PLANAR STRONGLY WELL-COVERED GRAPHS

Michael R. Pinter *
Belmont University  Nashville, Tennessee 37212 USA

Introduction.

Plummer [11] introduced the concept of a well-covered graph in 1970. A graph is well-covered if every maximal independent set (with respect to set inclusion) in the graph is also a maximum independent set. Various subclasses of well-covered graphs have been studied (see, for example, [1] - [7], [10], and [12] - [14] ). We consider the subclass which we call strongly well-covered graphs. A strongly well-covered graph $G$ is a well-covered graph with the additional property that $G-e$ is also well-covered for every edge $e$ in $G$. By making use of (i) structural characteristics of strongly well-covered graphs and (ii) the theory of Euler contributions (for planar graphs), we show that there are only four planar strongly well-covered graphs.

Preliminaries.

From the definition, strongly well-covered graphs remain well-covered upon deletion of any edge. Well-covered graphs which remain well-covered upon deletion of any vertex (called 1-well-covered) have previously been studied by several authors (see [10], [13] and [14] ). It is interesting to note that a strongly well-covered graph fails to remain well-covered if any vertex is deleted. The following theorem is proved in [10].

Theorem 1. If $G (G \neq K_1$ or $K_2)$ is strongly well-covered, then for all vertices $v$ in $G$ the graph $G-v$ is not well-covered.

Two structural characteristics which we need are stated in the following two theorems. The proof of 3-connectedness proceeds by induction on the independence number. See [9] or [10] for proofs.

Theorem 2. If $G$ is strongly well-covered, $G \epsilon \{K_1,K_2,C_4\}$, then $\delta \geq 4$.

Theorem 3. Suppose $G$ is strongly well-covered, $G \epsilon \{K_1,K_2,C_4\}$. Then $G$ is 3-connected.

Next we state a lemma which we will frequently use later. See [91 or [10] for the proof.

* work partially supported by ONR Contracts #N00014-85-K-0488 and #N00014-91-J-1142.
Lemma 4. Suppose $G$ is well-covered. Also suppose that $S$ is an independent set and $x$ is a point in $G$ such that (i) $x \in S$ and $x \sim v$ for exactly one $v$ in $S$, and (ii) $S$ dominates $N[x]$, the closed neighborhood of $x$. Then $G-e$ is not well-covered, where $e = vx$.

Let $G_v$ be the subgraph of $G$ obtained from $G$ by deleting a vertex $v$ and all its neighbors. The next lemma states that if the vertex $a$ is isolated in the graph $G_v$, then the vertices $a$ and $v$ must have the same set of neighbors in $G$. The proof is by induction on the independence number; see [9] or [10].

Lemma 5. Suppose $G$ is connected and strongly well-covered and $v$ is a point in $G$ such that $G_v$ has an isolated point $a$. Then $N_G(a) = N_G(v)$.

Planar Strongly Well-covered Graphs.

For the remainder of this paper, we restrict ourselves to planar strongly well-covered graphs. For graphs drawn in the plane, we say two faces are adjacent if they share an edge. If a face $F$ contains vertex $v$, we say $F$ is incident to $v$. The size of a face is the number of vertices it contains. We refer to the order and sizes of the faces incident to a vertex $v$ as the face configuration at $v$.

In the next two lemmas, we consider points of degree four and five, respectively, in planar strongly well-covered graphs.

Lemma 6. Suppose $G$ is strongly well-covered planar and 3-connected. If $G$ has a point of degree four which is on a triangular face, then $G$ is the octahedron graph (see Figure 1).

![Figure 1](image-url)
Proof. Suppose $v$ is a point of degree four in $G$ and $v$ is on a triangular face. Let $N(v) = \{u_1, u_2, u_3, u_4\}$. Note that $\delta \geq 4$ by Theorem 2.

Case 1. Suppose the face configuration at $v$ is $(3,3,3,3)$. Let $u_1u_2v, u_2u_3v, u_3u_4v$ and $u_4u_1v$ be the faces.

If $u_1 \sim u_3$, then $\{u_1\}$ dominates $N[v]$. By Lemma 4, the graph $G-vu_1$ is not well-covered. This contradicts the assumption that $G$ is strongly well-covered. So $u_1$ is not adjacent to $u_3$.

Thus, there exists $w \sim u_1$ such that $w \notin \{u_2, u_3, u_4, v\}$.

If $w$ is not adjacent to $u_3$, then $\{w, u_3\}$ dominates $N[v]$, $w$ is not adjacent to $v$ and $u_3 \sim v$. This leads to a contradiction via Lemma 4. So $w \sim u_3$.

Let $z \sim u_2$ such that $z \notin \{u_1, u_3, u_4, v\}$. If $z \neq w$, then $\{z, u_4\}$ is independent and dominates $N[v]$. $z$ is not adjacent to $v$ and $u_4 \sim v$. By Lemma 4, this is a contradiction. Thus $z = w$; that is, $w \sim u_2$ and $\deg(u_2) = 4$. Similarly, $w \sim u_4$ and $\deg(u_4) = 4$. It then follows that $\deg(u_1) = 4 = \deg(u_3)$. Hence, $G$ is the graph given in Figure 1.

Case 2. Suppose the face configuration at $v$ is $(3,3,3,n)$, $n \geq 4$.

Assume the triangular faces are $u_2u_3v$, $u_3u_4v$ and $u_4u_1v$. Since $G$ is 3-connected, then $u_1$ is not adjacent to $u_2$.

If $u_1 \sim u_3$, then $\{u_3\}$ dominates $N[v]$, a contradiction by Lemma 4. So $u_1$ is not adjacent to $u_3$.

Since $\deg(u_1) \geq 4$, there exist points $a$ and $b$ adjacent to $u_1$ such that $\{a, b\} \cap \{v, u_2, u_3, u_4\} = \emptyset$.

If $a$ is not adjacent to $u_3$, then $\{a, u_3\}$ is independent and dominates $N[v]$, $a$ is not adjacent to $v$ and $u_3 \sim v$. By Lemma 4, we have a contradiction. So $a \sim u_3$ and, by symmetry, $b \sim u_3$.

Since $\deg(u_2) \geq 4$, there exists $z \sim u_2$ such that $z \notin \{v, u_3, b\}$.

Since $G$ is planar, $\{z, u_4\}$ is independent. Then $\{z, u_4\}$ dominates $N[v]$, $u_4 \sim v$ and $z$ is not adjacent to $v$, a contradiction by Lemma 4.

Thus, the face configuration $(3,3,3,n)$, $n \geq 4$, cannot occur.

Case 3. Suppose the cyclic face configuration is $(3,3,m,n)$, $m, n \geq 4$.

Assume the triangular faces are $u_2u_3v$ and $u_3u_4v$. Since $G$ is 3-connected, then $u_1$ is not adjacent to $u_2$ and $u_1$ is not adjacent to $u_4$.

If $u_1 \sim u_3$, then $\{u_3\}$ dominates $N[v]$, a contradiction by Lemma 4. So $u_1$ is not adjacent to $u_3$.

Thus, let $N(u_1) \supseteq \{v, a, b, c\}$, where $\{a, b, c\} \cap \{u_2, u_3, u_4\} = \emptyset$.

If $a$ is not adjacent to $u_3$, then $\{a, u_3\}$ is independent and dominates $N[v]$, $a$ is not adjacent to $v$ and $u_3 \sim v$. We obtain a contradiction via Lemma 4. So $a \sim u_3$; by symmetry, $b \sim u_3$, $c \sim u_3$. 

3
Without loss of generality, we can assume that $b$ is on the "outside" of cycle $au_1vu_4u_3$ and on the "outside" of cycle $u_1cu_3u_2v$ (see Figure 2). Since $\deg(u_2) \geq 4$, there exists $t - u_2$ such that $t \in \{v,c,u_3\}$. But then $\{b,t,u_4\}$ is independent and dominates $N[v]$, $u_4 \sim v$ and neither $b$ nor $t$ is adjacent to $v$. So by Lemma 4, we obtain a contradiction.

![Figure 2]

Hence, the cyclic face configuration $(3,3,m,n)$, $m,n \geq 4$, cannot occur.

Case 4. Suppose the cyclic face configuration at $v$ is $(3,m,3,n)$, $m,n \geq 4$, with triangular faces $u_1u_2v$ and $u_3u_4v$. Since $G$ is 3-connected, then $u_1$ is not adjacent to $u_4$ and $u_2$ is not adjacent to $u_3$.

Case 4.1. Suppose $u_1 \sim u_3$. If there exists $x \sim u_4$ $(x \not\in \{v,u_3\})$ such that $x$ is not adjacent to $u_1$, then $\{x,u_1\}$ is independent and dominates $N[v]$, $x$ is not adjacent to $v$ and $u_1 \sim v$. By Lemma 4, we have a contradiction. Thus, $N(u_1) \supseteq N(u_4)$. Similarly, $N(u_3) \supseteq N(u_2)$. By Lemma 5, it follows that $N(u_1) = N(u_4)$ and $N(u_3) = N(u_2)$. Since $u_1 \sim u_3$ and $G$ is planar, then $u_2$ is not adjacent to $u_4$. But $u_3 \sim u_4$, and so $N(u_3) \neq N(u_2)$, a contradiction.

Hence, $u_1$ is not adjacent to $u_3$. By symmetry, $u_2$ is not adjacent to $u_4$. Thus, there exist points $a$ and $b$ such that $a$ and $b$ are neighbors of $u_1$ and $\{a,b\} \cap \{v,u_2,u_3,u_4\} = \emptyset$.

Case 4.2. Suppose $a \sim u_2$. If $a$ is not adjacent to $u_3$, then $\{a,u_3\}$ is independent and dominates $N[v]$, a contradiction by Lemma 4. So $a \sim u_3$ and, similarly, $a \sim u_4$. By Lemma 5, it follows that $N(a) = N(v)$, and so $\deg(a) = 4$. 

4
Since \( \delta \geq 4 \) and \( G \) is planar, then \( \{u_1, u_4\} \) is a cutset for \( G \). Since \( G \) is 3-connected, we have a contradiction.

Hence, \( a \) is not adjacent to \( u_2 \). More generally, if \( x \sim u_1, x \neq v \), then \( x \) is not adjacent to \( u_2 \). By symmetry, if \( y \sim u_2, y \neq v \), then \( y \) is not adjacent to \( u_1 \). Since \( \deg(u_i) \geq 4 \) for all \( i \), there exist neighbors \( c \) and \( d \) of \( u_2 \) such that \( \{c, d\} \cap \{v, u_1\} = \emptyset \), and by the preceding sentence we note that \( \{a, b\} \cap \{c, d\} = \emptyset \).

Since \( G \) is planar, then \( x \) is not adjacent to \( y \) for some \( x \in \{a, b\}, y \in \{c, d\} \). Without loss of generality, suppose \( b \) is not adjacent to \( c \).

Case 4.3. Suppose \( c \sim u_3 \).

Case 4.3.1. If \( c \sim u_4 \), then \( \{c, u_1\} \) is independent and dominates \( N[v] \), a contradiction by Lemma 4. So \( c \) is not adjacent to \( u_4 \).

Case 4.3.2. If \( b \) is not adjacent to \( u_4 \), then \( \{b, c, u_4\} \) is independent and dominates \( N[v] \), a contradiction. So \( b \sim u_4 \).

Case 4.3.3. If \( b \sim u_3 \), then \( \{b, u_2\} \) dominates \( N[v] \), a contradiction. Thus, \( b \) is not adjacent to \( u_3 \).

Case 4.3.4. Suppose \( u_4 \sim x \) for all \( x \in N(u_1) \cup u_2 \). Then \( \{u_2, u_4\} \) is independent and dominates \( N[v] \), a contradiction by Lemma 4. So there exists \( x \sim u_1, x \neq u_2 \), such that \( x \) is not adjacent to \( u_4 \).

If \( x \) is not adjacent to \( c \), then \( \{c, x, u_4\} \) is independent and dominates \( N[v] \), a contradiction. So \( x \sim c \).

By symmetry of the points \( u_1 \) and \( u_2 \), there exists \( y \sim u_2, y \neq u_1 \), such that \( y \) is not adjacent to \( u_3 \). Since \( x \sim c \), then \( \{b, y, u_3\} \) is independent. Since \( \{b, y, u_3\} \) dominates \( N[v] \), we arrive at a contradiction via Lemma 4.

Thus, \( c \) is not adjacent to \( u_3 \) and, by symmetry, \( b \) is not adjacent to \( u_4 \).

If \( c \sim u_4 \), then \( \{b, c, u_3\} \) is independent and dominates \( N[v] \), a contradiction by Lemma 4. So \( c \) is not adjacent to \( u_4 \). By symmetry, \( b \) is not adjacent to \( u_3 \). Thus, \( \{b, c, u_3\} \) is independent and dominates \( N[v] \). We obtain a contradiction from Lemma 4.

Hence, the cyclic face configuration \((3, m, 3, n), m, n \geq 4\), cannot occur.

From Cases 1 through 4, we see that the only other possibility is that \( v \) has exactly one triangle in its face configuration.

Case 5. Suppose \( v \) has face configuration \((3, l, m, n), l, m, n \geq 4\), with \( u_1 u_2 v \) as the face triangle at \( v \). Since \( G \) is 3-connected, \( u_2 \) is not adjacent to \( u_3 \), \( u_3 \) is not adjacent to \( u_4 \) and \( u_4 \) is not adjacent to \( u_1 \).

Suppose \( u_1 \sim u_3 \). As in Case 4.1, we have \( N(u_1) \supseteq N(u_4) \).

Then by Lemma 5, it follows that \( N(u_1) = N(u_4) \). But \( u_1 \sim u_3 \) and \( u_4 \)
is not adjacent to \( u_3 \), a contradiction. Thus, \( u_1 \) is not adjacent to \( u_3 \). By symmetry, \( u_2 \) is not adjacent to \( u_4 \).

Let \( w \sim u_3 \) with \( w \in \{ v, u_1, u_2, u_4 \} \). Suppose \( w \sim u_4 \). If \( w \) is not adjacent to \( u_1 \), then \( \{ w, u_1 \} \) is independent and dominates \( N[v] \), a contradiction. So \( w \sim u_1 \) and, by symmetry, \( w \sim u_2 \). Thus, \( N(w) = N(v) \) by Lemma 5. Since \( \delta \geq 4 \) by Theorem 2, then \( \{ u_1, u_4 \} \) is a cutset for \( G \), contradicting 3-connectedness.

Hence, \( w \) is not adjacent to \( u_4 \) and so \( N(u_3) \cap N(u_4) = \{ v \} \).

Since \( G \) is planar and \( \delta \geq 4 \), then there exist points \( x \) and \( y \) such that \( x \sim u_3 \), \( y \sim u_4 \) and \( x \) is not adjacent to \( y \), where \( v \notin \{ x, y \} \). Suppose \( y \sim u_2 \). If \( y \) is not adjacent to \( u_1 \), then \( \{ x, y, u_1 \} \) is independent and dominates \( N[v] \), a contradiction. So \( y \sim u_1 \). But then \( \{ y, u_3 \} \) is independent and dominates \( N[v] \), a contradiction. So \( y \) is not adjacent to \( u_3 \). By symmetry, \( x \) is not adjacent to \( u_1 \).

If \( y \) is not adjacent to \( u_1 \), then \( \{ x, y, u_1 \} \) is independent and dominates \( N[v] \), a contradiction. So \( y \sim u_1 \) and, by symmetry, \( x \sim u_2 \).

Suppose \( z \in N(u_2) \) implies \( z \sim u_3 \). Then \( \{ u_1, u_3 \} \) dominates \( N[u_2] \), \( u_1 \sim u_2 \) and \( u_3 \) is not adjacent to \( u_2 \). By Lemma 4, we obtain a contradiction. So there exists \( z \in N(u_2) \) such that \( z \) is not adjacent to \( u_3 \).

Let \( a \) and \( b \) be neighbors of \( u_4 \) such that \( \{ a, b \} \cap \{ v, y \} = \emptyset \), and let \( c \) and \( d \) be neighbors of \( u_3 \) such that \( \{ c, d \} \cap \{ v, x \} = \emptyset \). From above, we know that \( \{ a, b, y \} \cap \{ c, d, x \} = \emptyset \) (see Figure 3).

![Figure 3](image-url)
Suppose \( a = z \) (that is, \( a \sim u_2 \)). Also suppose \( a \sim u_1 \). Since
\[ N(u_3) \cap N(u_4) = \{ v \}, \]
then \( \{ a,u_3 \} \) is independent. Also, \( \{ a,u_3 \} \)
dominates \( N[v] \). We obtain a contradiction via Lemma 4.

So \( a \) is not adjacent to \( u_1 \). Suppose \( a \sim t \) for all \( t \in N(u_3) - v \).
Then \( \{ a,v \} \) dominates \( N[u_3] \), a contradiction. So there exists some \( t \sim u_3 \), \( t \neq v \), such that \( t \) is not adjacent to \( a \). Since \( G \) is planar, then
\( \{ a,t,u_1 \} \) is independent. Since also \( \{ a,t,u_1 \} \) dominates \( N[v] \), we
obtain a contradiction via Lemma 4.

Thus, \( a \neq z \) and, by symmetry, \( b \neq z \).

Suppose there exists \( s \in \{ a,b \} \) such that \( s \sim u_1 \). Since \( G \) is
planar, then either \( s \) is not adjacent to \( z \) or \( y \) is not adjacent to \( z \). Say \( s \)
is not adjacent to \( z \). Then \( \{ s,z,u_3 \} \) is independent and dominates
\( N[v] \), a contradiction. If \( y \) is not adjacent to \( z \), then we obtain a
similar contradiction.

Thus, \( \{ s,z,u_3 \} \) implies \( s \) is not adjacent to \( u_1 \). Likewise,
\( \{ t,c,d \} \) implies \( t \) is not adjacent to \( u_2 \).

If \( y \sim c \) or \( y \sim d \), then \( x \) is not adjacent to \( a \). Then \( \{ a,x,u_1 \} \) is
independent and dominates \( N[v] \), a contradiction. So \( y \) is adjacent to
neither \( c \) nor \( d \). But then \( \{ c,y,u_2 \} \) is independent and dominates
\( N[v] \), a contradiction by Lemma 4.

Therefore, the face configuration \( (3,l,m,n) \), \( l, m, n \geq 4 \), is not
possible.

Hence if \( G \) has a point of degree four which is on a triangular
face, then \( G \) is the octahedron graph given in Figure 1.

**Lemma 7.** Suppose \( G \) is strongly well-covered planar and 3-
connected. Then \( G \) cannot have a point of degree five with face
configuration \( (3,3,3,3,n) \), \( n = 3, 4, \) or \( 5 \).

**Proof.** Suppose \( G \) has a point \( v \) with \( \deg(v) = 5 \) and face
configuration \( (3,3,3,3,n) \), \( n = 3, 4, \) or \( 5 \). Let \( N(v) = \{ u_1,u_2,u_3,u_4,u_5 \} \). Let \( U_i = N(u_i) - N[v] \), for \( i = 1, ..., 5 \). Since \( u_i \) is
in a triangle for all \( i \), then it follows from Lemma 6 that \( \deg(u_i) \geq 5 \) for
all \( i \).

Case 1. Suppose \( n = 3 \). Suppose \( u_1 \sim u_3 \). If \( u_1 \sim u_4 \), then \( \{ u_1 \}
dominates \( N[v] \). By Lemma 4, we obtain a contradiction. So \( u_1 \)
is not adjacent to \( u_4 \).

Suppose there exists \( x \sim u_4 \) such that \( x \) is not adjacent to \( u_1 \).
Then \( \{ x,u_1 \} \) is independent and dominates \( N[v] \), \( u_1 \sim v \) and \( x \) is not
adjacent to \( v \). By Lemma 4, we obtain a contradiction.
Thus, \( N(u_1) \supseteq N(u_4) \). It follows from Lemma 5 that \( N(u_1) = N(u_4) \). Since \( u_1 \sim u_3 \) and \( G \) is planar, then \( u_2 \) is not adjacent to \( u_4 \). But \( u_1 \sim u_2 \) implies \( N(u_1) \neq N(u_4) \), a contradiction.

So \( u_1 \) is not adjacent to \( u_3 \). By symmetry, \( u_1 \) is not adjacent to \( u_4 \), \( u_2 \) is not adjacent to \( u_5 \), \( u_2 \) is not adjacent to \( u_4 \), and \( u_3 \) is not adjacent to \( u_5 \).

Case 1. Suppose \( U_3 \cap U_4 \neq \emptyset \). Let \( a \in U_3 \cap U_4 \). If \( a \) is not adjacent to \( u_1 \), then \( \{a, u_1\} \) is independent and dominates \( N[v] \), a contradiction. So \( a \sim u_1 \).

Case 1.1. Suppose \( a \sim u_2 \). If \( a \) is not adjacent to \( u_5 \), then \( \{a, u_5\} \) is independent and dominates \( N[v] \), a contradiction; so \( a \sim u_5 \).

Suppose \( x \in U_3 \) implies \( x \sim u_4 \) (that is, \( U_4 \supseteq U_3 \)). Then \( \{u_1, u_4\} \) dominates \( N[u_3] \), \( u_1 \) is not adjacent to \( u_3 \) and \( u_4 \sim u_3 \). By Lemma 4, we obtain a contradiction. Thus, there exists \( x \in U_3 \) such that \( x \) is not adjacent to \( u_4 \). Similarly, there exists \( y \in U_4 \) such that \( y \) is not adjacent to \( u_3 \).

If \( y \) is not adjacent to \( x \), then \( \{x, y, u_1\} \) is independent and dominates \( N[v] \), a contradiction. So \( y \sim x \) (see Figure 4). Since \( \deg(u_2) \geq 5 \), there exists \( t \sim u_2 \) such that \( t \in \{u_1, u_3, a, v\} \). In particular, \( \{t, x, u_5\} \) is independent. Since \( \{t, x, u_5\} \) also dominates \( N[v] \), we obtain a contradiction by Lemma 4.

![Figure 4](image_url)

Case 1.1.2. Thus, \( a \) is not adjacent to \( u_2 \). By symmetry, \( a \) is not adjacent to \( u_5 \). Suppose \( x \in U_2 \) implies \( x \sim a \). Then \( \{a, v\} \) dominates
N[u_2], v \sim u_2 and a is not adjacent to u_2. By Lemma 4, we obtain a contradiction.

Thus, there exists x \in U_2 such that x is not adjacent to a. But then \{a,x,u_3\} is independent and dominates N[v], a contradiction.

Case 1.2. Hence, U_3 \cap U_4 = \emptyset. By symmetry, U_j \cap U_{j+1} = \emptyset, for all i (addition mod 5). Since G is planar and deg(u_i) \geq 5 for all i, then there exist x \sim u_4 and y \sim u_3 such that x is not adjacent to y.
Suppose x \sim u_1. If x \sim z for all z \in U_5, then \{x,v\} is independent and dominates N[u_5], v \sim u_5 and x is not adjacent to u_5. By Lemma 4, we obtain a contradiction. Thus, there exists z \in U_5 such that x is not adjacent to z. But then \{x,z,u_3\} is independent and dominates N[v], a contradiction.

So x is not adjacent to u_1. By symmetry, y is not adjacent to u_1. Thus, \{x,y,u_1\} is independent and dominates N[v], a contradiction.

So n = 3 is not possible.

Case 2. Suppose n = 4. Let the 4-face at v be vu_4au_5. If a is not adjacent to u_2, then \{a,u_2\} is independent and dominates N[v], a contradiction. So a \sim u_2.

Suppose a \sim u_1. If a is not adjacent to u_3, then \{a,u_3\} is independent and dominates N[v], a contradiction. So a \sim u_3. Since deg(u_i) \geq 5 for all i, there exist x \sim u_4 such that x \in \{a,v,u_3\} and y \sim u_5 such that y \in \{a,v,u_1\}. Then \{x,y,u_2\} is independent and dominates N[v], a contradiction. Thus, a is not adjacent to u_1. By symmetry, a is not adjacent to u_3.

Suppose x \in U_3 implies x \sim a. Then \{a,v\} dominates N[u_3], v \sim u_3 and a is not adjacent to u_3. By Lemma 4, we have a contradiction. So there exists x \in U_3 such that x is not adjacent to a. But then \{a,x,u_1\} is independent (since G is planar) and dominates N[v], a contradiction.

Hence, n = 4 is not possible.

Case 3. Suppose n = 5. Let the 5-face at v be vu_4abu_5. Since G is 3-connected, then u_4 is not adjacent to u_5, b is not adjacent to u_4 and a is not adjacent to u_5.

Suppose u_4 and u_5 have a common neighbor w, w \neq v. If w is not adjacent to u_2, then \{w,u_2\} is independent and dominates N[v], a contradiction. So w \sim u_2. Since deg(u_3) \geq 5, there exists x \in U_3 such that x \neq w. Since G is planar, \{a,x,u_1\} is independent. Thus, \{a,x,u_1\} is independent and dominates N[v], a contradiction.

Hence, u_4 and u_5 don't have a common neighbor w, w \neq v.
Suppose \( u_1 \sim u_3 \). If \( u_1 \) is not adjacent to \( a \), then \( \{a,u_1\} \) is independent and dominates \( N[v] \), a contradiction. So \( u_1 \sim a \). But then \( \{b,u_3\} \) is independent and dominates \( N[v] \), a contradiction. Thus, \( u_1 \) is not adjacent to \( u_3 \).

Suppose \( a \sim u_2 \). Then \( \{b,u_3\} \) is independent. If \( b \sim u_1 \), then \( \{b,u_3\} \) dominates \( N[v] \), a contradiction. So \( b \) is not adjacent to \( u_1 \).

Suppose \( x \in U_1 \) implies \( x \sim b \). Then \( \{b,v\} \) dominates \( N[u_1] \), \( v \sim u_1 \) and \( b \) is not adjacent to \( u_1 \). By Lemma 4, we obtain a contradiction. Thus, there exists \( x \in U_1 \) such that \( x \) is not adjacent to \( b \). But then \( \{b,x,u_3\} \) is independent and dominates \( N[v] \), a contradiction.

Hence, \( a \) is not adjacent to \( u_2 \); by symmetry, \( b \) is not adjacent to \( u_2 \).

Suppose \( u_2 \sim u_4 \). If \( b \) is not adjacent to \( u_2 \), then \( \{b,u_2\} \) is independent and dominates \( N[v] \), a contradiction. So \( b \sim u_2 \). Let \( z \sim u_3 \) such that \( z \notin \{u_2,u_4,v\} \). Since \( G \) is planar, then \( \{a,z,u_1\} \) is independent. Since \( \{a,z,u_1\} \) dominates \( N[v] \), we arrive at a contradiction via Lemma 4.

So \( u_2 \) is not adjacent to \( u_4 \); by symmetry, \( u_2 \) is not adjacent to \( u_5 \).

Suppose \( x \in N(u_4) - a \) implies \( x \sim u_2 \). Then \( \{a,u_2\} \) dominates \( N[u_4] \), \( a \sim u_4 \) and \( u_2 \) is not adjacent to \( u_4 \). By Lemma 4, we obtain a contradiction. So there exists \( x \sim u_4 \), \( x \neq a \), such that \( x \) is not adjacent to \( u_2 \). Similarly, there exists \( y \sim u_5 \), \( y \neq b \), such that \( y \) is not adjacent to \( u_2 \). From above, \( x \neq y \). See Figure 5.

![Figure 5](image)

Suppose \( x \sim y \). Since \( G \) is planar, then either \( x \) is not adjacent to \( b \) or \( y \) is not adjacent to \( a \). Without loss of generality, assume \( x \) is not
adjacent to b. Then \( \{b, x, u_2\} \) is independent and dominates \( N[v] \), a contradiction from Lemma 4.

Thus, \( x \) is not adjacent to \( y \). Then \( \{x, y, u_2\} \) is independent and dominates \( N[v] \), a contradiction via Lemma 4.

Hence, \( n = 5 \) is not possible.

Thus, \( G \) cannot have a point \( v \) with \( \deg(v) = 5 \) and face configuration \((3, 3, 3, 3, n)\), \( n = 3, 4 \) or 5.

Lebesgue [8] developed the theory of Euler contributions for planar graphs. The Euler contribution of a vertex \( v \), \( \phi(v) \), is defined as the quantity \( \phi(v) = 1 - \frac{1}{2}\deg(v) + \sum(1/x_i) \), where the sum is taken over all faces \( F_i \) incident to \( v \) and \( x_i \) is the size of \( F_i \). If \( |F(G)| \) denotes the number of faces in the plane graph \( G \), then it follows that

\[
\Sigma_v \phi(v) = |V(G)| - |E(G)| + |F(G)|.
\]

Here the sum is taken over all vertices \( v \) in \( G \). Since Euler's formula for plane graphs says \( |V(G)| - |E(G)| + |F(G)| = 2 \), then we have \( \Sigma_v \phi(v) = 2 \). Thus, \( \phi(v) \) must be positive for some \( v \) in \( G \). From the definition of \( \phi(v) \), it follows easily that \( \phi(v) \leq 0 \) whenever \( \deg(v) \geq 6 \). Thus, if \( \phi(v) > 0 \), then \( \deg(v) \leq 5 \).

As a consequence of the two previous lemmas and the theory of Euler contributions, we find all 3-connected planar strongly well-covered graphs in the following theorem.

**Theorem 8.** Suppose \( G \) is strongly well-covered planar and 3-connected. Then \( G \) is the octahedron graph shown in Figure 1.

**Proof.** From Theorem 2, \( \delta \geq 4 \). Suppose \( v \) is a point in \( G \) with positive Euler contribution; that is, \( \phi(v) > 0 \). Then \( \deg(v) = 4 \) or 5.

If \( \deg(v) = 4 \), then \( \phi(v) = 1 - (1/2)(4) + \sum(1/x_i) = -1 + \sum(1/x_i) \), where the sum is taken over all faces incident to \( v \). For \( \phi(v) \) to be positive, \( \sum(1/x_i) \) must be greater than 1. Thus, \( v \) must lie on a triangular face in order for \( \phi(v) \) to be positive. From Lemma 6, this can only occur if \( G \) is the graph given in Figure 1.

If \( \deg(v) = 5 \), then \( \phi(v) = 1 - (1/2)(5) + \sum(1/x_i) = -3/2 + \sum(1/x_i) \), where the sum is taken over all faces incident to \( v \). For \( \phi(v) \) to be positive in this case, \( \sum(1/x_i) \) must be greater than \( 3/2 \). Thus, \( v \) must
have a face configuration of the form (3,3,3,3,n), n = 3, 4 or 5. But from Lemma 7, this cannot occur.

From Theorem 3, we know that all strongly well-covered graphs on more than four points are 3-connected. Thus, we conclude in the following corollary that there are exactly four planar strongly well-covered graphs.

**Corollary 9.** The only planar strongly well-covered graphs are $K_1$, $K_2$, $C_4$ and the octahedron graph shown in Figure 1.

**References.**