ON CONSTRUCTING SOME STRONGLY WELL-COVERED GRAPHS

by

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Abstract

A graph is well-covered if every maximal independent set is a maximum independent set. If a well-covered graph $G$ has the additional property that $G-e$ is also well-covered for every line $e$ in $G$, then we say the graph is strongly well-covered. We exhibit a construction which produces strongly well-covered graphs with arbitrarily large (even) independence number. The construction is in terms of a lexicographic graph product.
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INTRODUCTION

A set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$. A set of independent points which attains the maximum size is referred to as a maximum independent set. A set $S$ of independent points in a graph is maximal (with respect to set inclusion) if the addition to $S$ of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [13] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a well-covered graph. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Campbell [2] characterized all cubic well-covered graphs with connectivity at most two, and Campbell and Plummer [3] proved that there are only four 3-connected cubic planar well-covered graphs. Royle and Ellingham [16] have recently completed the picture for cubic well-covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph are called very well-covered graphs. The subclass of very well-covered graphs was characterized by Staples [17] and includes all well-covered trees and all well-covered bipartite graphs. Independently, Ravindra [14] characterized bipartite well-covered graphs and Favaron [6] characterized the very well-covered graphs. Recently, Dean and Zito [4] characterized the very well-covered graphs as a subset of a more general (than well-covered) class of graphs.
A set $S$ of points in a graph dominates a set $V$ of points if every point in $V-S$ is adjacent to at least one point of $S$. Finbow and Hartnell [7] and Finbow, Hartnell, and Nowakowski [8] studied well-covered graphs relative to the concept of dominating sets. Finbow, Hartnell, and Nowakowski have also obtained a characterization of well-covered graphs with girth at least five [9].

A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph is strongly well-covered if and only if the deletion of any line from the graph leaves a graph which is also well-covered. A well-covered graph is in the class $W_2$ if and only if any two disjoint independent sets in the graph can be extended to disjoint maximum independent sets. Staples [18] showed that a well-covered graph is 1-well-covered if and only if it is in $W_2$. For the remainder of this paper, we use the $W_2$ nomenclature instead of referring to 1-well-covered graphs.

The class of well-covered graphs contains all complete graphs and all complete bipartite graphs of the form $K_{n,n}$. The only cycles which are well-covered are $C_3$, $C_4$, $C_5$, and $C_7$. We note that all complete graphs (except $K_1$) are also in $W_2$, but no complete bipartite graphs (except $K_{1,1}$) are in $W_2$. The cycles $C_3$ and $C_5$ are the only cycles in $W_2$. Also note that the only complete graphs which are strongly well-covered are $K_1$ and $K_2$, the only complete bipartite graphs which are strongly well-covered are $K_{1,1}$ and $K_{2,2}$, and $C_4$ is the only strongly well-covered cycle.

In [12], we show that a strongly well-covered graph with more than four points has minimum degree at least four and is 3-connected. Also, we show that all strongly well-covered graphs other than $K_1$ and $K_2$ have girth at most four, where the girth of a graph is the size of a smallest cycle in the graph and a graph with no cycles has infinite girth. In this paper we construct strongly well-covered graphs with triangles and strongly well-covered graphs with girth four.
PRELIMINARY RESULTS

Unless otherwise stated, we assume all graphs are connected. Note that a disconnected graph is a $W_2$ graph (strongly well-covered graph) if and only if each of its components is a $W_2$ graph (strongly well-covered graph). For notation and terminology not defined here, see [1].

For a point $v$ in a graph $G$, let $N[v] = N(v) \cup \{v\}$. Define $G_v$ to be the graph induced by $G-N[v]$. In other words, $G_v$ is the graph that remains after deleting $v$ and all of its neighbors. In [12], the author shows that if $G$ is a strongly well-covered graph and $G$ is not complete, then for all points $v$ in $G$, the graph $G_v$ cannot contain a component which is a line. Campbell and Plummer [3] proved the following very useful necessary condition for a graph to be well-covered. We will use this later to verify a construction.

**Theorem 1.** If a graph $G$ is well-covered and is not complete, then $G_v$ is well-covered for all $v$ in $G$. Moreover, $\alpha(G_v) = \alpha(G) - 1$.

Recall from earlier that if $G$ is a $W_2$ graph, then for all points $v$ the graph $G-v$ is well-covered (since a $W_2$ graph is 1-well-covered). On the other hand, we show in [12] that strongly well-covered is a sufficient condition for $G$ to have the property that for all points $v$ the graph $G-v$ is not well-covered. We state this here as Theorem 2. As a consequence, $K_2$ is the only strongly well-covered graph which is also a $W_2$ graph.

**Theorem 2.** If $G$ ($G \neq K_1$ or $K_2$) is strongly well-covered, then for all points $v$ in $G$ the graph $G-v$ is not well-covered.
Next, we state the characterization of the strongly well-covered graphs with independence number two, as given in [12]. This characterization will be quite helpful in building strongly well-covered graphs with independence number larger than two.

**Theorem 3.** Suppose $G$ is well-covered with $\alpha(G) = 2$. Then $G$ is strongly well-covered if and only if $G$ is $(|V(G)| - 2)$-regular.

If $G \neq K_2$ is well-covered and $e = uv$ is a line in $G$, consider maximal independent sets in the graph $G-e$. Suppose $J$ is a maximal independent set in $G-e$ which does not contain at least one endpoint of $e$ (that is, $J \cap \{u,v\} \neq \{u,v\}$). Then it follows that $J$ is a maximal independent set in $G$. Since $G$ is well-covered, then $|J| = \alpha(G)$. Thus, every maximal independent set in $G-e$ which does not contain at least one endpoint of $e$ has size $\alpha(G)$. Consequently, to show that $G-e$ is well-covered it suffices to show that every maximal independent set in the graph $G-e$ which contains both endpoints of $e$ has size $\alpha(G)$.

**A CONSTRUCTION**

For our construction, we use a product of well-covered graphs. Suppose $H$ is a graph with $n$ points and $\{G_i\}, i = 1, \ldots, n$, is a family of disjoint graphs. Associate one member of $\{G_i\}$ with each point of $H$. We assume $V(H) = \{v_1, \ldots, v_n\}$ and $G_i$ is associated with $v_i$ for all $i$. We define the lexicographic product graph of $H$ and $\{G_i\}$, denoted $H \circ (G_1, \ldots, G_n)$, as follows: $V( H \circ (G_1, \ldots, G_n) ) = V(G_1) \cup \ldots \cup V(G_n)$ and $E(H \circ (G_1, \ldots, G_n) ) = E(G_1) \cup \ldots \cup E(G_n) \cup \{xy: x \in V(G_i), y \in V(G_j) \text{ and } v_i - v_j \text{ in } H\}$.

If every member of the family $\{G_i\}$ is the same graph $G$, then the lexicographic product consists of replacing each point of $H$ with a copy of the graph $G$ and joining the
copies as indicated above. In this special case, we denote the lexicographic product by $H \circ G$.

Topp and Volkmann [19] considered several different types of products of well-covered graphs. In particular for the lexicographic product of well-covered graphs, they proved a theorem which implies the following theorem.

**Theorem 4.** If $H$ is well-covered and \{G$_i$\}, $i = 1, \ldots, |V(H)|$, is a family of well-covered graphs with $\alpha(G_i) = \alpha(G_j)$ for all $i$ and $j$, then $H \circ (G_1, \ldots, G_{|V(H)|})$ is well-covered. Moreover, $\alpha(H \circ (G_1, \ldots, G_{|V(H)|})) = \alpha(H) \alpha(G_1)$.

In the next theorem, we give an additional condition on a well-covered graph $H$ which is sufficient to obtain a strongly well-covered lexicographic product graph.

**Theorem 5.** Suppose $H$ is a well-covered graph with the following additional property: if $e = uv$ is a line in $H$, then $H_{uv}$ is a well-covered graph and $\alpha(H_{uv}) = \alpha(H) - 1$, where $H_{uv}$ is defined to be the graph $H - (N[u] \cup N[v])$.

Let $|V(H)| = n$ and $\{G_i\}, i = 1, \ldots, n$, be a family of strongly well-covered graphs with $\alpha(G_i) = 2$ for all $i$ and each $G_i$ is connected or $2K_1$. Let $L = H \circ (G_1, \ldots, G_n)$. Then $L$ is strongly well-covered, and $\alpha(L) = 2\alpha(H)$.

**Proof.** By Theorem 4, the lexicographic product graph $L$ is well-covered and $\alpha(L) = 2\alpha(H)$. Let $V(H) = \{u_1, u_2, \ldots, u_n\}$. Note the following about the structure of the lexicographic product graph: $V(L)$ is the union of $V(G_i)$, for $i = 1, \ldots, n$. If $u_i \sim u_j$ in $H$, then $x \sim y$ in $L$ for all $x \in V(G_i)$, for all $y \in V(G_j)$. If $u_i$ is not adjacent to $u_j$ in $H$ ($i \neq j$), then $x$ is not adjacent to $y$ in $L$ for all $x \in V(G_i)$, for all $y \in V(G_j)$. Also, $a \sim b$ in $G_i$ if and only if $a \sim b$ in $L$, for all $a$ and $b$ in $V(G_i)$.

We proceed to show that $L$ is strongly well-covered. Suppose $e$ is a line in $L$. Then either $e$ corresponds to a line in $H$, or $e$ corresponds to a line in some $G_j$. 
Case 1. Suppose $e = xy$ corresponds to a line in $G_j$, for some $j$. Since $G_j$ is strongly well-covered with $\alpha(G_j) = 2$, then $\{x,y\}$ is a maximum independent set in the graph $G_j-e$. We consider the graph $L-e$.

To this end, consider the graph $H_{u_j} = H - N[u_j]$ (a subgraph of $H$). By Theorem 1, graph $H_{u_j}$ is well-covered and $\alpha(H_{u_j}) = \alpha(H) - 1$. Let $S_j$ be the subgraph of $L$ corresponding to the components of $H_{u_j}$. Observe that $S_j$ is a lexicographic product graph itself. Then since $H_{u_j}$ is well-covered, by Theorem 4 the graph $S_j$ is well-covered and $\alpha(S_j) = 2\alpha(H_{u_j}) = 2(\alpha(H) - 1)$.

Suppose $J$ is a maximal independent set in $L-e$ such that $J \supseteq \{x,y\}$. Since $\{x,y\}$ is a maximum independent set in $G_j-e$, then $J' = J - \{x,y\}$ must be contained in $S_j$. Since $S_j$ is well-covered, each component of $S_j$ is well-covered and it follows that $|J'| = \alpha(S_j) = 2(\alpha(H) - 1)$. Thus, $|J| = 2\alpha(H)$. So a maximal independent set in $L-e$ which contains the endpoints of $e$ has size $2\alpha(H)$. Thus, every maximal independent set in $L-e$ has size $2\alpha(H)$ and hence is a maximum independent set in $L-e$. Therefore, $L-e$ is well-covered.

Case 2. Suppose $e$ corresponds to the line $u_iu_j$ in $H$. Say $e = xy$, where $x \in V(G_i)$ and $y \in V(G_j)$.

By hypothesis, $H_{u_j}$ is well-covered and $\alpha(H_{u_iu_j}) = \alpha(H) - 1$. Suppose $J \supseteq \{x,y\}$ is maximal independent in $L-e$. Let $S_{ij}$ be the subgraph of $L$ corresponding to $H_{u_iu_j}$. Observe that $S_{ij}$ is a lexicographic product graph itself. Since $H_{u_iu_j}$ is well-covered, then by Theorem 4 the graph $S_{ij}$ is well-covered with $\alpha(S_{ij}) = 2\alpha(H_{u_iu_j}) = 2(\alpha(H) - 1)$. Let $J' = J - \{x,y\}$. Then $J'$ is contained in $S_{ij}$ and is maximal independent in $S_{ij}$. Thus, $|J'| = 2(\alpha(H) - 1)$, and so $|J| = 2\alpha(H)$. Hence, a maximal independent set in $L-e$ which contains $\{x,y\}$ necessarily has size $2\alpha(H)$. Since $L$ is well-covered, then every maximal independent set in $L-e$ has size $2\alpha(H)$. Thus, $L-e$ is well-covered.

From Cases 1 and 2, we conclude that $L-e$ is well-covered for all lines $e$ in $L$. Therefore, $L$ is strongly well-covered.
Note in Theorem 5 that $G_i$ is allowed to be disconnected. In this case, $G_i$ must be $2K_1$ since $\alpha(G_i) = 2$, the graphs $K_1$ and $K_2$ are the only complete strongly well-covered graphs, and from above, for every point $v$ in $G$, the graph $G_v$ cannot contain a component which is a line.

Although the condition in Theorem 5 is very restrictive, there are well-covered graphs which satisfy the condition and, hence, lead to the construction of infinite families of strongly well-covered graphs. We now give five such infinite families based on the five well-covered graphs shown in Figure 1.

**Corollary 6.** Suppose $H$ is one of the five graphs in Figure 1 and $\{G_i\}$, $i = 1, \ldots, |V(H)|$, is a family of strongly well-covered graphs with $\alpha(G_i) = 2$ and each $G_i$ is connected or $2K_1$. Then $H_0(G_1, \ldots, G_{|V(H)|})$ is strongly well-covered.

**Proof.** If $H$ is one of the five graphs in Figure 1, it can be shown that $H$ is well-covered, and for any line $uv$ in $H$, the graph $H_{uv} = H - (N[u] \cup N[v])$ is well-covered with $\alpha(H_{uv}) = \alpha(H) - 1$. By Theorem 5, it follows that $H_0(G_1, \ldots, G_{|V(H)|})$ is strongly well-covered.
We stated earlier that a strongly well-covered graph has girth at most four. From the following corollary, we are assured of the existence of strongly well-covered graphs with girth exactly four.

Corollary 7. If $H$ is a triangle-free well-covered graph which satisfies the conditions in Theorem 5, then $H \circ 2K_1$ is a girth 4 strongly well-covered graph.

**Proof.** If $H$ is triangle-free, then $H \circ 2K_1$ is also triangle-free. Clearly $H \circ 2K_1$ has 4-cycles. The result then follows immediately from Theorem 5.

For example, the graph in Figure 2 is $C_5 \circ 2K_1$. This graph was found by Royle [15] with the aid of a computer, and independently by the author.

![Figure 2](image_url)

**STRONGLY WELL-COVERED GRAPHS VIA $W_2$ GRAPHS OF GIRTH FOUR**

From the graphs given in Figure 1, we can construct strongly well-covered graphs with $\alpha \leq 8$. In order to construct strongly well-covered graphs with arbitrarily large independence number, we turn to the family of $W_2$ graphs of girth 4. First, we prove the following lemma about $W_2$ graphs of girth 4, which will allow us to use Theorem 5 to construct families of strongly well-covered graphs.
Lemma 8. Suppose $H$ is a $W_2$ graph of girth 4 and $e = uv$ is a line in $H$. Let $H_{uv}$ be the graph $H - (N[u] \cup N[v])$. Then $H_{uv}$ is well-covered and $\alpha(H_{uv}) = \alpha(H) - 1$.

Proof. Suppose $e = uv$ is a line in $H$. Let $U = N(u) - v$ and $V = N(v) - u$. Since $H$ has no triangles, then $U \cap V = \emptyset$.

Suppose $J$ is a maximal independent set in the graph $H_{uv}$. Clearly $|J| < \alpha(H)$. We wish to show that $|J| = \alpha(H) - 1$. We assume to the contrary that $|J| < \alpha(H) - 1$.

If $J$ dominates $V$, then $J \cup \{u\}$ is maximal independent in $H$. Since $|J \cup \{u\}| < \alpha(H)$ and $H$ is well-covered, we have a contradiction. Thus, $J$ does not dominate $V$.

Hence, there exists a point $y$ such that $y \in V$ and $J$ does not dominate $y$ (see Figure 3).

![Figure 3](image)

Note that $N(y) - v$ is contained in $V(H_{uv}) \cup U$, since $H$ has no triangles. Therefore, $(J \cup \{u\}) \cap N(y) = \emptyset$, $J \cup \{u\}$ is independent, and $J \cup \{u\}$ dominates $N(y)$. It follows that $J \cup \{u\}$ and $\{y\}$ are disjoint independent sets in $H$ which cannot be extended to disjoint maximum independent sets in $H$, and so $H$ is not in $W_2$. This contradicts our hypothesis.

Thus, $|J| = \alpha(H) - 1$. Therefore, every maximal independent set in $H_{uv}$ has size $\alpha(H) - 1$. It follows that $H_{uv}$ is well-covered and $\alpha(H_{uv}) = \alpha(H) - 1$. \[\square\]
In [10] and [11], the author presents constructions which yield $W_2$ graphs of girth four with arbitrarily large independence number. Based on the $W_2$ graphs obtained from these constructions, we show in the following theorem that we can construct infinite families of strongly well-covered graphs with arbitrarily large (even) independence number.

**Theorem 9.** Suppose $H$ is a $W_2$ graph of girth 4 with $n$ points, and \{${G_i}$, $i = 1, ..., n$, is a family of strongly well-covered graphs with $\alpha(G_i) = 2$ and $G_i$ is connected or $2K_1$, for all $i$. Then the lexicographic product graph $H \circ (G_1, ..., G_n)$ is a strongly well-covered graph, and $\alpha( H \circ (G_1, ..., G_n) ) = 2\alpha(H)$.

**Proof.** From Lemma 8, if $e = uv$ is a line in $H$, then the graph $H_{uv}$ is well-covered and $\alpha(H_{uv}) = \alpha(G) - 1$. Thus, the graph $H$ satisfies the additional condition required of a well-covered graph in Theorem 5. It follows by Theorem 5 that $H \circ (G_1, ..., G_n)$ is strongly well-covered and $\alpha( H \circ (G_1, ..., G_n) ) = 2\alpha(H)$.

Recall that a $W_2$ graph $H$ has the property that for all points $v$ in $H$, the graph $H-v$ is well-covered, and a strongly well-covered $G$ has the property that for all points $v$ in $G$, the graph $G-v$ is not well-covered. Given this disparity between the two types of well-covered graphs, it is perhaps surprising that the lexicographic product of a $W_2$ graph and a family of strongly well-covered graphs as produced in Theorem 9 will yield a strongly well-covered graph.

If $H$ is a $W_2$ graph of girth 4, then $H \circ 2K_1$ is strongly well-covered by Theorem 9. Clearly, $H \circ 2K_1$ has girth 4. Since there are infinitely many $W_2$ graphs of girth 4, it follows that there are infinitely many girth 4 strongly well-covered graphs.

The graphs given in Figure 4 are the strongly well-covered lexicographic product graphs $H_1 \circ 2K_1$ and $H_2 \circ 2K_1$, where $H_1$ and $H_2$ are planar $W_2$ graphs of girth 4 with eight points and eleven points, respectively (see [10] for a discussion of planar $W_2$ graphs of.
girth 4). Each of these graphs has points with degree four. Hence, the lower bound of four for the minimum degree in a strongly well-covered graph (mentioned above) is sharp.
A line in a graph $G$ is a **critical** line if its removal increases the independence number. A **line-critical** graph is a graph with only critical lines. Staples proved in [17] that a triangle-free $W_2$ graph is line-critical.

In searching for well-covered graphs $H$ such that $H_0(G_1, ..., G_{IV(H)})$ is strongly well-covered, for an appropriate family of graphs $(G_i)$, we discovered the following necessary condition on $H$.

**Theorem 10.** Suppose $(G_i), i = 1, ..., n$, is a family of strongly well-covered graphs with $\alpha(G_i) = 2$, for all $i$. If $H$ is a well-covered graph on $n$ points and $H_0(G_1, ..., G_n)$ is strongly well-covered, then $H$ is line-critical.

**Proof.** Assume to the contrary that $e = uv$ is not a critical line in $H$. Thus, $\alpha(H-e) = \alpha(H)$. Let $L = H_0(G_1, ..., G_n)$. Let $e' = u,v_j$ be a line in $L$ corresponding to the line $e$ in $H$, with $u \in V(G_i), v_j \in V(G_j)$ ($i \neq j$). Since $\alpha(H-e) = \alpha(H)$, then there exists a maximal independent set $J$ in $H-e$ which contains $(u,v)$ such that $|J| \leq \alpha(H)$. So $J-\{u,v\}$ dominates $H_{uv}$ and is contained in $V(H_{uv})$. For $x \in J-\{u,v\}$, we have $x \in V(G_m)$ for some $m, m \neq \{i,j\}$. Since $\alpha(G_m) = 2$, there exists maximum independent set $I_x \supseteq \{x\}$ in $G_m$ with $|I_x| = 2$. Let $I = \bigcup \{I_x: x \in J-\{u,v\}\}$. So $I$ is in $V(L)$. Since $\alpha(H) = 2$ and $|I_x| = 2$, then $|I| \leq 2(2) = 4$. But then $I \cup \{u,v\}$ is maximal independent in $L-e'$, and $|I| \cup \{u,v\} \leq 2 \alpha(H) - 2 < 2 \alpha(H)$. Since $\alpha(L) = 2 \alpha(H)$ by Theorem 4 and $L$ is assumed to be strongly well-covered, we have a contradiction. 

However, if $H$ is line-critical, then $H_0(G_1, ..., G_{IV(H)})$ is not necessarily strongly well-covered. In fact, being line-critical and in $W_2$ are not **sufficient** conditions to ensure that $H_0(G_1, ..., G_{IV(H)})$ is strongly well-covered. If $H$ is the line-critical $W_2$ graph in Figure 5, then $H_02K_1$ is not strongly well-covered. Note that the graph $H_{uv} = H-(N[u] \cup N[v])$ is not well-covered.
REFERENCES


