COATING SELECTION PROGRAM

Theory

by Frederick A. Costello, Thomas P. Harper, and Barbara Aston

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JULY 2, 1996
ABSTRACT

A rational and direct method has been developed for selecting the optical coating pattern for the external surface of a spacecraft, such that the spacecraft temperatures are as close as possible to the midpoint of their preselected ranges. The temperature control is maintained passively by radiation and conduction, using no active control devices. The complete range in mission environments is considered in the optimization procedure.

The selection technique has been programmed for use on the GE 600 Series, IBM 7094, or any other computer that uses the standard IBM Fortran IV system.
FORWARD

This document is a three-part final report on the Coating Selection Program. Part I describes the theory and basis for selection of: (1) the iteration scheme used to solve the heat balance equations; and (2) the optimization scheme. Part II describes the computer program, including the details of each subroutine and the details of input and output. Part II therefore includes the user's manual. Part III presents the Program Listing. Parts I, II and III have been revised (Revision A) under a contract extension to incorporate descriptions of the first two month's usage, as well as a new Program Listing with improvements as found advisable during these two months.

The work reported herein was sponsored by the National Aeronautics and Space Administration and was monitored by Mr. Conrad Mook, of NASA-Headquarters, and Mr. Robert Kidwell, Jr., of NASA's Goddard Space Flight Center.

The chief contributors to the work reported were Mr. Frederick A. Costello, Engineer, who developed the techniques, Mr. Thomas P. Harper, Analyst, who converted the techniques to computer form, both of the Re-entry Systems Department's Thermodynamics Technology Component, and Miss Barbara Aston, Programmer-Analyst, from the Spacecraft Department, who programmed the technique for computer usage.
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1.0 INTRODUCTION

The use of optical coatings to control the temperatures of satellites has been exploited since the first successful orbital flight in 1958 (Explorer). Since then, coating materials have been developed to the point where coatings are available that give any desired emittance value between 0.1 and 0.9 for any desired absorptance value between 0.1 and 0.9. What has lagged, however, is the development of a systematic approach to selecting coating patterns for the external surface of the vehicle, such that the satellite temperatures are passively maintained as close as possible to the optimum temperature. It was in response to this developmental need that the present work was undertaken. The work discussed in this report constitutes a generalization of the work performed by Costello, Harper, and Cline.

For the uninitiated, an example may prove useful in illustrating the effect of proper coating-pattern selection. Figure 1 shows the environmental conditions for a typical satellite, as well as the optimum coating pattern and resulting temperatures. It is seen that using two different coatings rather than one narrows the temperature excursions from ±7°F to ±5°F from the desired 70°F. This result is not as dramatic as those obtained for more complicated shapes. Further improvement is still possible, but the point is adequately illustrated.

The solution to simple coating selection problems, as shown in Figure 1, is indeed difficult, but the complexity and difficulty increase rapidly as the number of surfaces increases and as the number of critical components increases. For realistic satellite designs, it is usually necessary to include the entire conduction and radiation heat transfer networks; consequently, the selection process is obscured by the various interactions and by the number of degrees of freedom (twice the number of external surfaces). Such complexity demands the use of a high-speed computational device. From the beginning, then, the present work has been directed toward the development of an IBM 7094 digital-computer program that would assist the designer in the coating selection process.

In the following sections, the coating-selection problem is formulated in mathematically precise terms. The mathematical development of the minimization scheme is presented in Sections 3.0, 4.0 and 5.0. Applications are considered in Section 6.0. The work is summarized in Section 7.0 and future areas for research are cited in Section 8.0.

It should be recognized that operating experience is important to the efficient usage of a computer program. The present program has, as all programs do, a variety of input constants that affect such things as iteration procedures and optimization sequences. An extention to the original contract enabled a study of the input constants. The study was conducted by analyzing several vehicle designs, three of which are shown in Section 6. The operating experience obtained from the study along with recommended procedures for use of the program is summarized in the User's Manual. It is felt that the study, permitted by the contract extension, has enhanced significantly the benefits of the program.
SOLAR RAY

\(30^\circ\)

\(\text{(POSITION A)}\)

\(S_1\)

\(E_2, A_2\)

\[\begin{array}{|c|c|c|c|}
\hline
\text{Orbit} & \text{Heat Fluxes} \\
\hline
\text{Solar Angle} & \text{Side 1} & \text{Side 2} & \text{Albedo Flux} \\
& \text{Solar Flux} & \text{Earth Flux} & \text{Flux} \\
\hline
0^\circ & 140.0 & 70.0 & 50.5 \\
30^\circ & 121.0 & 70.0 & 43.5 \\
\hline
\end{array}\]

\(\text{Coating Limit}\)

\[\begin{array}{c}
0.1 \leq \alpha \leq 0.9 \\
0.1 \leq \epsilon \leq 0.9
\end{array}\]

\(T = 70^\circ F\) is desired

**Single Coating**

\[(140 + 50.5) \alpha + (70) \epsilon = 2 \epsilon \sigma T^4\]

\[(121 + 43.5) \alpha + (70) \epsilon = 2 \epsilon \sigma T^4\]

Best Solution

\[\begin{array}{ccc}
\alpha & \epsilon & T \\
0.9 & 0.8 & \boxed{63-77^\circ F}
\end{array}\]

**Double Coating**

\[140 \alpha_1 + 50.5 \alpha_2 + 70 \epsilon_2 = (\epsilon_1 + \epsilon_2) \sigma T^4\]

\[121 \alpha_1 + 43.5 \alpha_2 + 70 \epsilon_2 = (\epsilon_1 + \epsilon_2) \sigma T^4\]

Best Solution

\[\begin{array}{ccccc}
\alpha_1 & \alpha_2 & \epsilon_1 & \epsilon_2 & T \\
0.4 & 0.5 & 0.345 & 0.5 & \boxed{65-75^\circ F}
\end{array}\]

Figure 1. Example of Two-Coating Advantages
2.0 PROBLEM STATEMENT

As intimated in the Introduction, the purpose of the present work was to develop a computer program that would determine the optical coating pattern that minimizes temperature excursions from some preselected temperatures for all possible environmental conditions. More precisely, we define the excursion parameter, $\beta_{ik}$ as

$$
\beta_{ik} = \max \left[ \frac{T_{ik} - T_{iU}}{T_{iU} - T_{iL}} \right] \left[ \frac{T_{iL} - T_{ik}}{T_{iU} - T_{iL}} \right]
$$

(2.1)

Where $T_{ik}$ is the temperature of the $i^{th}$ network element (node) in the $k^{th}$ orbit, and $T_{iU}$ and $T_{iL}$ are the upper and lower allowable temperature limits of the $i^{th}$ node. We wish to find the surface coating properties of infra-red emittance, $\epsilon$, and solar absorptance, $\alpha$, such that

$$
\beta = \max_{i,k} \beta_{ik} = \min_{i,k}
$$

(2.2)

subject to the constraints that $\alpha$ and $\epsilon$ are within a preselected allowable range.

From the above definitions, it is obvious that $(-1/2)$ is the absolute minimum value of $\beta_{ik}$ and that $\beta \leq 0$ implies that all nodes are within their allowable temperature limits.

Physically, the above formulation implies that the coating system is selected to minimize the temperature excursion of the node that is furthest from the midpoint of its allowable temperature range.

Other Possible Criteria

The above criterion, Equation (2.2), was selected from a number of alternates, the most interesting of which were

$$
\gamma_1 = \sum_{i,k} (\beta_{ik})^2 = \min_{i,k}
$$

(2.3)

$$
\gamma_2 = \sum_i (C_i \alpha_i + \alpha_i \epsilon_i) = \min_{i,k}
$$

(2.4)

subject to the additional constraints $\beta_{ik} \leq 0$

Equation (2.3) was the most convenient from a mathematical viewpoint, but $\gamma_1$ could be a minimum when one temperature was so far out of range as to make the solution ridiculous.
Equation (2.4) resembles a linear programming formulation and is physically the most satisfying. However, the equations are significantly non-linear, and the optimum may lie at large distances from the vertices of the restraint volume.

**Time-Dependent Solutions**

Inherent in the above problem statement is the assumption that the critical temperature can be adequately calculated using time-averaged environmental heat fluxes. This is an important and restrictive assumption, complicated by the fact that the time constants of each node are different and vary non-linearly with the node temperature. In Section 8.0, further study is recommended in this area. In the meantime, however, the average-flux procedure should serve as a useful guide in selecting the optimum coating pattern. The average-flux analysis can be made quite accurate if the emittance is kept low and/or the node heat capacities kept high, so that all the time constants are significantly greater than, say, the orbit period.

**Development of Solution**

The solution to Equation (2.2) can be conveniently, but not completely, divided into two steps: (1) solving for the $T_{ik}$'s, given a set of $\alpha$'s and $\epsilon$'s; and (2) selecting new values of $\alpha$ and $\epsilon$ to reduce $\delta^*$. The first problem is discussed in Section 3.0; the second, in Sections 4.0 and 5.0.
3.0 SOLVING THE HEAT-BALANCE EQUATIONS

The node temperatures are governed by the heat-balance equations:

\[ \sum_j K_{ij} (T_{ik} - T_{jk}) + \sum_j \sigma R_{ij} (T_{ik}^4 - T_{jk}^4) = \]

\[ = \alpha_i (K_{Si} S_{ik} + K_{Al} A_{ik}) + \epsilon_i (K_{Ei} E_{ik} - a_i \sigma T_{ik}^4) + K_{Q_i} Q_{ik} \]

\[ \text{incident flux} \quad \text{incident flux} \quad \text{incident re-emission flux} \quad \text{internal heat generation} \]

absorbed solar energy net absorbed IR energy

where

\[ \begin{align*}
  i, j &= \text{node numbers} = 1, 2, \ldots, N + S \\
  k &= \text{orbit, or time-interval, number} = 1, 2, \ldots, q
\end{align*} \]

and where the nomenclature is described in Appendix D (Section 9.0).

Equation (3.1) can be written in the more compact form

\[ \sum_j \bar{A}_{ij} T_j + \sum_j M_{ij} T_j^4 = \alpha_i S_i + \epsilon_i E_i - \epsilon_i a_i \sigma T_i^4 + Q_i \]  

\[ i = 1, 2, \ldots, N \]

where for convenience we have dropped the \( k \) (orbit number) subscript. In discussing solutions to Equation (3.2), it is frequently necessary to examine the symmetrically linearized form:

\[ \sum_j L_{ij} T_j = C_i \]  

\[ i = 1, 2, \ldots, N \]

where

\[ L_{ij} = \begin{dcases} 
  -K_{ij} - \sigma R_{ij} (\tilde{T}_i + \tilde{T}_j) (\tilde{T}_i^2 + \tilde{T}_j^2) & \text{if } i \neq j \\
  \sum_{j=1}^{N+S} L_{ij} + 4 a_i \epsilon_i \sigma T_i^3 & \text{if } i = j \end{dcases} \]
\[ C_i = \alpha_i (K_{Si} S_i + K_{Ai} A_{i}) + \epsilon_i (K_E E_i + 3 a_i \sigma \tilde{T}_i^4) K_{Qi} Q_i \]

where once again k has been dropped.

Many methods for solving the linear system, Equation (3.3), have been devised. A convenient and up-to-date summary is given by Fox (3). These methods have been adapted to the present set of equations and the speeds of convergence compared. In addition, two new matrix-inversion methods have been examined. In the following paragraphs, each method is described. All methods make use of the residual vector, \( r_i \):

\[ r_i^{(n)} = \left[ \alpha_i S_i + \epsilon_i E_i - \epsilon_i a_i \sigma T_i^4 + Q_i - \sum_j K_{ij} T_j^{(n)} - \sum_j M_{ij} (T_j^{(n)})^4 \right] \]

(3.5)

When the exact solution is obtained, \( r_i = 0 \) for all values of "i".

**Gauss-Seidel Methods**

Applied to the linear system, Equation (3.3), the extrapolated Gauss-Seidel method gives:

\[ T_i^{(n+1)} = \frac{C_i - \sum_{j<i} L_{ij} T_j^{(n+1)} - \sum_{j>i} L_{ij} T_j^{(n)}}{L_{ii}} \]

\[ T_i^{(n+1)} = \frac{1}{\alpha_n} T_i^{'} + \frac{\alpha_n^{-1}}{\alpha_n} T_i^{(n)} \]

In terms of the residual vector, this may be written

\[ T_i^{(n+1)} = T_i^{(n)} + \frac{r_i^{(n)} + \sum_{j<i} L_{ij} (T_j^{(n)} - T_j^{(n+1)})}{\alpha_n L_{ii}} \]  

(Symmetrically Linearized Gauss-Seidel)

(3.6)

The factor \( \alpha_n \) is the extrapolation (\( \alpha_n < 1 \)) or interpolation (\( \alpha_n > 1 \)) factor.

Equation (3.6) was used directly for the present non-linear case, with the residual vector defined by Equation (3.5).

In the non-stationary iteration process, \( \alpha_n \) is governed by the equation

\[ \alpha_n = 1 - \frac{T^{(n+1)} - T^{(n)}}{T^{(n)} - T^{(n-1)}} \]
However, from the experience of Kaplan (4), it is necessary to restrict $\alpha$ between 1/2 and 1 to obtain convergence for the heat-balance equations. For the present case, $\alpha^*$ was taken as 1.0, a constant.

Some saving in computation can be realized if the linearization process applied to Equation (3.2) takes a Taylor-series form:

$$\sum_j \bar{A}_{ij} T_j + \sum_j M_{ij} T_j^4 = \sum_j \left[ (\bar{A}_{ij} + 4M_{ij} \bar{T}_j^3 ) T_j - 3 M_{ij} \bar{T}_j^4 \right]$$

In this case, the iteration formula becomes

$$T_i^{(n+1)} = T_i^{(n)} + \frac{r_i^{(n)} - \sum_{j \neq i} \left[ (\bar{A}_{ij} + 4M_{ij} \bar{T}_i^3 ) \left( T_j^{(n+1)} - T_j^{(n)} \right) \right]}{\bar{A}_{ii} + 4M_{ii} \bar{T}_i^3}$$

(3.7)

**Maximum Rate of Descent**

Applied to Equation (3.3), the maximum rate of descent method described by Fox (3) gives:

$$T_i^{(n+1)} = T_i^{(n)} + \frac{\sum_j r_j^{(n)} r_j^{(n)}}{\sum_{i,j} L_{ij} r_i^{(n)} r_j^{(n)}} r_i^{(n)}$$

(3.8)

This was carried over directly, using Equation (3.5) to calculate $r_j$.

**Conjugate Gradient Method**

Fox gives for the linear system

$$T_i^{(n+1)} = T_i^{(n)} + \frac{\sum_j w_j^{(n)} r_j^{(n)}}{\sum_{i,j} L_{ij} w_i^{(n)} w_j^{(n)}} w_i^{(n)}$$

(3.9)

where

$$w_i^{(n)} = \begin{cases} r_i^{(n)} & n = 0 \\ r_i^{(n)} - \frac{\sum_{i,j} L_{ij} r_i^{(n)} w_j^{(n-1)}}{\sum_{i,j} L_{ij} w_i^{(n-1)} w_j^{(n-1)}} w_i^{(n-1)} & n \geq 1 \end{cases}$$

Once again, using Equation (3.5) to define $r_j$, Equation (3.9) can be used in the non-linear case.
Newton-Raphson Method

The Newton-Raphson method uses Equation (3.5) directly, taking

\[ r_i^{(n+1)} = r_i^{(n)} + \sum_j \left( \frac{\partial r_i}{\partial T_j} \right)^{(n)} \left[ T_j^{(n+1)} - T_j^{(n)} \right] \]

Setting \( r_i^{(n+1)} \) equal to zero gives

\[ T_i^{(n+1)} = T_i^{(n)} - \sum_j N_{ij}^{(n)} r_j^{(n)} \]

where \( \{N_{ij}\} \) is the inverse of the matrix \( \left( \frac{\partial r_i}{\partial T_j} \right) \) where

\[ \left( \frac{\partial r_i}{\partial T_j} \right)^{(n)} = \begin{cases} -\bar{A}_{ij} - 4 M_{ij} T_j^3 & \text{if } i \neq j \\ -4 \epsilon_i a_i \sigma T_j^3 - \bar{A}_{ii} - 4 M_{ij} T_j^3 & \text{if } i = j \end{cases} \]

This method requires a matrix inversion at each iteration and was found slower than the matrix-inversion method described next.

One way of alleviating the difficulty of inverting a matrix each time is to use an approximate inverse in the form

\[ (\tilde{E} + \epsilon)^{-1} = \left[ \tilde{E} \left( I + \tilde{E}^{-1} \epsilon \right) \right]^{-1} = \left[ I - \tilde{E}^{-1} \epsilon \right]^{-1} \tilde{E}^{-1} = (I - \tilde{E}^{-1} \epsilon) \tilde{E}^{-1} \]

Then the expression for \( T^{(n+1)} \) becomes, for \( \tilde{E}_{ij} = \left( \frac{\partial r_i}{\partial T_j} \right) T_j = T_j^{(o)} \) and \( \tilde{N} = \tilde{E}^{-1} \)

\[ T_i^{(n+1)} = T_i^{(n)} - \sum_j \tilde{N}_{ij}^{(n)} r_j^{(n)} + \sum_{j,k,l} \tilde{N}_{ij}^{(n)} \epsilon_j^{(n)} \tilde{N}_{kl}^{(n)} r_l^{(n)} \] (Expanded Inverse)

where

\[ \epsilon_{ij}^{(n)} = \begin{cases} -4 M_{ij} \left[ (T_j^{(n)})^3 - (T_j^{(o)})^3 \right] & \text{if } i \neq j \\ -4 \left[ \epsilon_i^{(o)} a_i \sigma + M_{ij} \right] \left[ (T_j^{(n)})^3 - (T_j^{(o)})^3 \right] & \text{if } i = j \end{cases} \]
Matrix Inversion Methods

A direct Matrix Inversion method of solving Equation (3.2) can be obtained simply by writing:

$$T_1^{(n+1)} = \left( \bar{A}'_{ij} \right)^{-1} \left\{ X_j T_j^{(n)} + \alpha_j S_j + \epsilon_j E_j - \epsilon_j a_j \sigma (T_j^{(n)})^4 + Q_j - \sum_k M_{jk} (T_k^{(n)})^4 \right\}$$

(3.10)

where $\bar{A}'_{ij}$ differs from $\bar{A}_{ij}$ in that a convergence factor, $X_i$, has been added, according to the convergence criteria of Appendix B (Section 9.0). Note that Equation (3.10) converges quite rapidly when the radiation terms are small.

Using the residual vector, Equation (3.10) can be written in the form

$$T_1^{(n+1)} = T_1^{(n)} + \sum_j J'_{ij} r_j^{(n)} \quad \text{(Direct Matrix Inversion)}$$

(3.11)

where

$$\begin{Bmatrix} J'_{ij} \\ \bar{A}'_{ij} \end{Bmatrix} = \left( \bar{A}'_{ij} \right)^{-1}$$

which is evaluated once per solution.

One improvement to Equation (3.11) can be obtained by writing Equation (3.2) in the form

$$\bar{A}''_{ij} = \begin{cases} \bar{A}_{ij} + M_{ij} \left[ T_i^{(o)} + T_j^{(o)} \right] \cdot \left[ (T_i^{(o)})^2 + (T_j^{(o)})^2 \right] = \bar{A}_{ij} + M_{ij} & i \neq j \\ - \sum_{j \neq i} \bar{A}''_{ij} + 4 \epsilon_i^{(o)} \sigma a_i (T_i^{(o)})^3 + X_i = \bar{A}_{ii} + M_{ii} & i = j \end{cases}$$

and

$$\sum_j \left( \bar{A}_{ij} T_j + M_{ij} T_j^4 + M_{ij} T_j^{(o)} \right) = \alpha_i S_i + \epsilon_i E_i - \epsilon_i \sigma a_i T_i^4 + Q_i + \sum_j \bar{M}_{ij} T_j^{(o)}$$
Then
\[ T^{(n+1)}_i = \sum_j J''_{ij} \left[ \alpha_j S_j + \epsilon_j E_j - \epsilon_j \sigma a_j (T^{(n)}_j)^4 + Q_j + \sum_k \tilde{M}_{jk} T^{(0)}_k \right] \]

In terms of the residual vector
\[ T^{(n+1)}_i = T^{(n)}_i + \sum_j J''_{ij} r^{(n)}_j \quad \text{(Linearized Matrix Inversion)} \quad (3.12) \]

\( J''_{ij} \) is evaluated once per solution.

**Comparison of Methods**

It is beyond the scope of this development program to develop the system of theorems required to compare the rates of convergence of the above methods of solving the heat-balance equations. Such a task is rendered difficult by the heterogeneous character of the iteration schemes. However, it is practical and important to evaluate the schemes numerically for typical satellite-type heat-balance equations. The methods are compared in Tables I and II.

There are several parameters that must be defined before the significance of the following numerical comparisons can be understood. Convergence is assumed to be completed if
\[ |T^{(n+1)}_i - T^{(n)}_i| \leq DT \quad i = 1, 2, \ldots, N \]

and
\[ |r^{(n+1)}_i - r^{(n)}_i| \leq DR \quad i = 1, 2, \ldots, N \]

In addition, those methods that use \( \tilde{T} \), the linearizing temperature, update \( \tilde{T} \) (i.e., let \( T = T^{(n+1)} \) every time
\[ |T^{(n+1)}_i - T^{(n)}_i| \leq DTR \quad i = 1, 2, \ldots, N \]

Those methods using the convergence parameter, \( X \), calculate the value of \( X \) using \( T_{\text{MAX}} \) and/or \( T_{\text{SOL}} \).
### TABLE I. SOLUTION TIMES FOR ITERATION SCHEMES

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<th>Iteration Scheme</th>
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<td>1.6</td>
</tr>
</tbody>
</table>

#### NOTES

1. **Problems Used Were:**
   - Pure Radiation ($h_A \approx 1.0$ BTU/hr°F between adjacent nose)
   - Pure conduction, except external surfaces which used pure radiation ($h_A \approx 1.0$)
   - Combined conduction/radiation; $h_A \approx kA/x = 1.0$

In all cases, 8, 16, and 64 node problems were considered. The geometry was a cube broken into equally sized cubes. The 8 node case used a 2 x 2 x 2 configuration; the 16, a 2 x 2 x 4; and the 64, a 4 x 4 x 4.

2. In general, $DTR = DR = DT = 1.0$ for the above cases. A 20% to 30% improvement was made in all methods but (1) and (2) by setting $DTR = 0.0$; that is, by not correcting the original linearization.

3. Methods 1, 2, and 3 require specification of $T_{MAX}$ to assure convergence. The correlated time is good for $T_{MAX} \leq 1.3 T_{SOL}$. A 20% increase in running time was experienced for $T_{MAX} = 1.5 T_{SOL}$.

4. See Table II for actual running times and additional notes.
### TABLE II. TIMING DATA FOR ITERATION SCHEMES

<table>
<thead>
<tr>
<th>Problem</th>
<th>Nodes</th>
<th>ΔT (%)</th>
<th>Iteration Scheme</th>
<th>Millisecond (Times In Nodes)</th>
<th>1.8 (ΔT)</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(Times In Nodes)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>8</td>
<td>2.6</td>
<td>1.40 3.08 1.13</td>
<td>1.40 2.52 1.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>2.4</td>
<td>1.28 3.37 1.21</td>
<td>1.77 2.67 1.53</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>2.4</td>
<td>2.34 4.77 1.63</td>
<td>2.32 3.08 1.73</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>15.6</td>
<td>1.45 2.89 0.92</td>
<td>1.18 1.05 1.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>15.6</td>
<td>1.55 3.06 0.95</td>
<td>1.10 1.02 1.36</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>15.6</td>
<td>2.25 4.56 1.45</td>
<td>1.70 1.19 1.55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>2.4</td>
<td>0.84 1.96 1.12</td>
<td>1.40 2.50 1.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>2.4</td>
<td>0.563 0.723 1.45</td>
<td>1.69 2.56 1.12</td>
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</tr>
<tr>
<td></td>
<td>64</td>
<td>2.4</td>
<td>0.482 0.63 1.49</td>
<td>2.31 3.07 1.30</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>15.6</td>
<td>0.66 0.79 1.58</td>
<td>1.05 1.32 1.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>15.6</td>
<td>0.416 0.68 1.36</td>
<td>1.32 1.29 1.02</td>
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</tr>
<tr>
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<td>64</td>
<td>15.6</td>
<td>0.446 0.66 1.5</td>
<td>1.83 1.47 1.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>8</td>
<td>2.4</td>
<td>1.68 3.08 1.68</td>
<td>2.24 3.48 1.40</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>2.4</td>
<td>1.69 2.98 1.77</td>
<td>2.81 3.70 1.37</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>64</td>
<td>2.4</td>
<td>2.19 4.24 2.42</td>
<td>4.13 4.70 1.47</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>8</td>
<td>5.9</td>
<td>1.56 2.92 1.56</td>
<td>2.14 3.30 1.36</td>
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<td>16</td>
<td>5.9</td>
<td>1.50 2.90 1.56</td>
<td>2.40 3.24 1.28</td>
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<td>5.9</td>
<td>2.17 4.39 3.24</td>
<td>3.36 4.46 1.48</td>
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<td></td>
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<tr>
<td></td>
<td>8</td>
<td>15.6</td>
<td>1.53 2.76 1.32</td>
<td>1.72 1.45 1.45</td>
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<td>15.6</td>
<td>1.44 2.87 1.51</td>
<td>2.04 1.36 1.36</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>15.6</td>
<td>2.13 3.96 2.05</td>
<td>2.84 1.53 1.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
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<td>15.6</td>
<td>0.526 1.32 4.48(1)</td>
<td>6.97(1) 4.21(1) 1.58</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>15.6</td>
<td>0.378 1.28 6.65(2)</td>
<td>12.3(2) 8.69(2) 2.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>15.6</td>
<td>0.375 0.673 7.63(2)</td>
<td>1.91(3) 15.2(2) 2.15(1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**NOTES:**

1. For these runs, DT = 1.0 and DR = 0.5, but the converged solutions were 3 to 7°F in error.

2. For these runs, DT = 1.0 and DR = 0.5, but the runs were terminated for being unsuccessful after 100 iterations. Solutions are normally obtained in 10-20 iterations.

3. For this run, DT = DR = 1.0, but the converged solution was about 80°F in error.

4. ΔT is the change (in %) in equilibrium temperature, based on the initial temperature in R, from the initial temperature to the exact final temperature (TSOL).

5. The geometry used was a cube broken into equally sized cubes.
Examination of Tables I and II reveals that Method (2) is the fastest in nearly every case. One notable exception is the conduction dominated case (B), where Method (1) was faster. The reason for this is that (2) used too conservative a stability factor for these particular cases. If the normal values had been used (see Appendix B), Method (2) would have shown the approximately 20% improvement over Method (1) that is evident in the other cases analyzed.

One other case was studied that is not reported on the tables. This was a 67-node model of an ablative heat shield for a satellite/re-entry vehicle. For this problem, due to the peculiarities of the conduction matrix, \( A_{ij} \), only Methods (1) and (2) converged, the times correlating well with those shown on the tables. The other methods were no more than 1% of the way toward a solution after 100 iterations, indicating that their convergence times would be a factor of one thousand greater than the times indicated on Tables I and II.

The choice of methods, therefore, is clearly in favor of Method (2).
4.0 GEOMETRY OF THE $\beta$ SURFACES

Of paramount importance in the development of any optimization scheme is fore-
knowledge of the geometry of the criterion surface. For example, if there is
more than one minimum point, a method must be devised for first finding one
minimum, then search for the second, and so on, to determine which point is the
absolute minimum. In the following paragraphs, unimodality and other properties
of the $\beta$-surface are examined.

Monotonic $\beta_{ik}$

The first important characteristic of $\beta$ is that it is composed of segments of mono-
tonic surfaces. That it is composed of segments of surfaces can be seen from the
definition of $\beta$ (see Equation 2.2). That $\beta_{ik}$ is composed of two monotonic branches
can be seen as follows:

The heat-balance equation can be written:

$$
\sum_j \overline{A}_{ij} T_j + \sum_j M_{ij} T_j^4 = \alpha_1 (K_{Si} S_i + K_{Ai} A_i) + \epsilon_1 (K_{Ei} E_i - \sigma a_i T_i^4) + K_{Q_i} Q_i
$$

(4.1)

This equation can be linearized in $T_4^4$ according to the scheme

$$
\sum_j \overline{A}_{ij} T_j = \sum_{j \neq i} \overline{A}_{ij} (T_j - T_i) + (\overline{A}_{ii} + \sum_{j \neq i} \overline{A}_{ij}) T_i
$$

$$
T_j - T_i = \frac{T_j^4 - T_i^4}{(T_j + T_i)(T_j^2 + T_i^2)}
$$

Then Equation (4.1) becomes

$$
\sum_{j=1}^{N} N_{ij} T_j^4 = \alpha_1 (K_{Si} S_i + K_{Ai} A_i) + \epsilon_1 (K_{Ei} E_i - \sigma a_i T_i^4) + K_{Q_i} Q_i - \frac{3}{4} (\overline{A}_{ii} + \sum_{j \neq i} \overline{A}_{ij}) T_i + \sum_{j=N+1}^{N+S} M_{ij} T_j^4 - \frac{3}{4} \sum_{j=N+1}^{N+S} \overline{A}_{ij} T_j
$$

(4.2)
where
\[
N_{ij} = \begin{cases} 
M_{ij} + \bar{A}_{ij} (\bar{T}_1 + \bar{T}_j) \left( \bar{T}_1^2 + \bar{T}_j^2 \right)^{-1} & i \neq j \\
\sum_{i \neq j} N_{ij} + \frac{1}{4} \bar{T}_1^3 \left( \bar{A}_{ii} + \sum_{i \neq j} A_{ij} \right) & i = j
\end{cases}
\]  

(4.3)

Note that since \(N_{ij} \leq 0\) for \(i \neq j\) and \(N_{ii} = -\sum_{j \neq i} N_{ij}\),
\[
\left\{N_{ij}\right\}^{-1} \geq 0
\]  

(4.4)

This property of \(\left\{N_{ij}\right\}^{-1}\) holds regardless of the value of the \(\bar{T}_1\)'s, since \(\bar{T}_1 \geq 0\).

Equation (4.2) carries on the right-hand side one term involving \(\bar{T}_1\). This term vanishes if (a) there is no conduction, or (b) there are no constant-temperature nodes to which heat is conducted.

Equation (4.2) gives
\[
\sum_{j=1}^{N} N_{ij} \frac{\partial T_j^4}{\partial \epsilon_{ij}} = \delta_{ij} \left( K_{E1} E_1 - \sigma a_1 T_1^4 \right) - \epsilon_i \sigma a_1 \frac{\partial T_1^4}{\partial \epsilon_{ij}}
\]  

(4.5)

\[
\sum_{j=1}^{N} N_{ij} \frac{\partial T_j^4}{\partial \alpha_{ij}} = \delta_{ij} \left( K_{S1} S_1 + K_{A1} A_1 \right) - \epsilon_i \sigma a_1 \frac{\partial T_1^4}{\partial \alpha_{ij}}
\]  

(4.6)

\[
\sum_{j=1}^{N} N_{ij} T_j^4 = \epsilon_i \left( K_{E1} E_1 - \sigma a_1 T_1^4 \right) + K_{Q1} Q_1 - \frac{3}{4} \sum_{j=N+1}^{N+S} A_{ij}(T_j - \bar{T}_1) + \frac{N+S}{j=N+1} M_{ij} T_j^4
\]  

(4.7)

Since \(\left\{N_{ij} + \epsilon \delta_{ij}\right\}\) has the same properties of \(\left\{N_{ij}\right\}\),
\[
\left\{N_{ij} + \epsilon \delta_{ij}\right\}^{-1} \equiv \left\{P_{ij}\right\} \geq 0
\]  

4-2
Therefore, from Equation (4.6), $T^A_i$ is monotonically increasing with $\alpha_i$. To determine how $T^A_i$ varies with $\epsilon_i$, define

$$Y_i = \sigma a_i T^A_i - K_{ei} E_i$$

Then

$$\frac{\partial Y_i}{\partial \epsilon_i} = \sigma a_i \frac{\partial T^A_i}{\partial \epsilon_i}$$ (4.8)

and

$$\sum_j \left( \frac{N_{ij}}{\sigma a_j} + \epsilon_i \delta_{ij} \right) \frac{\partial Y_j}{\partial \epsilon_i} = \delta_{i1} Y_1$$ (4.9)

$$\sum_j \left( \frac{N_{ij}}{\sigma a_j} + \epsilon_i \delta_{ij} \right) Y_j = \left[ \alpha_1 (K_{s1} s_1 + K_{a1} A_1) - \sum_{j=1}^{N} \frac{N_{ij}}{\sigma a_j} E_j + K_{q1} Q_1 - \frac{3}{4} \sum_{j=N+1}^{N+S} \bar{A}_{ij} (T_j - \tilde{T}_i) + \sum_{j=N+1}^{N+S} M_{ij} T^4_j \right] = (RHS)_i$$ (4.10)

Therefore, from Equation (4.9)

$$\frac{\partial Y_k}{\partial \epsilon_i} = P_{ik} Y_i$$

or

$$\frac{\partial Y_k}{\partial \epsilon_i} = \sum_{i=1}^{N} P_{ik} P_{i1} (RHS)_i$$ (4.11a)

Since $\{P_{ij}\} \geq 0$, Equation (4.11) indicates that $Y_k$ increases or decreases with $\epsilon_i$ depending on the signs of the right-hand-sides.

If $(K_{s1} s_1 + K_{a1} A_1) \neq 0$ for all values of "i", then Equation (4.6) shows that no $T^A_j$-surface has a horizontal point. If $(K_{s1} s_1 + K_{a1} A_1) = 0$ for all "i", Equation (4.11a) must be examined, along with the definition of $(RHS)_i$, Equation (4.10).
\[
\frac{\partial Y_k}{\partial \epsilon_L} = \sum_{i=1}^{N} P_{k,i}^L P_{L,i} \left[ Q_1 - \sum_{j=1}^{N} N_{ij} \frac{E_j}{\sigma a_j} - \frac{3}{4} \sum_{j=N+1}^{N+S} \bar{K}_{ij}(T_j - \bar{T}_i) - \sum_{j=N+1}^{N+S} M_{ij} T_j^4 \right]
\]

(4.11b)

Within the accuracy of the linearization, the expression for (RHS) is a constant, independent of the values of \(\alpha\) and \(\epsilon\). Its sign, therefore, will not change over a given \(Y_k\) (or \(T_k^4\)) surface. Since \(\left\{P_{k,i}^L\right\}\) is positive, \(\frac{\partial Y_k}{\partial \epsilon_L}\) also will not change sign on the \(Y_k\) surface.

The conclusion to be drawn from the above is that the \(Y_k\), and therefore \(T_k^4\) and \(T_k\) (see Equation (4.8)), have no extremes inside the boundaries of any given \(Y_k\) surface.

A similar analysis linearizing \(T^4\) into \(T\) leads to the same conclusion. Trying to solve the non-linear case directly is frustrated by the lack of an explicit expression for \(Y_k\) in Equation (4.9).

Consider next a composite surface, \(\beta\), defined by

\[
\beta = \max_{i,k} \beta_{ik}
\]

where

\[
\beta_{ik} = \max \left[ \left( \frac{\sigma T_{ik}^4 - \sigma T_{iD}^4}{\sigma T_{iU}^4 - \sigma T_{iL}^4} \right), \left( \frac{\sigma T_{iD}^4 - \sigma T_{ik}^4}{\sigma T_{iU}^4 - \sigma T_{iL}^4} \right) \right]
\]

\[
= \max \left[ \left( \frac{(Y_{ik} - E_{ik})/a_1 - \sigma T_{iD}^4}{\sigma T_{iU}^4 - \sigma T_{iL}^4} \right), \left( \frac{\sigma T_{iD}^4 - (Y_{ik} - E_{ik})/a_1}{\sigma T_{iU}^4 - \sigma T_{iL}^4} \right) \right]
\]

\[
= \max \left[ (C_1 Y_{ik} - C_2); (C_2 - C_1 Y_{ik}) \right]
\]

Define

\[
\beta_{ikU} = C_1 Y_{ik} - C_2
\]

\[
\beta_{ikL} = C_2 - C_1 Y_{ik}
\]

Now the \(\beta_{ikL}\) and \(\beta_{ikU}\) have the same properties of the \(Y_k\)'s analyzed above. Equations (4.6) and (4.11) imply that the minimum value of \(\beta\) must occur at the intersection of at least two \(\beta_{ik}\) surfaces.
Unimodality of $\beta$

The next objective is to show that the preceding properties lead to the conclusion that there can be only one relative minimum point. Suppose there were two, say $\beta_1$ and $\beta_2$. Then around each of these points

$$\delta \beta > 0$$

Therefore there is a relative maximum, $\beta_3$, lying along a continuous but broken path following the axes between Points 1 and 2. Then around $\beta_3$

$$\delta \beta \leq 0$$

(4.13)

Consider a line through Point 3 parallel to the $x_i$ axis. A variation in $\beta$ along the axis is designated $\delta \beta_1$. Now the condition (4.13) implies that

$$\beta(x_1 + \delta x_1) = \beta(x_1) + \left( \frac{\partial \beta}{\partial x_1} \right) \delta x_1$$

or

$$\delta \beta^+_1 = \left( \frac{\partial \beta}{\partial x_1} \right)^+ \delta x_1$$

$$\delta \beta^-_1 = -\left( \frac{\partial \beta}{\partial x_1} \right)^- \delta x_1$$

Although $\beta$ is discontinuous at Point 3, the surfaces that intersect there are sections of infinite, continuous, and monotonic surfaces, so that

$$\left( \frac{\partial \beta}{\partial x_1} \right)^+ = -\left( \frac{\partial \beta}{\partial x_1} \right)^-$$

Therefore, if $\delta \beta^+_1 > 0$, $\delta \beta^-_1 < 0$, contradicting Equation (4.13). Therefore there cannot be two discrete minima. However, if $\delta \beta = 0$, there may be an infinite number of minima connected by a continuous path.

Concavity of $\beta$

A surface is simple or concave if for any two points on the $\beta$-surface, $\beta_1$, and $\beta_2$,

$$\beta \left[ \lambda \bar{x}_1 + (1 - \lambda) \bar{x}_2 \right] \leq \lambda \beta \left[ \bar{x}_1 \right] + (1 - \lambda) \beta \left[ \bar{x}_2 \right]$$

If we consider a line along the $\epsilon$ axis, Equation (4.11) shows that $\beta_{ik}$ is either convex or concave. Therefore $\beta$ has segments that are concave and other segments that are convex. $\beta$ is therefore not strictly concave or convex.

Figure 2 shows a typical $\beta_{ik}$ surface, from which the $\beta$ surface is constructed. Figure 3 shows a typical map for a one-node/one-external-surface problem.
Figure 2. Section of a Typical $\beta_{ik}$ Surface

<table>
<thead>
<tr>
<th>ORBIT</th>
<th>S</th>
<th>E</th>
<th>Q</th>
<th>$\beta_U$</th>
<th>$\beta_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>75</td>
<td>75</td>
<td>75</td>
<td>$\alpha = (2/3 \beta + 1) \epsilon - 2/15$</td>
<td>$\alpha = (1/2 - 2/3 \beta) \epsilon - 2/15$</td>
</tr>
<tr>
<td>B</td>
<td>50</td>
<td>75</td>
<td>75</td>
<td>$\alpha = (\beta + 1/2) \epsilon - 0.2$</td>
<td>$\alpha = (1/2 - \beta) \epsilon - 0.2$</td>
</tr>
<tr>
<td>C</td>
<td>25</td>
<td>75</td>
<td>75</td>
<td>$\alpha = (2 \beta + 3) \epsilon - 0.8$</td>
<td>$\alpha = (1 - 2 \beta) \epsilon - 0.8$</td>
</tr>
</tbody>
</table>

Figure 3. Map of $\beta$ Surface for One-Node Case
5.0 OPTIMIZATION SCHEMES

Several optimization schemes were considered for use in the present program. These included:

(a) Linear Programming
(b) Pattern Search
(c) Miguel's Poor-Man's Ridge Follower
(d) Variational Methods
(e) Hill-climbing (Maximum Rate of Descent, or MRD)

Methods (b) and (c), which are described in Wilde's book: "Optimum Seeking Methods," can fail to find the minimum due to the extremely sharp ridges (discontinuous derivatives) characteristic of the $\beta$ - surfaces.

Method (d), used in many trajectory optimization schemes, is frustrated by the fact that $\beta$ has discontinuous derivatives. It is possible, however, to redefine $\beta$ and consider the independent variables to be $\alpha_i$, $\epsilon_i$, and $T_i$, subject to the constraints

$$T_{iL} \leq T_i \leq T_{iU}$$
$$\alpha_{iL} \leq \alpha_i \leq \alpha_{iU}$$
$$\epsilon_{iL} \leq \epsilon_i \leq \epsilon_{iU}$$

This leads to a rapid solution if the solution satisfies these limits within the constraints of the heat balance equation.

The most severe disadvantage of the variational scheme is the number of computations required if the constraints cannot be met. For example, a three-node problem, each with independent values of $\alpha$, $\epsilon$, and $T$, has $N = 9$ independent variables. Considering both upper and lower bounds on the variables, the heat-balance equation must be solved $S$ times, where

$$S = 2^{(N-1)} \sum_{j=1}^{N} \frac{N!}{(N-j)! \ j!}$$

$$= 2^{9} \sum_{j=1}^{9} \frac{9!}{(9-j)! \ j!}$$

$$= 13180$$

In the MRD scheme, $S$ is on the order of 100 to 200. Since most of the computation time is spent solving the heat-balance equation, the variational scheme need not be considered further.
Method (a), although quite powerful in linear problems, depends on the accuracy of linearization for nonlinear problems. It is anticipated, on the basis of past experience, that some of the external-surface temperatures will range from say 300° to 500° due to orbit-to-orbit changes in the relative position of the sun. Linearization, therefore appears dangerous. In addition, if there is no feasible solution, a rather extensive sensitivity analysis would be necessary to determine what must be done to obtain a feasible solution.

Suppose, however, that these objections were overcome. The linear-programming time per iteration would be on the order of

\[
\text{Time (μ sec)} = 1000 \, p \, N^2 + 61 \, N^3 \, (1 + p) + 400 \, p \, q \, r \\
+ \frac{(2 \, rp + 2q)(2 \, rp + 4q)^2}{12} \times 10^3
\]

where

\[
N = \text{number of nodes} \\
p = \text{number of orbits} \\
q = \text{number of external nodes} \\
r = \text{number of critical nodes}
\]

For reasonably sized problems, the last term dominates. In fact, one finds the time nearly proportional to \((rp)^3\). The following table gives some time estimates for \(q\) equal to 10.

<table>
<thead>
<tr>
<th>(r)</th>
<th>(p)</th>
<th>Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>130</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1060</td>
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<td>20</td>
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<td></td>
<td>10</td>
<td>8450</td>
</tr>
<tr>
<td>30</td>
<td>5</td>
<td>5300</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>28500</td>
</tr>
</tbody>
</table>

For problems of this general size, MRD would take an estimated 3,600 seconds.

From the foregoing discussion, it may be concluded that the MRD approach appears most promising due to the highly non-linear character of the equations, the long running times for LP systems with no feasible solutions, and the comparable running times between LP and MRD for systems of common complexity and with feasible solutions. Hence, MRD was selected for the present program.

The MRD method is described in many texts \(^5\). It involves selecting a starting point (a set of \(\alpha_i\)s and \(\varepsilon_i\)s) and calculating the criterion function (\(g\)) as well as

\(^5\)
the values of \( \frac{\partial \beta}{\partial \alpha} \) and \( \frac{\partial \beta}{\partial \epsilon} \) at this point. Using the derivatives, the direction in which \( \frac{\partial \beta}{\partial s} \) is a maximum can be determined where

\[
(ds)^2 = \sum_j \left[ (d \epsilon_j)^2 + (d \alpha_j)^2 \right]
\]

Then for a given \( \Delta s \), \( \Delta \beta \) will be a maximum, and \( \beta \) will decrease more than if any other direction had been used (provided \( \Delta s \) is small enough). Thus a step-by-step procedure is used to decrease \( \beta \), where each step is taken in the local maximum-rate-of-descent direction.

MRD converges to the optimum very slowly when it encounters a sharp ridge, such as are characteristic of the \( \beta \)-surface defined by Equation (2.1). Several ridge-following techniques are described in the literature, but these are frustrated by the extreme sharpness of the present ridges. None use the ridges to assist in finding the solution. Therefore, a method called TREND was devised for the present problem.

In TREND, use is made of the fact that the \( \beta \) surfaces are monotonic, so that when a ridge is crossed, say from Point 1 to Point 2, one of the subscripts of \( \beta_{ik} \) changes (or a switch from upper to lower bound occurs). If the ridge is immediately re-crossed to, say Point 3, this indicates that the points being studied lie in a valley. A vector extrapolation from Point 1 through Point 3 gives an improved value of \( \beta \) (see Appendix A). Figure 4 illustrates the process. The vector \( \overrightarrow{P1P3} \) is in the downward direction of the valley and indicates the trend of the valley. In many test cases, it has been found that TREND results in a time saving of 1.5:1 to 10:1 over the conventional MRD methods.

The logic diagram for the computer program developed in the present work, using the TREND procedures, is shown in Appendix C (Section 9.0).

![Figure 4. Illustration of TREND Method](image)

5-3
6.0 APPLICATIONS

FICTITIOUS SATELLITE

As an example of the use of the foregoing theory, the problem of optimizing the coating pattern of a cylindrical, horizontally stabilized earth satellite will be considered. Figures 5 and 6 show the details of the satellite geometry, the orbits considered, and the arrangement of the coating patches.

Table III shows the heat fluxes (BTU/hr-ft²) on each patch for the range in solar angles shown on Figure 6. These are the average heat fluxes over one orbit period.

The details of the input/output procedures and the actual computer program are shown in Section 5 of Part II of this final report. The results, however, are summarized on Table IV. The importance of the optimization process is obvious on examination of Table IV. Even for an optimized single-coating pattern, the temperature excursions from the desired mean are 50% greater than can be attained with the multi-patch coating.

NASA EPE-D

A second example was supplied by Mr. Robert Kidwell of NASA-GSFC. This was a mathematical model of the NASA EPE-D (Explorer XXVI) Satellite. The nodal breakdown is shown on Figure 7 and a photograph of the actual satellite is shown on Figure 8. The orbit of this satellite is such that the heat flux received from the earth is negligible. The significant parameter is the angle between the solar ray and the spin axis. In this case, instead of orbit number, the different sets of heat fluxes represent different angles between solar ray and spin axis.

The engineers at GSFC had had some experience with the temperature response of satellites of this configuration. These investigators undertook a trial-and-error procedure using considerable engineering judgement to select a suitable coating pattern for the EPE-D. Mr. Kidwell estimates that at least 4 weeks and 4 hours of computer time were required to achieve the desired results. Table V shows the temperature ranges attained, along with the allowable ranges. Only the critical nodes are shown on this table.

The challenge was to determine if the computer program could be used to improve on this design and the design procedure. The Coating Selection Program did devise two better coating patterns. The results of the one-week study are also summarized on Table V. A total of approximately 40 minutes of computer time was used. This shows that the program can reduce costs by a factor of four.

NASA IMP-C

A third example was again supplied by Mr. Robert Kidwell of NASA-GSFC. This was a mathematical model of the NASA IMP-C satellite. The nodal breakdown is shown in Figure 9, and a photograph of the actual satellite is shown in Figure 10. The Coating Selection Program devised a better coating pattern than the one devised by the engineers at GSFC. The study using the Coating Selection Program took one week and 50 minutes of computer time which also represents a significant gain over the original design procedure used at GSFC. The results are summarized in Table VI.
**Figure 5. Vehicle Geometry**

RADIANT INTERCHANGE FACTOR: \( \tau_A = 15.71 \text{ ft}^2 \)

(BASED ON TOTAL EXTERNAL SURFACE AREA OF 157 \( \text{ft}^2 \))

**Figure 6. Orbit and Patch Geometry**

RANGE IN SOLAR
FLUX ANGLES = \( \pm 60^\circ \)
FROM OVERHEAD

VIEW AA - 4 PATCH
VIEW AA - 8 PATCH
### TABLE III. ORBITAL HEAT FLUXES

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TABLE IV. RESULTS OF COATING-OPTIMIZATION STUDY

Specifications: Allowable $\alpha - \epsilon$ Range: $0.1 \leq \epsilon \leq 0.9$, $0.1 \leq \alpha \leq 0.9$
Allowable Temperature Range:
Skin Temperatures: $200^\circ R$ to $800^\circ R$
Internal Temperature: $525^\circ R$ to $545^\circ R$

Results:

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## TABLE V: SUMMARY OF EPE-D DESIGNS

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<th>Temperature Ranges</th>
<th>Predicted Limits</th>
<th>Coatings Used</th>
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<td>Using Computer Solution</td>
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| NODE | Allowable Limits | Coatings Used | By NASA | Using Computer Solution |            |
|------|------------------|---------------|---------|-------------------------|            |
|      |                  |               | #1     | #2                      | #3         |
| 1    |                  | .278 .260    | .350 .849 | 152 .118 | .97 .86     |            |
| 2    |                  | .170 .112    | .121 .032 | 149 .115 | .391 .826   |            |
| 3    |                  | .170 .112    | .121 .034 | 153 .116 | .424 .81    |            |
| 4    |                  | .170 .112    | .123 .038 | 155 .118 | .474 .776   |            |
| 5    |                  | .320 .260    | .129 .040 | 150 .119 | .510 .768   |            |
| 6    |                  | .310 .240    | .236 .232 | 167 .196 | .561 .457   |            |
| 7    | \(\alpha=0.12\), \(\epsilon=0.03\) | .310 .240    | .254 .20  | 168 .199 | .556 .560   |            |
| 8    | \(\alpha=0.36\), \(\epsilon=0.26\) | .310 .240    | .256 .179 | 168 .198 | .679 .574   |            |
| 9    | \(\alpha=0.97\), \(\epsilon=0.86\) | .970 .850    | .946 .853 | 207 .166 | .902 .793   |            |
| 10   | \(\alpha=0.12\), \(\epsilon=0.03\) | .970 .850    | .944 .859 | 146 .123 | .365 .667   |            |
| 11   | \(\alpha=0.36\), \(\epsilon=0.26\) | .970 .850    | .941 .859 | 170 .147 | .75 .644    |            |
| 12   | \(\alpha=0.97\), \(\epsilon=0.86\) | .160 .110    | .143 .108 | 163 .183 | .705 .599   |            |
| 13   | \(\alpha=0.12\), \(\epsilon=0.03\) | .310 .240    | .123 .039 | 153 .123 | .431 .792   |            |
| 14   | \(\alpha=0.36\), \(\epsilon=0.26\) | .350 .850    | .123 .039 | 144 .116 | .359 .764   |            |

### NOTES:

1. Computer solutions differed only in their starting points, which were as follows:
   - Case 1: \(\alpha=0.35, \epsilon=0.85\) for node 1; \(\alpha=0.97, \epsilon=0.86\) for nodes 9, 10, 11; \(\alpha=0.12, \epsilon=0.03\) for all other nodes.
   - Case 2: \(\epsilon=0.12\) for all nodes
   - Case 3: \(\alpha=0.97, \epsilon=0.86\) for node 1; \(\alpha=0.35, \epsilon=0.85\) for all other nodes

2. The third computer case terminated without attaining an acceptable solution. It appears that if the step size had not been reduced so rapidly, a better solution would have been attained.
Figure 9. Nodal Breakdown For IMP-C

Figure 10. NASA's IMP-C
### TABLE VI: SUMMARY OF IMP-C DESIGN

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6-8
7.0 CONCLUSIONS

The theoretical foundation and a detailed logic diagram have been developed from which a digital computer program has been devised that will select a space-vehicle external-coating pattern in such a way that the internal temperatures of the vehicle can be maintained as constant as possible in a purely passive manner.

The program has been demonstrated on several vehicle designs, three of which are shown in Section 6.0 of this report. It is seen that the gain of using a multiple-patch-external coating amounts to a factor of 2 in temperature variations from orbit to orbit. The coating pattern is derived in a direct method in a few minutes time on a digital computer with approximately 2-microsecond access time (such as an IBM 7094). The added expense of the enhanced design is seen to be quite small.

The details of the program will be found in Part II: User's Manual and Program Description.
8.0 RECOMMENDATIONS FOR FURTHER WORK

Time-Dependent Solutions

The present work assumes that the critical temperatures can be calculated using average heat fluxes, that is, that the component temperature deviates little from the average temperature. This assumption is good if the node heat capacitance is high compared to the equivalent thermal conductance between the node and its environment so that the time constant

\[
\tau_{ci} = \frac{(WC_{p_i})}{L_{il}}
\]

(8.1)

(see Equation 3.4) is much greater than the time interval, \( \tau \), for which the average heat fluxes are calculated. If \( \tau_{ci} \) is much greater than one orbit period, no difficulties arise.

For thin external surfaces, \( \tau_{ci} \) is usually quite small, so that temperature deviations from the mean are large. It is therefore important to determine the true average temperatures and the magnitude of the deviation from the mean.

For transient heat flow, Equation (3.2) becomes

\[
(WC_{p_i}) \frac{dT_i}{d\theta} + \sum_j A_{ij} T_j + \sum_j M_{ij} T_j^4 = D_i(\theta) - \epsilon_i a_i \sigma T_i^4
\]

(8.2)

where

\[
D_i(\theta) = \alpha_i S_1 + \epsilon_i E_1 + Q_i
\]

Defining \( \bar{X} = \frac{1}{\theta_1 - \theta_0} \int_{\theta_0}^{\theta_1} X \, d\theta \), integration of Equation (8.2) gives

\[
(WC_{p_i}) \left[ T_i(\theta_1) - T_i(\theta_0) \right] + \sum_j A_{ij} T_j + \sum_j M_{ij} T_j^4 = \bar{D}_i - \epsilon_i a_i \sigma \bar{T}_i^4
\]

If the temperature history is periodic of period \( \tau_P \), then, if \( \theta_1 = \theta_0 + \tau_P \), the heat-capacitance term is zero. If, in addition, \( A_{ij} = 0 \), the average fluxes will give the true average \( T^4 \)s. If \( A_{ij} \neq 0 \), then, since

\[
\bar{T}_i^4 \neq \bar{T}_i^4
\]
neither the true average $T^4$ nor $T$ are calculated. This is a problem that should be considered in future work. One approach would be to devise form factors, $\gamma$, such that

$$\frac{T^4}{\gamma} = \gamma T^4$$

However, the development of this concept is beyond the scope of the present work.

Convergence of Iterative Solution

The heat-balance equation was solved for several numerical examples using seven iterative schemes. The fastest of the methods was then selected for the CSP. The rates of convergence should be investigated and compared on a more mathematical and rigorous basis.

Sub-optimizations of each iteration scheme should also be conducted. For example, one could determine if $X$ could be made smaller for iteration schemes (1) and (2) (Table I), or one might add $X$ to the L matrix of method (7) to determine if this aids convergence in such cases as Problem D (Table II).

"Typical" Orbits

In selecting the orbits used to obtain the optimum coating pattern, the usual approach is to take the extremes in, say, $\beta$-angles (i.e., the angle between solar ray and orbit plane), and then take a "few" orbits in between. This would give representative orbits for a uniformly coated vehicle. However, when the coating pattern is optimized for the particular orbits given in the input, it becomes questionable whether the particular orbits still represent extreme cases.

For most spacecraft, the environmental heat fluxes can be expressed as a function of one or two variables ($\beta$-angle, solar view-angle, orbit position, etc.), so a solution to the problem of properly selecting the orbits should be attainable.

Program Enlargement

The program is now limited to 34 nodes. For many satellites this is not adequate. The program can be enlarged if a LINK (chain) procedure is used, so that tape and drum storage is used.

Tolerances

The present program uses the nominal values of $\alpha$ and $\epsilon$. The effects of tolerances in these values on the space vehicle temperatures must be calculated separately. This extra calculation could be incorporated in the optimization program.

Heaters and Shutters

The present program considers only those vehicles in which no auxiliary temperature control device is used. As such, it determines the best control that can be attained without such devices. The next step in determining the optimum temperature control system would be to include heaters and shutters (sometimes called louvres, vanes, or variable-surface-property devices).
9.0 APPENDICES

Appendix A. Mathematical Analysis of the TREND Step

The mathematical basis for TREND can be stated as follows:

Given two intersecting hypersurfaces, A and B. Given also a point 'a' on A and a point 'b' on B. 'b' lies on the projection of the maximum-rate-of-descent line of A through 'a', the projection being taken on the hyperplane A (x_i) = 0. The projected distance from 'a' to 'b' is 'Δs'. Given also a point 'c' on A having the same relationship to 'b' as 'b' does to 'a'.

We wish to prove that A (c) ≤ A(ā). If this is the case, then B (d) ≤ B(b), where 'd' is found using the maximum-rate-of-descent line from c. This process can be continued until a minimum value of A or B is reached. This will prove that the maximum-rate-of-descent method will converge to the minimum. For TREND, we must show also that A (e) ≤ A(e), where e = c - (c - a)/|c - a| Δs.

Consider first the maximum rate of descent. The point b is given by

\[ b_i = a_i - \frac{A_{x_i}(Δs)}{\sqrt{\sum_i (A_{x_i})^2}} \]

where the independent variables are taken as x_i. Also

\[ c_i = b_i = -\frac{B_{x_i}(Δs)}{\sqrt{\sum_i (B_{x_i})^2}} \]

Therefore

\[ c_i - a_i = -\left[ \frac{A_{x_i}}{\sqrt{\sum_i (A_{x_i})^2}} + \frac{B_{x_i}}{\sqrt{\sum_i (B_{x_i})^2}} \right] Δs \]

Now

\[ A(\bar{c}) = A(\bar{a}) + \sum_i A_{x_i}(\bar{a}) \cdot (c_i - a_i) + \ldots \]

\[ \leq A(\bar{a}) - \sum_i \left[ \frac{A_{x_i}^2}{\sqrt{\sum_i A_{x_i}^2}} + \frac{A_{x_i} B_{x_i}}{\sqrt{\sum_i B_{x_i}^2}} \right] Δs \]
Therefore

\[ A(\hat{c}) \leq A(\vec{a}) \iff \sqrt{\sum A_{xi}^2} \sum_i \left[ \frac{A_{xi}^2}{\sqrt{\sum A_{xi}^2}} + \frac{A_{xi} B_{xi}}{\sqrt{\sum B_{xi}^2}} \right] \geq 0 \]

Now \( \sqrt{\sum A_{xi}^2} \geq 0 \) and \( \frac{A_{xi}}{\sqrt{\sum A_{xi}^2}} \) is the \( i \)th component of a unit vector from \( \vec{a} \), say

\[ \vec{u}_1 = \sum_i \frac{A_{xi}}{\sqrt{\sum A_{xi}^2}} \hat{e}_i \]

Similarly

\[ \vec{u}_2 = \sum_i \frac{B_{xi}}{\sqrt{\sum B_{xi}^2}} \hat{e}_i \]

Now

\[ \vec{u}_1 \cdot \vec{u}_1 = 1 \]

\[ \vec{u}_1 \cdot \vec{u}_2 = \cos \theta \]

where \( \theta \) is the angle between \( \vec{u}_1 \) and \( \vec{u}_2 \). Therefore

\[ \vec{u}_1 \cdot \vec{u}_1 + \vec{u}_1 \cdot \vec{u}_2 = 1 + \cos \theta \geq 0 \]

or

\[ \sum_i \left[ \left( \frac{A_{xi}^2}{\sqrt{\sum A_{xi}^2}} \right)^2 + \frac{A_{xi} B_{xi}}{\sqrt{\sum A_{xi}^2 \sum B_{xi}^2}} \right] = 1 + \cos \theta \geq \theta \]

Therefore \( A(\hat{c}) \approx A(\vec{a}) \), provided \( \Delta s \) is small enough that the first term of the Taylor's expansion is representative of the function \( A \) (no more than say 1/2\% in error). If the radius for which the single term expansion is adequate, is designated \( R \), it is required that \( |\hat{c} - \vec{a}| \leq R \), or
\[
\sum_i (c_i - a_i)^2 = \sum_i \left( \frac{A_{xi}}{\sqrt{\sum_i A_{xi}^2}} + \frac{B_{xi}}{\sqrt{\sum_i B_{xi}^2}} \right)^2 (\Delta s)^2 \leq R^2 \tag{A-1}
\]

The maximum value of the term inside the brackets is 4, so it is sufficient if

\[
\Delta s \leq \frac{R}{2} = 0.5R \tag{A-2}
\]

In TREND, we wish to state also that \( A(\bar{c}) \leq A(\bar{a}) \), where

\[
e_i - c_i = \frac{(c_i - a_i)}{\sqrt{\sum_i (c_i - a_i)^2}} (\Delta s)
\]

Now if \((\Delta s)\) is chosen small enough

\[
A(\bar{c}) = A(\bar{a}) + \sum_i A_{xi} (\bar{a}) (e_i - a_i) + \ldots
\]

\[
\approx A(\bar{a}) + \left[ 1 + \frac{\Delta s}{\sqrt{\sum_i (c_i - a_i)^2}} \right] \sum_i A_{xi} (\bar{a}) (c_i - a_i)
\]

since

\[
e_i - a_i = e_i - c_i + c_i - a_i = (c_i - a_i) \left( \frac{e_i - c_i}{c_i - a_i} + 1 \right)
\]

Therefore

\[
A(\bar{c}) - A(\bar{a}) = \frac{\Delta s}{\sqrt{\sum_i (c_i - a_i)^2}} \sum_i A_{xi} (\bar{a}) (c_i - a_i)
\]

\[
= \frac{\Delta s}{\sqrt{\sum_i (c_i - a_i)^2}} \left[ A(\bar{c}) - A(\bar{a}) \right] \leq 0
\]

which proves the validity of TREND. The \((\Delta s)\) must be chosen such that

\[
|\bar{e} - \bar{a}| \leq R
\]
or
\[ \sum_i (e_i - a_i)^2 \leq R^2 \]

or
\[ \left[ 1 + \frac{\Delta s}{\sqrt{\sum (c_i - a_i)^2}} \right]^2 \sum_i (c_i - a_i)^2 \leq R^2 \]

but
\[ \sum_i (c_i - a_i)^2 \leq 4 (\Delta s)^2 \quad \text{From Equation (A-1)} \]

\[
\therefore \sum_i (c_i - a_i)^2 + 2(\Delta s) \sqrt{\sum (c_i - a_i)^2 + (\Delta s)^2} = 4(\Delta s)^2 + 4(\Delta s)^2 + (\Delta s)^2 = 9 (\Delta s)^2
\]

Therefore, \( \Delta s \) must be such that
\[ \Delta s \leq \frac{1}{3} R = 0.333R \quad \text{(A-3)} \]

It is seen by comparison between Equations (A-2) and (A-3) that TRENDS will be slower than MRD if the \( A_{x_i} \)'s equal the \( B_{x_i} \)'s; that is, if the two steps of MRD are approximately in the same direction. However, if \( c_i \neq a_i \), so that \( \sum (c_i - a_i)^2 = \epsilon^2 \), MRD will move with net step sizes on \( A \) of \( \epsilon \) and convergence will be slow. TRENDS, under the same circumstances will proceed with a net step size of \( (\epsilon + \Delta s) \).
Appendix B. Convergence of the Inverse-Matrix Iteration Methods

To analyze the convergence of the matrix inversion iteration schemes (Methods 1 and 2 of Table 1), it is necessary to linearize the radiation terms. Let

$$
\overline{R}_{ij} = \begin{cases} 
- \sigma R_{ij} \left( \tilde{T}_i + \tilde{T}_j \right) \left( \tilde{T}_i^2 + \tilde{T}_j^2 \right) & i \neq j \\
N + S \sum_{j=1}^{N+S} \overline{R}_{ij} + 4 a_1 \epsilon_i \sigma \tilde{T}_i^3 & j \neq i 
\end{cases} \quad (B-1)
$$

Then Equation (3.2) becomes

$$
\sum_{j=1}^{N+S} \left( \overline{A}_{ij} + \overline{R}_{ij} \right) T_j = C_i
$$

where $C_i$ is defined as in Equation (3.4).

Method (1) solves this equation in the form

$$
T^{(n+1)} = (\overline{A} + X)^{-1} \left( C - (\overline{R} - X) T^{(n)} \right) \quad (B-2)
$$

Convergence of the iteration process depends, then, on the spectral radius, $\rho(Z)$, of

$$
Z = (\overline{A} + X)^{-1} (X - \overline{R}) \quad (B-3)
$$

being less than one.

Note that

$$
\begin{align*}
(\overline{A} + X)_{ii} & \geq - \sum_{j=1}^{N} \overline{A}_{ij} \quad \text{and} \quad \overline{A}_{ij} \leq 0 \quad i \neq j \\
(\overline{R})_{ii} & \geq - \sum_{j=1}^{N} \overline{R}_{ij} \quad \text{and} \quad \overline{R}_{ij} \leq 0 \quad i \neq j
\end{align*} \quad (B-4)
$$

$$
X_{ii} \geq 0 \quad \text{and} \quad X_{ij} = 0 \quad i \neq j
$$

The matrix $(\overline{A} + X)^{-1}$ is positive by virtue of Equations (B-4). If, in addition

$X_{ii} \geq R_{ii}$, $(X - \overline{R})$ is positive so that, since

$$
(\overline{A} + \overline{R}) = (\overline{A} + X) - (X - \overline{R}) \quad \text{with} \quad (\overline{A} + \overline{R})^{-1} \geq 0, \quad \text{(B-5)}
$$

$$
\left[ (\overline{A} + X) - (X - \overline{R}) \right] \quad \text{is a regular splitting of the matrix} \quad (\overline{A} + \overline{R}). \quad \text{Then by Theorem 2.2 of Varga(6)}
$$

$$
\rho(Z) = \frac{\rho \left[ (\overline{A} + \overline{R})^{-1} (X - \overline{R}) \right]}{1 + \rho \left[ (\overline{A} + \overline{R})^{-1} (X - \overline{R}) \right]} < 1 \quad (B-6)
$$
so that convergence is assured if \( X_{ii} = \overline{R}_{ii} \). By Theorem 2.7 of Varga, it is evident that \( X_{ii} = \overline{R}_{ii} \) gives the minimum value of \( \rho (Z) \) and, therefore, the most rapid convergence.

Relating the foregoing to the non-linear equation shows that a sufficient condition for convergence is that

\[
X_{ii} \geq - \sum_{j=1}^{N+S} \sigma R_{ij} (T_i + T_j) \max (T_i^2 + T_j^2) \max + 4 a_1 \epsilon \sigma T_{imax}^3 \tag{B-7}
\]

and that most rapid convergence will occur if

\[
X_{ii} = - \sum_{j=1}^{N+S} \sigma R_{ij} (T_i + T_j) (T_i^2 + T_j^2) + 4 a_1 \epsilon \sigma T_i^3 \tag{B-8}
\]

where \( T_k \) and \( X_{ii} \) are evaluated at each iteration. This latter criteria can be approximated by using \( T_i \) as \( T_i \), solution.

It is evident that if Equation (B-8) is satisfied, Equation (B-7) will, in general, be violated. Although it has not been shown in a mathematically rigorous fashion, experience with many numerical cases indicates that the criteria expressed in Equation (B-7) can be relaxed somewhat so that convergence will occur if

\[
X_{ii} = \max \left[ \frac{1}{2} (\overline{R}_{ii} - A_{ii}); \sum_{j \neq i} |\overline{A}_{ij}|; \sum_{j \neq i} (\overline{R}_{ij} + \overline{A}_{ij}) - \overline{A}_{ii} \right] \tag{B-9}
\]

where \( \overline{R}_{ij} = \overline{R}_{ij} \) evaluated with \( T_i \) and \( T_j \) at their maximum anticipated values.

Combining Equations (B-8) and (B-9) gives the following criteria on \( X_{ii} \):

\[
X_{ii} = \max \left[ \frac{1}{2} (\overline{R}_{ii} - A_{ii}); \sum_{j \neq i} |\overline{A}_{ij}|; \sum_{j \neq i} (\overline{R}_{ij} + \overline{A}_{ij}) - \overline{A}_{ii} ; \overline{R}_{ii}^0 \right] \tag{B-10}
\]

(Method 1)

where \( \overline{R}_{ii}^0 = \overline{R}_{ii} \) evaluated with \( T_i \) and \( T_j \) at their anticipated solution values.

The analysis of Method (2) follows the same pattern as for Method (1). The counterpart of Equation (B-10) is found by replacing \( \overline{R} \) in Equation (B-10) by \( \overline{R} - \overline{R}^0 \).

\[
X_{ii} = \max \left[ \frac{1}{2} (\overline{R}_{ii} - \overline{R}_{ii}^0 - A_{ii}); \sum_{j \neq i} |\overline{A}_{ij}|; \sum_{j \neq i} (\overline{R}_{ij}^* + \overline{R}_{ij}^0 + \overline{A}_{ij}) - \overline{A}_{ii} ; 0 \right] \tag{B-11}
\]

(Method 2)
LOGIC DIAGRAM FOR TREND

READ $k_j$, $r_j$, $s_j$, $A_j$, $F_j$, $S_j$, $T_j$, $T_{Uj}$, $T_{Lj}$, $\Delta s_j$, $\Delta s_{\min}$, $K_j$, $K_{\Delta}$, $K_{\epsilon_j}$, $\sigma - \epsilon$ limits, $\sigma_T$, $T_{\max}$, $T_{sol}$

Let $r = 1$, $g = 1$; $\sigma - \epsilon$ limit no. 5; $\sigma - \epsilon$ limit no. 1

Start

MRD 1

Solve for $T_{ik}$'s. calculate $s_{ik}$'s. Select $i$, $k, l$ such that $s_{ikl} = \max_s, \sum_{i, k, l}$. Let $\beta = \beta_{ikl}

Let $l_{\text{max}} = 1$, $k_{\text{max}} = k$, $L_{\text{max}} = 1$, $r_{\text{max}} = r_j$, $\Delta s = \Delta s_j$, $AL_{ir} = \beta_{ir}$, $EPS_{ir} = \epsilon_{ir}$, $\gamma_{r} = k$, $E_{i} = 1$, $L_{r} = 1$

MRD 2

Calculate $\frac{\Delta \beta}{\Delta r}$, $\delta r$, $\sigma r$. Let $\alpha_{ig} = \Delta \alpha_{ig}$, $\gamma_{ig} = \Delta \gamma_{ig}$, $r = r + 1$

Calculate new $\sigma - \epsilon$: $\sigma_{ig} = \sigma_{ig} + \Delta \alpha_{ig}$, $\epsilon_{ig} = \epsilon_{ig} + \Delta \gamma_{ig}$

Let $\lambda_{i} = \epsilon_{ig}$, $\mu_{i} = \alpha_{ig}$, $\sigma_{i} = \beta_{ir}$, $YY_{i} = \gamma_{i}$

go to LIM

ALSR = $\gamma_{(g+1)}$, EPSR = $\epsilon_{(g+1)}$

Solve for $T_{ik}$'s. Calculate $s_{ik}$'s. Select $i$, $k, l$ such that $s_{ikl} = \max_s, \sum_{i, k, l}$

Are $i = l_{\text{max}}$, $k = k_{\text{max}}$, $L = L_{\text{max}}$?

Yes

No

Go To Step

Let $\lambda = \lambda_{i}$, $\mu = \mu_{i}$, $\sigma = \sigma_{ikl}$, $\sigma_{i} = \beta_{ikl}$, $YY_{i} = \gamma_{i}$

$g = 1$, $r = 1$

END

NOTES:

Subscripts are generally used as follows:

1 designates Node Number
k designates Orbit
g designates Iteration
r designates MRD Point
l designates Upper Or Lower Limit

9-7
Let $KS_r = k; IS_r = i; LS_r = 1$

Does $KS_r = KS_j, LS_r = LS_j, IS_r = IS_j$ for $j = r-1, r-2, \ldots, 1$?

No \hspace{1cm} Yes

Go To MRD 2

Let $j$, for the first value of $j$ for which $IS_j = IS_r$, equal $q$

Let $\Delta \alpha_{ig} = ALS_{ir} - ALS_{iq}, \Delta \epsilon_{ig} = EPS_{ir} - EPS_{iq}$

Trend

Let $\Delta \alpha_{ig} = \frac{(\Delta \alpha_{ig}) (\Delta s)}{\sqrt{\sum (\Delta \alpha_{ig})^2 + \sum (\Delta \epsilon_{ig})^2}}, \Delta \epsilon_{ig} = \frac{(\Delta \epsilon_{ig}) (\Delta s)}{\sqrt{\sum (\Delta \alpha_{ig})^2 + \sum (\Delta \epsilon_{ig})^2}}$

Let $\Delta \alpha_{lo} = \Delta \alpha_{ig}, \Delta \epsilon_{lo} = \Delta \epsilon_{ig}$

Is $(r - q) \geq 3$?

No \hspace{1cm} Yes

Go To Test 2

Trend 2

Let $\sigma_{i(g+1)} = \sigma_{ig} + \Delta \alpha_{ig}, \epsilon_{i(g+1)} = \epsilon_{ig} + \Delta \epsilon_{ig}$

Go To LIM

Solve for $T_{ik}$'s, calculate $\beta_{ikl}$'s, Select $i, k, l$ such that $\beta_{ikl} = \max_{i, k, l} \beta_{ikl}$

Let $IT = i, KT = k, LT = l$

Is $\beta_{ikl} \geq \beta$?

Yes \hspace{1cm} No

NOTE: An Alternate Trend 2 is given in Appendix D

Go To BACK \hspace{1cm} Go To ADVANCE
STEP

Is $\beta_{ikl} > \beta$ ?

No

Let $\Delta s = USMRD \cdot \Delta s$; $r = 1; g = g + 1; KS_1 = k$

$LS_1 = 1; I_1 = I; ALS_{11} = \alpha_{ig}; EPS_{11} = \epsilon_{ig}$

Yes

Let $\Delta s = DSMRD \cdot \Delta s$

Is $\Delta s < \Delta s_{\min}$

Yes

$\Delta \alpha_{ig} = 0; 4 \Delta \alpha_{ig}; \Delta \epsilon_{ig} = 0; 4 \Delta \epsilon_{ig}$

$ALS_{ir} = EPS_{ir} = KS_r = I_1 = LS_r = 0$

For $r = 2, 3, 4, \ldots$

No

Stop

Advance

$I_{\max} = 1$

$L_{\max} = 1$

$K_{\max} = k$

Go To MRD2

For $r = 2, 3, 4, \ldots$

$\beta = \beta_{ikl}$

Let $\beta = \beta_{ikl} g = g + 1; I_1 = I_1; KS_1 = KS_1; LS_1 = LS_1$

$I_{\max} = 1; K_{\max} = k; L_{\max} = 1$

Go To CNTR

Let $\Delta \alpha_{ig} = \Delta \alpha_{(g-1)} + USTRN \cdot \Delta \alpha_{io}$

$\Delta \epsilon_{ig} = \Delta \epsilon_{(g-1)}$

$USTRN \cdot \Delta \epsilon_{io}$

$ALS_{ir} = EPS_{ir} = KS_r = I_1 = LS_r = 0$

For $r = 2, 3, 4, \ldots$

$\beta = \beta_{ikl}$

$\epsilon_{ig}$

$\epsilon_{ig}$

RK = 1

Go To TREDN 2
Let $\Delta \alpha_{lg} = 0.4 \Delta \alpha_{lg} \quad \Delta \epsilon_{lg} = 0.4 \Delta \epsilon_{lg}$

$c_{l(g+1)} = c_{lg} + \Delta \alpha_{lg} \quad c_{l(g+1)} = c_{lg} + \Delta \epsilon_{lg}$

Solve for $T_{ik}^{*}$, calculate $\beta_{ikl}^{*}$, select $i, j, k, l$ such that $\beta_{ikl}^{*} = \max \beta_{ikl}$; let $I = i, K = k, L = l, L = 1$

Is $\beta_{ikl}^{*} = \beta_{I}$?

Yes

Is $R = 1$?

Yes

Let $\Delta s = 2.5 + 0.35n \cdot \sqrt[3]{(\Delta \alpha_{lg}) + \sum_{i} (\Delta \epsilon_{lg})}$

$ALS_{lr} = EPS_{lr} = IS_{r} = KS_{r} = LS_{r} = 0$ For $r = 2, 3, 4, \ldots$

Let $g = g+1$

Is $\Delta s < (\Delta s)_{\min}$?

No

Stop

Let $\alpha_{l(g+1)} = \alpha_{lg} - (c_{lg})$ $\Delta \epsilon_{lg}$ $l_{max} = IT$

$g = g+1$

$K_{max} = KT$

$E_{r} = IT, KS_{r} = KT, LS_{r} = LT, ALS_{lr} = \alpha_{lg}, EPS_{lr} = \epsilon_{lg}, L_{max} = LT$

Go to MRD 2

TEST 2

Calculate $c_{lg}^2, 3\beta, \Delta \alpha_{lg}, \Delta \epsilon_{lg}$ (As for MRD 2)

Is $\sum_{l} (\Delta \alpha_{lg}) (\Delta \alpha_{lg}) + (\Delta \epsilon_{lg}) (\Delta \epsilon_{lg}) \leq 0$?

Yes

Let $r = r + 1$

$\Delta \alpha_{lg} = \Delta \alpha_{lg}$

$\Delta \epsilon_{lg} = \Delta \epsilon_{lg}$

Go to CNTR
### Appendix D. Nomenclature (Excepted and Noted)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Suggested Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>incident albedo flux</td>
<td>BTU/hr ft²</td>
</tr>
<tr>
<td>( \overline{A} )</td>
<td>conduction matrix term</td>
<td>BTU/hr (^{0})F</td>
</tr>
<tr>
<td>C</td>
<td>defined by Equation (3, 4)</td>
<td>BTU/hr</td>
</tr>
<tr>
<td>D</td>
<td>defined by Equation (8, 2)</td>
<td>BTU/hr</td>
</tr>
<tr>
<td>E</td>
<td>incident earth flux</td>
<td>BTU/hr ft²</td>
</tr>
<tr>
<td>( \tilde{E}_{ij} )</td>
<td>((\partial r_i/\partial T_j)) at ( T_i = T_j^0 )</td>
<td>BTU/hr (^{0})F</td>
</tr>
<tr>
<td>J</td>
<td>inverse matrix of ( \overline{A} )</td>
<td>hr (^{0})F/BTU</td>
</tr>
<tr>
<td>( K_{Ai} )</td>
<td>multiplier of incident albedo flux for node 'i'</td>
<td>ft²</td>
</tr>
<tr>
<td>( K_{Ei} )</td>
<td>multiplier of incident earth flux for node 'i'</td>
<td>ft²</td>
</tr>
<tr>
<td>( K_{Qi} )</td>
<td>multiplier of internal heat generation</td>
<td>BTU/watt hr</td>
</tr>
<tr>
<td>( K_{ij} )</td>
<td>conductance from node 'i' to node 'j'</td>
<td>BTU/hr (^{0})F</td>
</tr>
<tr>
<td>( K_{Si} )</td>
<td>multiplier of incident solar flux for node 'i'</td>
<td>ft²</td>
</tr>
<tr>
<td>L</td>
<td>defined by Equation (3, 4)</td>
<td>BTU/hr (^{0})F</td>
</tr>
<tr>
<td>M</td>
<td>radiation-matrix term</td>
<td>BTU/hr ft² ((^{0}R)^4)</td>
</tr>
<tr>
<td>N</td>
<td>defined by Equation (4, 3)</td>
<td>BTU/hr ((^{0}R)^4)</td>
</tr>
<tr>
<td>( P )</td>
<td>inverse of ( N_{ij} + \epsilon_i \delta_{ij} )</td>
<td>hr ((^{0}R)^4)/BTU</td>
</tr>
<tr>
<td>Q</td>
<td>internal heat generation</td>
<td>watts</td>
</tr>
<tr>
<td>( \overline{R} )</td>
<td>defined by Equation (9, 2, 1)</td>
<td>BTU/hr (^{0})F</td>
</tr>
<tr>
<td>( R_{ij} )</td>
<td>product of radiant interchange factor between nodes 'i' and 'j' and area of node 'i'</td>
<td>ft²</td>
</tr>
<tr>
<td>(RHS)</td>
<td>defined by Equation (4, 10)</td>
<td>BTU/hr</td>
</tr>
<tr>
<td>S</td>
<td>incident solar flux</td>
<td>BTU/hr ft²</td>
</tr>
<tr>
<td>T</td>
<td>absolute temperature</td>
<td>(^{0})R</td>
</tr>
<tr>
<td>( (WC_{p})_i )</td>
<td>heat capacitance of node 'i'</td>
<td>BTU/(^{0})F</td>
</tr>
<tr>
<td>X</td>
<td>stability parameter in matrix inversion methods</td>
<td>BTU/hr (^{0})F</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Suggested Units</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>Y</td>
<td>$\sigma a_i T_i^4 \cdot K_{EI} E_i$</td>
<td>BTU/hr</td>
</tr>
<tr>
<td>$a_i$</td>
<td>re-radiating area for node 'i' (external area)</td>
<td>ft$^2$</td>
</tr>
<tr>
<td>c</td>
<td>weighting factor, Equation (2.3)</td>
<td>-</td>
</tr>
<tr>
<td>d</td>
<td>weighting factor, Equation (2.4)</td>
<td>-</td>
</tr>
<tr>
<td>r</td>
<td>residual vector (Equation 3.5)</td>
<td>BTU/hr</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>solar absorptance</td>
<td>-</td>
</tr>
<tr>
<td>$\alpha_n$</td>
<td>iteration scheme extrapolation parameter</td>
<td>-</td>
</tr>
<tr>
<td>$\beta$</td>
<td>temperature deviation parameter</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>criterion function, Equations (2.3) and (2.4)</td>
<td>-</td>
</tr>
<tr>
<td>$\tilde{\gamma}$</td>
<td>form factor, $(T_i^3)/(\bar{T})^4$</td>
<td>-</td>
</tr>
<tr>
<td>$\delta_{ij}$</td>
<td>Kronecker delta (1 if i=j; 0 if i≠j)</td>
<td>-</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>infrared emittance</td>
<td>-</td>
</tr>
<tr>
<td>$\theta$</td>
<td>time</td>
<td>hrs</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>interpolation parameter</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Stephen-Boltzmann constant ($0.1713 \times 10^{-8}$)</td>
<td>BTU/hr ft$^2$ ($^\circ$R)$^4$</td>
</tr>
<tr>
<td>$\tau_{ci}$</td>
<td>$(WC_p \chi)/L_{1i}$</td>
<td>hrs</td>
</tr>
</tbody>
</table>

**Subscripts**

- $i$ usually node number
- $iL$ corresponding to lower limit on $T_i$
- $iU$ corresponding to upper limit on $T_i$
- $k$ usually orbit number

**Superscript**

- n iteration number
10.0 REFERENCES


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