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A NEW METHOD OF EVALUATING DYNAMIC RESPONSE OF COUNTER-FLOW AND PARALLEL-FLOW HEAT EXCHANGERS

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A NEW METHOD OF EVALUATING DYNAMIC RESPONSE OF COUNTER-FLOW AND PARALLEL-FLOW HEAT EXCHANGERS

By Henry M. Paynter[1] and Yasundo Takahashi[2]

SYNOPSIS

From the exact solutions for the frequency response of counter-flow and parallel-flow heat exchangers, successive parameters are calculated which give direct information for the heat exchangers regarding transient responses as well as frequency responses. The numerical evaluations of the parameters from the design data of heat exchangers are generally very simple, although the formulae themselves appear somewhat involved. Good coincidence with measured transient responses is demonstrated on an example.

INTRODUCTION

One of the authors has published[3,4] analytical solutions of heat exchangers. But numerical evaluations of these results were not simple, especially for tubular heat exchangers (Fig. 1) because they involved distributed parameter systems. A new method developed by the other author[5] can be applied to these cases to obtain a numerical basis for dynamic response calculations. Thus, for example, an estimation of the transient response, which otherwise would have required complicated calculations[6], can be made very easily from the design parameters listed below.

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3 "Transfer Function Analysis of Heat Exchangers," by Y. Takahashi in Automatic and Manual Control, edited by A. Tustin, Butterworth Sci. Pub., 1952, p. 235.

4 "Regeluechnizche Eigenschaften dar Glerch - und Gegenstromwärmeaus-tauschern," by Y. Takahashi. Regelungstechnik Heft 2, 1 Jarg., 1953, p.32.

5 "On An Analogy Between Stochastic Processes and Certain Dynamic Systems," by H. M. Paynter. (Forthcoming ASME Paper).

6 Conduction of Heat in Solids, by H. S. Carslaw and J. C. Jaeger, Oxford, 1950, pp. 325-330.

NOMENCLATURE

A = surface area of tube walls (ft^2)

$a_1 = \frac{kA}{W_1 c_1}$ $a_2 = \frac{kA}{W_2 c_2}$ (dl), when both are equal, $a = a_1 = a_2$

$a_1' = \frac{\alpha_1 A_1}{W_1 c_1}$ $a_2' = \frac{\alpha_2 A_2}{W_2 c_2}$ $a_s = \frac{\alpha_s A_s}{W_2 c_2}$ (dl)

B = intermediate parameter (dl)

b $= b_1 + b_2$, $b_1 = \frac{\alpha_1 A_2}{C_h v_1}$ $b_2 = \frac{\alpha_2 A_2}{C_h v_1}$ $b_s = \frac{\alpha_s F_s}{C_s v_1}$ (dl)

C = tube or shell heat capacity per unit length along the flow (Btu/ft.deg.F)

c = specific heat of fluid (Btu/lb. deg.F)

D = intermediate parameter (dl)

E = ditto (dl)

F = ditto (dl)

f = ditto (dl)

G = transfer functions

g = intermediate parameter (dl)

H = total length of flow distance in the heat exchanger (ft)

h = running length along tube side fluid (ft)

K = intermediate parameter (dl)

k = overall coefficient of heat transfer (Btu/ft^2.min.deg.F)

L = H/v = distance-velocity lag of fluids (min)

M = intermediate parameter (dl)

n = mumbers of lags (dl)

r = v_1/v_2 (dl)

s = complex variable of Laplace Transformation (dl)

T_a = skew time of step response relative to L_1 (dl)

T_d = dead time of lag-delay model relative to L_1 (dl)

T_1 = time constant of lag model relative to L_1 (dl)

T_m = mean delay of step response relative to L_1 (dl)

T_r = time constant of root lag model relative to L_1 (dl)

T_s = dispersion time of step response relative to L_1(dl)

t = running time (min)

v = fluid velocity (ft/min)

W = flow rate (lb/min)

x = h/H (dl)

α = film coefficient of heat transfer (Btu/ft² min.deg.F)

α = coefficient of skew (dl)

$\beta = (a_1 + a_2)/2$

δ = ln (Steady state change in output/Steady state change in input)

$\varepsilon = (a_1 - a_2)/2$

θ = temperature of fluid (deg.F)

μ = coefficient of variance (dl)

$\rho = a/(1 + a)$

$\tau = t/L_1$ (dl)

ϕ = pipe temperature(deg.F)

ω = circular frequency

<u>Subscripts</u> :

1 = tube side

2 = shell side

h = tube

s = shell

BASIC ASSUMPTIONS

1) System parameters are uniform and constant.
2) Complete mixing in crosswise directions of each flow.
3) The heat conductivities of walls are either infinite in directions at right angles to the flow or alternatively assumed to be included in film coefficients, and zero in flow directions.
4) There are no internal sources or sinks of heat.
5) Pure counter- and parallel-flows (fig. 1) are considered.

FUNDAMENTAL EQUATIONS

The system parameters necessary for dynamic response analysis under the stated assumptions are the following fifteen (see also fig. 1) :

Flow rates of fluids = W_1 , W_2

Specific heats of fluids = c_1 , c_2

Surface areas = A_1 , A_2 , A_s

Film coefficients = α_1 , α_2 , α_3

Film velocities = v_1, v_2

Flowing distance = H

Solid heat capacities = C_h , C_s

These are conveniently grouped into the dimensionless forms (defined above);

a_1' , a_2' , a_s , b_1 , b_2 , b_s , r

Four of them are also conveniently grouped into the following dimensionless forms[7] for d-c gain calculations and other purposes;

---

7. These are also given in the following forms:

$$a_1 = \frac{kA}{W_1 c_1} \quad , \qquad a_2 = \frac{kA}{W_2 c_2}$$

where k (Btu/mn-ft^2-F) is the overall coefficient of heat transmission of the heating surface. This form can be introduced by means of the well-known law of heat transmission, which under the assumption stated above, is written as:

$$1/kA = 1/\alpha_1 A_1 + 1/\alpha_2 A_2$$

where 1/kA is the equivalent resistance to heat transmission

$$\frac{a_1' b_2}{b} = a_1 \quad , \quad \frac{a_2' b_1}{b} = a_2$$

where $b = b_1 + b_2$. Almost all these parameters have been necessary for the conventional steady state design of heat exchangers; for example, for mean temperature difference calculations.

The running time t(min) and running distance along the tube-side fluid h(ft) are also expressed in the following dimensionless forms;

$$\tau = t/L_1 \quad , \quad x = h/H$$

Now the simultaneous equations to be solved are:

$$\begin{aligned}
&\frac{\partial \theta_1}{\partial \tau} + \frac{\partial \theta_1}{\partial x} = a_1' \; (\phi_h - \theta_1) \\
&\frac{\partial \phi_h}{\partial \tau} = b_1 \; (\theta_1 - \phi_h) + b_2 \; (\theta_2 - \phi_h) \\
&\frac{\partial \theta_2}{\partial \tau} \pm \frac{\partial \theta_2}{\partial x} = a_2' \; (\phi_h - \theta_2) + a_s(\phi_s - \theta_2) \\
&\frac{\partial \phi_s}{\partial \tau} = b_s(\theta_2 - \phi_s)
\end{aligned} \qquad (1)$$

In these equations, θ_1 and θ_2 are tube-side and shell-side fluid temperatures, ϕ_h and ϕ_s are tube and shell temperatures, the double symbol $\pm$ is - for counter flow (Fig. 2) and + for parallel flow (Fig. 3).

In the following treatments, the Laplace transform solutions of Equation (1) are expanded in the following form:

$$G(s) = e^{\delta - T_m s + \frac{T_s^2}{2} s^2 - \frac{T_a^3}{b} s^3 + \cdots} \qquad (2)$$

where the parameters, δ, T_m, T_s, and T_a are given in terms of system constants listed above. The symbol s is the complex variable of the Laplace transformation. The value and significance of this representation has been indicated elsewhere (5). However, one may say in summary that δ measures the steady-state amplitude ratio between response and disturbance, T_m measures the mean time delay between response and disturbance, T_s defines the dispersion or attenuation and T_a the assymmetry or phase non-linearity. This characterization is very efficient for any physical process, such as those treated here, where the step response is monotonic non-decreasing in time.

COUNTER-FLOW

The Laplace transform solution (transfer function) of Equation (1) is:

$$G(s) = \frac{g_{1\cdot 2}}{\frac{f_1 + f_2}{2} + \frac{\sqrt{(f_1 + f_2)^2 - 4g_1g_2}}{2} \coth \frac{\sqrt{(f_1 + f_2)^2 - 4g_1g_2}}{2}} \qquad (3)$$

where

$$f_1 = \frac{a_1' \, (b_2+s)}{b + s} + s$$

$$f_2 = \frac{a_2' \, (b_1 + s)}{b + s} + s \quad r + \frac{a_s}{b_s + s}$$

$$g_1 = \frac{a_1' b_2}{b + s}, \qquad g_2 = \frac{a_2' b_1}{b + s}$$

and

$g_{1,2} = g_1$ when G(s) is defined as

$\left(\begin{array}{l}\text{Outlet temperature} \\ \text{of tube side fluid}\end{array}\right) / \left(\begin{array}{l}\text{Inlet temperature} \\ \text{of shell side fluid}\end{array}\right)$ Case 1 (Fig.2)

$g_{1,2} = g_2$ for the G(s) of

$\left(\begin{array}{l}\text{Outlet temperature} \\ \text{of shell side fluid}\end{array}\right) / \left(\begin{array}{l}\text{Inlet temperature} \\ \text{of tube side fluid}\end{array}\right)$ Case 2 (Fig. 2)

Now, the parameters of Equation (2) are determined by expanding (2) and (3) in series in s and comparing the corresponding terms. The expansion is easier for $G^{-1}(S)$ than G(S). The results yield a solution for the new parameters in the symbolic form;

$$\begin{aligned} \delta &= f_o \, (a_1', a_2', a_s', b_1, b_2, b_s, r) \\ T_m &= f_1 \, (a_1', a_2', a_s', b_1, b_2, b_s, r) \\ T_s &= f_2 \, (a_1', a_2', a_s', b_1, b_2, b_s, r) \\ T_a &= f_3 \, (a_1', a_2', a_s', b_1, b_2, b_s, r) \end{aligned} \qquad (4)$$

Details of the algebraic reduction procedure are given in Appendix 1. The most simple relation is given when solid capacities are neglected, and when the Wc values and the velocities are equal for both fluids. For this case, taking $\rho = a/(1+a)$, where $a = \frac{kA}{Wc}$, as abscissa, the new parameters T_m, T_s, T_a are plotted in terms of conventional relative statistical measures, in which we define coefficients in the form:

$$\text{Coefficient of variance:} \qquad \mu = \frac{T_s}{T_m}$$

$$\text{Coefficient of skew:} \qquad \alpha = \frac{T_a^{\,3}}{T_s^{\,3}} \tag{5}$$

A zero value of μ means that the time distribution has no dispersion about the mean T_m; a zero value of α signifies that the distribution is symmetric about the mean T_m. From the plot we can observe directly that when $\rho = 0$, which occurs for small sizes, with low overall efficiencies, the time distribution for a step disturbance is symmetric ($\alpha = 0$) and has the quickest response (minimum values of T_m and μ). As ρ increases, T_m, μ, and α, all increase with μ and α, becoming infinitely large as the length of the exchanger becomes infinite.

PARALLEL FLOW

The Laplace transform solution of Equation (1) is:

$$G(s) = g_{1,2}\, e^{-\frac{f_1 + f_2}{2}} \left[\frac{2}{\sqrt{(f_1 - f_2)^2 + 4g_1g_2}} \right] \sinh\left[\frac{\sqrt{(f_1-f_2)^2 + 4g_1g_2}}{2} \right] \tag{6}$$

The symbols f_1, f_2, g_1, g_2 and $g_{1,2}$ are the same as defined on counter-flow, see also Fig. 3.

From this equation the parameters of equation (2) are determined in the same symbolic form as equation (4). Details of algebraic reduction procedure and typical special cases are given in Appendix 2.

NUMERICAL EXAMPLE

As an example of the application of the formulas above to engineering practice, one can consider the special instances of counter-flow and parallel-flow exchangers with the following assumed characteristics:

$a_1 = 1.5$	$a_1' = 6$	$b = 27$	$a_s = 4$
$a_2 = 1.5$	$a_2' = 2$	$r = 3$	$b_s = 3$

STEP RESPONSE IN TUBE INLET TEMPERATURE

Counter Flow: Exchanger

Applying the above values to Equations (9), (12), and (13) we obtain

$$e^{\delta} = 1.65$$

$$T_m = 2.28$$

$$T_s = 2.16$$

$$T_a = 3.08$$

The measure of spread T_s and the assymmetry measure T_a can, as before, be expressed in terms of dimensionless coefficients, namely:

$$\text{Variation} \quad \mu = T_s/T_m = 0.948$$

$$\text{Skew} \quad \alpha = (T_a/T_s)^3 = 2.90$$

These values can be compared, for example to those for a unit lag, whose transform has the form:

(Lag) $$G_a(s) = \frac{1}{1 + T_\ell s}$$

where $\mu = 1$ and $\alpha = 2$. The variation μ can be matched by adding a suitable time delay term, since the transform for this case becomes

(Lag + Delay) $$G_b(s) = \frac{e^{-T_d s}}{1 + T_\ell s}$$

for which $T_m = T_d + T_\ell$

$$T_s = T_\ell$$

$$T_a = \sqrt[3]{2}\, T_\ell$$

giving $\mu = T_s/T_m = T_\ell / T_d + T_\ell$

$$\alpha = T_a^{\,3}/T_s^{\,3} = 2$$

Thus a lag-delay model with $\mu = 0.95$ and $\alpha = 2$ would have the form shown in Fig. 5.

However, the value of α for the heat exchanger indicates a curve more skewed than that expressed by a unit lag. Such a function is found in what the statisticians would call a "chi-square" distribution of one degree of freedom with the transform:

(Root-Lag) $$G_c(s) = \frac{1}{\sqrt{1 + T_r s}}$$

having the parameters

$$T_m = T_r/2$$

$$T_s = T_r/\sqrt{2} \qquad \mu = \sqrt{2} = 1.414$$

$$T_a = T_r \qquad \alpha = 2\sqrt{2} = 2.838$$

If this root-lag function is delayed in addition, with the transform:

(Root-lag-delay) $G_d(s) = e^{-T_d s} / \sqrt{1 + T_r s}$

and parameters

$$T_m = T_d + \frac{1}{2} T_r$$

$$T_s = T_r / \sqrt{2} \qquad \mu = \sqrt{2}/(1 + 2 \frac{T_d}{T_r})$$

$$T_a = T_r \qquad \alpha = 2 \sqrt{2}$$

Thus a distribution curve of this form with $\mu = 0.95$ and $\alpha = 2.828$ is also sketched in Fig. 5.

There are many other possible distributions, all with the same values T_m, T_s, T_a, but differing in higher order terms. These will all, in general, give reasonable approximations to the dynamic response of the given system, and many are susceptible to ready calculation. In the present instance, those shown in Fig. 5 come directly from functions which are readily available in tabular form and also easily realized in computing networks, etc.

Parallel Flow Exchanger

The general formulas (18) to (21) give

$$e^{\delta} = 0.475$$

$$T_m = 2.81$$

$$T_s = 1.10$$

$$T_a = 1.02$$

with the coefficients

$$\mu = T_s / T_m = 1.10/2.81 = 0.391$$

$$\alpha = (T_a / T_s)^3 = (1.02/ 1.10)^3 = 0.795$$

Now, it is readily shown (3) that an n-lag cascade (a chi-square distribution of 2n degrees of freedom) has the coefficients

$$\mu = 1/\sqrt{n}\ , \qquad \alpha = 2/\sqrt{n}$$

In the special case of n = 6.5, with a transfer characteristic of the form:

$$G_d\ (s) = \frac{1}{(1 + T_\ell\, s)^{6.5}}$$

The coefficients become

$$\mu = 0.392, \quad \alpha = 0.784$$

and the mean time T_m is given by

$$T_m = 6.5\ T_\ell$$

Accordingly, a reasonable approximation to the parallel flow step response characteristic can be found in the chi-square distribution of 13 degrees of freedom with $T_\ell = 2.81/6.5 = 0.432$. This is indicated in Fig. 6 and compared with a lag-delay approximation with $\mu = 0.391$ and $\alpha = 2$.

FREQUENCY RESPONSE RESULTS

In terms of the same representation used for transient response, the frequency responses of heat exchangers may be found directly.

Thus, if

$$G(s) = e^{\delta - T_m s + \frac{1}{2} T_s{}^2 s^2 - \frac{1}{6} T_a{}^3 s^3 + \ldots}$$

$$= e^{\delta \underbrace{+ \frac{1}{2} T_s{}^2 s^2 + \ldots\ldots\ldots\ldots}_{\text{Even}}}\ \ e^{-T_m s \underbrace{- \frac{1}{6} T_a{}^3 s^3 \ldots}_{\text{Odd}}}$$

Then, with $s = j\omega$

$$\text{Amplitude } \left| G(\omega) \right| = e^{\delta - \frac{1}{2} T_s{}^2 \omega^2 + \ldots}$$

$$\text{Phase } \angle G(\omega) = - T_m \omega + \frac{1}{6} T_a{}^3 \omega^3 - \ldots$$

so that the set of parameters δ, T_m, T_s, T_a, ... describe the frequency characteristics simply and uniquely. However, specification of <u>only</u> the first few parameters merely defines the <u>low</u> frequency behavior, and as the frequency ω is increased, more of the time constants will be required to characterize the behavior.

However, one can proceed as was done in estimating the step response, by picking a suitable model using the low frequency constants alone, and thus extrapolate the response to high frequencies under the tacit assumption that the model so chosen will behave at least roughly like the prototype at higher frequencies.

Then for the counterflow exchanger, there has been plotted in Fig. 7 the predicted frequency response characteristics for the two previously determined models, namely

COUNTER-FLOW EXCHANGER

	Lag-Delay Model	Root-Lag-Delay Model
Amplitude Ratio	$1/\sqrt{1 + T_\ell^2\omega^2}$ $= 1/\sqrt{1 + 4.66\omega^2}$	$1/\sqrt[4]{1 + T_r^2\omega^2}$ $= 1/(1 + 9.40\omega^2)^{0.25}$
Phase	$T_d\omega + \tan^{-1} T_\ell\omega$ $= 0.12\omega + \tan^{-1} 2.16\omega$	$T_d\omega + \frac{1}{2}\tan^{-1} T_r\omega$ $= 0.75\omega + 0.5\tan^{-1} 3.07\omega$

In a directly similar fashion, the response characteristics for the parallel flow exchanger have also been plotted in Fig. 8 from the formulas below.

PARALLEL-FLOW EXCHANGER

	Lag-Delay Model	Multi-Lag Model
Amplitude Ratio	$1/\sqrt{1 + T_\ell^2\omega^2}$ $= 1/\sqrt{1 + 1.21\omega^2}$	$1/(1 + T_r^2\omega^2)^{13/4}$ $= 1/(1 + 0.19\omega^2)^{3.25}$
Phase	$T_d\omega + \tan^{-1} T_\ell\omega$ $= 1.71\omega + \tan^{-1} 1.10\omega$	$6.5\tan^{-1}(T_r\omega)$ $= 6.5\tan^{-1}(0.43\omega)$

EXPERIMENTAL CONFIRMATION

Experiments under carefully controlled conditions have been made previously by one of the authors (1,2) upon a heat exchanger model used both in counter flow and parallel flow. These yielded among other results the response in the shell stream outlet temperature to disturbances in the tube stream inlet temperature. These disturbances involved both stepwise and sinusoidal changes.

Results of some of these model tests are indicated in Fig. 9 to 12. It is important to stress that the step responses and the frequency responses represent data from independent test procedures and therefore represent, in a certain sense at least, independent physical data.

From blueprint data and direct measurements of the model the basic physical constants were obtained. These correspond precisely to the data assumed in the numerical examples of the previous paragraphs. However, the surface conductance constants α_1 and α_2 were back-figured, at least in part, from the calculated steady-state (aero frequency) temperature ratios. Moreover, the distance velocity lag was estimated from measurements only with tolerable accuracy at $L_1 = 0.6$ minutes with a probable error of at least 0.05 minutes.

With these restrictions understood, the predicted and measured step and frequency responses are depicted in Fig. 9 to 12.

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APPENDIX 1.

For counter-flow, the parameters of equation (2) are given by;

$$\left.\begin{aligned} \delta &= -\ln D_o \\ T_m &= D_1/D_o \\ T_s &= \sqrt{(D_1/D_o)^2 - 2D_2/D_o} \\ T_a &= \sqrt[3]{\frac{1}{3}\left(\frac{D_1}{D_o}\right)^3 - \left(\frac{D_1}{D_o}\right)\left(\frac{D_2}{D_o}\right) + \left(\frac{D_3}{D_o}\right)} \end{aligned}\right\} \qquad (7)$$

etc.

The D_o , D_1 , D_2 , D_3 are;

$$\left.\begin{aligned} D_o &= \frac{1}{a_{1,2}}\,(M_o + B_o) \\ D_1 &= \frac{1}{a_{1,2}}\,(M_1 + B_1 + \frac{B_o}{b}) \\ D_2 &= \frac{1}{a_{1,2}}\,(M_2 + B_2 + \frac{B_1}{b}) \\ D_3 &= \frac{1}{a_{1,2}}\,(M_3 + B_3 + \frac{B_2}{b}) \end{aligned}\right\} \qquad (8)$$

etc.

where $a_{1,2} = a_1$ for case 1, $a_{1,2} = a_2$ for case 2 in Fig. 2. The M and B are:

$$\begin{aligned} M_o &= \frac{1}{2}\,(a_1 + a_2) \\ M_1 &= \frac{1}{2}\left[\frac{a_1' + a_2'}{b} + 1 + r + \frac{a_s}{b_s}\right] \\ M_2 &= \frac{1}{2}\left\{\frac{1}{b}\,(1 + r + \frac{a_s}{b_s}) - \frac{a_s}{b_s^2}\right\} \\ M_3 &= \frac{1}{2}\,\left(\frac{a_s}{b_s^2}\right)\,\left(\frac{1}{b_s} - \frac{1}{b}\right) \end{aligned} \qquad (9)$$

etc.

$$
\begin{aligned}
B_o &= \mathcal{E} \coth \mathcal{E} \\
B_1 &= \mathcal{E} E_1 \left(\coth \mathcal{E} - \mathcal{E} \operatorname{csech}^2 \mathcal{E}\right) \\
B_2 &= \mathcal{E}\left\{E_2 \left(\coth \mathcal{E} - \mathcal{E} \operatorname{csech}^2 \mathcal{E}\right) + \mathcal{E} E_1^2 \left(\mathcal{E} \coth \mathcal{E} - 1\right)\right\} \\
B_3 &= \mathcal{E}\left\{E_3 \left(\coth \mathcal{E} - \mathcal{E} \operatorname{csech}^2 \mathcal{E}\right) + 2 E_1 E_2\, \mathcal{E} \operatorname{csech}^2 \mathcal{E} \left(\mathcal{E} \coth \mathcal{E} - 1\right)\right. \\
&\quad \left. + E_1^3\, \mathcal{E}^2 \operatorname{csech}^2 \mathcal{E} \left(\coth \mathcal{E} - \mathcal{E} \coth^2 \mathcal{E} + \frac{\mathcal{E}}{3}\right)\right\}
\end{aligned}
\tag{10}
$$

etc.

where $\mathcal{E} = (a_1 - a_2)/2$.

The E_1 , E_2 , ... in Equation (6) are:

$$
\begin{aligned}
E_1 &= \frac{K_1}{2K_o} , \quad E_2 = \frac{K_2}{2K_o} - \frac{K_1^2}{8K_o^2} \\
E_3 &= \frac{K_3}{2K_o} - \frac{K_1 K_2}{4K_o^2} + \frac{K_1^3}{K_o^3}
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
K_o &= 4\mathcal{E}^2 \\
K_1 &= -\frac{8\mathcal{E}^2}{b} + 2(a_1 + a_2)\left\{\frac{a_1' + a_2'}{b} + \left(1 + r + \frac{a_s}{b_s}\right)\right\} \\
K_2 &= \frac{12\mathcal{E}^2}{b} + \left\{\left(1 + r + \frac{a_s}{b_s}\right) + \frac{2(a_1' + a_2')}{b}\right\}\left\{\left(1 + r + \frac{a_s}{b_s}\right)\right. \\
&\quad \left. - \frac{2(a_1 + a_2)}{b}\right\} + \left(\frac{a_1' + a_2'}{b}\right)^2 - 2\frac{a_s}{b_s^2}(a_1 + a_2)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
K_3 &= -\frac{16\mathcal{E}^2}{b^3} + 6\,\frac{a_1' + a_2'}{b^3}(a_1 + a_2) - 2\,\frac{(a_1' + a_2')^2}{b^3} \\
&\quad + 2\,\frac{a_s}{b_s^3}(a_1 + a_2) - 2\,\frac{a_s}{b_s^2 b}\left\{(a_1' + a_2') - (a_1 + a_2)\right\} \\
&\quad + \frac{2}{b^2}\left\{(a_1 + a_2) - (a_1' - a_2')\right\}\left(1 + r + \frac{a_s}{b_s}\right) \\
&\quad - 2\,\frac{a_s}{b_s^2}\left(1 + r + \frac{a_s}{b_s}\right)
\end{aligned}
$$

Given the system parameters, we can evaluate M and K, and from K we can find E, hence B. Applying these M and B, the required parameters δ, T_m, T_s, and T_a are found by Equation (7), (8). These procedures and relations get simpler for special cases as follows:

Special Case 1:

If $\varepsilon \doteq 0$, that is, $a_1 \doteq a_2$, the Equation (10) may be rewritten as

$$\begin{aligned} B_o &= 1 \\ B_1 &= K_1/12 \\ B_2 &= K_2/12 - K_1^2/720 \\ B_3 &= K_3/12 - K_1K_2/360 + K_1^3/30240 \end{aligned} \qquad (13)$$

Special Case 2:

If we neglect the effects of solid capacities, the Equations (11) and (12) reduce to:

$$\begin{aligned} E_1 &= (a_2 + \varepsilon)\ (1 + r)/(2\varepsilon^2) \\ E_2 &= \frac{(1 + r)^2}{8\varepsilon^2}\left\{1 - \frac{(a_2 + \varepsilon)^2}{\varepsilon^2}\right\} \\ E_3 &= \frac{(1 + r)^3}{16\varepsilon^4}\left[-(a_2 + \varepsilon) + \frac{(a_2 + \varepsilon)^2}{\varepsilon^2}\right] \end{aligned} \qquad (14)$$

Special Case 3:

If $\varepsilon \doteq 0$ in Special Case 2, the Equation (10) may be directly given by:

$$\begin{aligned} B_o &= 1 + \varepsilon^2/3 \\ B_1 &= \frac{(a_2 + \varepsilon)\ (1 + r)}{2}\left[\frac{2}{3} - \frac{4}{45}\varepsilon^2\right] \\ B_2 &= \frac{(1 + r)^2}{4}\left\{\frac{1}{3} - \frac{4}{45}(a_2 + \varepsilon)^2 - \frac{2}{45}\varepsilon^2 + \frac{8}{315}\varepsilon^2(a_2 + \varepsilon)^2\right\} \\ B_3 &= \frac{(1 + r)^3}{8}(a_2 + \varepsilon)\left[-\frac{4}{45} + \frac{16}{945}(a_2 + \varepsilon)^2 + \frac{8}{315}\varepsilon^2 \right. \\ &\qquad \left. - \frac{2078}{288225}(a_2 + \varepsilon)^2\ \varepsilon^2\right] \end{aligned} \qquad (15)$$

Special Case 4:

If $\mathcal{E} = 0$ in Special Case 3, i.e., no solid capacities and $a_1 = a_2 = a$, the final results are directly given by:

$$e^{-\delta} = \frac{1 + a}{a}$$

$$e^{-\delta} T_m = (1 + r)\left(\frac{1}{2a} + \frac{1}{3}\right)$$

$$e^{-\delta}\left(-\frac{T_s^2}{2} + \frac{T_m^2}{2}\right) = (1 + r)^2\left(\frac{1}{12a} - \frac{a}{45}\right) \tag{16}$$

$$e^{-\delta}\left(\frac{T_a^3}{6} - \frac{T_m T_s^2}{2} + \frac{T_m^3}{6}\right) = -(1 + r)^3\left(\frac{1}{90} - \frac{2a^2}{94}\right)$$

Special Case 5:

Same as Special Case 4, and $r = 1$. Let us denote $e^{\delta} = \frac{a}{1 + a} = \rho$, then we have:

$$T_m = 1 + \frac{1}{3}\rho$$

$$T_s^2 = \frac{1}{3} + \frac{4}{3}\rho + \frac{\rho^2}{9} + \frac{8\rho^2}{45(1 - \rho)} \tag{17}$$

$$T_a^3 = \frac{32}{15}\rho + \frac{4}{3}\rho^2 + \frac{2\rho^3}{27} + \frac{8\rho^2}{15(1 - \rho)} + \frac{8\rho^3}{45(1 - \rho)} + \frac{32\rho^3}{315(1 - \rho)^2}$$

The values of these terms are shown in Fig. 4.

APPENDIX 2.

For parallel-flow from Equation (6) the parameters of Equation (2) are determined as follows:

$$e^{\delta} = a_{1,2} \frac{\sinh \beta}{\beta} e^{-\beta}$$

$$T_m = M_1 + \frac{1}{b} - F_1$$

$$\frac{T_s^2}{2} = -M_2 + F_2 - \frac{1}{2}(F_1^2 - \frac{1}{b^2}) \tag{18}$$

$$\frac{T_a^3}{6} = M_3 - F_3 + F_1F_2 + \frac{F_1}{3b^2} - \frac{2}{3b} F_1^2 - \frac{1}{3b^3}$$

where $a_{1,2} = a_1$ for Case 1, $a_{1,2} = a_2$ for Case 2 in Fig. 3, and $\beta = (a_1 + a_2)/2$.

The M and F in these results are:

$$M_1 = \frac{1+r}{2} + \frac{a_s}{2b_s} + \frac{a_1' + a_2'}{2b} - \frac{\beta}{b}$$

$$M_2 = \frac{a_s}{2b_s^2} - \frac{a_1' + a_2'}{2b^2} + \frac{\beta}{b^2} \tag{19}$$

$$M_3 = \frac{a_s}{2b_s^3} + \frac{a_1' + a_2'}{2b^3} - \frac{\beta}{b^3}$$

$$F_1 = E_1 (\beta \coth \beta - 1)$$

$$F_2 = E_2 (\beta \coth \beta - 1) - E_1^2 \left[\beta \coth \beta - (1 + \frac{\beta^2}{2})\right] \tag{20}$$

$$F_3 = E_3 (\beta \coth \beta - 1) - 2E_1E_2\left[\beta \coth \beta - (1 + \frac{\beta^2}{2})\right]$$

$$+ A_1^3 \left[(1 + \frac{\beta^2}{6}) \beta \coth \beta - (1 + \frac{\beta^2}{2})\right]$$

The E_1, E_2, ... in Equation (20) are defined in the same form as Equation (11), but K_o, K_1, ... in it are given in place of Equation (12) as follows:

$$K_o = 4\beta^2$$

$$K_1 = 2(a_1 - a_2)(1 - r) - 2(a_1 - a_2)\frac{a_s}{b_s} + \frac{2}{b}\left[(a_1 - a_2)(a_1' - a_2') - 4\beta^2\right]$$

$$K_2 = \left[(1-r-\frac{a_s}{b_s}) + \frac{a_1'-a_2'}{b}\right]^2 + (a_1-a_2)\left[2\frac{a_s}{b_s} - \frac{2}{b}(1-r-\frac{a_s}{b_s}) + \frac{4(a_1'-a_2')}{b^2}\right] + \frac{12\beta^2}{b^2} \tag{21}$$

$$K_3 = 2\frac{a_s}{b_s^2}(1-r-\frac{a_s}{b_s}) - 2(a_1-a_2)\frac{a_s}{b_s^3} + \frac{2}{b}\frac{a_s}{b_s^2}\left\{(a_1'-a_2') - (a_1-a_2)\right\} + \frac{2}{b^2}(1-r-\frac{a_s}{b_s})(a_1-a_2) - \frac{16\beta^2}{b^3} + \frac{6}{b^3}(a_1-a_2)(a_1'-a_2') - \frac{2}{b^3}(a_1'-a_2')$$

Special Case 1:

No solid capacities,

$$e^{\delta} = e^{-\beta} a_{1,2}\frac{\sinh\beta}{\beta}$$

$$T_m = \frac{1+r}{2} - \frac{(a_1-a_2)(1-r)}{4\beta^2}(\beta\coth\beta - 1) \tag{22}$$

$$\frac{T_s^2}{2} = \frac{(1-r)^2}{8\beta^2}\left[1 - \frac{(a_1-a_2)^2}{\beta^2}\right](\beta\coth\beta - 1) + \frac{(1-r)^2(a_1-a_2)^2}{32\beta^2}(1-\beta^2\operatorname{csech}^2\beta)$$

$$T_a^3 = \frac{(a_1-a_2)^3(1-r)^3}{64\beta^5}\left[1 + \frac{\beta^2}{3} - \beta\coth\beta\right]\coth\beta + \frac{(a_1-a_2)(1-r)^3}{32\beta^4} \times \left[1 - \frac{(a_1-a_2)^2}{\beta^2}\right]\left(\beta^2\operatorname{csech}^2\beta - 1\right) + \frac{(1-r)^3(a_1-a_2)}{32\beta^4} \times \left[1 - \frac{(a_1-a_2)^2}{4\beta^2}\right](\beta\coth\beta - 1)$$

Special Case 2:

Same as Case 1, and r = 1.

$$e^{\delta} = a_{1,2}e^{-\beta}\frac{\sinh\beta}{\beta}$$

$$T_m = 1 \tag{23}$$

$$T_s = 0, \quad T_a = 0$$

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Fig. 12 Computed and experimental frequency responses for a parallel-flow exchanger

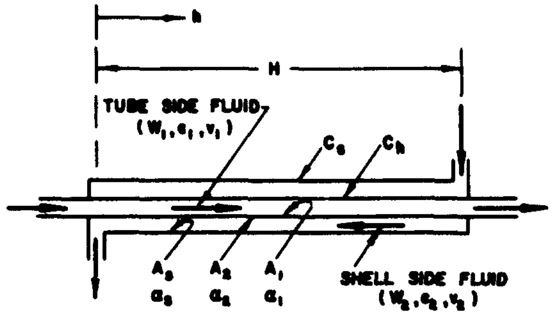

Fig.1

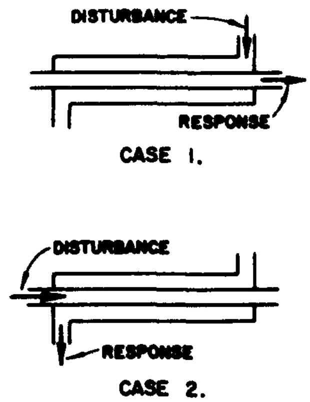

Fig.2

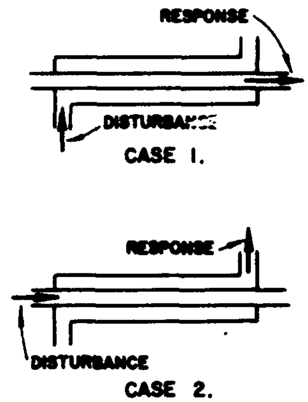

Fig.3

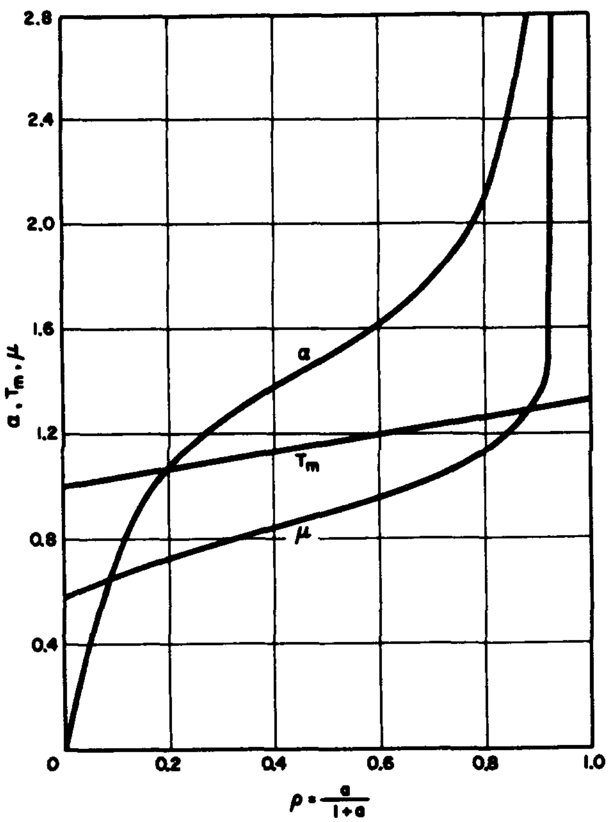

Fig.4

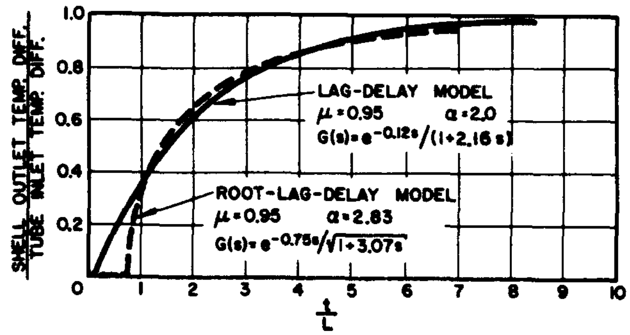

Fig.5

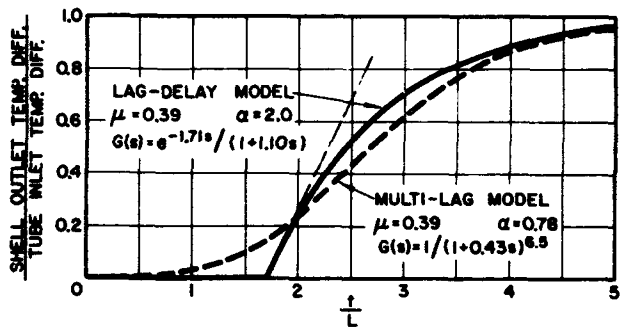

Fig.6

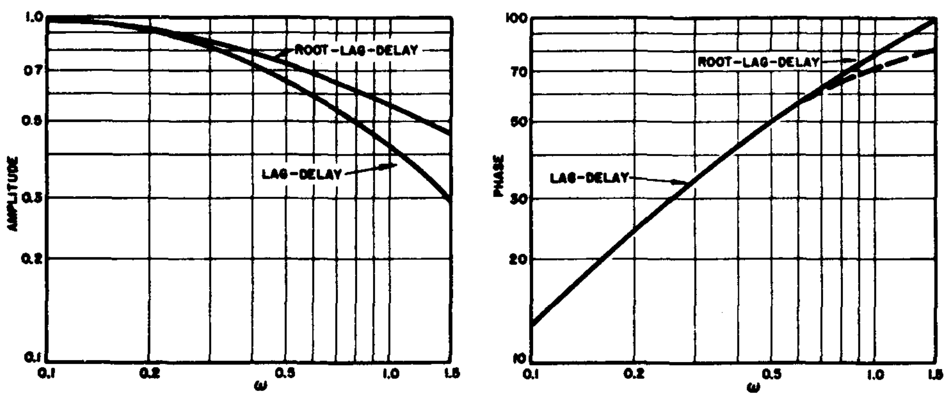

Fig.7

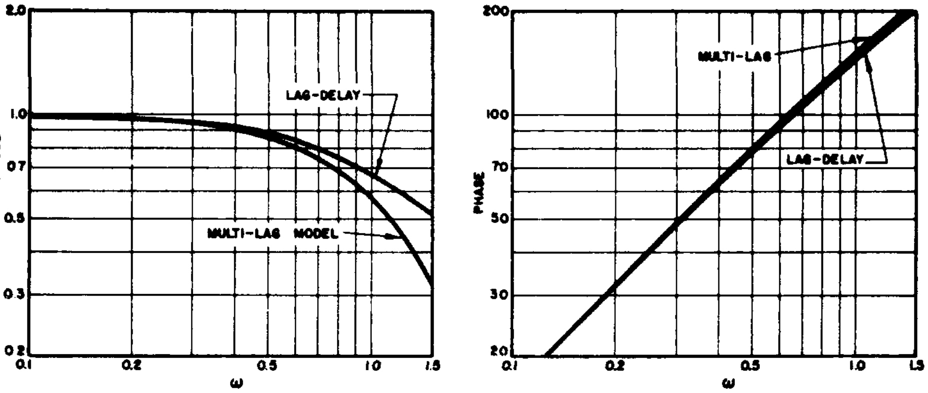

Fig.8

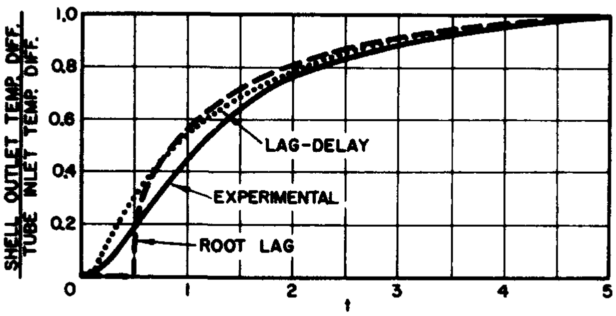

Fig.9

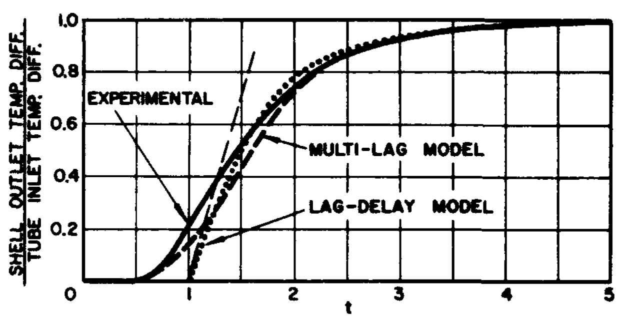

Fig.10

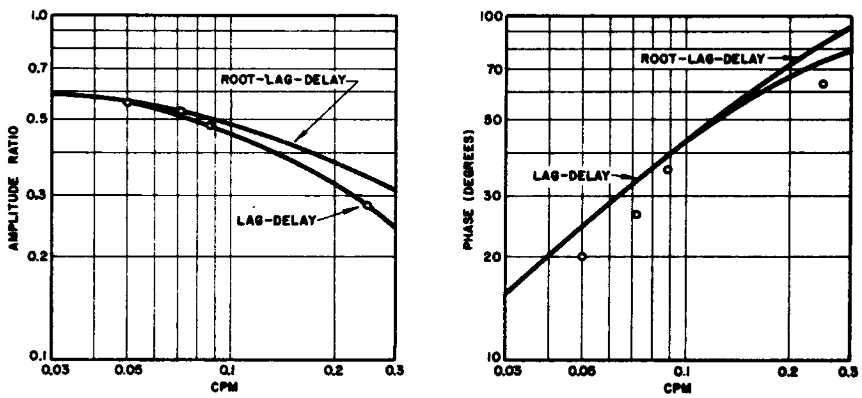

Fig.11

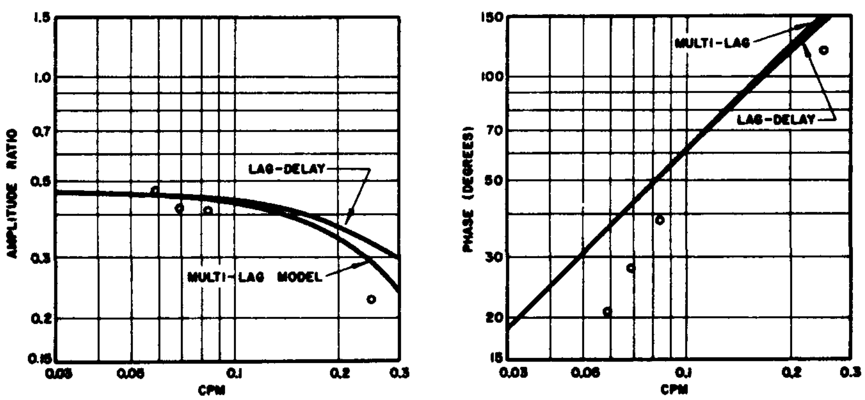

Fig.12