CF2=C12**JP
C
DJP=DJPS(INP)
IF (RTHE .EQ. 0.) THEN
IF (N .LT. 0) THEN
CETHN=CETHN+DAH*CF2*(CKLN(IPW1)-CI1*CKLN(IPW3))
CEPHN=CEPHN+DAH*CF2*(CI1*CKLN(IPW1)-CKLN(IPW3))
ELSE
CETHN=CETHN+DAH*CF2*(CKLN(IPW1)+CI1*CKLN(IPW3))
CEPHN=CEPHN+DAH*CF2*(CI1*CKLN(IPW1)-CKLN(IPW3))

109
END IF
END IF
IF (RTHE .EQ. PI) THEN
IF (N .LT. 0) THEN
CETHN=CETHN+DAH*DF1*CKLN(IPW1)+CI1*CKLN(IPW3)
CEPHN=CEPHN+DAH*DF1*CI1*CKLN(IPW1)-CKLN(IPW3)
ELSE
CETHN=CETHN+DAH*DF1*CKLN(IPW1)-CI1*CKLN(IPW3)
CEPHN=CEPHN+DAH*DF1*CI1*CKLN(IPW1)+CKLN(IPW3)
END IF
END IF
2200 CONTINUE
END IF
3000 RETURN
END

SUBROUTINE KLERN(TX,CKLN,CIW)

C*****************************************************************************
C This subroutine adjusts equivalent current K and L on inner and outer
C surfaces respectively.
INCLUDE 'REALPF.INC'
INCLUDE 'CMPXTP.INC'
INCLUDE 'MAINDM.INC'
C
DIMENSION CKLN(KCERN),CIW(KCERN)
DIMENSION CKLNP(KCERN),CKLN(KCERN)
DIMENSION CRQP31(KCERN,KCERN),CRQP32(KCERN,KCERN)
DIMENSION CKHPH(61,61),CKZP(61,61),CLPHP(61,61),
+ CLZP(61,61),CKPHN(61,61),CKZN(61,61),
+ CLPHN(61,61),CLZN(61,61)
COMMON /CKLMTX/ CKHPH,CKZP,CLPHP,CLZP,CKPHN,CKZN,CLPHN,CLZN
COMMON /CRQPMP/ CRQP31,CRQP32
COMMON /CRMTMP/ NMAX,MAXG,MINMAX1,MINMAX2,IKERN,ICRN,MAXQGB,
+ MXQOG
COMMON /INPUT3/ RTHE,RELT,RPHI,RELP,NTHTAO,NPHI,THESEN,THECOS,
+ RHI
COMMON /INPUT4/ T2,T1,IS,NYSN
C
DO 400 IX=1,ICRN
CKLNP(IX)=ZERO
CKLN(IX)=ZERO
400 CONTINUE
PH1=DN*RPHI
CPI=CONE*COS(PH1)+CI1*SIN(PH1)
C
DO 600 IX=1,ICRN
DO 500 IY=1,ICRN
CKLNP(IX)=CKLNP(IX)+CRQP31(IX,IY)*CKLN(IY)
500 CONTINUE
600 CONTINUE
C
DO 800 IX=1,ICRN
DO 700 IY=1,ICRN
CKLN(IX)=CKLN(IX)+CRQP32(IX,IY)*CKLN(IY)
700 CONTINUE
800 CONTINUE
C
DO 1600 IY=-30.30
DF=IF
DPI=DF*PI/32.D0
DNPHI=DN*RPHI
CPI=CONE*DCOS(DNPHI)+CI1*DSIN(DNPHI)
DO 1500 JZ=-30.30
DJZ=JZ
DZ=DJZ/32.D0
DV=DCOS(DZ)
SINV=DSIN(DV)

110
J. SUBROUTINE RCSAREA

SUBROUTINE RCSAREA(CESTH, CESPH)
C******************************************************************************
C This subroutine computes cross section per projected area in all direction.
C******************************************************************************
C INCLUDE 'REALTP, INC'
C INCLUDE 'CMXPRT, INC'
C
REAL*4 ARCSPPA1, ARCSPPA2, APHASE1, APHASE2
COMMON /GCONST/ DKH, DKA, HH, HA, HSQ, DASQ, HNSQ, DASQN, HHSQ, DASQH,
+ RHAH, RHASH, DHA, DAH
COMMON /INPUT3/ RTHE, RDELT, RPHI, RDELP, NTHTAO, NPHI, THESIN, THECOS,
+ RPHI
C
ESTH=ABS(CESTH)
ESPH=ABS(CESPH)
ESTSQ=ESTH*ESTH
EPSHSQ=ESPH*ESPH
IF (RTHE .EQ. RHPI) THEN
  RCSPPA1=PI*ESTSQ/DKH/DKA
  RCSPPA2=PI*EPSHSQ/DKH/DKA
ELSE IF ((RTHE .EQ. ZERO) .OR. (RTHE .EQ. PI)) THEN
  RCSPPA1=4.D0*ESTSQ/DASQ
  RCSPPA2=4.D0*EPSHSQ/DASQ
ELSE
AP = ABS(QUR * DASQ * THECOS) + ABS(DKH * DXA * THESIN / PI)
RCSSPA1 = ESTEMQ / AP
RCSSPA2 = ESMH2Q / AP
END IF
ARCSSPA1 = RCSSSPA1
ARCSSPA2 = RCSSSPA2
C
ER1 = REAL(CESTH)
EI1 = IMAG(CESTH)
ER2 = REAL(CESPH)
EI2 = IMAG(CESPH)
PHASE1 = DATAN2(EIF, ER1)
PHASE2 = DATAN2(EIF, ER2)
APHASE1 = PHASE1
APHASE2 = PHASE2
WRITE (21, *) ARCSSPA1, APhASE1, ARCSSPA2, APhASE2
C
RETURN
END
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