CONVECTION ALGORITHMS FOR MAGNETOSPHERIC PARTICLES

Kevin J. Kerns, Capt, USAF

10 August 1993

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**Abstract:**
This report documents the implementation of the convection model used for the research by Kerns, Hardy, and Gussenhoven titled "Modelling of Convection Boundaries Seen by CRRES in 120 eV to 28 keV Particles." This research is to be published in the Journal of Geophysical Research, Space Physics in 1993. The convection model assumes an Earth centered dipole magnetic field with equipotential field lines and a static cross tail electric field. Convection results from EXB drifts and magnetic field gradient and curvature drifts. Algorithms are provided to calculate drift paths of particles as well as to calculate the energy range of particles at a specified position that are on open drift paths to the magnetospheric tail. The algorithms presented in this document are optimized to converge to a solution quickly for any particle energy, position, and equatorial pitch angle. These algorithms are implemented in the CRRES survey software called SHOWLEPA.EXE which was used in support of the research.

**Subject Terms:**
- Plasma sheet
- Magnetospheric convection
- Electron
- Proton
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Convection Algorithms for Magnetospheric Particles

1. INTRODUCTION

This report documents in detail the implementation of the convection model used in the research conducted by Kerns, Hardy, and Gussenhoven [1993]. The convection model assumes an Earth-centered dipole magnetic field with equipotential field lines, and it assumes a static cross tail electric field of the form \( V = \gamma L \sin(\phi) \) where \( \gamma \) is a positive real number, \( L \) is L-shell (see McIlwain [1961]), and \( \phi \) is local time. Convection occurs because of \( E \times B \) drifts and magnetic field gradient and curvature drifts. The model is used to calculate drift paths of particles as well as to calculate the range of kinetic energy for which ions and electrons move on drift paths open to the magnetospheric tail. Drift paths are calculated using conservation of the first two adiabatic invariants [Chen, 1984] and conservation of energy. Electrons are shown to be on open drift paths if the kinetic energy, \( W_e \), is \( 0 < W < W_c \) where \( W_c \) is the electron cutoff energy, and ions are on open drift paths if \( W_l < W < W_u \) where \( W_l \) and \( W_u \) are the ion lower and upper cutoff energies respectively. The algorithms presented in this document are optimized to converge to a solution quickly for any positive real value of \( \gamma \), for any particle energy, position, and equatorial pitch angle, and for any cross-tail field strength. These algorithms are used by the CRRES survey software called "SHOWLEPA.EXE" which was developed by this author at the Phillips Laboratory in support of the research by Kerns, Hardy, and Gussenhoven [1993].

Implementations of equations are given in algorithms that are listed in text boxes separate from the main text. These algorithms are written in pseudo-code similar to Pascal to make them readable to the non-programmer and useable to the programmer; comments are given in italics and are not considered part of the instructions or code. These are added to make the code more readable. Constants are defined at the top of any procedure or function using the " = " symbol, and each constant represents a value that does not change. Constants are used to reduce unnecessary math to make the code execute more quickly. All variables are assumed to contain real values unless specifically identified as integers, and the variables required for any given function or procedure are listed near the top of the code for that procedure or function. Algorithms use logical, not

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mathematical, statements so that the statement "i=i+1", which is nonsense in mathematics, means to increment the value contained in the variable called "i" by 1. Subscripts are part of variable names, and are used to make code more readable. The one exception to the use of subscripts is with the table of values for the function I(y) given in Algorithm 1; here subscripts are used to identify individual elements of the table. Function and procedure names are given in bold letters, and all parameters passed to these use the naming convention given in Algorithm 1. The repeat block until condition loop implements a block of code once, and then continues to re-implement the same block until the given condition is true. The while condition block iterates the next block of code for as long as the given condition is true. The if condition then code statement implements the block of code only if the condition is true, and the if condition then code1 else code2 statement implements the "code1" block of code if the condition is true and the "code2" block of code otherwise. All blocks of code that are longer than one line, with the exception of those bounded by "repeat ... until" loops, are marked by "begin" and "end" statements. Indentation is used to make loops and blocks more readable. Nested parentheses are made more readable by using "{{ ( .. ) }}".

2. CONVECTION MODEL

In this section the model used to calculate the trajectories of central plasma sheet

Algorithm 1 Global constants and variables used by other algorithms, and names of parameters passed to functions and procedures given in other algorithms.

Global constants used in all calculations and algorithms:
- \( w = 7.272 \times 10^6 \) radians per second, Earth's spin rate
- \( a = 6.378 \times 10^8 \) meters, Earth's radius
- \( B_0 = 3.1 \times 10^{-7} \) tesla, magnetic field at Earth's Equator
- \( I = I(i/N); 0 \leq i \leq N \) \( I() \) defined in Algorithm 2; "i" is element index of table
- \( N = 1000 \) \( I_p \) is first element, and \( I_n \) is last element of table
- \( \delta_w = 0.01 \) \( R_e \) error allowed in calculating L-Shells
- \( \delta_w = 0.001 \) eV, error allowed in calculating cutoff energies
- \( W_{step} = 10000 \) eV, energy step used in cutoff solutions

Global variables used in all calculations and algorithms:
- \( A \) volts/meter, from Equation 2
- \( y \) from Equation 2
- \( L_0 \) from Equation 7

Standardized names of parameters passed to functions and procedures:
- \( \gamma \) sine of equatorial pitch angle, \( 0.0 \leq \gamma \leq 1.0 \), (Equations 3 and 4)
- \( s \) sine of local time; \( \sin(n/12 \cdot L) \), \( -1.0 \leq s \leq 1.0 \)
- \( W \) kinetic energy in eV
- \( L \) L-shell in \( R_e \)
- \( \mu \) first adiabatic invariant constant (Equation 3)
- \( J \) second adiabatic invariant constant (Equation 4)
- \( U \) total energy of particle in eV (Equation 5)

- electron boolean; true if particle is an electron and false if it is an ion.

2
(CPS) particles in the inner magnetosphere under conditions of steady convection is described. The model follows the same method and assumptions used by Volland [1973]. It assumes a dipole geomagnetic field aligned with the Earth’s spin axis and having equipotential field lines. The co-rotation electric field potential is then taken as

\[ V_\omega = B_0 a^2 \omega L^{-1}, \]  

(1)

where \( B_0 \) is the strength of the equatorial magnetic field at the surface of the Earth, \( a \) is the Earth’s radius, \( \omega \) is the Earth’s spin rate, and \( L \) is L-shell in \( R_E \). The cross-tail potential is assumed to be of the form given by Volland [1973]

\[ V_\gamma = a A L^\gamma \sin(\phi) \]  

(2)

where \( A \) scales the field strength, \( \gamma \) determines the rate of decrease in potential with decreasing altitudes, and \( \phi \) is the local time. The model is used to determine the range of kinetic energy of ions and electrons in which the particles convect on open drift paths from the tail. The bounds of this range are referred to as cutoff energies, and these are a function of the position at which the particle is measured as well as the parameters \( A \) and \( \gamma \) of Eq. (2). Cutoff energies for steady convection have been introduced previously in other works including Stern [1975] and Southwood and Kaye [1979]. In all examples given in this report, \( \gamma = 2 \), after Stern [1975], and the values of the constants in Eqs. (1) and (2) are given in Algorithm 1. The values of \( A \) and \( \gamma \) are given as global variables in Algorithm 1, and these are used by the subsequent algorithms. These values are scaled to represent different magnetospheric conditions. In addition to the values of the constants used, Algorithm 1 contains the standard names of parameters passed to the functions given in subsequent algorithms.

Algorithm 2 Implementation of \( I(y) \) given in Equation 4.

FUNCTION DEFINITION OF \( I(y) \)  
local constants required for \( I(y) \):
\[ \alpha = 1.380173 \] \[ \beta = \frac{1}{2} \alpha - 0.3702402 \] \[ a_1 = 0.055 \quad a_2 = -0.037 \] \[ a_3 = -0.074 \quad a_4 = 0.056 \] \[ b_0 = 2 \alpha \] \[ b_1 = 3 a_1 \] \[ b_2 = 6 a_2 \] \[ b_3 = 48 - 2 \alpha - 3 a_1 - 6 a_2 + 6 a_4 \] \[ b_4 = -6 a_1 \] \[ b_5 = 2 (\beta - a_3) \] \[ b_6 = -4 \beta \]  

local variables required for \( I(y) \):
\( \gamma_L \), \( \gamma_C \), \( I \)  
implementation of \( I(y) \):
if \( y = 0.0 \) then
\( I = b_0 \)
else if \( y = 1.0 \) then
\( I = 0.0 \)
else
begin conditional block
\( \gamma_L = \ln(y) \)  
\( \gamma_C = \exp(\frac{1}{2}\gamma_L) \)  
\( I = b_0 + (b_1 + (b_2 + (b_3 + b_4 \cdot \gamma_L + b_5 \cdot \gamma_C) \cdot \gamma_C) \cdot \gamma_C + b_6 \cdot \gamma^2) \)
end conditional block

return value of \( I \)
Particle motion is assumed to be a combination of $E \times B$ drift and magnetic field gradient and curvature drift. Trajectories are determined by conservation of energy and conservation of the first two adiabatic invariants. From these the three following constants of motion are defined:

$$\mu = y^2 W L^3$$  \hspace{1cm} (3)

$$J = W^v L I(y)$$  \hspace{1cm} (4)

$$U = W + q[aAL^v \sin(\phi) - L^{-1} B_0 a^2 \omega]$$  \hspace{1cm} (5)

where $\mu$ and $J$ are proportional to the first two adiabatic invariants, $U$ is the total energy, $y$ is the sine of the equatorial pitch angle, $W$ is kinetic energy, and $q$ is charge. Ions are assumed to be singly ionized so that $q = \pm 1$ for ions and electrons respectively. The function $I(y)$ in Eq. (4) is an integration function of the bounce motion, and the approximation given by Ejiri [1978] is used for the implementation shown in Algorithm 2. Equation (5) is the sum of the kinetic energy and potential energy [Eq. (1) and (2)].

3. CALCULATION OF DRIFT PATHS

Valid drift paths of particles are the set of all points where Eqs. (3), (4), and (5) give $\mu = \mu_M$, $J = J_M$, and $U = U_M$. These constants of motion are for a particle with measured kinetic energy $W = W_M$ at a position $L = L_M$, $\sin(\phi) = \sin(\phi_M)$, and $y = y_M$. Algorithm 3 shows the implementation of these equations. Here the function $I(y)$ from Eq. (4) is replaced by a look-up table with $N + 1$ equally spaced elements, and the value of $I(y)$ is found by interpolating between the two nearest table elements. This table is used to make the algorithms faster.

The simplest case of convection is for zero kinetic energy particles ($W_M = 0$), and this is presented first. From Eqs. (3) and (4), $\mu_M = J_M = 0$. and conservation of energy [Eq. (5)] requires that these particles drift on equipotential lines given by

$$V = aAL^v \sin(\phi) - L^{-1} B_0 a^2 \omega$$  \hspace{1cm} (6)

Figure 1 shows the equipotentials given by this equation for $A = 7.46 \times 10^5$ V/m. In this figure, noon is towards the top of the page and each hour of LT is marked by a radial line. The half shaded circle represents the Earth, and the concentric circles show L-shells 2 through 9 $R_E$. The heavy lines show equipotentials from -45 kV to 5 kV at 5 kV intervals, and these are symmetric about the dawn-dusk line. In this example, the three lines shown closest to the Earth (-45 to -35 kV) are closed, and zero kinetic energy particles on these lines are trapped and drift around the Earth in an Eastward direction. The equipotential lines further from the Earth are open, and zero kinetic energy particles on these are not trapped, but drift sunward from the tail until they encounter the magnetopause.
Algorithm 3 Implementation of Equations 3, 4, and 5.

FUNCTION DEFINITION OF \( p(y, W, L) \)

implementation of \( p(y, W, L) \):
\[
\text{return value of } (y \cdot L)^2 \cdot L - W
\]

FUNCTION DEFINITION OF \( J(y, W, L) \)

Equation 4.

local variables required for \( J(y, W, L) \):
\[
I, \quad \text{value of } l(y) \text{ interpolated from the array } l, \text{ (Algorithm 1)}
\]
\[
i, \quad \text{index of element in } l, \text{ to be used for interpolating } l,
\]

implementation of \( J(y, W, L) \):
\[
\begin{align*}
\text{if } y = 0.0 \text{ then} & \\
I, = I_0 \\
\text{else if } y = 1.0 \text{ then} & \\
I, = I_n \\
\text{else} & \\
\text{begin conditional block} & \\
I, = N \cdot y & \quad N \text{ is maximum index of } l \text{ (Algorithm 1)} \\
i, = \text{trunc}(l, l) & \quad \text{trunc returns the largest integer less than} \\
l, = I_0 - i & \quad \text{or equal to } I_0 \text{,} \\
l, = l, \cdot (1 - l, \cdot 1) + l, + 1, & \quad I, \text{ now contains interpolated value of } l(y) \\
\text{end conditional block} \\
\text{return value of } W^3 \cdot L \cdot l, & \\
\end{align*}
\]

FUNCTION DEFINITION OF \( U(\text{electron}, W, L, s) \)

Equation 5.

local variables required for \( U(\text{electron}, W, L, s) \):
\[
k = B_0 \cdot a^2 \cdot \omega
\]

implementation of \( U(\text{electron}, W, L, s) \):
\[
\begin{align*}
\text{if electron then} & \\
\text{see Algorithm 1} & \\
\text{return value of } W - a \cdot A \cdot s \cdot L^r + k/L & \quad q = -1 \\
\text{else} & \\
\text{return value of } W + a \cdot A \cdot s \cdot L^r \cdot k/L & \quad q = +1 \\
\end{align*}
\]

The zero energy Alfvén layer, \( L_0 \), is the last closed drift path for zero kinetic energy particles. In Figure 1, \( L_0 \) is tear-drop shaped and lies along the -30 kV equipotential, and it bounds the shaded region that surrounds the Earth. \( L_0 \) at dusk (1800) is a stagnation point for \( W=0 \) particles because here the corotation electric field is equal and opposite to the cross-tail field resulting in no electric field drift; Ejiri [1978] derives the L-shell of this point, \( L_0 \), as:

\[
L_0 = \left( \frac{\omega a B_0}{\gamma A} \right)^{\frac{1}{1+1}}
\]  

(7)

The value of \( L_0 \) is used, instead of the value of \( A \), to indicate the cross-tail field strength in the remaining examples because \( A \) can be determined from Eq. (7), and its value scales with the distance of \( L_0 \) from the Earth. \( L_0 = 4.58 \text{ Re} \) in Figure 1.

The L-shell of \( L_0 \) can be determined for any local time by using Eqs. (6) and (7). The potential, \( V_2 \), of \( L_0 \) is found by substituting \( L_0 \) for \( L \), and -1 for \( \sin(\phi) \) in Eq. (6). As is evident in Figure 1, \( L_0 \) is at maximum L-shell at dusk, so that \( L_0 \) is always greater than or equal to the L-shell of \( L_0 \) for the given local time. Solving for the partial of \( V \)
Figure 1 Equipotentials for $A=7.46 \times 10^{-5}$ V/m [Eq. (2)]. Radial distance shows L-shell in $R_E$, and angular distance shows local time. The shaded region is bounded by $L_0$.

[Eq. (6)] with respect to $L$ and setting the result equal to zero gives

$$L_{\text{wshL}=0} = \left[ \frac{\omega A B_0}{-\gamma A \sin(\phi)} \right]^{\frac{1}{\gamma+1}}$$

(8)

This equation shows that the slope of $V$ as a function of $L$ in Eq. (7) does not change sign on the dawn side ($\sin(\phi) \geq 0$) because $\partial V/\partial L$ cannot equal zero. On the dusk side ($\sin(\phi) < 0$), Eq. (8) shows that the L-shell at which the slope changes sign ($\partial V/\partial L = 0$) must be greater than $L_0$ [Eq. (7)]. Because the slope does not change sign in the region where the valid solution exists ($0 \leq L \leq L_0$), a simple binary search can be used to determine the L-shell at a given local time with the same potential as that for $L_0$ at dusk. The starting limits of L-shell for the search are 0 and $L_0$, and the implementation is shown in Algorithm 4. This algorithm assumes that the values of $A$, $\gamma$, and $L_0$ have been
Algorithm 4 Calculation of the L-shell of $S_o$ for a local time specified by sin($\phi$).

**FUNCTION DEFINITION OF L$_2$(s)**

local constants required for L$_2$(s):
$$k = -B_o \cdot a^2 \cdot \omega$$

local variables required for L$_2$(s):
$$k_y, V, V'_2, \Delta L_x, L_x$$

implementation of L$_2$(s):

1. Potential of $S_o$ is found from Equation 6
2. These multiplications are done before the loop
3. Highest possible value for $L_x$ is zero.
4. If $s > -0.999999$ then prevent round off error from causing a run-away loop.

Repeat the following loop:

- $V = k/L_x - a \cdot A \cdot L_0'$
- $\Delta L_x = \frac{1}{2} \cdot \Delta L_x$ the step size halves during each iteration
- If ($V > V'_2$) then
  - $L_x = L_x - \Delta L_x$ $L_x$ should be smaller
- Else
  - $L_x = L_x + \Delta L_x$ $L_x$ should be larger

Until $\Delta L_x < \delta_i$ The loop is terminated once sufficient accuracy is reached, and the value of $L_x$ is returned.

the previous determined. A similar algorithm not shown in this report was used to plot the equipotential lines in Figure 1. Algorithm 5 shows the implementation of Eq. (7) to set the value of the global variable $L_o$, when $A$ and $\gamma$ are known. It also shows a simple method of determining the values of $L_o$ and $A$ when only $\gamma$ is known so that $S_o$ includes a specified L-shell and local time. It uses a binary search to determine the value of $A$ that results in a value of L$_2$(s) (Algorithm 4) equal to $L$.

When $W > 0$, Eqs. (3), (4), and (5) must be solved simultaneously to determine the drift paths of a particle measured with the constants of motion, $\mu_M$, $J_M$, and $U_M$. Equations (3) and (4) are first combined to solve for $y$ as a function of L-shell.

$$I(y) y^{-1} = J_M \mu_M^{-\gamma} L^{\mu} = Q$$  (9)

Equation (9) can be solved analytically if the function $I(y)$ is assumed to be linear such that $I(y) = a + by$. The solution of $y$ is then

$$y = a_i (Q - b_i)^{-1}$$  (10)

The values of $a_i$ and $b_i$ are determined using the two elements of table I (Algorithm 1) nearest to the solution of $y$. This method leads to an iterative solution which is illustrated in Figure 2. The two heavy lines show the function $I(y)$ and $yQ$, where $Q = 1.69$. The intersection of these two lines is the solution to Eq. (9). At the beginning of the iteration, $y$ is assumed to equal 1.0, and $I(y)$ is approximated by a line at Point A. This approximation is shown by the thin line through Point A. Equation (10) gives $y = 0.467$ at the intersection of this approximation with $yQ$. A new line is used to approximate $I(y)$ at Point B using the new value for $y$. This approximation yields Point C at $y = 0.512$, and then converges on the next iteration to Point D at $y = 0.513$. $W$ is then determined using
Algorithm 5 Calculation of the global variables \( L_0 \) and \( A \).

PROCEDURE DEFINITION OF SetLo
This sets the global variable \( L_0 \) after \( y \) and \( A \) have been set (see Algorithm 1).

local constants required for SetLo:
\[ k = \omega_a \cdot B_0 \]

implementation of SetLo:
\[ L_0 = \left[ \frac{k}{(y-A)^{1/3}} \right] \]

Equation 7

PROCEDURE DEFINITION OF SetLo&A(L, s)
This sets the global variables \( A \) and \( L_0 \) when \( y \) is known (see Algorithm 1) so that \( L_0 \) includes the point indicated by \( L \) and \( s \). A binary search is used.

local constants required for SetLo&A(L, s):
\[ k = \omega_a \cdot B_0 \]

local variables required for SetLo&A(L, s):
\[ \Delta A, \Delta L \]

implementation of SetLo&A(L, s):
\[ A = \frac{k}{(y-L_s)^{1/3}} \]
from Equation 7; the maximum value of \( A \) is found when \( L \) is assumed to be \( L_0 \) at dusk.
\[ \Delta L = L - L_2(s) \]
from Algorithm 4.
\[ \Delta A = A \]
This is the starting step size of search.

while abs(\( \Delta L \)) >= \( \delta \), repeat the following loop:

begin while loop
\[ \Delta A = \frac{1}{2} \Delta A \]
The step size is halved during each iteration.
if \( \Delta L < 0 \) then
\[ A = A + \Delta A \]
A was too small.
else
\[ A = A - \Delta A \]
A was too large.
SetLo
\[ \Delta L = L - L_2(s) \]
from Algorithm 4.
end while loop

Eq. (3). For a given L-shell and local time, \( y \) and \( W \) are first determined from Eqs. (3) and (4) using the above process, and then \( U \) is determined using Eq. (5). The given L-shell is a valid solution if the difference, \( \Delta U \), between \( U_m \) and \( U \) is zero. Algorithm 6 uses this process to solve for \( \Delta U \) for a measured position and energy. Valid solutions of \( L \) for each local time give the drift path of the particle.

In summary, the drift paths of a particle are determined by calculating the three constants of motion, \( \mu_\perp, J_\perp, U_\perp \), for its kinetic energy and position in space using Eqs. (3), (4), and (5). Using these constants of motion, Eqs. (3) and (4) are solved simultaneously to give kinetic energy as a function of L-shell. All L-shells at a given local time for which the difference between \( U_m \) and \( U \) [Eq. (5)] is zero are on the drift path of the particle.

4. CALCULATION OF ELECTRON CUTOFF ENERGY, \( W_C \)

In a manner similar to that used to calculate drift paths, the range in energy of electrons coming from the tail that should be observed at a given position in the magnetosphere can be determined using Eqs. (3), (4), and (5). It is known that electrons
with $W > 0$ execute a gradient curvature drift in the direction of the corotation drift. For increasing kinetic energy this gradient curvature drift comes to dominate the electric field drift, and since this drift has no radial component, the electron trajectory becomes more circular and eventually closes. For any position outside of $\mathcal{L}_o$, electrons measured with $W=0$ are on open drift paths as shown in Figure 1, and electrons measured with $W \to \infty$ are on closed circular drift paths because the magnetic drift completely dominates the electric field drift. For the position where the electron is observed, there is a cutoff energy, $W_c$, at which the drift trajectory changes from open to closed. Electrons convecting from the tail with energies ranging from 0 to $W_c$ should be observed at that position. $W_c$ is not defined for positions inside $\mathcal{L}_o$ where all electrons move on closed drift paths.

Figure 3 illustrates the electron cutoff energy, $W_c$. Six sets of electron drift paths in the equatorial plane are shown in this figure, and these are calculated using the method presented in Section 3 and Algorithm 6. The constants of motion are determined for electrons with one of six different values of $W_M$ measured at the white circle with $L_M=6.0$ $R_E$, $y_M=0.90$, and $\sin(\phi_M)=-0.95$. The six values of $W_M$ are given on the right side of the figure, and are labeled A to F. In this example $L_o=5$ $R_E$, and the shaded region is bound by $\mathcal{L}_o$. Line A shows the path for electrons with $W_M=0$. In this case there is a closed eastward path around the Earth inside of $\mathcal{L}_o$, and there is a separate
Algorithm 6 Calculation of $\Delta U$ for given constants of motion and position.

FUNCTION DEFINITION OF $\Delta U(p, J, U, \text{electron}, L, s)$

local constants required by $\Delta U(p, J, U, \text{electron}, L, s)$:

- $k = B_0; a^2; \omega$

local variables required for $\Delta U(p, J, U, \text{electron}, L, s)$:

- $W, y$: kinetic energy (eV) and sine of equatorial pitch angle
- $a_i, b_i$: linear coefficients of $l_i$
- $i, j$: integer indices of $l_i$

implementation of $\Delta U(p, J, U, \text{electron}, L, s)$:

if $\mu = 0.0$ then
    $W = (J/L/|l_i|)^2$ \hspace{1cm} \text{Equation 4.}
else if $J = 0.0$ then
    $W = \mu/L^3$ \hspace{1cm} \text{Equation 3.}
else
    begin conditional block
    \hspace{1cm} $i = N$ \hspace{1cm} $N$ is from Algorithm 1.
    \hspace{1cm} $j = 0$ \hspace{1cm} this ensures that $i \neq j$ the first time.
    \hspace{1cm} $Q = J \cdot (L/|\mu|)^k$ \hspace{1cm} \text{Equation 9.}
    while $i \neq j$ repeat the following loop: Check for convergence
    \hspace{1cm} \hspace{1cm} begin while loop
    \hspace{1cm} \hspace{1cm} $j = i$
    \hspace{1cm} \hspace{1cm} $b_i = N \cdot (|l_i| - |l_{i-1}|)$ \hspace{1cm} slope between $l_i$ and $l_{i-1}$.
    \hspace{1cm} \hspace{1cm} $a_i = l_i \cdot b_i; i/N$ \hspace{1cm} zero offset.
    \hspace{1cm} \hspace{1cm} $y = a_i / (Q - b_i)$ \hspace{1cm} \text{Equation 10.}
    \hspace{1cm} \hspace{1cm} $i = \text{trunc}(N \cdot y) + 1$ \hspace{1cm} update index with new value of $y$.
    \hspace{1cm} \hspace{1cm} end while loop
    \hspace{1cm} $W = \mu \cdot (y \cdot L)^2 \cdot L$ \hspace{1cm} \text{Equation 3.}
    end conditional block

if electron then
    return value of $U - W + a \cdot A \cdot s \cdot L' - k/L \quad \text{Equation 5, } q = -1$
else
    return value of $U - W - a \cdot A \cdot s \cdot L' + k/L \quad \text{Equation 5, } q = +1$

sunward path on the dusk side of the Earth that is open to the tail. The point of measurement is on the open path. Line B shows the path for $W_e=0.6$ keV electrons. Here the inner and outer paths join at a single stagnation point on the dusk meridian near 5.2 RE, and the point of measurement is located on the open outer open path. Line C shows paths for $W_e=3.8$ keV electrons. Because the inner and outer paths have joined, only one path exists, and this is open to the tail and crosses the dawn-dusk meridian on the dawn side of the Earth. Line D shows the path for electrons with $W_e=W_C$. Here a closed eastward path joins at a single point on the dusk meridian near 6.5 RE with an open sunward path. This path looks very similar to that of Line B, but here the point of measurement is on the closed inner path of Line D. Line D shows the transition trajectory between open and closed drift paths, because all electrons with $W_e<W_C$ are on open paths while higher energies, as shown by Lines E and F, place the point of measurement on closed eastward paths. In the last two cases, the open paths from the tail and the closed eastward paths are completely separate.

To calculate $W_C$ for a measured location, the value of $W_e$ is determined that yields a single solution of $\Delta U=0$ on the dusk meridian as shown by Line D in Figure 3.
Figure 3  Convection paths of electrons measured at the white circle for six values of $W_M$. The shaded region is bounded by $\mathcal{L}_e$, and arrows show the direction of drift along the paths.

In Figure 4, Algorithm 6 is used to calculate $\Delta U$, and each curve shows $\Delta U$ versus L-shell at dusk for the same six values of $W_M$ and for the same measured location given for Figure 3. The peak values of $\Delta U$ at dusk are referred to as $\Delta U_{p}$. For Curves A, E, and F, $\Delta U_{p}>0$, so there are two solutions to $\Delta U=0$ at dusk for these energies. Because there are two solutions, two separate paths are seen at dusk in Figure 3 for Lines A, E, and F. For Curve C, $\Delta U_{p}<0$, so that there are no solutions to $\Delta U=0$, and no paths are seen at dusk in Figure 3 for Line C. For Curves B and D, $\Delta U_{p}=0$, so that both have exactly one solution for $\Delta U=0$ at dusk. Both Lines B and D in Figure 3 have exactly one path on the dusk meridian, and their respective inner and outer paths join at this point. Curve D shows the proper solution for $W_c$, as was discussed in the previous paragraph, because it places the point of measurement on the inner path, which is closed for $W_M=W_c$ and open for $W_M<W_c$. When two values of $W_M$ exist for which $\Delta U_{p}=0$, $W_c$ is equal to the greater value.
\[ A, W_u=0.00\text{keV} \]
\[ B, W_u=0.56\text{keV} \]
\[ C, W_u=3.83\text{keV} \]
\[ D, W_u=8.40\text{keV}=W_c \]
\[ E, W_u=9.00\text{keV} \]
\[ F, W_u=13.83\text{keV} \]

Figure 4 $\Delta U$ as a function of L-shell at dusk ($\sin(\phi)=-1$) for the same conditions, measured location, and energies as those given for Figure 3.

In order to determine the value of $\Delta U_p$, the L-shell at dusk at which the peak occurs is identified, and this L-shell is referred to as $L_p$, so that $\Delta U_p=\Delta U(L_p)$. $L_p$ occurs at the L-shell where $\partial \Delta U/\partial L=0$ at dusk. $\Delta U=U_{\mu'}U$, and $U$ comes from Eq. (5) with $q=-1$ and $\sin(\phi)=-1$, so that

\[ \frac{\partial \Delta U}{\partial L}(L=L_p) = -\frac{\partial W}{\partial L}(L=L_p) - \gamma aAL_p^{-1} + B_0 a^2 \omega L_p^2 = 0 \quad (11) \]

The term $\partial W/\partial L$ is derived from the simultaneous solution of Eqs. (3) and (4) as is done in Algorithm 6. From Eq. (3),

\[ \frac{\partial W}{\partial L} = -3 \mu \gamma y^{-2}L^{-4} - 2 \frac{\partial y}{\partial L} \mu \gamma y^{-3}L^{-3} \quad (12) \]

The $\partial y/\partial L$ term can be derived from Eq. (10) as

\[ \frac{\partial y}{\partial L} = -\frac{\partial Q}{\partial L} q_i(Q-b)^{-2} \quad (13) \]

and the $\partial Q/\partial L$ term can be derived from Eq. (9) as
Algorithm 7  Calculation of $\partial W/\partial L$ as a function of $L$-shell at dusk for a given set of constants of motion.

FUNCTION DEFINITION OF $\partial W/\partial L(\mu, J, L)$

local variables required for $\partial W/\partial L(\mu, J, L)$:
- $D_w$ value of $\partial W/\partial L$ returned by function.
- $i, j$ integer indices to $L$.
- $a_i, b_i$ linear coefficients to $L$.
- $Q, D_o, y, D_y$

implementation of $\partial W/\partial L(\mu, J, L)$:

if $\mu = 0$ then
- $D_w = -2[J/(L-I_{1,r})]^2/L$ derived from Equation 4.
else if $J = 0$ then
- $D_w = -3\mu/L^4$ derived from Equation 3.
else
begin conditional block
- $i = N$ $N$ is from Algorithm 1.
- $j = 0$ this ensures that $i \neq j$ the first time.
- $Q = J \cdot (L/\mu)^{x}$ Equation 9.
- $D_o = \frac{1}{2}Q/L$ Equation 14.
while $j \neq i$ repeat the following loop:
begin while loop
- $j = i$
- $b_i = N \cdot (l_i - l_{1,r})$ slope between $l_i$ and $l_{1,r}$.
- $a_i = l_i - b_i i/N$ zero offset.
- $y = a_i / (Q - b_i)$ Equation 10.
- $i = \text{trunc}(y \cdot N) + 1$ update index with new value of $y$.
end while loop
- $D_y = -a_i D_o / (Q - b_i)^2$ Equation 13.
- $D_w = -\mu y^3 L^4 (2 D_y L + 3 y)$ Equation 12.
end conditional block

return the value of $D_w$

$$\frac{\partial Q}{\partial L} = \frac{1}{2} Q L^{-1} \tag{14}$$

The solution to $\partial W/\partial L$ is implemented in Algorithm 7. If either $\mu$ or $J$ is equal to zero then $\partial W/\partial L$ is solved from either Eq. (4) or Eq. (3). Otherwise, the iterative search shown for Algorithm 6 is used to solve for $y$ and $W$, and Eqs. (12), (13), and (14) are used to solve for $\partial W/\partial L$.

There is no analytic solution to Eq. (11) when Eq. (14) is substituted for $\partial W/\partial L$, so a numeric solution is used. First $\partial W/\partial L$ is approximated by a simple form which gives an analytic solution to Eq. (11). Inspection of the two extreme cases, $\mu_m = 0$ and $J_m = 0$, of Algorithm 6 shows that $\partial W/\partial L \approx kL^{-m}$ where $m$ ranges from 3 to 4. This is approximated by the form

$$\frac{\partial W}{\partial L} = c_1 L^{-2} + c_2 L^{-3} \tag{15}$$

This form is used because it yields a quadratic equation when it is substituted into Eq.
Algorithm 8 Calculation of $L_p$ for given values of $\mu$ and $J$ for electrons.

FUNCTION DEFINITION OF PeakWc($\mu, J$)

local constants required for PeakWc($\mu, J$):

$k = B_0 a^2 \omega$

local variables required for PeakWc($\mu, J$):

$L, L_1, L_2$ These hold the current solutions of L-shell
$D, D_1, D_2$ These hold $\partial W/\partial L$ for the current solutions of L-shell
$c_1, c_2, Q_a, Q_b, Q_0$

implementation of PeakWc($\mu, J$):

$L_1 = L_0$

Find two starting points for the iteration.

$L_0$ is usually within a few $R_e$ of the final solution.

$L_2 = L_1 + 1.0$

$L_2$ is usually within a few $R_e$ of the final solution.

$D_1 = \partial W/\partial L(\mu, J, L_1)$

First coefficient of Equation 16.

$L_1 = L_0$

Second coefficient of Equation 16.

$L_2 = L_1 + 1.0$

$L_2$ is usually within a few $R_e$ of the final solution.

$D_2 = \partial W/\partial L(\mu, J, L_2)$

Value inside of root in Equation 17.

$Q_a = -\gamma a A$

First coefficient of Equation 16.

$D_2 = \partial W/\partial L(\mu, J, L_2)$

New value for $\partial W/\partial L$ is computed.

if abs($L_2 - L_1$) < abs($L - L_2$) then

Replace whichever solution is farthest from new solution.

$L_2 = L$

begin conditional block

$D_2 = D$

end conditional block

else

begin conditional block

$L_1 = L$

$D_1 = D$

end conditional block

end while loop

return value of $L$

(11). Substitution of Eq. (15) into Eq. (11), and multiplying both sides by $L^*+3$ gives

$$0 = -\gamma a A L_p^{*+2} + (B_0 a^2 \omega - c_1) L_p^{*+1} - c_2$$

The solution of interest is

$$L_p = \left( -\frac{Q_b - (Q_a^2 - 4 Q_a Q_c)^{1/2} - c_1}{2 Q_a} \right)^{(1/2) Q_a^{*+1}}$$

where $Q_a = -\gamma a A$, $Q_b = B_0 a^2 \omega - c_1$, and $Q_c = -c_2$. The values of $c_1$ and $c_2$ are first determined from two points at $L = L_0$ and $L = L_0 + 1$. An estimate of $L_p$ is determined using Eq. (17), and this new value is used to replace the initial estimated value of L-shell farthest from the new estimate. New values for $c_1$ and $c_2$ are determined which then give a better estimate of $L_p$. This process is repeated until it converges to a solution, usually in four to five iterations. Algorithm 8 shows this method of determining $L_p$. 14
Figure 5 illustrates Algorithm 8 for the solution of $L_p$ for the case $W_M = W_C$, Curve D of Figure 4. The bold line shows $\partial \Delta U / \partial L$ as a function of L-shell at dusk. $L_p$ is located where this line crosses zero (Point T). Points P and Q at $L = L_0$ and $L = L_0 + 1$ are used to determine initial values for $c_1$ and $c_2$ [Eq. (15)]. Line u, which is a thin solid line, shows the approximation to $\partial \Delta U / \partial L$ by using Eq. (15). Because the approximation is very close to the actual solution, Line u', which is a thin dashed line, shows the error between the approximation and the actual solution exaggerated by a factor of 50. Equation (17) identifies the L-shell of the zero intersect of Line u at Point R. The second iteration uses Points Q and R to determine $c_1$ and $c_2$. This approximation is shown by Line v with the exaggerated error shown by Line v'. Here the approximation is more accurate near the solution. Point S, which is the zero intercept of Line v, is used with Point R for the final iteration shown by Line w. Line w' shows that the approximation is very good near the solution shown by Point T. The iteration is stopped here because the difference between the Points S and T is sufficiently small.

Because there are potentially two values of $W_M$ for which $\Delta U_p = 0$, as shown by Curves B and D in Figure 4, it is necessary to identify a lower limit to $W_M$ which is always larger than the smaller solution to $\Delta U_p = 0$ and is less than or equal to the larger, correct solution. In Figure 4, all of the curves have the same value of $\Delta U$ at $L = 6.0\ R_e$ because this is the value of $L_M$. When $L = L_M$, conservation of the first two adiabatic
invariants [Eqs. (3) and (4)] requires that \( y' = y_m \) and \( y = y_m \). Substituting Eq. (5) into \( \Delta U = U_m - U \) yields

\[
\Delta U(L = L_m) = -aAL_m^r [1 + \sin(\phi_m)]
\]  

(18)

and this shows that \( \Delta U(L = L_m) \leq 0 \). As can be seen by looking at Curve C, \( \Delta U(L = L_m) \) is the minimum possible value of \( \Delta U_p \). The value of \( W_m \) that gives \( \Delta U_p = \Delta U(L = L_m) \) is referred to as \( W_m \) and is obtained by setting \( \partial \Delta U/\partial L(L = L_m) \) to zero. Since \( U_m \) is a constant, the equation is

\[
\frac{\partial \Delta U}{\partial L}(L = L_m) = -\frac{\partial W}{\partial L}(L = L_m) - \gamma aAL_m^{r-1} + B_0 a^2 \omega L_m^2 = 0
\]  

(19)

For two different electrons, labeled 1 and 2, if \( L_1 = L_2 \) and \( y_1 = y_2 \) then Eqs. (3) and (4) require that \( \mu_1 W_1 = \mu_2 W_2 \) and \( J^2_1 W_1 = J^2_2 W_2 \) respectively. With these requirements, Eq. (9) shows \( Q_2 = Q_1 \), Eq. (14) shows \( \partial Q_2/\partial L = \partial Q_1/\partial L \), and Eq. (13) shows \( \partial y_2/\partial L = \partial y_1/\partial L \). Equation (12) then shows that

\[
W_2 = W_1 \left[ \frac{\partial W_1}{\partial L} \right]^{-1}
\]  

(20)

To solve for \( W_m \), \( \partial W/\partial L(L = L_m) \) from Eq. (19) is substituted into \( \partial W_1/\partial L \) in Eq. (20). Any energy, \( W_m \), can be used for \( W_1 \) provided that \( \partial W_1/\partial L \) uses the constants of motion from \( L = L_m, y = y_m, \phi = \phi_m \), and \( W = W_m \). This gives

\[
W_0 = W_m \left[ B_0 a^2 \omega L_m^2 - \gamma aAL_m^{r-1} \right]^{-1}
\]  

(21)

Substitution of Eq. (7) into Eq. (21) shows that \( W_o > 0 \) for \( L_m > L_o \), and \( W_o < 0 \) for \( L_m < L_o \). When \( W_o > 0 \), two values of \( W_o \) may exist for which \( \Delta U_p = 0 \), and \( W_o \) is between these two values. In Figure 4, \( W_o \) is the energy used for Curve C, and Curves B and D use the lower and upper values of \( W_m \) for which \( \Delta U_p = 0 \). When \( W_o \leq 0 \), only one solution can exist because kinetic energy cannot be negative.

\( W_c \) is determined by identifying the value of \( W_m > W_o \) that gives \( \Delta U_p = 0 \). Algorithm 9 shows the implementation of the iterative search used to find \( W_c \) for a position given as \( y = y_m, L = L_m, \) and \( \sin(\phi) = s_m \). If \( L_m < L_o \), then the location \( L_m, s_m \) is outside of \( \mathcal{L}_o \) if \( U \) for \( W_m = 0 \) electrons is less than \( U \) of \( \mathcal{L}_o \). This can be seen in Figure 1 in which the contour lines show negative \( U \) because kinetic energy is zero and \( q \) is negative. If the measured position is inside of \( \mathcal{L}_o \), then \( W_c \) is given the value of -1 to indicate that \( W_c \) is not defined. The initial lower value of \( W_m \) for the search is given by Eq. (21). This equation gives a negative solution if \( L_m < L_o \), and for these cases an initial lower value of \( W_m = 0 \) is used. The initial upper value is chosen arbitrarily to be 10 keV greater than the lower value. A third point is picked by assuming the first two points lie on a parabola, and the slope of the parabola is zero at the lower point. If \( W_o < 0 \), a third point is simply picked halfway between the first two points. A parabola is then used to
**Algorithm 9** Calculation of electron cutoff energy, $W_c$, for a position given by $L$, $y$, and $\sin(\phi)$.

FUNCTION DEFINITION OF $W_c(y, L, s)$

local constants required for $W_c(y, L, s)$:

\[ k = B \cdot \omega \cdot a^2 \]

local variables required for $W_c(y, L, s)$:

\[ U_0, \mu_1, J_1, W_0, W_1, W_2, \Delta U_0, \Delta U_1, \Delta U_2, Q_A, Q_B, Q_c \]

implementation of $W_c(y, L, s)$:

\[ \mu_1 = \mu(y, 1, L) \quad \text{Value of } \mu \text{ at } y, L, s \text{ if } W = 1 \text{ eV} \; \text{ Eqn 3} \]

\[ J_1 = J(y, 1, L) \quad \text{Value of } J \text{ at } y, L, s \text{ if } W = 1 \text{ eV} \; \text{ Eqn 4} \]

\[ U_0 = U(\text{true}, 0, L, s) \quad \text{Value of } U \text{ at } y, L, s \text{ if } W = 0 \text{ eV} \; \text{ Eqn 5} \]

if ($L < L_o$) and $[\Delta U(0, 0, U_0, \text{true}, L_o, -1)] > 0$ then satellite is inside of $\mathcal{L}_o$

return value of -1 and exit function -1 indicates that $W_c$ does not exist

Equation 21 finds initial starting point of search.

if $L > L_o$ then

\[ W_c = W_{\text{step}}(k/L^2 - y \cdot a \cdot A \cdot L^r) \]

\[ \Delta U_0 = \Delta U(\mu_1, W_0, J_1, W_0^n, U_0 + W_0, \text{true}, \text{Peak}W_c(\mu_1, W_0, J_1, W_0^n), -1) \]

The second starting point is offset from the first by $W = \Delta U_0$.

\[ W_1 = W_0 + W_{\text{step}} \]

\[ \Delta U_1 = \Delta U(\mu_1, W_1, J_1, W_1^n, U_0 + W_1, \text{true}, \text{Peak}W_c(\mu_1, W_1, J_1, W_1^n), -1) \]

A parabolic approximation is used to find the third point. It uses the points $W_0$ and $W_1$, and assumes the slope at $W_c$ is zero.

if $W_0 = 0$ then

\[ W_2 = \frac{1}{2}W_1 \]

Slope at $W_c$ is not zero if $W_0 = 0$; use midpoint.

else

\[ W_2 = W_0 + W_{\text{step}} \cdot [(\Delta U_0 - \Delta U_1)]^n \]

This is zero intercept of parabola.

\[ \Delta U_2 = \Delta U(\mu_1, W_2, J_1, W_2^n, U_0 + W_2, \text{true}, \text{Peak}W_c(\mu_1, W_2, J_1, W_2^n), -1) \]

Reorder $W_0$, $W_1$, and $W_2$ so that $|\Delta U_0| < |\Delta U_1| < |\Delta U_2|$

if $|\Delta U_1| < |\Delta U_0|$ then

swap values of $\Delta U_1$ with $\Delta U_0$ and $W_1$ with $W_0$

else $|\Delta U_2| < |\Delta U_1|$ then

swap values of $\Delta U_2$ with $\Delta U_1$ and $W_2$ with $W_1$

if $|\Delta U_2| < |\Delta U_0|$ then

swap values of $\Delta U_2$ with $\Delta U_0$ and $W_2$ with $W_0$

while $|\Delta U_0 - W_0| > \delta W (W_0 + W_1)$ repeat following loop:

\[ Q_a = (\Delta U_0 \cdot W_2 - W_1) + \Delta U_1 \cdot (W_0 - W_2) + \Delta U_2 \cdot (W_0 - W_1) / \]

\[ (W_0^2 - (W_2 - W_1) + W_1^2 - (W_0 - W_2) + W_2^2 - (W_1 - W_0)) \]

\[ Q_b = (\Delta U_0 \cdot \Delta U_1 \cdot Q_A \cdot (W_2^2 - W_1^2)) / (W_0 \cdot W_1) \]

\[ Q_c = \Delta U_0 - W_0 \cdot (Q_a + Q_A \cdot W_0) \]

\[ \Delta U_2 = \Delta U_1 \]

Shift point 1 to point 2

\[ W_2 = W_1 \]

and point 0 to point 1,

\[ \Delta U_1 = \Delta U_0 \]

and find new point 0.

\[ W_1 = W_0 \]

\[ W_0 = \frac{1}{2} \cdot [Q_a + (Q_a^2 - 4 \cdot Q_A \cdot Q_a)\cdot W_2] / Q_A \]

\[ \Delta U_0 = \Delta U(\mu_1, W_0, J_1, W_0^n, U_0 + W_0, \text{true}, \text{Peak}W_c(\mu_1, W_0, J_1, W_0^n), -1) \]

end while loop

return value of $W_0$

* These lines can be made more efficient by calculating $\mu_1 \cdot W_n$ and $J_1 \cdot W_n^n$ only once for each line instead of twice.
approximate the relationship between $\Delta U_p$ and $W_M$ using the three points, and it is used to solve for a new value of $W_M$. The point that is farthest from $\Delta U_p=0$ is replaced by the new solution for $W_M$ in the next iteration. The iteration is continued until the value of $W_C=W_M$ is known to sufficient accuracy, usually in about three iterations.

Figure 6 illustrates the solution for $W_C$ for the conditions given for Figures 3 and 4, and Points A through F have the same energy as do those in the previous figures. The x axis shows $W_M$, the y axis shows $\Delta U_p$, and the dark curve shows the relationship between the two. The initial lower boundary at Point C is identified using Eq. (21), and the initial upper boundary at Point F is found by adding 10 keV to the energy of Point C. Point E is found using the zero intercept of the parabola formed from Points C and F with a slope of zero at Point C (shown by thin dotted line). The zero intercept of the parabola formed from Points C, E and F (thin solid line) is very close to the true solution shown by Point D. The process converges to Point D on the next iteration.

5. CALCULATION OF ION LOWER CUTOFF ENERGY, $W_L$

Ions move in a more complex manner than do electrons because the magnetic gradient and curvature drift of ions is westward which is counter to that of the corotation electric field drift. Ions with zero kinetic energy experience no gradient and curvature
drift, and so move identically to electrons with zero kinetic energy. Therefore, an ion inside of \( \mathcal{L}_0 \) with \( W = 0 \) moves eastward on a closed drift path. As \( W \to \infty \), the gradient curvature drift completely dominates the electric field drift so that the ion moves westward on a closed circular drift path. The drift paths for intermediate kinetic energies transition from closed eastward paths to closed westward paths, and many of these paths are open to the tail. The bounding limits to the range of kinetic energy open to the tail are \( W_L \) for the lower bound and \( W_H \) for the upper bound. Ions with \( W < W_L \) move on closed eastward drift paths, and ions with \( W > W_H \) move on closed westward drift paths. \( W_L \) is not defined for positions outside of \( \mathcal{L}_0 \) because here no ions can move on closed eastward drift paths. The two ion cutoff energies are discussed separately, \( W_L \) in this section, and \( W_H \) in Section 6.

Figure 7 illustrates the ion lower cutoff energy, \( W_L \). Four sets of ion drift paths in the equatorial plane are shown in this figure; they are calculated using the method

\[
\begin{align*}
A & : 0.0 \text{keV} \\
B & : 1.0 \text{keV} = W_L \\
C & : 7.2 \text{keV} \\
D & : 9.0 \text{keV}
\end{align*}
\]

Figure 7 Convection paths of singly charged ions measured at the white circle for four values of \( W_M \). The shaded region is bounded by \( \mathcal{L}_0 \), and arrows show the direction of drift along the paths.
presented in Section 3 and Algorithm 6. The constants of motion are determined for ions with one of four different values of $W_M$ measured at the white circle with $L_m=3.5 \text{ R}_E$, $y_m=0.90$, and $\sin(\phi_m)=0.95$. In each case, $W_M < W_H$. The four values of $W_M$ are given on the right side of the figure and are labeled A to D. In this example $L_c=6 \text{ R}_E$, and the shaded region is bounded by $L_o$. Line A shows the path for ions with $W_M=0$. For these ions, there is a closed eastward path around the Earth inside of $L_o$, and there is a separate sunward path that is open to the tail on the dusk side of the Earth. The point of measurement is on the closed path whereas it is on the open path for electrons with $W_M=0$. Line B shows the path for ions with $W_M=W_L$. Here a closed eastward path joins at a single point on the dusk meridian with an open sunward path. This path looks very similar to that for electrons with $W_m=W_L$ except that the paths join inside of $L_o$. This is the transition trajectory between open and closed drift paths, because all ions with $W_M < W_L$ are on closed paths while higher energies, as shown by Lines C and D, place the point of measurement on paths open to the tail. In the last two cases, the open paths from the tail and the closed eastward paths join to form a single path that crosses the dawn-dusk meridian on the dawn side of the Earth. Lines C and D also show two closed westward drift paths inside of the point of measurement. Electron drift paths have no equivalent to these paths because the electron gradient curvature drift is eastward.

$W_L$ is calculated in a manner similar to that of $W_c$. In Figure 8, Algorithm 6 is used to calculate $\Delta U$, and four curves show $\Delta U$ versus L-shell at dusk for the same four values of $W_M$ and for the same measured location given for Figure 7. Comparison of this

![Figure 8](image_url)

**Figure 8** $\Delta U$ as a function of L-shell at dusk ($\sin(\phi)=-1$) for the same conditions, measured location, and energies as those given for Figure 7. The crosses mark the location of $\Delta U_p$. 

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figure to Figure 4 shows the more complex nature of the ions. In Figure 4 there is one peak for each electron energy. In Figure 8, peaks can be seen in Curves A, B, and C at the crosses, but the peaks are negative, and a peak does not exist for Curve D. Also, Curves C and D show an extra solution to \( \Delta U = 0 \) at L-shells between 2 and 3 RE. These two inner solutions form the closed westward drift paths for Lines C and D of Figure 7. The peak values of \( \Delta U \) at dusk are referred to as \( \Delta U_p \). In cases where a peak does not exist, \( \Delta U_p \) is defined at the point were the slope minimizes as is shown by the cross on Curve D. For Curve A, \( \Delta U_p < 0 \), so there are two solutions to \( \Delta U = 0 \) at dusk for this energy. Because there are two solutions, there are two separate paths seen at dusk in Figure 7 for Line A. For Curves C and D, \( \Delta U_p > 0 \), so that there are no solutions other than the inner closed westward drift path, and only this path is seen at dusk in Figure 7 for Lines C and D. \( \Delta U_p = 0 \) for Curve B, so Line B of Figure 7 has exactly one

![Algorithm 10 Calculation of \( L \), at dusk for given values of \( \mu \) and \( J \) for ions.](image)

FUNCTION DEFINITION OF \( \text{PeakWL} (\mu, J) \)

local constants required for \( \text{PeakWL} (\mu, J) \):

\[ k \equiv -B_0 - \alpha^2 \cdot \omega \]

local variables required for \( \text{PeakWL} (\mu, J) \):

\[ L, L_1, L_2, D, D_1, D_2, c_1, c_2, Q_a, Q_b, Q_0 \]

implementation of \( \text{PeakWL} (\mu, J) \):

\[ L_1 = L_0 \quad \text{Find two starting points for the iteration.} \]
\[ D_1 = \partial W / \partial L (\mu, J, L_1) \quad L_0 \text{is usually within a few } RE \text{of the final solution.} \]
\[ L_2 = L_1 - 1.0 \]
\[ D_2 = \partial W / \partial L (\mu, J, L_2) \]
\[ Q_a = \gamma \cdot \alpha \cdot A \quad \text{First coefficient for Equation 23.} \]

while abs(\( L_2 - L_1 \)) > \( \delta_l \) repeat the following loop:

begin while loop

\[ c_2 = \left( L_1^2 \cdot D_1 - L_2^2 \cdot D_2 \right) / \left( L_1^2 - L_2^2 \right) \quad \text{Solution to Equation 15} \]
\[ c_1 = L_1^2 \cdot D_1 - c_2 \cdot L_1 \quad \text{coefficients.} \]
\[ Q_0 = k - \alpha \]
\[ Q_0 = Q_a^2 + 4 \cdot Q_a \cdot c_2 \quad \text{Value inside of root in Equation 23.} \]
if \( Q_0 < 0 \) then

\[ Q_0 = 0 \quad \text{If no peak is found then search for minimum slope.} \]
\[ L = \left( \frac{1}{2} (-Q_0 + \sqrt{Q_0^2 + 4 Q_a}) \right) / \left( Q_a \right)^{1/2} \quad \text{Equation 23 yields better value for } L. \]
\[ D = \partial W / \partial L (\mu, J, L) \quad \text{New value for } \partial W / \partial L \text{ is computed.} \]
if abs(L - L_1) < abs(L - L_2) then

begin conditional block

\[ L_2 = L \]
\[ D_2 = D \]
end conditional block
else

begin conditional block

\[ L_1 = L \]
\[ D_1 = D \]
end conditional block

end while loop

return value of L

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solution for ∆U = 0 at dusk, and the inner and outer paths join at this point.

To determine the value of ∆Uₚ, the L-shell at dusk at which the peak occurs is identified. This L-shell is referred to as Lₚ, so that ∆Uₚ = ∆U(Lₚ). Lₚ occurs at the L-shell where ∂∆U/∂L = 0 at dusk. ∆U = Uₘ - U, and U comes from Eq. (5) with q=1 and sin(ϕ)=-1, so that

$$\frac{\partial \Delta U}{\partial L}(L=L_p) = -\frac{\partial W}{\partial L}(L=L_p) + \gamma aA L_p^{-1} - B_0 a^2 \omega L_p^{-2} = 0$$  \hfill (22)

This solution is identical to Eq. (11) except for sign changes in the second and third terms. It is solved iteratively as is Eq. (11) by using Eq. (15) except that the solution of interest is

$$L_p = \left[ \frac{-Q_0 + (Q_a^2 - 4 Q_4 Q_c)^{1/2}}{2 Q_a} \right]^{1/(γ+1)}$$  \hfill (23)

where Qₐ = γaA, Qₐ = -Bₐa²ωc₁, and Qₐ = -c₂. The values of c₁ and c₂ of Eq. (15) are first determined from initial estimates of L = Lₗ₀ and L = Lₗ₋₁. An estimate of Lₚ is determined using Eq. (23), and this new value is used to replace the initial estimated value of L-shell farthest from the new estimate. New values for c₁ and c₂ are determined giving a better estimate of Lₚ. If no peak is found, such as in the case shown by Curve D of Figure 8, the term in Eq. (23) inside of the radical is set to zero, so that Eq. (23) identifies the L-shell of minimum slope. This process is repeated until it converges to a solution, usually in four to five iterations. Algorithm 10 shows this method of determining Lₚ.

Figure 9 illustrates Algorithm 10 for the solution of Lₚ for the case Wₘ = Wₗ, Curve B of Figure 8. The bold line shows ∂∆U/∂L as a function of L-shell at dusk. Lₚ is located where this line crosses zero (Point S). Points P and Q at L = Lₗ₀ and L = Lₗ₋₁ are used to determine initial values for c₁ and c₂ [Eq. (15)]. Line u, which is a thin solid line, shows the approximation to ∂∆U/∂L using Eq. (15). Because the approximation is very close to the actual solution, Line u', which is a thin dashed line, is shown with the error between the approximation and the actual solution exaggerated by a factor of 500. Equation (23) identifies the L-shell of the zero intersect of Line u at Point R. The second and final iteration uses Points P and R to determine c₁ and c₂. This approximation is shown by Line v with the exaggerated error shown by Line v'. Point S is the zero intercept of Line v. The iteration is stopped here because the difference between the Points R and S is sufficiently small.

Wₗ is determined by identifying the value of Wₘ which gives ∆Uₚ = 0. The initial value used to determine Wₗ is found in a manner similar to that for Wₗ given by Eq. (21). Because the sign of q is changed, there are sign changes in Eq. (21), giving

$$W_0 = W_m \left( \gamma a A L_m^{-1} - B_0 a^2 L_m^{-2} \right) \left[ \frac{\partial W}{\partial L}(W=W_m, L=L_m) \right]^{-1}$$  \hfill (24)
Algorithm 11 shows the implementation of the iterative search used to find $W_L$ for a position given as $y=y_M$, $L=L_M$, and $\sin(\phi)=s_M$. If $L_M<L_0$ then the location, $L_M$, $s_M$, is outside of $\mathcal{L}_0$ if $U$ for $W_M=0$ is greater than $U$ of $\mathcal{L}_0$. This can be seen in Figure 1 where the contour lines are equal to $U$ since $q$ is positive. If the measured position is outside of $\mathcal{L}_0$, then $W_L$ is given the value of -1 to indicate that $W_L$ is not defined. The initial upper value of $W_M$ for the search is given by Eq. (24). $L<L_0$ because $W_L$ is only defined for positions inside of $\mathcal{L}_0$. Substitution of Eq. (7) into Eq. (24) shows that Eq. (24) gives a positive result when $L<L_0$. The initial lower value is set to zero. A line is used to approximate the relationship between $\Delta U_p$ and $W_M$ using two points and is used to solve for a new value of $W_M$. The point farthest from $\Delta U_p=0$ is replaced by the new solution for $W_M$ for the next iteration. The iteration is continued until the value of $W_L=W_M$ is known to sufficient accuracy, usually in about three iterations.

Figure 10 illustrates the solution for $W_L$ for the conditions given for Figures 7 and 8, and Points A through D have the same energy as do those in the previous figures. The x axis shows $W_M$, the y axis shows $\Delta U_p$, and the dark curve shows the relationship between the two. The curve is shown as a dotted line on the right side of the figure where a peak does not exist at $\Delta U_p$ such as for that shown by Curve D in Figure 8. The initial upper boundary at Point C is identified using Eq. (24), and the initial lower boundary at Point A is at $W_M=0$. The thin solid line connecting Points A and C shows
Algorithm 11 Calculation of ion lower cutoff energy, $W_L$, for a position given by $L$, $y$, and $\sin(\phi)$.

**FUNCTION DEFINITION OF $W_L(y, L, s)$**

local constants required for $W_L(y, L, s)$:

$k = B_0 \cdot \omega \cdot a^2$

local variables required for $W_L(y, L, s)$:

$U_0$, $\mu$, $J$, $W$, $W_u$, $\Delta U$, $\Delta U_0$, $\Delta U_1$

implementation of $W_L(y, L, s)$:

1. $\mu = \mu(y, 1, L)$ Value of $\mu$ at $y, L, s$ if $W = 1$ eV; $\mu = \mu \cdot W$ Eqn 3
2. $J = J(y, 1, L)$ Value of $J$ at $y, L, s$ if $W = 1$ eV; $J = J \cdot W$ Eqn 4
3. $U_0 = U(false, 0, L, s)$ Value of $U$ at $y, L, s$ if $W = 0$ eV; $U = U_0 + W$ Eqn 5
4. $W_0 = 0$
5. $\Delta U_0 = \Delta U(0, 0, U_0, false, L_0, -1)$
6. if $(L > L_0)$ or $(\Delta U_0 > 0)$ then satellite is outside of $L_0$ return value of -1 and exit function -1 indicates that $W_L$ does not exist
7. $W_1 = W_{\text{esc}}(y \cdot a \cdot L^{-1} \cdot k/L^2) \left/ \partial W/\partial L(\mu \cdot W_{\text{esc}}, J \cdot W_{\text{esc}}^*, L) \right.$ Equation 24
8. $\Delta U_1 = \Delta U(\mu \cdot W_1, J \cdot W_1^*, U_0 + W, false, \text{Peak}W(\mu \cdot W_1, J \cdot W_1^*), -1)$
9. while $\text{abs}(W_1 - W_0) > \delta_W(W_0 + W_1)$ repeat following loop:
   begin while loop
   10. $W = W_0 \cdot (W_1 - W_0)/(\Delta U_1 \cdot \Delta U) \Delta U$
   11. $\Delta U = \Delta U(\mu \cdot W_1, J \cdot W_1^*, U_0 + W, false, \text{Peak}W(\mu \cdot W_1, J \cdot W_1^*), -1)$
   12. if $\text{abs}(\Delta U_1) < \text{abs}(\Delta U_0)$ then
      begin conditional block
      13. $W_0 = W$
      14. $\Delta U_0 = \Delta U$
      end conditional block
   else
      begin conditional block
      16. $W_1 = W$
      17. $\Delta U_1 = \Delta U$
      end conditional block
   end while loop

return value of $W$

* These lines can be made more efficient by calculating $\mu \cdot W_n$ and $J \cdot W_n^*$ only once for each line instead of twice.

the linear approximation to the relationship. The zero intercept of this approximation is very close the actual solution, shown by Point B. The process converges to Point B in one to two more iterations.

6. **CALCULATION OF ION UPPER CUTOFF ENERGY, $W_H$**

Figure 11 illustrates the ion upper cutoff energy, $W_H$. Six sets of ion drift paths in the equatorial plane are shown in this figure, and these are calculated using the method presented in Section 3 and Algorithm 6. The constants of motion are determined for ions with one of six different values of $W_H$ measured at the white circle with $L_H = 3.5$ RE, $y = 0.90$, and $\sin(\phi)H = 0.95$. In each case, $W_H > W_L$. The six values of $W_H$ are given on the right side of the figure and are labeled A to G. The letter C is omitted so that the energies shown in this figure will match the energies used in following figures. In this example $L_0 = 6$ RE, and the shaded region is bounded by $L_0$. Line A shows the path for
ions with $W_M = 5.4$ keV. In this case there is a closed westward path around the Earth inside of $L_o$, and there is a separate sunward path open to the tail on the dawn side of the Earth. This path is similar to Lines C and D of Figure 7. The point of measurement is on the path open to the tail. Line B shows the path for ions with $W_M = 9.4$ keV. This energy is labeled $W_L$ to indicate the stagnation energy which is discussed in Section 7. Here the inner closed westward path joins with the outer open path at a single point on the dawn meridian near 3.2 RE, and the point of measurement is located on the outer open path. Line D shows the path for ions with $W_M = 10.8$ keV. Here the inner westward path is joined with the outer open path to form a single open path that crosses the dawn-dusk meridian on the dusk side of the Earth. Line E shows the path for ions with $W_M = W_H$. This path is similar to Line B except that the point of measurement is now located on the inner closed path. This is the transition trajectory between open and closed drift paths, because all ions with $W_M < W_H$ are on open paths while higher energies, as shown by Lines F and G, place the point of measurement on closed westward paths. In the last two cases, the open paths from the tail are separate from the open sunward paths.

$W_L$ is calculated in a manner similar to $W_C$ except that $\Delta U$ is evaluated to find a single solution at dawn instead of dusk. Figure 12 shows $\Delta U$ as a function of L-shell at dawn for the conditions given for Figure 11. Algorithm 6 is used to calculate $\Delta U$, and each curve shows $\Delta U$ versus L-shell at dawn for the same six values of $W_M$ and for the
Figure 11 Convection paths of singly charged ions measured at the white circle for six values of $W_H$. The shaded region is bounded by $S_{po}$, and arrows show the direction of drift along the paths.

The peaks in this figure look very similar to those seen in Figure 4 for electrons at dusk. Curve E shows the solution for $W_H$ because it gives a single solution of $\Delta U = 0$ at dawn while placing the point of measurement on the inner closed path. The peak values of $\Delta U$ at dawn are referred to as $\Delta U_p$.

In order to determine the value of $\Delta U_p$, the L-shell $L_p$ at dawn at which the peak occurs is identified, so that $\Delta U_p = \Delta U(L_p)$. $L_p$ occurs at the L-shell where $\partial \Delta U / \partial L = 0$ at dawn. $\Delta U = U - U_0$, and $U$ comes from Eq. (5) with $q = 1$ and $\sin(\phi) = 1$, so that

$$\frac{\partial \Delta U(L=L_p)}{\partial L} = -\frac{3W(L=L_p)}{\partial L} - \gamma a A L r^{-1} - B_s a^2 \omega L^2 = 0 \quad (25)$$

This solution is identical to Eq. (22) except for the sign change in the second term. It is solved iteratively as is Eq. (11) by using Eq. (15), except that the solution of interest
Figure 12 ΔU as a function of L-shell at dawn (sin(φ)=1) for the same conditions, measured location, and energies as those given for Figure 11.

is that given by Eq. (17) where $Q_A = -\gamma aA$, $Q_b = -B_0 a^2 \omega - c_1$, and $Q_c = -c_2$. The values of $c_1$ and $c_2$ of Eq. (15) are first determined from initial points at $L=L_0$ and $L=L_0-1$. An estimate of $L_p$ is determined using Eq. (17), and this new value is used to replace the initial estimated value of L-shell farthest from the new estimate. New values for $c_1$ and $c_2$ are determined, which then give a better estimate of $L_p$. This process is repeated until it converges to a solution, usually in four to five iterations. Algorithm 12 shows this method of determining $L_p$.

Figure 13 illustrates Algorithm 12 for the solution of $L_p$ for the case $W_H=W_H$. Curve E of Figure 12. The bold line shows $\partial \Delta U / \partial L$ as a function of L-shell at dawn. $L_p$ is located where this line crosses zero (Point U). Points P and Q at $L=L_0$ and $L=L_0-1$ are used to determine initial values for $c_1$ and $c_2$ [Eq. (15)]. Line u, which is a thin solid line, shows the approximation to $\partial \Delta U / \partial L$ by using Eq. (15). Because the approximation is close to the actual solution, Line u', which is a thin dashed line, is shown with the error between the approximation and the actual solution exaggerated by a factor of 50. Equation (17) is used to identify the zero intersect of Line u at Point R. The second iteration uses Points Q and R to give Line v which yields Point S. This method is iterated two more times to yield Points T and U before it stops.

$W_H$ is determined by identifying the value of $W_H$ which gives $\Delta U_p = 0$. The initial value used to determine $W_H$ is found in a manner similar to that for $W_C$ given by Eq.
Algorithm 12 Calculation of $L_p$ at dawn for given values of $\mu$ and $J$ for ions.

FUNCTION DEFINITION OF PeakWH($\mu$, $J$)
local constants required for PeakWH($\mu$, $J$):
    $k = -B_0 \cdot a^2 \cdot \omega$

local variables required for PeakWH($\mu$, $J$):
    $L_1$, $L_2$, $L_3$, $D_1$, $D_2$, $L_4$
These hold the current solutions of L-shell
    $c_1$, $c_2$, $Q_A$, $Q_b$, $Q_a$

implementation of PeakWH($\mu$, $J$):
    $L_1 = L_0$
    $D_1 = \partial W/\partial L(\mu, J, L_1)$
    $L_2 = L_1 - 1.0$
    $D_2 = \partial W/\partial L(\mu, J, L_2)$
    $Q_A = -\gamma \cdot a \cdot A$

while abs($L_2 - L_1$) > $\delta$, repeat the following loop:
    begin
    $c_2 = (L_2^2 \cdot D_1 - L_2 \cdot D_2) / (L_1^2 - L_2 \cdot L_1)$
    $c_1 = L_1^2 \cdot D_1 - c_2 \cdot L_1 \cdot L_1$
    $Q_b = k - c_1$
    $Q_A = Q_a^2 + 4 \cdot Q_A \cdot c_2$
    $L = [\frac{1}{2}(-Q_b - Q_A) / Q_A]^{1/2} + 1$
    $D = \partial W/\partial L(\mu, J, L)$

if abs($L - L_1$) < abs($L - L_2$) then
    begin
    conditional block
    L_2 = L
    D_2 = D
    end conditional block
else
    begin
    conditional block
    L_1 = L
    D_1 = D
    end conditional block
end while loop
return value of $L$

(21). Because the sign of $q$ is changed and the sign of $\sin(\phi)$ is changed, there are some
sign changes in Eq. (21), giving

\[ W_0 = -W_{\infty} \left\{ \gamma a A L_{\infty}^{-1} + B_0 \omega a^2 L_{\infty}^{-2} \right\} \left[ \frac{\partial W}{\partial L}(W=W_{\infty}, L_{\infty}) \right]^{-1} \]

Algorithm 13 shows the implementation of the iterative search used to find $W_M$ for a
position given as $y=y_M$, $L=L_M$, and $\sin(\phi)=s_M$. The initial lower value of $W_M$ for the
search is given by Eq. (26). This equation always gives a positive solution. The initial
upper value is chosen arbitrarily to be 10 keV greater than the lower value. A third point
is picked by assuming that the first two points lie on parabola, and that the slope of the
parabola is zero at the lower point. A parabola is used to approximate the relationship
between $\Delta U_p$ and $W_M$ using three points, and it is used to solve for a new value of $W_M$.
The point that is farthest from $\Delta U_p=0$ is replaced by the new solution for $W_M$ for the
next iteration. The iteration is continued until the value of $W_M=W_{\infty}$ is known to
Figure 13 Slope of Curve E in Figure 12. The bold line is the actual slope, and the thin solid lines are approximations using Equation 15. The dashed lines exaggerate the error between the approximations and actual slope by a factor of 50.

sufficient accuracy, usually in about three iterations.

Figure 14 illustrates the solution for \( W_s \) for the conditions given for Figure 13. Points A through G have the same energy as those in the previous figure. The x axis shows \( W_s \), the y axis shows \( \Delta U_e \), and the dark curve shows the relationship between the two. The initial lower boundary at Point D is identified using Eq. (26), and the initial upper boundary at Point G is found by adding 10 keV to the energy of Point D. Point F is found using the zero intercept of the parabola formed from Points D and G with a slope of zero at Point D (shown by thin dotted line). The zero intercept of the parabola formed from these three points (thin solid line) is very close to the true solution shown by Point E. The process converges to Point E on the next iteration.

7. CALCULATION OF ION STAGNATION ENERGY, \( W_s \)

The ion stagnation energy, \( W_s \), is the energy for which ions on open drift paths switch from crossing the dawn-dusk meridian on the dawn side to crossing on the dusk side. This energy is called a stagnation energy because ions with this energy move through a stagnation point on the dawn side of the Earth, even though they may still be on open drift paths. The significance of \( W_s \) is discussed in Kerns et al. [1993].
Algorithm 13 Calculation of ion upper cutoff energy, $W_u$, for a position given by $L$, $y$, and $\sin(\phi)$.

**FUNCTION DEFINITION OF $W_u(y, L, s)$**

**local constants required for $W_u(y, L, s)$:**
- $k = B_0 \cdot \omega \cdot a^2$

**local variables required for $W_u(y, L, s)$:**
- $U_0$, $\mu$, $J_1$, $W_0$, $W_1$, $\Delta U_0$, $\Delta U_1$, $\Delta U_2$, $Q_A$, $Q_b$, $Q_c$

**implementation of $W_u(y, L, s)$:**

1. $\mu = \mu(y, 1, L)$  
   \text{Value of $\mu$ at $y, L, s$ if $W=1$ eV; $\mu = \mu_1 \cdot W$}  \text{ Eqn 3}

2. $J_1 = J(y, 1, L)$  
   \text{Value of $J$ at $y, L, s$ if $W=1$ eV; $J = J_1 \cdot W$}  \text{ Eqn 4}

3. $U_0 = U(\text{false}, 0, L, s)$  
   \text{Value of $U$ at $y, L, s$ if $W=0$ eV; $U = U_0 + W$}  \text{ Eqn 5}

4. $W_0 = -W_{\text{step}} \cdot k \cdot (L)^2 + y \cdot a \cdot A \cdot L^3$ / $\partial W / \partial L | \mu_1 \cdot W, J_1 \cdot W, L$  \text{ Equation 26}

5. $\Delta U_0 = \Delta U(\mu_1 \cdot W_0, J_1 \cdot W_0, U_0 + W_0, \text{false}, \text{Peak}$ \text{$W_u(\mu_1 \cdot W_0, J_1 \cdot W_0, 1))$}  \text{1}  
   \text{The second starting point is offset from the first by $W_{\text{step}}$.}

6. $W_1 = W_0 + W_{\text{step}}$

7. $\Delta U_1 = \Delta U(\mu_1 \cdot W_1, J_1 \cdot W_1, U_0 + W_0, \text{false}, \text{Peak}$ \text{$W_u(\mu_1 \cdot W_1, J_1 \cdot W_1, 1))$}  \text{1}  
   \text{A parabolic approximation is used to find the third point. It uses the points $W_0$ and $W_1$, and assumes the slope at $W_0$ is zero.}

8. $W_2 = W_0 + W_{\text{step}} \cdot [\Delta U_0 / (\Delta U_1 \cdot \Delta U_2)]^k$  
   \text{This is the zero intercept of the parabola.}

9. $\Delta U_2 = \Delta U(\mu_1 \cdot W_2, J_1 \cdot W_2, U_0 + W_2, \text{false}, \text{Peak}$ \text{$W_u(\mu_1 \cdot W_2, J_1 \cdot W_2, 1))$}  \text{1}  
   \text{Reorder $W_0$, $W_1$, and $W_2$ so that $|\Delta U_0| < |\Delta U_1| < |\Delta U_2|$.}

10. if $\text{abs}(\Delta U_0) < \text{abs}(\Delta U_1)$ then
      swap values of $\Delta U_0$ with $W_0$ and $W_1$ with $W_0$

11. if $\text{abs}(\Delta U_1) < \text{abs}(\Delta U_2)$ then
      swap values of $\Delta U_2$ with $W_0$ and $W_1$ with $W_2$

12. if $\text{abs}(\Delta U_2) < \text{abs}(\Delta U_0)$ then
      swap values of $\Delta U_0$ with $W_0$ and $W_1$ with $W_0$

13. while $\text{abs}(W_1 - W_0) > \delta W (W_0 + W_2)$ repeat following loop:

   **begin while loop**
   
   14. $\Omega_A = \left[ \Delta U_0 \cdot (W_2 \cdot W_1) + \Delta U_1 \cdot (W_0 - W_2) + \Delta U_2 \cdot (W_1 - W_0) \right] /$
   \[ (W_0 - W_2) \cdot (W_2 - W_1) + W_2 \cdot (W_0 - W_1) \]

   15. $\Omega_b = \left[ \Delta U_0 \cdot A_{\Omega, 1} \cdot (W_0^2 - W_2^2) \right] / (W_0 - W_1)$

   16. $\Omega_c = \Delta U_0 - W_0 \cdot (Q_b + Q_c \cdot W_0)$

   17. $\Delta U_2 = \Delta U_1$  \text{Shift point 1 to point 2}

   18. $W_2 = W_1$  \text{and point 0 to point 1,}

   19. $\Delta U_1 = \Delta U_0$  \text{and find new point 0.}

   20. $W_1 = W_0$

   21. $W_0 = W_1$  \text{1}  
      \text{Quadratic Formula}

   22. $\Delta U_0 = \Delta U(\mu_1 \cdot W_0, J_1 \cdot W_0, U_0 + W_0, \text{false}, \text{Peak}$ \text{$W_u(\mu_1 \cdot W_0, J_1 \cdot W_0, 1))$}  \text{1}  
      \text{end while loop}

   return value of $W_0$

* These lines can be made more efficient by calculating $\mu_1 \cdot W_n$ and $J_1 \cdot W_n$ only once for each line instead of twice.

11 illustrates the ion stagnation energy, $W_s$. In this figure Lines A, B, and D place the point of measurement on an open drift path. Line A, which has $W_s < W_5$ crosses the dawn-dusk meridian on the dawn side of the Earth. Line D, which has $W_s > W_5$ crosses the dawn-dusk meridian on the dusk side of the Earth. Line B shows $W_s = W_5$, and here the closed inner path meets the open outer path at a single point at dawn. Although the point of measurement is on an open field line, ions convecting from the tail move through the dawn stagnation point to reach the point of measurement, and cannot convect to the point of measurement.
Figure 14 Relationship between $\Delta U_p$ and $W_M$ for the conditions given for Figure 13. This illustrates the solutions for $W_H$ and $W_S$.

$W_S$ is calculated in a manner identical to that of $W_H$ except that Eq. (26) gives the initial upper value instead of the initial lower value of $W_M$ in the search. Also, the quadratic formula uses the negative solution to find the zero intercept of the approximating parabolas. The upper solution to $\Delta U_p=0$ is shown by Curve E in Figure 12 where $W_M=W_H$. The lower solution to $\Delta U_p=0$ is shown by Curve B, and here $W_M=W_S$. Curve D uses the value given by Eq. (26) for $W_M$, and as can be seen in the figure, $W_S$ is less than this value. The location of the peaks are determined using Algorithm 12.

Algorithm 14 shows the implementation of the iterative search used to find $W_S$ for a position given as $y=y_M$, $L=L_M$, and $\sin(\phi)=s_M$. The initial upper value of $W_M$ for the search is given by Eq. (26). This equation always gives a positive solution. The initial lower value is set to one-half the initial upper value. A third point is picked by assuming the first two points lie on parabola, and the slope of the parabola is zero at the upper point. If $W_M$ for the third point is less than or equal to zero, then $W_M$ for this point is set to one-half the lower value. A parabola is used to approximate the relationship between $\Delta U_p$ and $W_M$ using three points, and it is used to solve for a new value of $W_M$. The point which is farthest from $\Delta U_p=0$ is replaced by the new solution for $W_M$ for the next iteration. The iteration is continued until the value of $W_S=W_M$ is known to sufficient accuracy, usually in about three iterations.
Algorithm 14 Calculation of ion stagnation energy, $W_s$, for a position given by $L$, $y$, and $\sin(\phi)$.

FUNCTION DEFINITION OF $W_s(y, L, s)$

local constants required for $W_s(y, L, s)$:

$k = B_0 \cdot w \cdot a^2$

local variables required for $W_s(y, L, s)$:

$U_0, \mu_1, J_1, W_0, W_1, W_2, \Delta U_0, \Delta U_1, \Delta U_2, Q_A, Q_0, Q_c$

implementation of $W_s(y, L, s)$:

$\mu_1 = \mu(y, 1, L)$ Value of $\mu$ at $y, L, s$ if $W = 1$ eV; $\mu = \mu_1 \cdot W$ Eqn 3

$J_1 = J(y, 1, L)$ Value of $J$ at $y, L, s$ if $W = 1$ eV; $J = J_1 \cdot W$ Eqn 4

$U_0 = U(\text{false}, 0, L, s)$ Value of $U$ at $y, L, s$ if $W = 0$ eV; $U = U_0 + W$ Eqn 5

$W_0 = -W_{\text{step}} \cdot \left(\frac{k \cdot L^2 + \gamma \cdot a \cdot a \cdot L^2}{8 \cdot W_{\text{step}} \cdot (\mu_1 \cdot W_{\text{step}}, J_1 \cdot W_{\text{step}}^n, L) \cdot \text{Equation 26}}\right)$

$\Delta U_0 = \Delta U(\mu_1 \cdot W_0, J_1 \cdot W_0^n, U_0 + W_0, \text{false}, \text{Peak}_W(\mu_1 \cdot W_0, J_1 \cdot W_0^n), 1)$ *

The second starting point is offset from the first by $W_0$.

$W_1 = \frac{1}{2} W_0$

$\Delta U_1 = \Delta U(\mu_1 \cdot W_1, J_1 \cdot W_1^n, U_0 + W_1, \text{false}, \text{Peak}_W(\mu_1 \cdot W_1, J_1 \cdot W_1^n), 1)$ *

A parabolic approximation is used to find the third point. It uses the points $W_0$ and $W_1$, and assumes the slope at $W_0$ is zero.

$W_2 = W_0 - \left(\frac{W_0 \cdot W_1 \cdot \text{null} \cdot \text{null} + \Delta U_0 \cdot (\Delta U_0 \cdot \Delta U_1) \cdot \text{null} \cdot \text{null}}{W_0 \cdot W_1 \cdot \text{null} \cdot \text{null}}\right)$ This is zero intercept of parabola.

if $W_2 \leq 0$ then

$W_2 = \frac{1}{2} W_1$

$\Delta U_2 = \Delta U(\mu_1 \cdot W_2, J_1 \cdot W_2^n, U_0 + W_2, \text{false}, \text{Peak}_W(\mu_1 \cdot W_2, J_1 \cdot W_2^n), 1)$ *

Reorder $W_0, W_1, and W_2$ so that $|\Delta U_0| < |\Delta U_1| < |\Delta U_2|$

if abs($\Delta U_1$) < abs($\Delta U_0$) then

swap values of $\Delta U_0$ with $\Delta U_1$ and $W_1$ with $W_0$

if abs($\Delta U_2$) < abs($\Delta U_1$) then

swap values of $\Delta U_2$ with $\Delta U_1$ and $W_2$ with $W_1$

if abs($\Delta U_1$) < abs($\Delta U_0$) then

swap values of $\Delta U_0$ with $\Delta U_1$ and $W_0$ with $W_1$

while abs($W_1 - W_0$) > $\delta_w \cdot (W_0 + W_1)$ repeat following loop:

begin while loop

$Q_A = [\Delta U_0 \cdot (W_2 - W_1) + \Delta U_1 \cdot (W_0 - W_2) + \Delta U_2 \cdot (W_1 - W_0)] / \left[ W_0^2 \cdot (W_2 - W_1) + W_1^2 \cdot (W_0 - W_2) + W_2^2 \cdot (W_1 - W_0) \right]$

$Q_c = \Delta U_0 \cdot W_0 \cdot (Q_A + Q_A \cdot W_0)$

$\Delta U_2 = \Delta U_1$, Shift point 1 to point 2

$W_2 = W_1$, and point 0 to point 1,

$\Delta U_1 = \Delta U_0$, and find new point 0.

$W_1 = W_0$

$W_0 = \frac{1}{2} \cdot (1 \cdot Q_A - (Q_0^2 - 4 \cdot Q_A \cdot Q_c)) / Q_A$ Quadratic Formula

if $W_0 \leq 0$ then

$W_0 := \frac{1}{2} W_1$

$\Delta U_0 = \Delta U(\mu_1 \cdot W_0, J_1 \cdot W_0^n, U_0 + W_0, \text{false}, \text{Peak}_W(\mu_1 \cdot W_0, J_1 \cdot W_0^n), 1)$ *

end while loop

return value of $W_0$

* These lines can be made more efficient by calculating $\mu_1 \cdot W_n$ and $J_1 \cdot W_n^n$ only once for each line instead of twice.

Figure 14 illustrates the solution for $W_s$ for the conditions given for Figure 13. The initial upper boundary at Point D is identified using Eq. (26), and the initial lower boundary at Point A is one-half the energy of Point D. Point C is found using the zero intercept of the parabola formed from Points D and A with a slope of zero at Point D.
(shown by thin dotted line). The zero intercept of the parabola formed from these three points (thin solid line) is very close to the true solution shown by Point B. The process converges to Point B on the next iteration.


