PLANAR REGULAR ONE-WELL-COVERED GRAPHS

Michael R. Pinter *
Belmont University
Nashville, TN 37212

Abstract

An independent set in a graph is a subset of vertices with the property that no two of the vertices are joined by an edge, and a maximum independent set in a graph is an independent set of the largest possible size. A graph is called well-covered if every independent set that is maximal with respect to set inclusion is also a maximum independent set. If \( G \) is a well-covered graph and \( G - v \) is also well-covered for all vertices \( v \) in \( G \), then we say \( G \) is 1-well-covered. By making use of a characterization of cubic well-covered graphs, it is straightforward to determination all cubic 1-well-covered graphs. Since there is no known characterization of \( k \)-regular well-covered graphs for \( k \geq 4 \), it is more difficult to determine the \( k \)-regular 1-well-covered graphs for \( k \geq 4 \). The main result in this regard is the determination of all 3-connected 4-regular planar 1-well-covered graphs.

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Introduction

A set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the independence number of G and is denoted by α(G). A set of independent points which attains the maximum size is referred to as a maximum independent set. A set S of independent points in a graph is maximal (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [12] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a well-covered graph. Equivalently, a well-covered graph is one in which every independent set can be extended to a maximum independent set. Sankaranarayana and Stewart [15] and, independently, Chvátal and Slater [3], have shown that determining if a given graph G is not well-covered is an NP-complete problem. Hence, determining if a graph is well-covered is in the class of problems referred to as co-NP-complete. What is not known is whether or not well-covered is an NP-complete property.

The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. The subclasses covered include cubic well-covered graphs ([1], [2] and [14]), well-covered graphs whose independence number is exactly one-half the size of the graph ([16], [4], [5]), well-covered graphs with girth at least five [6], well-covered graphs without 4-cycles and 5-cycles [7], and products of well-covered graphs [18].

Staples ([16] and [17]) introduced two subclasses of well-covered graphs which she called 1-well-covered and W2. A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph G is in the class W2 if and only if any two disjoint independent sets in G can be extended to two disjoint maximum independent sets. Some other results for graphs in W2 were obtained in [11].

In this paper, we primarily consider 1-well-covered planar regular graphs. Campbell characterized the cubic planar well-covered graphs in [1]; however, the technique he employed becomes very cumbersome when applied to planar 4-regular or 5-regular well-covered graphs. For this reason, we focus on the one-well-covered graphs. The primary result is stated in Theorem 13.

Preliminary Results

Staples [16] proved an equivalency between two seemingly different subclasses of well-covered graphs, which we state as the following theorem.

Theorem 1. Suppose G is well-covered. Then G is 1-well-covered if and only if G \subseteq W2.

Since we will appeal mostly to the notion of extending two disjoint independent sets to disjoint maximum independent sets, henceforth we use the W2 nomenclature instead of referring to 1-well-covered graphs.

Consider a graph G which is not complete and point v in G. By deleting v and its neighbors, we obtain a subgraph of G. Specifically, we define the subgraph Gv = G-N[v]. Campbell [1] proved the following very useful necessary condition for a graph to be well-covered.

Theorem 2. If a graph G is well-covered and is not complete, then Gv is well-covered for all v in G. Moreover, \alpha(Gv) = \alpha(G) - 1.
We prove in Theorem 3 that we have a similar necessary condition for a well-covered graph to be in \( W_2 \).

**Theorem 3.** If a graph \( G \) is in \( W_2 \) and \( G \) is not complete, then \( G_v \) is in \( W_2 \) for all \( v \) in \( G \).

**Proof.** Let \( v \) be a point in \( G \). Since \( G \) is not complete, then \( G_v \neq \emptyset \). By Theorem 2, graph \( G_v \) is well-covered and \( \alpha(G_v) = \alpha(G) - 1 \). Suppose \( I_1 \) and \( I_2 \) are disjoint independent sets in \( G_v \). Then \( I_1 \cup \{v\} \) is an independent set in \( G \), as is \( I_2 \cup \{v\} \). Since \( G \) is in \( W_2 \), there exists maximum independent set \( J_1 \supseteq I_1 \cup \{v\} \) such that \( J_1 \cap I_2 = \emptyset \). Since \( I_2 \cup \{v\} \) and \( J_1-v \) are disjoint independent sets in \( G \), then there exists maximum independent set \( J_2 \supseteq I_2 \cup \{v\} \) such that \( J_2 \cap (J_1-v) = \emptyset \). Hence, \( J_2-v \) and \( J_1-v \) are disjoint independent sets in \( G_v \). Since \( |J_i| = \alpha(G) \), then \( |J_i-v| = \alpha(G) - 1 \), for \( i = 1, 2 \). Thus, \( J_1-v \) contains \( I_1 \), \( J_2-v \) contains \( I_2 \), and \( J_1-v \) and \( J_2-v \) are disjoint maximum independent sets in \( G_v \). So any two disjoint independent sets in \( G_v \) can be extended to disjoint maximum independent sets in \( G_v \). By definition of the class \( W_2 \), we conclude that \( G_v \in W_2 \). \( \square \)

The next lemma will play a significant role for us. We will use it to eliminate many graphs from consideration as possible \( W_2 \) graphs.

**Lemma 4.** Suppose \( G \) contains an independent set \( S \) and point \( v \in S \) such that (i) \( S \cup \{v\} \) is independent, and (ii) if \( y \in N(v) \), then \( y \sim x \) for some \( x \in S \) (that is, \( S \) dominates \( N(v) \)). Then \( G \) is not in \( W_2 \).

**Proof.** If \( G \) is not well-covered, then \( G \) is not in \( W_2 \). If \( G \) is well-covered, then from conditions (i) and (ii), we have that \( S \cap N(v) = \emptyset \) and \( S \) dominates \( N(v) \). Thus, \( S \) and \( \{v\} \) are disjoint independent sets in \( G \) which don't extend to disjoint maximum independent sets in \( G \). Therefore, \( G \) is not in \( W_2 \). \( \square \)

For graphs drawn in the plane, we say two faces are adjacent if they share a line. If a face \( F \) contains point \( v \), we say \( F \) is incident to \( v \). The size of a face is the number of points it contains. We refer to the order and sizes of the faces incident to a point \( v \) as the face configuration at \( v \). To reduce the number of face configurations considered, we will use the theory of Euler contributions. Lebesgue [8] developed the theory of Euler contributions for planar graphs and Ore [9] and Ore and Plummer [10] used the theory to study plane graph colorings.

**The Euler contribution of a point** \( v \), \( \phi(v) \), **is defined as the quantity** \( \phi(v) = 1 - (1/2)\deg(v) + \sum(1/x_i) \), **where the sum is taken over all faces** \( F_i \) **incident to** \( v \) **and** \( x_i \) **is the size of** \( F_i \). **If** \( \deg(G) \) **denotes the number of faces in the plane graph** \( G \), **then it follows that** \( \phi(v) = IV(G) - IE(G) + IF(G) \). **Here the sum is taken over all points** \( v \) **in** \( G \). **Since Euler's formula for plane graphs says** \( IV(G) - IE(G) + IF(G) = 2 \), **then we have** \( \phi(v) = 2 \). **Thus,** \( \phi(v) \) **must be positive for some** \( v \) **in** \( G \). **If** \( \phi(v) > 0 \), **we say** \( v \) **is a point with positive Euler contribution.**

**Cubic** \( W_2 \) **Graphs**

Consider the three graph fragments given in Figure 1. Note that fragments A and B each have four semi-lines and fragment C has two semi-lines.
Let $W$ be the family of cubic graphs obtained from fragments A, B and C by placing any number of the fragments in a cycle or path configuration and then joining the left-hand semi-lines of one fragment to the right-hand semi-lines of the fragment on its left. Since crossing the lines joining one fragment to another gives a graph which is isomorphic to the graph obtained without crossing the lines, then we can assume the lines do not cross.

Building on the work of Campbell [1], Royle and Ellingham [14] proved that, with a few small exceptions, all cubic well-covered graphs belong to $W$. We state their result in Theorem 5.

**Theorem 5:** All cubic well-covered graphs, except for the 6 graphs in Figure 2, belong to $W$. Moreover, all graphs in $W$ are well-covered.

Using the characterization of cubic well-covered graphs given in Theorem 5, in the next theorem we determine all of the cubic $W_2$ graphs.

**Theorem 6:** The only cubic $W_2$ graphs are $K_4$ and the triangular prism.

**Proof.** Of the 6 exceptional cubic graphs given in Figure 2, only $K_4$ is a $W_2$ graph. For each of the other five graphs, it is straightforward to find two disjoint independent sets which don’t extend to disjoint maximum independent sets in $G$. We omit the details.

Suppose $G$ is a graph in the family $W$. Then $G$ is obtained by connecting fragments A, B and C in paths or cycles.
Case 1. Suppose G contains fragment A. If $a_1 \sim a_5$ and $a_3 \sim a_6$, then G is the triangular prism. It is easily verified that the triangular prism is a $W_2$ graph.

Suppose $|V(G)| > 6$. Without loss of generality, let $x \sim a_5$ and $y \sim a_6$, where $x$ and $y$ are not in the original A fragment. Then $x \sim y$ and $\{y, a_2\}$ is independent. Thus, $\{y, a_2\}$ and $\{a_5\}$ don't extend to disjoint maximum independent sets in G. So $G \not\in W_2$.

Case 2. Suppose G contains fragment B. If $b_1 \sim b_4$ and $b_5 \sim b_6$, then $(b_3, b_4)$ and $(b_5)$ don't extend to disjoint maximum independent sets in G. So $G \not\in W_2$.

Suppose $|V(G)| > 8$. Without loss of generality, let $x \sim b_4$ and $y \sim b_5$, where $x$ and $y$ are not in the original B fragment. Then $x \sim y$ and $\{y, b_2\}$ is independent. Thus, $\{y, b_2\}$ and $(b_3)$ don't extend to disjoint maximum independent sets in G. So $G \not\in W_2$.

Case 3. Suppose G contains fragment C. Then $|V(G)| > 6$. Let $x \sim c_1$ and $y \sim c_6$ such that $x$ and $y$ are not in the original C fragment. Then $x \sim y$ and $\{y, c_3\}$ is independent. Thus, $\{y, c_3\}$ and $(c_1)$ don't extend to disjoint maximum independent sets in G. So $G \not\in W_2$.

Therefore, $K_4$ and the triangular prism are the only cubic $W_2$ graphs.

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4-regular Planar $W_2$ Graphs

We now turn our attention to 4-regular $W_2$ graphs. Since no characterization of 4-regular well-covered graphs is known (unlike the situation for cubic well-covered graphs), we focus most of our efforts on only the planar 3-connected 4-regular $W_2$ graphs. But first we show in Theorem 7 that no 4-regular $W_2$ graph has a cutpoint.

**Theorem 7.** Suppose G is 4-regular and in $W_2$. Then G is 2-connected.

**Proof.** Assume to the contrary that G has a cutpoint $v$. Since G is 4-regular, then $G-v$ must have exactly two components, say $G_1$ and $G_2$, each containing two neighbors of $v$. Let $N(v) \cap G_1 = \{a_1, b_1\}$ and $N(v) \cap G_2 = \{a_2, b_2\}$. Define $A_1, A_2, B_1$ and $B_2$ as follows: $A_i = (N(a_i) \cap G_i) - \{b_i\}$, $B_i = (N(b_i) \cap G_i) - \{a_i\}$, for $i = 1, 2$. Let $y_1 \in B_1$.

Case 1. Suppose there exist points $u_1 \in A_1$, $y_1 \in B_1$, $u_2 \in A_2$ and $y_2 \in B_2$ such that $u_1$ is not adjacent to $y_1$ (possibly $u_1 = y_1$) and $u_2$ is not adjacent to $y_2$ (possibly $u_2 = y_2$). Then $\{u_1, u_2, y_1, y_2\}$ is independent and so $\{u_1, u_2, y_1, y_2\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, a contradiction since G is in $W_2$.

Case 2. So either every $u_1 \in A_1$ is adjacent to every $y_1 \in B_1$, or every $u_2 \in A_2$ is adjacent to every $y_2 \in B_2$. Without loss of generality, assume every $u_1 \in A_1$ is adjacent to every $y_1 \in B_1$. Let $z \in A_1$. Note that $z$ is not adjacent to $b_1$. Thus, $\{u_1, u_2, y_1, y_2\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, a contradiction since G is in $W_2$.

Therefore, G cannot have a cutpoint.

The following four lemmas will be helpful in determining the 3-connected 4-regular planar $W_2$ graphs.

**Lemma 8.** Suppose G is 3-connected 4-regular and planar. Suppose v is a point in G with face configuration $(3,3,x,y)$, $x, y \geq 3$, where two triangles incident to v share a line. If two triangles at v are $u_1u_2v$ and $u_2u_3v$, then $u_1$ is not adjacent to $u_3$.

**Proof.** Assume to the contrary that $u_1 \sim u_3$. Let $u_4$ be the fourth neighbor of v (see Figure 3). If $u_1$ has its fourth neighbor on one side of triangle $u_1u_2v$ and $u_3$ has its fourth neighbor on the other side of triangle $u_1u_3v$, then either $\{v, u_1\}$ or $\{v, u_3\}$ is a cutset of G. This contradicts the 3-connected assumption. Thus, $u_1$ and $u_3$ each have their fourth neighbor on the same side of triangle $u_1u_2v$, and so either v or $u_2$ is a cutpoint for G. This again contradicts the 3-connected assumption.
The next three lemmas are fairly obvious; hence, we omit proofs. Lemma 11 says that two faces in a 3-connected planar graph which are incident to the same point either have only that point in common or they are adjacent faces at the point and share only a line.

**Lemma 9.** Suppose $G$ is 3-connected 4-regular and planar. Suppose $F_4 = vu_4 \ldots u_1$ is an $n$-face at $v$, $n \geq 3$, and $F_1 = vu_1u_2$ is a triangular face at $v$ such that $F_4$ and $F_1$ share the line $vu_1$. If $x \in F_4$ such that $x \notin \{v,u_1\}$, then $x$ is not adjacent to $u_2$.

**Lemma 10.** Suppose $G$ is 3-connected and planar. Suppose $x$ and $y$ are non-consecutive points on a face of $G$. Then $x$ is not adjacent to $y$.

**Lemma 11.** Suppose $G$ is planar and 3-connected. Suppose $v$ is a point of $G$ with incident faces $F_1, F_2, \ldots, F_n$.

(i) If $F_i$ and $F_j$ share a line $xv$ ($i \neq j$), then $F_i \cap F_j = xv$.

(ii) If $F_i$ and $F_j$ do not share a line of the form $xv$, for any $x \in N(v)$, then $F_i \cap F_j = \{v\}$.

In the following lemmas, we will repeatedly use Lemma 4. In particular, if $S$ and $v$ are an independent set and point, respectively, which satisfy the hypotheses of Lemma 4, we will say that $S$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. If $G$ is assumed to be a $W_2$ graph, then we will have a contradiction.

For the next lemma only, we don't require $G$ to be planar.

**Lemma 12.** Suppose $G$ is 3-connected 4-regular and in $W_2$. If $G$ has a 4-wheel configuration at a point, then $G$ is $K_5$.

**Proof.** Assume $v$ is a point in $G$ with $N(v) = \{u_1, u_2, u_3, u_4\}$, and triangles $u_1u_2v$, $u_2u_3v$, $u_3u_4v$ and $u_4u_1v$ forming a 4-wheel configuration at $v$.

Suppose $u_1 \sim u_3$. If $u_2$ is not adjacent to $u_4$, then $\{u_2, u_4\}$ is a cutset for $G$. So $u_2 \sim u_4$. It follows that $G$ is $K_5$.

Suppose $u_1$ is not adjacent to $u_3$. Let $x$ be the fourth neighbor of $u_3$. If $x \sim u_1$, then $\{u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is not adjacent to $u_1$.

Suppose $x \sim u_2$ and $x \sim u_4$. Then $\{x, u_1\}$ is a cutset for $G$ since $x$ is not adjacent to $u_1$. So we can assume either $x$ is not adjacent to $u_2$ or $x$ is not adjacent to $u_4$. Without loss of generality, assume $x$ is not adjacent to $u_2$. Since $G$ is 4-regular, there is a point $y$ such that $y \sim x$ and $y$ is not adjacent to $u_1$. Then $\{y, u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$.

Hence, $u_1$ must be adjacent to $u_3$, and so $G$ must be $K_5$. 

[Diagram of Figure 3]
We will attack the problem of finding all 3-connected 4-regular planar $W_2$ graphs using the theory of Euler contributions. In each of the next ten lemmas, we consider a particular face configuration at a point $v$. Afterwards, the result which we pursue will follow easily. We will implicitly use Lemma 11 in each of these ten lemmas.

**Lemma 12.2.** Suppose $G$ is 3-connected 4-regular planar and in $W_2$. If $G$ has a point $v$ with face configuration $(3,3,3,4)$, then $G$ is the graph given in Figure 4.

![Figure 4](image)

**Proof.** Suppose $v$ has face configuration $(3,3,3,4)$ with $N(v) = \{u_1, u_2, u_3, u_4\}$ and the 4-face at $v$ is $u_1vu_4x$ (see Figure 5).

![Figure 5](image)

From Lemma 8, $u_1$ is not adjacent to $u_3$ and $u_2$ is not adjacent to $u_4$. From Lemma 9, $x$ is not adjacent to $u_2$ and $x$ is not adjacent to $u_3$. From Lemma 10, $u_1$ and $u_4$ are not adjacent.

Let $z$ be the fourth neighbor of $u_2$. From above, $z \notin \{x, u_4\}$. Let $(u_4) = N(u_4) - \{x, v, u_3\}$.

Case 1. Suppose $z \sim u_4$. Since $x$ is adjacent to neither $u_2$ nor $u_3$, then there exists a point $s \sim x$ such that $s \neq z$. Then $(s, u_2)$ is independent and so $(s, u_2)$ and $(u_4)$ do not extend to disjoint maximum independent sets in $G$, a contradiction. Thus $z$ is not adjacent to $u_4$.

Case 2. Suppose $z \sim u_3$.

Case 2.1. If $x$ and $z$ are not adjacent, then $(x, z)$ and $(v)$ do not extend to disjoint maximum independent sets in $G$. So $x \sim z$.

Case 2.2. If $z \sim u_1$, then $(x, u_4)$ is a cutset for $G$. So $z$ and $u_1$ are not adjacent.

Let $m \sim u_1$ such that $m \notin \{x, v, u_2\}$. Since $G$ is planar, $m$ and $w$ are not adjacent (see Figure 6). If $z \sim m$, then $(x, u_4)$ is a cutset. So $z$ and $m$ are not adjacent. If $z \sim w$, then $(x, w)$ is a cutset. So $z$ and $w$ are not adjacent. But then $(z, w, m)$ is independent and so $(z, w, m)$ and $(v)$ don't extend to disjoint maximum independent sets in $G$, a contradiction.
Thus, \( z \) and \( u_3 \) are not adjacent.

Case 3. Suppose \( x \sim z \).

Case 3.1. Suppose \( z \) and \( u_1 \) are not adjacent. Let \( y \in (N(u_1) - \{x,v,u_2\}) \), and let \( Y = N(y) - u_1 \).

Case 3.1.1. Suppose there exists \( p \in Y \) such that \( p \) is not adjacent to \( z \). Then \( \{p,z,u_4\} \) is independent and so \( \{p,z,u_4\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \).

Case 3.1.2. Thus, \( p \in Y \) implies \( p \sim z \). If \( y \sim z \), then \( \{z,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) and \( z \) are not adjacent. But then \( \{z,v\} \) and \( \{y\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( x \sim z \) implies \( z \sim u_1 \). See Figure 7.

Case 3.2. Suppose \( w \) and \( u_3 \) are not adjacent. Let \( y \sim u_3 \), \( y \notin \{v,u_2,u_4\} \). From above, \( y \notin \{x,z\} \).

Case 3.2.1. If \( y \sim w \), then \( \{w,u_1\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) and \( w \) are not adjacent.

Case 3.2.2. Suppose \( z \sim y \). Let \( \{a,b\} = N(y) - \{z,u_3\} \). If \( w \sim a \) and \( w \sim b \), then \( \{w,u_2\} \) and \( \{y\} \) don't extend to disjoint maximum independent sets in \( G \). So, without loss of generality, assume \( w \) is not adjacent to \( a \). If \( a = x \) (that is, \( x \sim y \)), then \( \{y,u_4\} \) is a
cutset. So \( a \neq x \) and \( \{w,a,u_1\} \) is independent. But then \( \{w,a,u_1\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( z \) and \( y \) are not adjacent.

Case 3.2.3. Suppose \( z \sim w \). Then \( \{w,u_3\} \) is a cutset. So \( z \) and \( w \) are not adjacent.

Hence, \( \{z,w,y\} \) is independent and so \( \{z,w,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( x \sim z \) implies \( w \sim u_3 \).

Case 3.3. If \( z \) and \( w \) are not adjacent, then \( \{w,z\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( z \sim w \).

Thus, \( x \sim z \) implies \( z \sim w \).

Case 3.4. If \( x \) and \( w \) are not adjacent, then \( \{x,w\} \) is a cutset. So \( x \sim w \).

Hence, \( \{z,w,y\} \) is independent and so \( \{z,w,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( x \sim z \) implies \( w \sim u_3 \).

Consequently, if \( x \sim z \) then \( G \) must be the graph given in Figure 4.

Now, recall from earlier that the following sets are independent: \( \{x,u_2\}, \{x,u_3\}, \{z,u_3\}, \{z,u_4\}, \{u_2,u_4\}, \{u_1,u_3\}, \{u_1,u_4\} \). Thus there exists \( y \sim u_3 \) such that \( y \in \{x,z,v,u_1,u_2,u_4\} \). Since \( z \) and \( u_4 \) are not adjacent, it follows by symmetry that \( y \) and \( u_1 \) are not adjacent.

Case 4. If \( x \sim y \), then by symmetry and the argument given in Case 3 for \( x \sim z \), the only \( W_2 \) graph which can result is the graph obtained in Case 3.

Case 5. So we assume \( x \) is not adjacent to \( z \) and \( y \) is not adjacent to \( x \).

If \( y \) and \( z \) are not adjacent, then \( \{x,y,z\} \) is independent and so \( \{x,y,z\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \sim z \).

Suppose \( y \sim u_4 \). Since \( y \) is not adjacent to \( u_1 \), then there exists \( w \sim y \) such that \( w \in \{x,z,v,u_1,u_2,u_3,u_4\} \). If \( w \sim x \), then \( \{w,u_2\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \) and \( x \) are not adjacent.

Since \( G \) is 4-regular, there exist points \( s \) and \( t \) such that \( s \) and \( t \) are neighbors of \( x \) and \( (s,t) \cap \{v,y,z,u_1,u_2,u_3,u_4\} = \emptyset \). Suppose \( w \) and \( s \) are not adjacent. Then \( \{w,s,u_2\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \sim s \) and, similarly, \( w \sim t \) (see Figure 9). But then \( \{v,w\} \) and \( \{x\} \) don't extend to disjoint maximum independent sets in \( G \).
Hence, y and u₄ are not adjacent. By symmetry, z and u₁ are not adjacent. Thus there exists m ~ u₁ such that mε {x,y,z,v,u₁,u₂,u₃,u₄}. If m ~ u₄, then {z,u₄} and {u₁} don't extend to disjoint maximum independent sets in G. So m and u₄ are not adjacent.

Suppose m ~ y. Then there exists a point n ~ u₄ such that {n,z,u₁} is independent, where nε {x,v,u₃}. But then {n,z,u₁} and {u₃} don't extend to disjoint maximum independent sets in G. So m and y are not adjacent (see Figure 10).

From above, we see that (m,y,u₄) is independent. Then (m,y,u₄) and (u₂) don't extend to disjoint maximum independent sets in G.

Therefore, the graph shown in Figure 2.5 is the only 3-connected 4-regular planar W₂ graph with the (3,3,3,4) face configuration.

**Lemma 12.3.** Suppose G is 3-connected 4-regular planar and in W₂. If v is a point in G, then v cannot have face configuration (3,3,3,5).

**Proof.** Assume to the contrary that v has face configuration (3,3,3,5). Let N(v) = {u₁,u₂,u₃,u₄} and the 5-face at v be abu₄v₁. From Lemma 8, u₁ is not adjacent to u₃ and u₂ is not adjacent to u₄. From Lemma 9, a is not adjacent to u₂, a is not adjacent to u₃, b is not adjacent to u₂, and b is not adjacent to u₃. From Lemma 10, a is not adjacent to u₄, u₁ is not adjacent to u₄, and b is not adjacent to u₁.
Thus, there exists $x \sim u_4$ such that $x \notin \{a,b,v,u_1,u_2,u_3\}$. By symmetry, there exists $y \sim u_1$ such that $y \notin \{a,b,v,u_2,u_3,u_4\}$ (we do not exclude the possibility that $y = x$).

Case 1. Suppose $a \sim x$. Then $\{a,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$. So $a$ is not adjacent to $x$. By symmetry, $y$ is not adjacent to $b$.

Let \( \{p\} = N(u_2) \setminus \{v,u_1,u_3\} \).

Case 2. If $p = x$ (that is, $x \sim u_2$) or $p \sim a$, then $\{a,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in $G$. So $p \neq x$ and $p$ and $a$ are not adjacent.

Case 3. Suppose $p \sim x$.

Case 3.1. Suppose $p \sim u_3$. If $x \sim u_1$, then $\{p,t,u_1\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$, where $t \sim b$ such that $t \notin \{a,u_4\}$. So $x$ is not adjacent to $u_1$.

Thus $\{x,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$.

Hence, $p$ is not adjacent to $u_3$.

Case 3.2. Suppose $x \sim u_3$.

Case 3.2.1. If $x \sim b$ or $x \sim u_1$, then $\{b,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is adjacent to neither $b$ nor $u_1$.

Thus, there exists $z \sim x$ such that $z \notin \{a,b,v,u_1,u_2,u_3,u_4\}$.

Case 3.2.2. If $z$ is not adjacent to $a$, then $\{a,z,u_2\}$ is independent and so $\{a,z,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$. So $z \sim a$.

Case 3.2.3. If $z \sim b$, then $\{z,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$. So $z$ is not adjacent to $b$.

Case 3.2.4. If $z$ is not adjacent to $u_1$, then $\{b,z,u_1\}$ is independent and so $\{b,z,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$. So $z \sim u_1$. But then $\{p,z\}$ is a cutset for $G$.

Thus, $x$ is not adjacent to $u_3$. So there exists $w \sim u_3$ and $m \sim w$ such that $w \notin \{v,u_2,u_4\}$ and $(w,m) \cap \{p,x\} = \emptyset$ (see Figure 11). But then $\{b,m,u_1\}$ is independent and so $\{b,m,u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$.

![Figure 11](image-url)

Hence, $p$ is not adjacent to $x$. Thus $\{p,x,a\}$ is independent. By symmetry, there exists $q \sim u_3$ such that $q \notin \{v,u_2,u_4, a, y\}$ and $q$ is not adjacent to $y$.

If any member of $\{p,x,a\}$ is adjacent to $u_3$, then $\{p,x,a\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $q \notin \{a,p,x\}$.

Suppose $x \sim u_1$ (that is, $x = y$). Then $\{p,t,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$, where $t \sim a$ such that $t \notin \{b,u_1\}$. Thus, $x$ is not adjacent to $u_1$; hence, $x \neq y$. See Figure 12.
Suppose \( p \sim q \). Then \( \{q,y,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( p \) and \( q \) are not adjacent. Suppose \( q \sim x \). Then \( \{x,u_1\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \). So \( q \) is not adjacent to \( x \) and, by symmetry, \( p \) is not adjacent to \( y \). If \( q \sim a \), then \( \{x,y,p,q\} \) is an independent set. Thus, \( \{x,y,p,q\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( q \) is not adjacent to \( a \), and it follows that \( \{a,x,p,q\} \) is independent. But then \( \{a,x,p,q\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Therefore, the face configuration \( (3,3,3,5) \) cannot occur.

Lemma 12.4. Suppose \( G \) is 3-connected 4-regular planar and in \( W_2 \). If \( v \) is a point in \( G \), then \( v \) cannot have face configuration \( (3,3,3,n) \), \( n \geq 6 \).

Proof. Assume to the contrary that \( v \) has face configuration \( (3,3,3,n) \), \( n \geq 6 \). Let \( N(v) = \{u_1,u_2,u_3,u_4\} \), and let the \( n \)-face at \( v \) be \( u_3cbdu_4v \). From Lemma 8, \( u_1 \) is not adjacent to \( u_3 \) and \( u_2 \) is not adjacent to \( u_4 \). From Lemma 9, \( a \) is not adjacent to \( u_1 \) and \( u_2 \) are not adjacent, \( c \) is not adjacent to \( u_1 \), \( a \) is not adjacent to \( u_2 \), \( b \) is not adjacent to \( u_2 \), \( c \) is not adjacent to \( u_2 \), and \( u_1 \) and \( b \) are not adjacent. From Lemma 10, \( a \) is not adjacent to \( c \), \( a \) is not adjacent to \( u_1 \), and \( c \) is not adjacent to \( u_4 \).

Now let \( s \sim u_2 \) such that \( s \notin \{v,u_1,u_3\} \).

Case 1. Suppose \( s \sim c \). Then \( \{c,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). Thus, \( s \) is not adjacent to \( c \).

Case 2. Suppose \( s \sim a \).

Case 2.1. If \( s \sim u_4 \), then \( \{c,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( s \) is not adjacent to \( u_4 \).

Let \( w \sim u_4 \) such that \( w \notin \{a,v,u_1\} \).

Case 2.2. If \( w \sim a \), \( w \sim s \) and \( w \sim u_1 \), then \( \{a,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). Thus there exists \( t \sim w \) such that \( t \notin \{a,s,u_1,u_4\} \). But then \( \{b,t,u_2\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( s \) is not adjacent to \( a \).

Case 3. If \( s \sim u_1 \), then \( \{a,s,c\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( s \) and \( u_1 \) are not adjacent.

Let \( t \sim u_1 \), where \( t \notin \{v,u_2,u_4\} \); by symmetry with \( s \), \( t \) is adjacent to neither \( a \) nor \( c \).

Case 4. Suppose \( s \sim t \).

Case 4.1. Suppose \( s \sim u_3 \).

Case 4.1.1. Suppose \( t \sim u_4 \). Let \( \{w\} = N(t) - \{s,u_1,u_4\} \). If \( a \sim w \), then \( \{a,u_2\} \) and \( \{t\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \) is not adjacent to \( w \).

Let \( N(a) = \{b,u_4\} = \{y_1,y_2\} \). If \( w \sim b \) and \( w \sim y_2 \), then \( \{w,v\} \) and \( \{a\} \) don't extend to disjoint maximum independent sets in \( G \). Thus there exists some \( x \sim a, x \neq \)
$u_4$, such that $x$ is not adjacent to $w$ (see Figure 13). But then $\{x, w, u_2\}$ is independent and so $\{x, w, u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$.

![Figure 13](image)

Case 4.1.2. So $t$ is not adjacent to $u_4$. Then $\{t, u_4, c\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in $G$.

Case 4.2. Hence, $s$ is not adjacent to $u_3$. It follows that $\{a, s, u_3\}$ is independent. Hence, $\{a, s, u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$.

Thus, $s$ is not adjacent to $t$. Then $\{s, t, a, c\}$ is independent and $\{s, t, a, c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$.

Therefore, the face configuration $(3, 3, 3, n)$, $n \geq 6$, cannot occur.

Lemma 12.5. Suppose $G$ is 3-connected 4-regular planar and in $W_2$. If $v$ is a point in $G$, then $v$ cannot have face configuration $(3, 3, 4, 4)$.

Proof. Assume to the contrary that $v$ has face configuration $(3, 3, 4, 4)$. Let $N(v) = \{u_1, u_2, u_3, u_4\}$.

Case 1. Suppose the cyclic order of the faces at $v$ is $(3, 4, 3, 4)$, with faces $u_1u_2v$, $u_2u_3v$, $u_3u_4v$ and $u_4u_1v$. By Lemma 9, $a$ is not adjacent to $u_2$, $a$ is not adjacent to $u_3$, $b$ is not adjacent to $u_1$, and $b$ is not adjacent to $u_4$.

If $a$ is not adjacent to $b$, then $\{a, b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $a \sim b$. Thus there exists $x \sim u_1$, $y \sim u_2$, $s \sim u_3$ and $t \sim u_4$ such that $\{x, y, s, t\} \cap \{a, b, v, u_1, u_2, u_3, u_4\} = \emptyset$.

If $x = y$ and $s = t$, then $\{x, s\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So either $x \neq y$ or $s \neq t$. Without loss of generality, assume $x \neq y$.

Suppose $x \sim y$. Then $\{y, u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is not adjacent to $y$.

If $s = t$, then $\{s, t, x, y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $s \neq t$. Since $s \neq t$ and by symmetry with $x$ and $y$, it follows that $s$ is not adjacent to $t$. But then $\{s, t, x, y\}$ is independent since $G$ is planar, and so $\{s, t, x, y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$ (see Figure 14).
Thus the cyclic face order (3,4,3,4) cannot occur.

Case 2. Suppose the cyclic face order is (3,3,4,4), with faces \( u_1u_2v, u_2u_3v, u_3vu_4v \) and \( u_4au_1v \). By Lemma 8, \( u_1 \) is not adjacent to \( u_3 \). By Lemma 9, \( a \) is not adjacent to \( u_2 \), \( b \) is not adjacent to \( u_1 \), and \( u_2 \) is not adjacent to \( u_4 \). By Lemma 10, \( u_3 \) is not adjacent to \( u_4 \) and \( u_3 \) is not adjacent to \( u_4 \).

Case 2.1. Suppose \( a \sim b \). Then there exists \( z \sim u_4 \) and \( w \sim z \) such that \( \{w,z\} \cap \{a,b\} = \emptyset \). Since \( G \) is planar, \( \{u_1, u_3, w\} \) is independent; hence, \( \{u_1, u_3, w\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). Thus, \( a \) is not adjacent to \( b \).

Let \( y \sim u_2 \) such that \( y \notin \{v, u_1, u_3\} \).

Case 2.2. Suppose \( y \sim a \).

Case 2.2.1. Suppose \( y \sim u_1 \) (see Figure 15).

Thus, either we have a \((3,3,3,4)\) face configuration at \( u_1 \), or there is a point inside triangle \( yau_1 \) or inside triangle \( yu_2u_1 \). From Lemma 12.2, point \( u_1 \) cannot have a \((3,3,3,4)\) face configuration. If there is a point inside triangle \( yau_1 \), then \( \{y, a\} \) is a cutset, contradicting 3-connectedness. If there is a point inside triangle \( yu_2u_2 \), then \( y \) is a cutpoint, contradicting 3-connectedness.

Case 2.2.2. Hence, \( y \) and \( u_1 \) are not adjacent (we are still assuming that \( y \sim a \)). Since \( y \) is not adjacent to \( u_1 \), there exists \( z \sim u_1 \) such that \( z \notin \{a, b, v, y, u_1, u_2, u_3, u_4\} \), and \( w \sim z \) such that \( w \notin \{a, y\} \). Then \( \{w, u_3, u_4\} \) is independent and so \( \{w, u_3, u_4\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \).
Hence, \( y \) is not adjacent to \( a \) and, by symmetry, \( y \) is not adjacent to \( b \). It follows that \( \{a, b, y\} \) is independent and so \( \{a, b, y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, the cyclic face order \((3,3,4,4)\) cannot occur. From Cases 1 and 2, we conclude that the face configuration \((3,3,4,4)\) cannot occur.

**Lemma 12.6.** Suppose \( G \) is 3-connected 4-regular planar and in \( W_2 \). If \( v \) is a point in \( G \), then \( v \) cannot have face configuration \((3,3,4,5)\).

**Proof.** Assume to the contrary that \( v \) has face configuration \((3,3,4,5)\). Let \( N(v) = \{u_1, u_2, u_3, u_4\} \).

Case 1. Suppose the cyclic order of the faces at \( v \) is \((3,3,4,5)\), with faces \( u_1u_2v, u_2u_3v, u_3ucv \) and \( u_4bav \). By Lemma 8, \( u_1 \) is not adjacent to \( u_3 \). By Lemma 9, \( u_2 \) is not adjacent to \( u_4 \), \( u_2 \) is not adjacent to \( a \), \( u_2 \) is not adjacent to \( b \), and \( u_2 \) is not adjacent to \( c \). By Lemma 10, \( u_1 \) is not adjacent to \( u_4 \), \( u_3 \) is not adjacent to \( u_4 \), and \( a \) is not adjacent to \( u_4 \).

Case 1.1. Suppose \( a \sim c \).

Case 1.1.1. Suppose \( c \sim u_1 \). Then \( \{u_2, u_3\} \) is a cutset for \( G \). So \( c \) is not adjacent to \( u_1 \).

Thus, there exists \( x \sim u_1 \) such that \( xy \in \{a, b, c, v, u_2, u_3, u_4\} \).

Case 1.1.2. If \( x \sim u_3 \), then \( \{x, u_2\} \) is a cutset for \( G \). So \( x \) is not adjacent to \( u_3 \).

Case 1.1.3. Suppose \( c \sim x \). Let \( m \sim u_3 \) such that \( m \in \{v, c, u_2\} \). Then \( \{b, m, u_1\} \) and \( \{c\} \) don't extend to disjoint maximum independent sets in \( G \). So \( c \) is not adjacent to \( x \).

Case 1.1.4. If \( x \sim u_2 \), then \( \{c, x\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( u_2 \) is not adjacent to \( x \).

Thus, there exists \( y \sim u_2 \) such that \( xy \in \{a, b, c, v, x, u_1, u_3, u_4\} \).

Case 1.1.5. If \( c \sim y \), then \( \{b, u_2\} \) and \( \{c\} \) don't extend. So \( c \) is not adjacent to \( y \).

Case 1.1.6. If \( x \) is not adjacent to \( y \), then \( \{c, x, y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \sim y \).

Case 1.1.7. Suppose \( y \sim u_3 \). If \( y \sim a \), then \( x \) is a cutpoint for \( G \). So \( y \) is not adjacent to \( a \). Thus, there exists \( z \sim y \) such that \( z \in \{a, b, c, v, x, u_1, u_2, u_3, u_4\} \). But then \( \{z, u_1, u_4\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \) (see Figure 16).

![Figure 16](image-url)
Thus, $y$ is not adjacent to $u_1$. Let $x \sim u_1$ such that $x \notin \{a,v,u_2\}$.

Case 1.2.2. Suppose $y$ is not adjacent to $x$. If $y$ is not adjacent to $c$, then $(x,y,c)$ and $(v)$ don't extend to disjoint maximum independent sets in $G$. So $y \sim c$. Then either $(u_1,a)$ or $(u_3,c)$ is a cutset for $G$ (see Figure 17).

![Figure 17](image)

Thus $y \sim x$. Since $G$ is 4-regular, $y$ is not adjacent to at least one of $u_3$ or $u_4$. Then either $(y,u_3)$ and $(u_1)$ or $(y,u_4)$ and $(u_1)$ don't extend to disjoint maximum independent sets in $G$.

Hence, $a$ is not adjacent to $v$.

Case 1.3. Suppose $y \sim c$.

Case 1.3.1. Suppose $y \sim u_3$ (see Figure 18).

![Figure 18](image)

Either we have a $(3,3,3,4)$ face configuration at $u_3$, or we have a point inside triangle $yu_2u_3$ or inside triangle $yuc$. From Lemma 12.2, we cannot have a $(3,3,3,4)$ face configuration at $u_3$. If there is a point inside triangle $yu_2u_3$, then $y$ is a cutpoint, contradicting 3-connectedness. If there is a point inside triangle $ycu_3$, then $(y,c)$ is a cutset, contradicting 3-connectedness.

Case 1.3.2. So $y$ is not adjacent to $u_3$. Let $m \sim u_3$ such that $m \notin \{v,c,u_2\}$ and let $z \sim m$ such that $z \notin \{c,y\}$ (see Figure 19). Then $(z,u_1,u_4)$ and $(u_3)$ don't extend to disjoint maximum independent sets in $G$. 

![Figure 19](image)
Hence, \( y \) is not adjacent to \( c \). It follows that \( \{a,y,c\} \) is independent and so \( \{a,y,c\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, the cyclic face configuration \((3,3,4,5)\) cannot occur.

Case 2. Assume the cyclic face configuration is \((3,4,3,5)\), with faces \( u_1u_2v, u_2u_3v, u_3u_4v \) and \( u_4bav_1v \). By Lemma 9, \( a \) is not adjacent to \( u_2 \), \( b \) is not adjacent to \( u_2 \), \( u_4 \) is not adjacent to \( u_2 \), \( c \) is not adjacent to \( u_1 \), \( a \) is not adjacent to \( u_3 \), \( b \) is not adjacent to \( u_3 \), and \( u_1 \) is not adjacent to \( u_3 \). By Lemma 10, \( u_4 \) is not adjacent to \( a \), \( u_1 \) is not adjacent to \( b \), and \( u_1 \) is not adjacent to \( u_4 \). So there exists \( y \sim u_4 \) such that \( y \notin \{a,b,c,v,u_1,u_2,u_3\} \).

Case 2.1. Suppose \( a \sim c \). Let \( x \sim u_1 \) such that \( x \notin \{a,v,u_2\} \). If \( c \sim x \) or \( c \sim b \), then \( \{u_1,u_4\} \) and \( \{c\} \) don't extend to disjoint maximum independent sets in \( G \). So \( c \) is adjacent to neither \( x \) nor \( b \). Thus, \( \{b,c,x\} \) is independent since \( G \) is planar. It follows that \( \{b,c,x\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( a \) is not adjacent to \( c \) and, by symmetry, \( b \) is not adjacent to \( c \).

Case 2.2. Suppose \( y \sim u_1 \). Let \( z \sim b \) such that \( z \notin \{a,y,u_4\} \). Then \( \{c,z,u_1\} \) is independent and so \( \{c,z,u_1\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( u_1 \).

Thus, there exists \( x \sim u_1 \) such that \( x \notin \{a,b,c,v,y,u_2,u_3,u_4\} \).

Case 2.3. Suppose \( y \sim c \).

Case 2.3.1. If \( x \sim c \), then \( \{u_1,u_4\} \) and \( \{c\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \) is not adjacent to \( c \).

Case 2.3.2. Suppose \( x \sim b \). If \( b \sim u_2 \), then \( \{a,x\} \) is a cutset for \( G \). So \( b \) is not adjacent to \( u_2 \).

Case 2.3.2.1. Suppose \( y \sim u_2 \). Let \( t \sim c \) such that \( t \notin \{y,u_2,u_3\} \). Then \( \{a,t,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( u_2 \).

Case 2.3.2.2. Suppose \( x \sim u_2 \)(see Figure 20).
(i) Suppose \( y \sim u_3 \). If \( y \sim x \), then \( \{a,b\} \) is a cutset for \( G \). So \( y \) is not adjacent to \( x \).
But then \( \{x,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).
(ii) Thus, \( y \) is not adjacent to \( u_3 \). So there exists \( z \sim u_3 \) and \( w \sim z \) such that \( z \in \{c,v,y,u_4\} \) and \( w \in \{c,y\} \). Then \( \{w,b,u_2\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \).
So \( x \) is not adjacent to \( u_2 \). Thus, there exists \( d \sim u_2 \) such that \( d \in \{a,b,c,v,x,y,u_1,u_3,u_4\} \) (see Figure 21).

Case 2.3.2.3. If \( b \) is not adjacent to \( d \), then \( \{b,d,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( b \sim d \). Then \( \{a,x\} \) is a cutset for \( G \).

Thus, \( x \) is not adjacent to \( b \). It follows that \( \{b,x,c\} \) is independent and so \( \{b,x,c\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( v \) is not adjacent to \( c \) and, by symmetry, \( x \) is not adjacent to \( c \).

Case 2.4. Suppose \( a \sim y \). Then \( \{b,x,c\} \) is independent since \( G \) is planar. Hence, \( \{b,x,c\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). Thus, \( a \) is not adjacent to \( v \).

So \( \{a,c,y\} \) is independent; thus, \( \{a,c,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, the cyclic face configuration \( (3,4,3,5) \) cannot occur. It follows that \( G \) cannot have a point with face configuration \( (3,3,4,5) \).
Lemma 12.7. Suppose $G$ is 3-connected 4-regular planar and in $W_2$. Then $G$ cannot have a point with face configuration $(3, 3, 4, n)$, $n \geq 6$.

**Proof.** Assume to the contrary that $v$ has face configuration $(3, 3, 4, n)$, $n \geq 6$. Let $N(v) = \{u_1, u_2, u_3, u_4\}$.

Case 1. Assume the cyclic face configuration is $(3, 4, 3, n)$. Let the faces at $v$ be $u_1u_2v$, $u_2bu_3v$, $u_3u_4v$ and $u_4dez$. . . $acu_1v$ ($e = a$ when $n = 6$). By Lemma 10, $u_1$ is adjacent to neither $u_4$ nor $d$, and $c$ is adjacent to neither $u_4$ nor $d$. By Lemma 9, $b$ is not adjacent to $u_1$, $b$ is not adjacent to $u_4$, $u_2$ is not adjacent to $u_4$, and $u_4$ is not adjacent to $u_3$.

Suppose $b \sim c$. Let $y \sim u_1$ such that $y \notin \{c, v, u_2\}$.

If $b \sim d$, then $\{u_1, u_4\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in $G$. So $b$ is not adjacent to $d$. If $b \sim y$, then $\{b, u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$. So $b$ is not adjacent to $y$. Since $b$ is not adjacent to $y$, then $\{b, d, y\}$ is independent and so $\{b, d, y\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$.

Thus, $b$ is not adjacent to $c$ and, by symmetry, $b$ is not adjacent to $d$. It follows that $\{b, c, d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$.

Hence, the cyclic face configuration $(3, 3, 4, n)$, $n \geq 6$, is not possible.

Case 2. Assume the cyclic face configuration is $(3, 4, 3, n)$, $n \geq 6$, with faces $u_1u_2v$, $u_2u_3v$, $u_3u_4v$ and $u_4dez$. . . $acu_1v$ ($e = a$ when $n = 6$). By Lemma 10, $u_1$ is adjacent to neither $u_4$ nor $d$, $c$ is adjacent to neither $u_4$ nor $d$ and $u_3$ is not adjacent to $u_4$. By Lemma 9, $b$ is not adjacent to $u_2$ and $x$ is not adjacent to $u_2$, for any $x$ in the $n$-face at $v$, $x \in \{v, u_1\}$. So there exists $s \sim u_2$ such that $s \in \{a, b, c, d, v, u_1, u_3, u_4\}$ and $s$ is not on the $n$-face at $v$.

Case 2.1. Suppose $b \sim d$. Let $y \sim u_4$ such that $y \notin \{b, d, v\}$, and $w \sim y$ such that $w \notin \{b, d\}$. If $e$ is not adjacent to $u_3$, then $\{e, w, u_3\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$. So $e \sim u_3$. Then $\{d, u_1\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$.

Thus, $b$ is not adjacent to $d$.

Case 2.2. Suppose $b \sim c$.

Case 2.2.1. If $b \sim u_1$, then $\{a, v\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in $G$. So $b$ is not adjacent to $u_1$.

Case 2.2.2. If $c \sim u_3$, then $\{u_1, u_2\}$ is a cutset for $G$. So $c$ is not adjacent to $u_3$.

Thus, there exist points $x$ and $t$ such that $x \sim u_3$, $t \sim u_1$ and $\{x, t\} \cap \{b, c, v, u_1, u_3, u_4\} = \emptyset$. If $x = t$, then $\{u_1, u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$. So $x \neq t$.

Case 2.2.3. Suppose $t \sim b$. Then $\{u_1, x, d\}$ is independent since $x \neq t$ and $G$ is planar. It follows that $\{u_1, x, d\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in $G$. So $t$ is not adjacent to $b$.

Case 2.2.4. Suppose $t \sim u_2$. Then $\{t, b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $t$ is not adjacent to $u_2$.

Since $G$ is 4-regular, there exists $z \sim t$ such that $z \notin \{c, x\}$ (see Figure 22). Thus $z$ is not adjacent to $u_3$, and so $\{a, z, u_3\}$ is independent. It follows that $\{a, z, u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$. 
Therefore, \( b \) is not adjacent to \( c \).

Case 2.3. Suppose \( s \sim c \).

Case 2.3.1. If \( s \sim b \), then either \( \{c,u_1\} \) or \( \{b,u_3\} \) must be a cutset of \( G \). So \( s \) is not adjacent to \( b \).

Case 2.3.2. If \( s \sim u_1 \), then \( \{s,b\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( s \) is not adjacent to \( u_1 \).

Let \( w \sim u_1 \) such that \( w \notin \{v,c,u_2\} \).

Case 2.3.3. If \( w \) is not adjacent to \( s \), then \( \{w,s,b\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \sim s \).

Case 2.3.4. If \( s \sim u_3 \), then \( \{b,u_1\} \) and \( \{s\} \) don't extend to disjoint maximum independent sets in \( G \). So \( s \) is not adjacent to \( u_3 \); hence, \( \{s,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \) (see Figure 23).

Thus, \( s \) is not adjacent to \( c \).

Case 2.4. Suppose \( s \sim b \).

Case 2.4.1. Suppose \( s \sim u_3 \). If \( s \) is not adjacent to \( d \), then \( \{s,c,d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( s \sim d \). Then there exist \( t \sim u_4 \) such that \( t \notin \{v,b,d,s\} \), and \( z \sim t \) such that \( z \notin \{b,d\} \). It follows that \( \{e,z,u_3\} \) is independent and \( \{e,z,u_3\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( s \) is not adjacent to \( u_3 \). So there exists \( w \sim u_3 \) such that \( w \notin \{s,b,v,u_2\} \).

Case 2.4.2. Suppose \( s \sim u_4 \). If \( s \) is not adjacent to \( w \), then \( \{s,w,c\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( s \sim w \). Then \( \{d,u_3\} \) and \( \{s\} \) don't extend to disjoint maximum independent sets in \( G \).

So \( s \) is not adjacent to \( u_4 \).
Case 2.4.3. Suppose $s \sim w$. Then $\{s,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$. So $s$ is not adjacent to $w$.

Let $W = N(w) - u_3$.

Case 2.4.4. Suppose $b \sim w$. Suppose $s \sim x$ for some $x \in W - b$. Then $\{v,x\}$ and $\{b\}$ don't extend to disjoint maximum independent sets in $G$. Let $x \in W - b$. Then $x$ is not adjacent to $s$ and so $\{s,x,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$.

So $b$ is not adjacent to $w$. Since $G$ is 4-regular, there exists $y \in W$ such that $y$ is not adjacent to $s$. But then $\{y,s,u_4\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in $G$ (see Figure 24).

![Figure 24](image)

Hence, $s$ is not adjacent to $b$. It follows that $\{s,b,c\}$ is independent and so $\{s,b,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$.

So the cyclic face configuration $(3,3,4,n), n \geq 6$, cannot occur. Thus, the face configuration $(3,3,4,n), n \geq 6$, cannot occur.

Lemma 12.8. Suppose $G$ is 3-connected 4-regular planar and in $W_2$. If $v$ is a point in $G$, then $v$ cannot have face configuration $(3,3,5,5)$.

Proof. Assume to the contrary that $v$ has face configuration $(3,3,5,5)$, with $N(v) = \{u_1,u_2,u_3,u_4\}$.

Case 1. Assume the cyclic face configuration at $v$ is $(3,5,3,5)$, with faces $u_1u_2v$, $u_2cdv$, $u_3u_4v$ and $u_4bau_1v$. By Lemma 9, $a$ is not adjacent to $u_2$, $a$ is not adjacent to $u_3$, $b$ is not adjacent to $u_2$, $b$ is not adjacent to $u_3$, $c$ is not adjacent to $u_1$, $c$ is not adjacent to $u_4$, $d$ is not adjacent to $u_1$, $d$ is not adjacent to $u_4$, $u_1$ is not adjacent to $u_3$, and $u_2$ is not adjacent to $u_4$. By Lemma 10, $a$ is not adjacent to $u_4$, $b$ is not adjacent to $u_1$, $c$ is not adjacent to $u_3$, $d$ is not adjacent to $u_2$, $u_1$ is not adjacent to $u_4$, and $u_2$ is not adjacent to $u_3$.

Hence, there exists $x \sim u_1$ such that $x \in \{a,b,c,d,v,u_2,u_3,u_4\}$.

Case 1.1. Suppose $a \sim c$.

Case 1.1.1. Suppose $x \sim u_2$. If $b$ is not adjacent to $d$, then $\{x,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $b \sim d$.

Let $s \sim u_3$ and $t \sim u_4$ such that $s \in \{d,v,u_4,b\}$ and $t \in \{b,v,u_3,d\}$.

Case 1.1.1.1. If $s = t$, then $\{s,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $s \neq t$.

Case 1.1.1.2. If $s$ is not adjacent to $t$, then $\{s,t,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $s \sim t$.

But then $\{a,s,u_2\}$ and $\{u_4\}$ don't extend to disjoint maximum independent sets in $G$. Therefore, the face configuration $(3,3,4,n), n \geq 6$, cannot occur. Thus, the face configuration $(3,3,4,n), n \geq 6$, cannot occur.
Case 1.1.2. Thus $x$ is not adjacent to $u_2$. Let $y \sim u_2$ such that $y \in \{v,c,u_1\}$. If $x$ is not adjacent to $y$, then we can proceed as in Case 1.1.1 to obtain a contradiction. So $x \sim y$ (see Figure 25).

![Figure 25]

Case 1.1.2.1. Suppose $x \sim a$. If $x \sim c$, then $y$ is a cutpoint for $G$. So $x$ is not adjacent to $c$. Thus, there exists $z \sim x$ such that $z \in \{a,y,u_1\}$. Then $\{z,u_2,u_4\}$ is independent and so $\{z,u_2,u_4\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in $G$.

Case 1.1.2.2. So $x$ is not adjacent to $a$. Since $G$ is 4-regular, $a$ is not adjacent to at least one of $u_3$ or $u_4$. Then either $\{a,x,u_4\}$ and $\{u_2\}$ or $\{a,x,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in $G$.

Thus, $a$ is not adjacent to $c$. By symmetry, $b$ is not adjacent to $d$.

Case 1.2. Suppose $b \sim c$.

Case 1.2.1. If $b \sim x$, then $\{a,x\}$ is a cutset for $G$. So $b$ is not adjacent to $x$.

Case 1.2.2. If $x \sim u_2$, then $\{x,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is not adjacent to $u_2$.

Let $y \sim u_2$ such that $y \in \{v,c,u_1\}$.

Case 1.2.3. If $y \sim x$, then $\{x,d,u_4\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in $G$. So $y$ is not adjacent to $x$.

Case 1.2.4. If $y$ is not adjacent to $b$, then $\{x,y,b,d\}$ is independent and so $\{x,y,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $y \sim b$. Then $\{b,x,u_3\}$ and $\{u_2\}$ don't extend to disjoint maximum independent sets in $G$.

Hence, $b$ is not adjacent to $c$ and, by symmetry, $a$ is not adjacent to $d$.

If $x$ is adjacent to any member of $\{b,c,u_3\}$, then $\{b,c,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is adjacent to no member of $\{b,c,u_3\}$.

Thus, there exists $z \sim u_3$ such that $z \in \{a,b,c,d,v,x,u_1,u_2,u_4\}$. By symmetry with $x$, it follows that $z$ is adjacent to neither $b$ nor $c$.

If $z$ is not adjacent to $x$, then $\{z,x,b,c\}$ is independent and so $\{z,x,b,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $z \sim x$.

Suppose $x \sim u_2$. If $x \sim d$, then $\{c,d\}$ is a cutset for $G$. So $x$ is not adjacent to $d$. Then $\{x,b,d\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is not adjacent to $u_2$.

Suppose $x \sim u_4$. There exists $t$ such that $t \in \{b,x,u_1\}$ and $\{t,c,u_4\}$ is independent. Then $\{t,c,u_4\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in $G$. So $x$ is not adjacent to $u_4$.

Hence, there exist points $p$ and $q$ such that $p \sim u_2$, $q \sim u_4$ and $(p,q) \cap \{a,b,c,d,v,x,z,u_1,u_2,u_3,u_4\} = \emptyset$. Since $z \sim x$ from above and $G$ is planar, then $p \neq q$ and $p$ is not adjacent to $q$. See Figure 26.
If \( p \sim d \), then \( \{d, u_4, a\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( p \) is not adjacent to \( d \) and, by symmetry, \( q \) is not adjacent to \( a \). Thus, \( \{a, d, p, q\} \) is independent; it follows that \( \{a, d, p, q\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, the cyclic face configuration \((3, 5, 3, 5)\) cannot occur.

Case 2. Assume the cyclic face configuration at \( v \) is \((3, 3, 5, 5)\), with faces \( u_1u_2v, u_2u_3v, u_3dcu_4v \) and \( u_4bau_1v \). By Lemma 8, \( u_1 \) is not adjacent to \( u_3 \). By Lemma 9, \( u_2 \) is not adjacent to \( u_2 \), \( a \) is not adjacent to \( u_2 \), \( b \) is not adjacent to \( u_2 \), \( c \) is not adjacent to \( u_2 \) and \( d \) is not adjacent to \( u_2 \). By Lemma 10, \( u_1 \) is not adjacent to \( u_4 \), \( u_3 \) is not adjacent to \( u_4 \), \( d \) is not adjacent to \( u_4 \), \( a \) is not adjacent to \( u_4 \), \( b \) is not adjacent to \( u_4 \), and \( c \) is not adjacent to \( u_4 \).

Thus, there exists \( w \sim u_4 \) such that \( w \notin \{a, b, c, d, v, u_1, u_2, u_4\} \).

Case 2.1. If \( d \sim u_1 \), then \( \{b, u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( d \) is not adjacent to \( u_1 \) and, by symmetry, \( a \) is not adjacent to \( u_1 \).

Case 2.2. Suppose \( w \sim a \).

Case 2.2.1. If \( a \sim c \), then \( \{a, u_3\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \) is not adjacent to \( c \).

Case 2.2.2. Suppose \( c \sim u_1 \). Since \( a \) is not adjacent to \( c \), there exists \( s \sim a \) such that \( s \notin \{b, w, u_1, c\} \). But then \( \{s, u_3, u_4\} \) is independent and so \( \{s, u_3, u_4\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( c \) is not adjacent to \( u_1 \).

Case 2.2.3. If \( w \sim u_3 \), then \( \{a, d, u_2\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \) is not adjacent to \( u_3 \).

Let \( t \sim u_3 \) such that \( t \notin \{v, d, u_2\} \).

Case 2.2.4. If \( c \sim t \), then \( \{c, u_1\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \). So \( c \) is not adjacent to \( t \).

Thus, there exists \( z \sim c \) such that \( z \notin \{d, u_4, a, u_1, u_2, w\} \) and \( z \) is not adjacent to \( u_3 \) (since \( G \) is 4-regular). See Figure 27.
Case 2.2.5. If $z \sim a$, then \{b,w\} is a cutset for $G$. So $z$ is not adjacent to $a$. Then \{a,z,u_3\} is independent and so \{a,z,u_3\} and \{u_4\} don't extend to disjoint maximum independent sets in $G$.

Hence, $w$ is not adjacent to $a$. By symmetry, $w$ is not adjacent to $d$.

Case 2.3. Suppose $a \sim d$. Then there exists $y \sim u_2$ such that $y \in \{a,b,c,d,v,w,u_1,u_3,u_4\}$.

Case 2.3.1. Suppose $a \sim y$. Then \{y,u_1\} is a cutset for $G$. So $a$ is not adjacent to $y$ and, by symmetry, $d$ is not adjacent to $y$.

Case 2.3.2. If $y \sim u_3$, then \{a,w,y\} is independent and so \{a,w,y\} and \{v\} don't extend to disjoint maximum independent sets in $G$. So $y$ is not adjacent to $u_3$ and, by symmetry, $y$ is not adjacent to $u_1$.

Thus, there exists $s \sim u_3$ such that $s \in \{a,b,c,d,w,v,y,u_1,u_2,u_4\}$.

Case 2.3.3. If $y \sim s$, then \{y,c,u_1\} and \{u_3\} don't extend to disjoint maximum independent sets in $G$. So $y$ is not adjacent to $s$.

Case 2.3.4. If $a$ is not adjacent to $s$, then \{a,y,w,s\} is independent and so \{a,y,w,s\} and \{v\} don't extend to disjoint maximum independent sets in $G$. So $a \sim s$.

Case 2.3.5. If $s \sim u_1$, then \{b,u_3\} and \{u_1\} don't extend to disjoint maximum independent sets in $G$. So $s$ is not adjacent to $u_1$.

So there exists $x \sim u_1$ such that $x \in \{a,v,u_2,y,s\}$ (see Figure 28).
Case 2.3.6. If \( y \sim x \), then \( \{y,u_3,b\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( x \); it follows that \( \{x,y,w,d\} \) is independent. Thus, \( \{x,y,w,d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( a \) is not adjacent to \( d \).

Case 2.4. If \( w \sim u_2 \), then \( \{a,d,w\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \) is not adjacent to \( a \).

Thus, there exists \( y \sim u_2 \) such that \( y \notin \{a,b,c,d,y,w,u_1,u_3,u_4\} \).

Case 2.5. If \( a \sim y \), then \( \{a,d,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \) is not adjacent to \( y \). By symmetry, \( d \) is not adjacent to \( y \).

Case 2.6. Suppose \( y \sim w \).

Case 2.6.1. Suppose \( y \sim u_1 \). If \( y \sim b \), then \( \{a,u_3,u_4\} \) and \( \{y\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( b \). Then \( \{b,d,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( y \) is not adjacent to \( u_1 \) and, by symmetry, \( y \) is not adjacent to \( u_3 \).

Case 2.6.2. If \( y \sim c \), then \( \{a,y,u_3\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( c \) and, by symmetry, \( y \) is not adjacent to \( b \).

Case 2.6.3. Consequently, \( y \) has two neighbors \( z_1 \) and \( z_2 \) such that \( \{z_1,z_2\} \cap \{a,b,c,d,w,v,u_1,u_2,u_3,u_4\} = \emptyset \). If \( a \sim z_1 \) and \( a \sim z_2 \), then \( \{u_1,z_1\} \) is a cutset for \( G \), for some \( i \). If \( d \sim z_1 \) and \( d \sim z_2 \), then \( \{u_3,z_1\} \) is a cutset for \( G \), for some \( i \). If \( z_1 \) is adjacent to neither \( a \) nor \( d \), for some \( i \), then \( \{z_1,a,d,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). Thus, without loss of generality, we can assume \( z_1 \sim a \) and \( z_2 \sim d \) (see Figure 29).

![Figure 29](image)

If \( z_1 \sim u_1 \), then \( \{b,y,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( z_1 \) is not adjacent to \( u_1 \).

Thus, there exist \( x \sim u_1 \) and \( t \sim x \) such that \( x \notin \{a,v,y,u_2,z_1\} \) and \( t \notin \{a,z_1\} \). But then \( \{t,b,u_3\} \) is independent and so \( \{t,b,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( y \) is not adjacent to \( w \); thus, the set \( \{a,y,d,w\} \) is independent. It follows that \( \{a,y,d,w\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

So the cyclic face configuration \((3,3,5,5)\) cannot occur. Therefore, the face configuration \((3,3,5,5)\) cannot occur. \( \Box \)

Lemma 12.9. Suppose \( G \) is 3-connected 4-regular planar and in \( W_2 \). If \( v \) is a point in \( G \), then \( v \) cannot have face configuration \((3,3,5,n)\), for \( n = 6 \) or \( 7 \).
Proof. Assume to the contrary that \( v \) has face configuration \((3,3,5,n)\), \( n = 6 \) or \( 7 \).
Let \( N(v) = \{u_1,u_2,u_3,u_4\} \).

Case 1. Suppose the cyclic face configuration is \((3,5,3,n)\), with faces \( u_1u_2v, u_2u_3v, u_3u_4v \) and \( u_4defcu_1v \) (\( e = f \) for the \( n = 6 \) case). By Lemma 9, \( a \) is not adjacent to \( u_1 \), \( b \) is not adjacent to \( u_1 \), \( a \) is not adjacent to \( u_4 \), \( b \) is not adjacent to \( u_4 \), \( e \) is not adjacent to \( u_2 \), \( f \) is not adjacent to \( u_2 \), \( c \) is not adjacent to \( u_3 \), \( d \) is not adjacent to \( u_3 \), \( e \) is not adjacent to \( u_3 \), \( f \) is not adjacent to \( u_3 \), \( u_2 \) is not adjacent to \( u_4 \), \( u_3 \) is not adjacent to \( u_4 \), and \( u_4 \) is not adjacent to \( u_1 \). By Lemma 10, \( u_1 \) is not adjacent to \( u_4 \), \( u_2 \) is not adjacent to \( u_4 \), \( u_3 \) is not adjacent to \( d \), \( a \) is not adjacent to \( u_3 \), \( b \) is not adjacent to \( u_2 \), \( c \) is not adjacent to \( u_4 \) and \( d \) is not adjacent to \( u_1 \).

Thus, there exists \( x \sim u_2 \) such that \( x \in \{a,b,c,d,e,f,v,u_1,u_3,u_4\} \).

Case 1.1. Suppose \( a \sim c \). Then there exists \( z \sim u_1 \) such that \( z \in \{a,b,c,d,e,f,v,u_1,u_3,u_4\} \).

Case 1.1.1. If \( a \sim z \), then \( \{a,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \) is not adjacent to \( z \).

Case 1.1.2. If \( z \sim a \) and \( z \sim u_2 \), then \( \{z,a\} \) is a cutset for \( G \). So \( z \) is not adjacent to at least one of \( c \) and \( u_2 \).

Since \( G \) is 4-regular, there exist points \( s \) and \( t \) adjacent to \( z \) such that \( \{s,t\} \cap \{a,c,u_2\} = \emptyset \). Now either \( a \) is not adjacent to \( t \) or \( a \) is not adjacent to \( s \). Say \( a \) is not adjacent to \( t \). Then \( \{a,t,u_3\} \) is independent and so \( \{a,t,u_3\} \) and \( \{u_1\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( a \) is not adjacent to \( c \). By symmetry, \( b \) is not adjacent to \( d \).

Case 1.2. Suppose \( x \sim u_3 \). Let \( t \sim b \) such that \( t \in \{a,x,u_3\} \). Then \( \{t,d,u_2\} \) is independent since \( G \) is planar. So \( \{t,d,u_2\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, \( x \) is not adjacent to \( u_3 \). Let \( y \sim u_3 \) such that \( y \in \{a,b,c,d,e,f,v,x,u_1,u_2,u_4\} \) (see Figure 30).

![Figure 30](image)

Case 1.3. Suppose \( x \sim c \). If \( b \) is not adjacent to \( c \), then \( \{b,c,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \); so \( b \sim c \). Let \( w \sim f \) such that \( w \in \{c,y\} \). Then \( \{u_2,u_3,w\} \) and \( \{c\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( x \) is not adjacent to \( c \). By symmetry, \( y \) is not adjacent to \( d \).

Case 1.4. Suppose \( b \sim x \). If \( b \) is not adjacent to \( c \), then \( \{b,c,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( b \sim c \) and \( \{a,x\} \) is a cutset for \( G \).

Thus, \( b \) is not adjacent to \( x \) and, by symmetry, \( a \) is not adjacent to \( y \).

Case 1.5. Suppose \( d \sim x \). If \( a \sim d \), then \( \{c,d,u_3\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \) is not adjacent to \( d \). Then \( \{a,c,d,y\} \) is independent and so \( \{a,c,d,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).
Thus, \(d\) is not adjacent to \(x\) and, by symmetry, \(c\) is not adjacent to \(y\).

If \(x \sim y\), then \(\{a,c,d,y\}\) is independent. So \(\{a,c,d,y\}\) and \(\{y\}\) don't extend to disjoint maximum independent sets in \(G\). Thus, \(x\) is not adjacent to \(y\) and it follows that \(\{c,d,x,y\}\) is independent. Hence, \(\{c,d,x,y\}\) and \(\{y\}\) don't extend to disjoint maximum independent sets in \(G\).

Thus, the cyclic face configuration \((3,5,3,n)\), \(n = 6\) or \(7\), is not possible.

Case 2. Suppose the cyclic face configuration is \((3,3,5,n)\), with faces \(\{u_1u_2v, u_2u_3v, u_3cdu_4v\}\) and \(u_4befau_1v\) \((e = f\) when \(n = 6\)). By Lemma 8, \(u_1\) is not adjacent to \(u_3\).

By Lemma 9, \(a\) is not adjacent to \(u_2\), \(b\) is not adjacent to \(u_2\), \(c\) is not adjacent to \(u_2\), \(d\) is not adjacent to \(u_2\), \(f\) is not adjacent to \(u_2\), and \(u_2\) is not adjacent to \(u_4\). By Lemma 10, \(a\) is not adjacent to \(u_4\), \(c\) is not adjacent to \(u_4\), \(u_1\) is not adjacent to \(u_4\), \(u_3\) is not adjacent to \(u_4\), \(a\) is not adjacent to \(b\), \(b\) is not adjacent to \(u_1\), and \(d\) is not adjacent to \(u_2\).

Thus, there exists \(y \sim u_2\) such that \(y \in \{a,b,c,d,e,f,v,u_1,u_3,u_4\}\).

Case 2.1. If \(a \sim u_3\), then \(\{d,u_1\}\) is independent and so \(\{d,u_1\}\) and \(\{u_3\}\) don't extend to disjoint maximum independent sets in \(G\). So \(a\) is not adjacent to \(u_3\) and, by symmetry, \(c\) is not adjacent to \(u_1\).

Case 2.2. Suppose \(b \sim c\). Since \(c\) is not adjacent to \(u_4\), then there exists \(w \sim u_4\) such that \(w \in \{b,c,d,v\}\). If \(w \sim c\), then \(\{w,d\}\) is a cutset for \(G\). So \(w\) is not adjacent to \(c\), and there exists \(s \sim w\) such that \(s \in \{b,c,d,u_4\}\).

Case 2.2.1. If \(c\) is not adjacent to \(s\), then \(\{c,s,u_2\}\) and \(\{u_4\}\) don't extend to disjoint maximum independent sets in \(G\). So \(c\) is not adjacent to \(s\) (see Figure 31).

Case 2.2.2. If \(w \sim b\), then let \(t \sim e\) such that \(t \neq b\). Then \(\{s,v,t\}\) and \(\{b\}\) don't extend to disjoint maximum independent sets in \(G\). So \(w\) is not adjacent to \(b\).

Thus, there exists \(z \sim w\) such that \(z \in \{b,c,d,v\}\). Then \(\{c,z,u_2\}\) is independent and so \(\{c,z,u_2\}\) and \(\{u_4\}\) don't extend to disjoint maximum independent sets in \(G\).

Therefore, \(b\) is not adjacent to \(c\).

Case 2.3. Suppose \(a \sim c\).

Case 2.3.1. Suppose \(y\) is not adjacent to \(u_1\). Let \(x \sim u_1\) such that \(x \in \{a,b,c,d,e,f,v,y,u_2,u_3,u_4\}\). If \(y \sim c\), then \(\{c,x,u_4\}\) and \(\{u_2\}\) don't extend to disjoint maximum independent sets in \(G\). So \(y\) is not adjacent to \(c\). If \(y \sim x\), then \(\{y,f,u_4\}\) and \(\{u_1\}\) don't extend to disjoint maximum independent sets in \(G\). So \(y\) is not adjacent to \(x\). If \(x \sim c\), then \(\{y,c,u_4\}\) and \(\{u_1\}\) don't extend to disjoint maximum independent sets in \(G\). So \(x\) is not adjacent to \(c\).

Thus, \(\{x,y,b,c\}\) is independent and so \(\{x,y,b,c\}\) and \(\{v\}\) don't extend to disjoint maximum independent sets in \(G\). So \(y \sim u_1\).
Case 2.3.2. If \( y \sim u_3 \), then \( \{y,d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( u_3 \).

Case 2.3.3. If \( y \sim c \), then \( \{y,u_3\} \) is a cutset for \( G \). So \( y \) is not adjacent to \( c \).

Thus, \( \{y,b,c\} \) is independent and so \( \{y,b,c\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( a \) is not adjacent to \( c \).

Case 2.4. Suppose \( b \sim u_3 \). Then there exists \( t \sim c \) such that \( t \notin \{d,u_3\} \), \( t \) is not adjacent to \( u_4 \), and \( \{t,u_1,u_4\} \) is independent. Thus, \( \{t,u_1,u_4\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \).

So \( b \) is not adjacent to \( u_3 \).

Case 2.5. If \( a \sim y \) or \( c \sim y \), then \( \{a,c,u_4\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is adjacent to neither \( a \) nor \( c \).

Case 2.6. Suppose \( b \sim y \). If \( y \sim u_4 \), then \( \{a,c,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( u_4 \) and there exists \( w \sim u_4 \) such that \( \{a,c,d,v,y,u_3\} \).

Case 2.6.1. Suppose \( y \sim w \). If \( y \sim u_3 \), then \( \{a,u_3,u_4\} \) and \( \{y\} \) don't extend to disjoint maximum independent sets in \( G \). So \( y \) is not adjacent to \( u_1 \). But then \( \{c,y,u_1\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \).

Case 2.6.2. So \( y \) is not adjacent to \( w \) (see Figure 32). If \( w \sim c \), then \( \{c,e,u_2\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \) is not adjacent to \( c \).

Then \( \{a,c,w,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( b \) is not adjacent to \( v \); it follows that \( \{a,b,c,y\} \) is independent and so \( \{a,b,c,y\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, the cyclic face configuration \( (3,3,5,n) \), \( n = 6 \) or 7, cannot occur. Therefore, the face configuration \( (3,3,5,n) \), \( n = 6 \) or 7, cannot occur.

Lemma 12.10. Suppose \( G \) is 3-connected 4-regular planar and in \( W_2 \). If \( v \) is a point in \( G \), then \( v \) cannot have face configuration \( (3,4,4,4) \).

Proof. Assume to the contrary that \( v \) has face configuration \( (3,4,4,4) \). Let \( N(v) = \{u_1,u_2,u_3,u_4\} \). Assume the faces at \( v \) are \( u_1u_2v \), \( u_2u_3v \), \( u_3u_4v \) and \( u_4u_1v \). By Lemma 9, \( a \) is not adjacent to \( u_2 \) and \( b \) is not adjacent to \( u_3 \).

If \( a \) is not adjacent to \( b \), then \( \{a,b\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \sim b \).

Let \( x \sim u_1 \), \( x \notin \{a,v,u_2\} \), and \( y \sim u_2 \), \( y \notin \{b,v,u_1\} \). If \( x = y \), then \( \{x,d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \neq y \). If \( x \) is not adjacent to \( y \), then \( \{x,y,d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \sim
y. Since \( G \) is planar, \( \{x,u_3\} \) is independent. Thus, \( \{x,u_3\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \).

Therefore, the face configuration \( (3,4,4,4) \) cannot occur.

**Lemma 12.11.** Suppose \( G \) is 3-connected 4-regular planar and in \( W_2 \). If \( v \) is a point in \( G \), then \( v \) cannot have face configuration \( (3,4,4,5) \).

**Proof.** Assume to the contrary that \( v \) has face configuration \( (3,4,4,5) \). Let \( N(v) = \{u_1,u_2,u_3,u_4\} \).

Case 1. Suppose the cyclic order of the faces is \( (3,4,5,4) \). Let the faces be \( u_1u_2v \), \( u_2bu_3v \), \( u_3cu_4v \) and \( u_4au_1v \).

By Lemma 9, \( a \) is not adjacent to \( u_2 \) and \( b \) is not adjacent to \( u_1 \). By Lemma 10, \( u_3 \) is not adjacent to \( u_4 \), \( d \) is not adjacent to \( u_3 \), and \( c \) is not adjacent to \( u_4 \).

If \( a \) is not adjacent to \( b \), then \( \{a,b\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \sim b \). Thus, there exist \( x \sim u_1 \) and \( y \sim u_2 \) such that \( \{x,y\} \cap \{a,b,c,d,v,u_1,u_2,u_3,u_4\} = \emptyset \).

If \( a \sim u_3 \), then \( \{y,a\} \) is independent and so \( \{y,a\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( a \) is not adjacent to \( u_3 \). Thus, there exists \( w \sim u_3 \) such that \( w \in \{a,b,c,d,v,x,y,u_1,u_2,u_4\} \).

If \( w \sim d \), then \( \{d,u_2\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \). So \( w \) is not adjacent to \( d \). If \( x = y \), then \( \{w,x,d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \neq y \). If \( x \) is not adjacent to \( y \), then \( \{d,x,y,w\} \) is independent since \( G \) is planar. Then \( \{d,x,y,w\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \sim y \). But then \( \{x,u_3\} \) and \( \{u_2\} \) don't extend to disjoint maximum independent sets in \( G \) (see Figure 33).

Thus, the cyclic face order \( (3,4,4,4) \) cannot occur.

Case 2. Suppose the cyclic order of the faces is \( (3,4,4,5) \). Let the faces be \( u_1u_2v \), \( u_2bu_3v \), \( u_3cu_4v \) and \( u_4au_1v \). By Lemma 9, \( u_1 \) is not adjacent to \( u_3 \) and \( a \) is not adjacent to \( u_2 \). By Lemma 10, \( b \) is not adjacent to \( u_3 \).

Suppose \( a \sim d \). Then there exist points \( z \) and \( w \) such that \( w \sim u_4 \), \( z \sim w \), \( w \in \{a,d,v\} \) and \( z \in \{a,d\} \). Since \( u_1 \) is not adjacent to \( u_3 \), then \( \{z,u_1,u_3\} \) is independent. Thus, \( \{z,u_1,u_3\} \) and \( \{u_4\} \) don't extend to disjoint maximum independent sets in \( G \). Hence, \( a \) is not adjacent to \( d \).

Suppose \( a \sim b \). Let \( x \sim u_2 \) such that \( x \in \{b,v,u_1\} \). If \( x \sim a \), then \( \{d,u_2\} \) and \( \{a\} \) don't extend to disjoint maximum independent sets in \( G \). So \( x \) is not adjacent to \( a \). But then \( \{a,d,x\} \) is independent and so \( \{a,d,x\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \). Thus, \( a \) is not adjacent to \( b \).
Suppose \( b - d \). Let \( z - u_3 \) such that \( z \in \{c, d, v\} \). From above, \( b \neq z \). If \( b - z \), then \( \{b, u_4\} \) and \( \{u_3\} \) don't extend to disjoint maximum independent sets in \( G \). So \( b \) is not adjacent to \( z \). Thus \( \{a, b, z\} \) is independent and so \( \{a, b, z\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Hence, \( b \) is not adjacent to \( d \). So \( \{a, b, d\} \) is independent. It follows that \( \{a, b, d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, the cyclic face order \( (3, 4, 4, 5) \) cannot occur. Therefore, the face configuration \( (3, 4, 4, 5) \) cannot occur.

Hence, \( b \) is not adjacent to \( d \). So \( (a, b, d) \) is independent. It follows that \( \{a, b, d\} \) and \( \{v\} \) don't extend to disjoint maximum independent sets in \( G \).

Thus, the cyclic face order \( (3, 4, 4, 5) \) cannot occur. Therefore, the face configuration \( (3, 4, 4, 5) \) cannot occur.

Now we are ready to state the main result of this paper in Theorem 13. In particular, there is only one 3-connected 4-regular planar \( W_2 \) graph.

**Theorem 13.** Suppose \( G \) is 3-connected 4-regular planar and in \( W_2 \). Then \( G \) is isomorphic to the graph in Figure 4.

**Proof.** Since \( G \) is 4-regular, then the Euler contribution for any point \( u \) in \( G \) is given by \( \phi(u) = 1 - \deg(u)/2 + \sum(1/x_i) = -1 + \sum(1/x_i) \), where the sum is taken over all faces \( F_i \) incident with \( u \) and \( x_i \) is the size of face \( F_i \). From the discussion earlier, we know that \( G \) must have a point with **positive** Euler contribution. Let \( v \) be a point in \( G \) with \( \phi(v) > 0 \).

Then \( \sum(1/x_i) > 1 \), where the sum is taken over the four faces \( F_1, F_2, F_3, F_4 \) incident with \( v \) and \( x_i \) is the size of \( F_i \), \( i = 1, 2, 3, \) or 4. The only solutions to the Diophantine inequality \( \sum(1/x_i) > 1 \) are:

- (a) \( (3, 3, 3, n) \), for \( n \geq 3 \);
- (b) \( (3, 3, 4, n) \), for \( 4 \leq n \leq 11 \);
- (c) \( (3, 3, 5, n) \), for \( 5 \leq n \leq 7 \);
- and (d) \( (3, 4, 4, n) \), for \( 4 \leq n \leq 5 \).

Thus, \( v \) must have one of the face configurations given in (a)-(d). By Lemmas 12.1 - 12.11, it follows that \( G \) must be the graph given in Figure 4.

**Open Questions**

Some questions related to the content of this paper remain open. They include the following:

1. Are there any exactly 2-connected planar 4-regular 1-well-covered graphs?
2. What are the planar 5-regular 1-well-covered graphs? The author conjectures that there are no such graphs (although there are known nonplanar 5-regular 1-well-covered graphs).
3. Can the 4-regular 1-well-covered graphs be characterized? (In a computer search on all regular graphs with at most 13 points, Royle [13] found that there are only nine 4-regular 1-well-covered graphs.)

**REFERENCES**


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