AD-A239 113

National Defence / Défense nationale

DEFENSIO PER SCIENTIAM

# FACTORIZATION OF THE DISCRETE NOISE COVARIANCE MATRIX FOR PLANS (U)

by

J. Chris McMillan





**DEFENCE RESEARCH ESTABLISHMENT OTTAWA**
REPORT NO. 1071

Canada

February 1991
Ottawa

National Défense
Defence nationale

# FACTORIZATION OF THE DISCRETE NOISE COVARIANCE MATRIX FOR PLANS(U)

by

**J. Chris McMillan**
*Navigation and Integrated System Section*
*Electronics Division*

## DEFENCE RESEARCH ESTABLISHMENT OTTAWA
REPORT NO. 1071

PCN
041LJ

February 1991
Ottawa

# ABSTRACT

The PLANS (Primary Land Arctic Navigation System), developed at DREO, optimally integrates a directional gyro/gyrocompass, an odometer, a 3-axis strapdown magnetometer, a GPS receiver, a Transit receiver, a baroaltimeter and a digital terrain elevation map, for the purpose of navigating a land vehicle in the Canadian Arctic under potentially adverse conditions. This report derives the exact form of the discrete driving noise covariance matrix $Q_k$ which is needed to propagate the covariance matrix in the Kalman filter used by PLANS. It is shown that the exact $Q_k$ does not have a Cholesky $UDU^T$ decomposition. However, a good approximation is shown to have the necessary decomposition for use in the Biermann-Agee-Turner formulation of the Kalman filter. This approximate decomposition is then found. A general result on the preservation of block diagonal form under $UDU^T$ decomposition is also proven.

# RÉSUMÉ

Le système de navigation terrestre PLANS, conçu au CRDO, intègre de façon optimale un gyroscope à deux modes (gyrocompas et directionnel), un odomètre, une sonde magnétique à trois axes, un récepteur GPS, un récepteur TRANSIT, un altimètre barométrique ainsi qu'un carte d'élévation numérique. PLANS a été conçu pour opérer dans l'arctique canadien à bord de véhicules terrestres. Ce rapport présente la formulation exacte de la matrice de covariance $Q_k$ nécessaire pour la propagation de la matrice de covariance du filtre Kalman utilisé par PLANS. Il est démontré que $Q_k$ ne peut être décomposé selon la méthode Cholesky $UDU^T$. Il est toutefois démontré qu'on peut obtenir d'une bonne approximation la décomposition nécessaire pour utiliser la formulation Biermann-Agee-Turner du filtre Kalman. Cette décomposition approximative est démontrée. Il est aussi démontré que la décomposition $UDU^T$ préserve la forme diagonale.

# EXECUTIVE SUMMARY

PLANS (Primary Land Arctic Navigation System) is a multi-sensor integrated navigation system developed at DREO. PLANS employs an 8 state Kalman filter to optimally integrate the sensor data from a Transit receiver, a GPS receiver, a gyrocompass/directional gyro, an odometer, a magnetometer and a baroaltimeter. In the process of deriving and implementing the Kalman filter equations, one of the many matrices that must be found is the discrete process noise covariance matrix $Q_k$ (also known as the driving noise covariance).

Initially, as is common practice, an approximation was used to evaluate this $Q_k$. During a detailed analysis of simulation results, the behaviour of the PLANS position error covariance matrix, P, under propagation (i.e. without position measurements from Transit or GPS) came under suspicion. This behaviour is governed solely by the state transition matrix $\Phi(t,\Delta t)$ (at time t over an interval $\Delta t$) and the driving noise covariance, $Q_k$. The discrete driving noise covariance matrix, $Q_k$, over the interval $\Delta t$, is itself defined by the continuous driving noise power spectral density matrix, Q, and the state transition matrix, $\Phi(t,\Delta t)$. Therefore these matrices came under special scrutiny. Since $\Phi(t,\Delta t)$ was already exact this then led to the desire for a more exact $Q_k$. The purpose of this report is therefore to derive the "exact" form of $Q_k$ (and find its Cholesky decomposition for use in PLANS).

Since PLANS employs a "square root" formulation of the Kalman filter equations for improved numerical stability, it is therefore necessary to find the Cholesky $UDU^T$ decomposition of $Q_k$ (where U is an upper triangular matrix and D is diagonal). For the original approximation, $Q_k$ was diagonal so that its decomposition was trivial. With the more exact $Q_k$ this is no longer the case. Furthermore the exact $Q_k$ is not constant, so that the use of a numerical routine to find its decomposition would require considerable computation, making an explicit decomposition highly desireable.

It is proven in this report that in general $UDU^T$ decomposition preserves block diagonal form, and therefore that the process of finding an exact decomposition of a large block diagonal matrix can be reduced to the much simpler problem of decomposing the smaller blocks. This is then applied to the $Q_k$ for PLANS, so that instead of having to decompose an 8x8 matrix, it is only necessary to decompose three 1x1 matrices (which is trivial), one 2x2 matrix and one 3x3 matrix. The 2x2 matrix is easily decomposed exactly. The 3x3 matrix however causes some difficulty. Its $UDU^T$ decomposition is easily enough found, however it is not a Cholesky decomposition because its D component has negative diagonal elements. This is not particularly surprising since the 3x3 matrix is not positive definite and therefore the existence of its Cholesky decomposition is not guaranteed.

It is thus shown that a Cholesky decomposition of the exact $Q_k$ is not possible, and an approximation is still required. An approximation is found which is exact for all but a few of the small off-diagonal terms of $Q_k$.

As it turned out, the "suspicious" behaviour of the error state covariance matrix P was not due to the inexactness of $Q_k$, but in fact could be explained by closer examination of the effect of $\Phi$. Although in hindsight this behaviour seems obviously correct, the suspicion arose because of the intuitive expectation that the position uncertainty (represented by bottom right 2x2 block of P, since the position error states are the last two elements of the state vector) should increase, or at least not decrease, in the absence of position measurements. Although it is true that the covariances of most elements of the state vector behave in this way (since they are independent Markov processes), it is not generally true for the position covariance. This is because of a geometric effect which can lead to a cancellation of errors in some situations, such that the position covariance locally decreases. In the case of PLANS this could be due to the effect of the speed and heading errors while returning to the starting point partially cancelling the errors accumulated during the outbound portion of the trip (a perfectly constant heading and speed error would perfectly cancel if movement were on a plane). Since this geometric effect is correctly modelled in the PLANS Kalman filter, the covariance behaves accordingly.

Although the "problem" that this effort was intended to solve turned out not to be a problem, a more exact form of the driving noise covariance is of course desireable in any case, and the general result on preservation of block diagonal form under decomposition is also quite useful.

# TABLE OF CONTENTS

# 1. INTRODUCTION

Reference [1] describes the multi-sensor integrated navigation system called PLANS (Primary Land Arctic Navigation System), along with the 8 state Kalman filter used for integration of the sensor data. In the process of deriving and implementing the Kalman filter equations one of the many matrices that must be found is the discrete process noise covariance matrix, $Q_k$ (also known as the driving noise covariance). In reference [2] an approximation was used to evaluate this $Q_k$. During a detailed analysis of simulation results, the behaviour of the PLANS position error covariance matrix, P, under propagation (i.e. without position measurements from Transit or GPS) came under suspicion. This behaviour is governed solely by the state transition matrix, $\Phi(t,\Delta t)$, and the driving noise covariance, $Q_k$, as follows:

$$P_{t+\Delta t} = \Phi(t,\Delta t)\, P_t\, \Phi^T(t,\Delta t) + Q_k \qquad (1)$$

where the discrete driving noise covariance matrix, $Q_k$, over the interval $\Delta t$ is defined by the continuous driving noise power spectral density matrix, Q, and the state transition matrix, $\Phi(t,\Delta t)$, as follows (see for example reference [5]):

$$Q_k = \int_0^{\Delta t} \Phi(t,\tau)\, Q\, \Phi^T(t,\tau)\, d\tau \qquad (2)$$

Therefore these matrices came under special scrutiny, which then led to the desire for a more exact $Q_k$, since $\Phi(t,\tau)$ was already exact. The purpose of this report is therefore to derive the "exact" form of $Q_k$ and find its Cholesky decomposition for use in PLANS.

Since PLANS employs a "square root" formulation of the Kalman filter equations for improved numerical stability (see reference [3]), it is therefore necessary to find the Cholesky $UDU^T$ decomposition of $Q_k$ (where U is an upper triangular matrix and D is diagonal). For the original approximation, $Q_k$ was diagonal so that its decomposition was trivial. With the more exact $Q_k$ this is no longer the case. Furthermore the exact $Q_k$ is not constant, so that use of a numerical routine to find its decomposition would require considerable computation, making an explicit decomposition highly desireable.

In fact it turns out that a Cholesky decomposition of the exact $Q_k$ is not possible, as will be shown below, and an approximation is still required. This approximation however only involves some of the small off-diagonal terms, and is exact for most terms of $Q_k$.

As it turned out, the "suspicious" behaviour of the error state covariance matrix P was not due to the inexactness of $Q_k$, but in fact could be explained by closer examination of the effect of $\Phi$. Although in hindsight this behaviour seems obviously correct, the suspicion arose because of the intuitive expectation that the position uncertainty (represented by bottom right 2x2 block of P, since the position error states are the last two elements of the state vector) should increase, or at least not decrease, in the absence of position measurements. Although it is true that the covariances of most elements of the state vector behave in this way (since they are independent Markov processes), it is not generally true for the position covariance. This is because of a geometric effect which can lead to a cancellation of errors in some situations, such that the position covariance locally decreases. In the case of PLANS this could be due to the effect of the speed and heading errors while returning to the starting point partially cancelling the errors accumulated during the outbound portion of the trip (a perfectly constant heading and speed error would perfectly cancel if movement were on a plane). Since this geometric effect is correctly modelled in the PLANS Kalman filter, the covariance behaves accordingly.

Although the "problem" that this effort was intended to solve turned out not to be a problem, a more exact form of the driving noise covariance is of course desireable in any case, and the general result of Appendix A is also quite useful.

# 2. THE CONTINUOUS Q MATRIX FOR PLANS

Since the discrete $Q_k$ matrix is found by integrating the continuous power spectral density matrix, Q, folded with the propagation matrix $\Phi(t,\tau)$, as in (2), we must first examine these two matrices.

As was shown in reference [2] the PLANS error state vector contains 5 independent first order Markov processes and three states which are derived from the integrals of these Markov processes. Thus the continuous power spectral density matrix Q has diagonal elements for each of these Markov processes, as follows:

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3)$$

where the $q_i$ are the constant values of the PSD's (power spectral densities) of the white driving noise for each of the individual Markov states. These can be expressed in terms of the standard Markov process error model parameters (correlation time Ti and steady state covariance pi ) as follows (see any standard text, such as [5]):

$$q_i = 2pi/Ti \qquad (4)$$

The PLANS error state transition matrix, $\Phi(t,\tau)$, is also derived in reference [2], where it is shown to be (at time t, over the interval $\tau$):

$$
\Phi(t,\tau) = \begin{bmatrix}
e^{-\tau/T1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-\tau/T2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-\tau/T3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-\tau/T4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & T4(1-e^{-\tau/T4}) & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-\tau/T6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\tau S\sin\theta & \tau S\cos\theta & 1 & 0 \\
0 & 0 & 0 & 0 & \tau S\cos\theta & \tau S\sin\theta & 0 & 1
\end{bmatrix} \qquad (5)
$$

where the Ti are Markov process correlation times (constants), S(t) is the vehicle speed, $\theta(t)$ is the vehicle heading and $\tau$ is the propagation period.

# 3. THE DISCRETE $Q_k$ MATRIX FOR PLANS

The discrete driving noise covariance matrix, $Q_k$, over the interval $\Delta t$, is defined by the continuous driving noise power spectral density matrix, Q, and the state transition matrix, $\Phi(t,\tau)$, as shown in equation (2):

$$Q_k = \int_0^{\Delta t} \Phi(t,\tau) Q \Phi^T(t,\tau)\, d\tau \tag{6}$$

where from equations (3) and (5) we can see that the integrand is:

$\Phi Q \Phi^T =$

$$\begin{bmatrix} e^{-\tau/T1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\tau/T2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\tau/T3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\tau/T4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T4(1-e^{-\tau/T4}) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\tau/T6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau S\sin\theta & \tau S\cos\theta & 1 & 0 \\ 0 & 0 & 0 & 0 & \tau S\cos\theta & \tau S\sin\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 & & & & & & & \\ & q_2 & & & & & \mathbf{0} & \\ & & q_3 & & & & & \\ & & & q_4 & & & & \\ & & & & 0 & & & \\ & \mathbf{0} & & & & q_6 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix} \Phi^T \tag{7}$$

Now from this we can already see that $Q_k$ will have the same block diagonal form as $\Phi$, namely three 1-blocks and a 5-block. As shown in Appendix A, this block diagonal form is also preserved under $UDU^T$ decomposition. Therefore the top three 1-blocks have trivial decompositions, since they are already diagonal. Thus for $i = 1,2,3$ we have:

$$Q_k(i,i) = \int_0^{\Delta t} e^{-\tau/Ti} q_i e^{-\tau/Ti}\, d\tau \qquad (8)$$

$$= \frac{-Ti}{2} q_i e^{-2\tau/Ti} \Big|_0^{\Delta t} \qquad (9)$$

$$= \frac{Ti}{2} q_i (1 - e^{-2\Delta t/Ti}) \qquad (10)$$

Now as seen in equation (4) above, for the steady state Markov process $x_i$, the magnitude of the PSD of the white driving noise is $q_i$, which is related to the steady state covariance pi and the correlation time Ti according to:

$$q_i = 2pi/Ti$$

Substituting this into equation *(10) gives, for i = 1,2,3:*

$$Q_k(i,i) = p_i(1 - e^{-2\Delta t/Ti}) \qquad (11)$$

We will now restrict our attention to the remaining 5-block. Since this is non-diagonal, it requires a non-trivial $UDU^T$ decomposition. Henceforth for simplicity Q, $Q_k$ and $\Phi$ shall refer to the corresponding 5-blocks rather than the full matrices. Thus, from (7) we have:

$$\Phi Q \Phi^T = \begin{bmatrix} e^{-\tau/T4} & 0 & 0 & 0 & 0 \\ T4(1-e^{-\tau/T4}) & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-\tau/T6} & 0 & 0 \\ 0 & -\tau S\sin\theta & \tau S\cos\theta & 1 & 0 \\ 0 & \tau S\cos\theta & \tau S\sin\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} q_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Phi^T \qquad (12)$$

$$
= \begin{bmatrix} q_4 e^{-\tau/T4} & 0 & 0 & 0 & 0 \\ q_4 T4(1-e^{-\tau/T4}) & 0 & 0 & 0 & 0 \\ 0 & 0 & q_6 e^{-\tau/T6} & 0 & 0 \\ 0 & 0 & \tau q_6 S\cos\theta & 0 & 0 \\ 0 & 0 & \tau q_6 S\sin\theta & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\tau/T4} & T4(1-e^{-\tau/T4}) & 0 & 0 & 0 \\ 0 & 1 & 0 & -\tau S\sin\theta & \tau S\cos\theta \\ 0 & 0 & e^{-\tau/T6} & \tau S\cos\theta & \tau S\sin\theta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}
$$

$$
= \begin{bmatrix} q_4 e^{-2\tau/T4} & q_4 T4(1-e^{-\tau/T4})e^{-\tau/T4} & 0 & 0 & 0 \\ q_4 T4(1-e^{-\tau/T4})e^{-\tau/T4} & q_4 T4^2(1-e^{-\tau/T4})^2 & 0 & 0 & 0 \\ 0 & 0 & q_6 e^{-2\tau/T6} & \tau q_6 S\cos\theta e^{-\tau/T6} & \tau q_6 S\sin\theta e^{-\tau/T6} \\ 0 & 0 & \tau q_6 S\cos\theta e^{-\tau/T6} & q_6 S^2\cos^2\theta\tau^2 & q_6 S^2\sin\theta\cos\theta\tau^2 \\ 0 & 0 & \tau q_6 S\sin\theta e^{-\tau/T6} & q_6 S^2\sin\theta\cos\theta\tau^2 & q_6 S^2\sin^2\theta\tau^2 \end{bmatrix} \tag{14}
$$

It is now clear that this $Q_k$ also has a block diagonal form, with a 2-block and a 3-block. Assuming that the time dependent terms (speed S and heading $\theta$) are constant over the integration interval, the above matrix can be explicitly integrated (as in equation (6)). For convenience we will label the individual terms as follows:

$$
Q_k \equiv \begin{bmatrix} q_{11} & q_{12} & 0 & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 & 0 \\ 0 & 0 & q_{33} & q_{34} & q_{35} \\ 0 & 0 & q_{43} & q_{44} & q_{45} \\ 0 & 0 & q_{53} & q_{54} & q_{55} \end{bmatrix} \tag{15}
$$

where by symmetry $q_{ij} = q_{ji}$. Then $q_{11}$ can be found as in equations (8) to (11) above:

$$
q_{11} = \int_0^{\Delta t} q_4 e^{-2\tau/T4} d\tau \tag{16}
$$

$$= \frac{-T4}{2} q_4 e^{-2\tau/T4} \Big|_0^{\Delta t}$$

$$= \frac{T4}{2} q_4 (1 - e^{-2\Delta t/T4}) \qquad (17)$$

$$= p4 (1 - e^{-2\Delta t/T4}) \qquad (18)$$

Similarly for the other components:

$$q_{12} = \int_0^{\Delta t} T4 q_4 (1 - e^{-\tau/T4}) e^{-\tau/T4} d\tau$$

$$= \frac{T4^2}{2} q_4 (1 - e^{-\tau/T4})^2 \Big|_0^{\Delta t}$$

$$= \frac{T4^2}{2} q_4 (1 - e^{-\Delta t/T4})^2$$

$$= T4\ p4 (1 - e^{-\Delta t/T4})^2 \qquad (19)$$

$$q_{33} = \int_0^{\Delta t} q_6 e^{-2\tau/T6} d\tau$$

$$= p6 (1 - e^{-2\Delta t/T6}) \qquad \text{(as in } q_{11}\text{)} \qquad (20)$$

$$q_{43} = \int_0^{\Delta t} q_6 S\cos\theta\,\tau e^{-\tau/T6} d\tau$$

$$= q_6 S\cos\theta \frac{e^{-\tau/T6}}{(1/T6)^2}(-\tau/T6 - 1)\Big|_0^{\Delta t}$$

$$= -T6 q_6 S\cos\theta[e^{-\Delta t/T6}(\Delta t + T6) - T6]$$

$$= 2T6p6S\cos\theta[1 - e^{-\Delta t/T6}(1 + \Delta t/T6)] \quad (21)$$

Similarly

$$q_{53} = \int_0^{\Delta t} q_6 S\sin\theta\ \tau e^{-\tau/T6}\ d\tau$$

$$= 2T6p6S\sin\theta[1 - e^{-\Delta t/T6}(1 + \Delta t/T6)] \quad (22)$$

$$q_{54} = \int_0^{\Delta t} q_6 S^2 \sin\theta\cos\theta\ \tau^2\ d\tau$$

$$= q_6 S^2 \sin\theta\cos\theta \frac{\tau^3}{3}\Big|_0^{\Delta t}$$

$$= q_6 S^2 \sin\theta\cos\theta\ \frac{\Delta t^3}{3} \quad (23)$$

$$q_{44} = \int_0^{\Delta t} q_6 S^2 \cos^2\theta\,\tau^2 d\tau$$

$$= q_6 S^2 \cos^2\theta \frac{\tau^3}{3}\Big|_0^{\Delta t}$$

$$= q_6 S^2 \cos^2\theta \frac{\Delta t^3}{3} \tag{24}$$

Similarly

$$q_{55} = \int_0^{\Delta t} q_6 S^2 \sin^2\theta \tau^2 d\tau$$

$$= q_6 S^2 \sin^2\theta \frac{\tau^3}{3}\Big|_0^{\Delta t}$$

$$= q_6 S^2 \sin^2\theta \frac{\Delta t^3}{3} \tag{25}$$

$$q_{22} = \int_0^{\Delta t} T4^2 q_4 (1 - e^{-\tau/T4})^2 d\tau$$

$$= T4^2 q_4 \int_0^{\Delta t} (1 - 2e^{-\tau/T4} + e^{-2\tau/T4}) d\tau$$

$$= T4^2 q_4 (\tau + 2T4e^{-\tau/T4} - \frac{T4}{2} e^{-2\tau/T4}) \Big|_0^{\Delta t}$$

$$= T4^2 q_4 (\Delta t + 2T4e^{-\Delta t/T4} - \frac{T4}{2} e^{-2\Delta t/T4} - 2T4 + \frac{T4}{2})$$

$$= T4^3 q_4 (\Delta t/T4 + 2e^{-\Delta t/T4} - \frac{1}{2} e^{-2\Delta t/T4} - \frac{3}{2})$$

$$= T4^2 (\frac{2\Delta t}{T4} + 4e^{-\Delta t/T4} - e^{-2\Delta t/T4} - 3) p4 \tag{26}$$

Now by substituting equations (18) through (26) into equation (15) we can write the discrete $5 \times 5$ $Q_k$ matrix as follows:

$$
Q_k = \begin{bmatrix}
p_4(1-e^{-2\Delta t/T4}) & T4p_4(1-e^{-\Delta t/T4})^2 & 0 & 0 & 0 \\
T4p_4(1-e^{-\Delta t/T4})^2 & T4^2p_4(\frac{2\Delta t}{T4}+4e^{-\Delta t/T4}-e^{-2\Delta t/T4}-3) & 0 & 0 & 0 \\
0 & 0 & p_6(1-e^{-2\Delta t/T6}) & 2T6S\cos\theta A p_6 & 2T6S\sin\theta A p_6 \\
0 & 0 & 2T6S\cos\theta A p_6 & \frac{2\Delta t^3}{3T6} S^2\cos^2\theta \, p_6 & \frac{2\Delta t^3}{3T6} S^2\sin\theta\cos\theta \, p_6 \\
0 & 0 & 2T6S\sin\theta A p_6 & \frac{2\Delta t^3}{3T6} S^2\sin\theta\cos\theta \, p_6 & \frac{2\Delta t^3}{3T6} S^2\sin^2\theta \, p_6
\end{bmatrix} \tag{27}
$$

where:

$$
A = 1 - e^{-\Delta t/T6}\left(1 - \frac{\Delta t}{T6}\right) \tag{28}
$$

As indicated in reference [1], the correlation times (T4 and T6) and steady state covariances (p4 and p6) for the Markov processes representing the error in the gyro drift rate and the odometer scale factor are assumed to be constants. Therefore it is easy to see how this process driving noise covariance matrix $Q_k$ behaves numerically for different discretization intervals $\Delta t$. Using the values given in [2], we have for $\Delta t = 60$ seconds:

$$
Q_k \cong \begin{bmatrix}
0.03p_4 & p_4 & 0 & 0 & 0 \\
p_4 & 40p_4 & 0 & 0 & 0 \\
0 & 0 & 0.03p_6 & 237p_6S\cos\theta & 237p_6S\sin\theta \\
0 & 0 & 237\,p_6S\cos\theta & 40\,p_6S^2\cos^2\theta & 40\,p_6S^2\sin\theta\cos\theta \\
0 & 0 & 237\,p_6S\sin\theta & 40p_6S^2\sin\theta\cos\theta & 40p_6S^2\sin^2\theta
\end{bmatrix} \tag{29}
$$

and for $\Delta t = 1$ second:

$$Q_k \cong \begin{bmatrix} 0.0005p_4 & 0.0003p_4 & 0 & 0 & 0 \\ 0.0003p_4 & 0.001p_4 & 0 & 0 & 0 \\ 0 & 0 & 0.0005p_6 & 0.0003\,p_6 S\cos\theta & 0.0003p_6 S\sin\theta \\ 0 & 0 & 0.0003\,p_6 S\cos\theta & 0.0002\,p_6 S^2\cos^2\theta & 0.0002\,p_6 S^2\sin\theta\cos\theta \\ 0 & 0 & 0.0003\,p_6 S\sin\theta & 0.0002p_6 S^2\sin\theta\cos\theta & 0.0002p_6 S^2\sin^2\theta \end{bmatrix} \tag{30}$$

This now allows us to see the relative significance (or insignificance) of the off-diagonal terms.

# 4. $UDU^T$ DECOMPOSITION OF $Q_k$

To decompose the $Q_k$ matrix for use in one of the numerically superior "square root" formulations of the Kalman filter (see for example reference [3]), we follow Bierman in using the Cholesky factorization method. For this we must find matrices U and D such that U is upper triangular with 1's on the diagonal, D is diagonal and positive semi-definite, and

$$Q_k = UDU^T \tag{31}$$

Since $Q_k$ is of block diagonal form, it can be easily shown that its square root has the same block diagonal form, and by simple extension so must its U factor (as is proven in Appendix A below). Therefore the factorization can be greatly simplified by performing it separately on the diagonal blocks of $Q_k$.

## 4.1. EXACT DECOMPOSITION OF THE 2X2 BLOCK

A general factorization for a 2x2 block can be found as follows. We equate the general matrix to the $UDU^T$ product, where U and D are of the required form:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} e & fd \\ 0 & f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (e+fd^2) & fd \\ fd & f \end{bmatrix} \tag{33}$$

We then solve for the unknown elements of U and D (e, f and d) as functions of the elements of the general matrix (a, b and c). Therefore, by inspection, the exact solution is:

$$\boxed{\begin{aligned} f &= c \\ d &= b/c \\ e &= a - b^2/c \end{aligned}} \tag{34}$$

This can now be used to find the decomposition of the 2x2 block of $Q_k$, as given in (27). Thus we take a, b and c are from equation (27) and substitute into (34). The resulting expressions can be simplified as follows.

$$f = c = Q_k(2,2)$$

$$= p4T4^2\left[\frac{2\Delta t}{T4} + 4e^{-\Delta t/T4} - e^{-2\Delta t/T4} - 3\right] \tag{35}$$

$$\therefore f \cong p4T4^2\left\{\frac{2\Delta t}{T4} + 4\left[1 - \frac{\Delta t}{T4} + \frac{(-\Delta t/T4)^2}{2} + \frac{(-\Delta t/T4)^3}{3!}\right]\right.$$

$$\left. - \left[1 - \frac{2\Delta t}{T4} + \frac{(-2\Delta t/T4)^2}{2} + \frac{(-2\Delta t/T4)^3}{3!}\right] - 3\right\}$$

(where we have used the first 4 terms of the Maclaurin series expansion for $e^x$)

$$= p4T4^2\left\{\frac{2\Delta t}{T4} + 4 - 4\frac{\Delta t}{T4} + 2\left[\frac{\Delta t}{T4}\right]^2 - \frac{2}{3}\left[\frac{\Delta t}{T4}\right]^3 - 1 + 2\frac{\Delta t}{T4} - 2\left[\frac{\Delta t}{T4}\right]^2 + \frac{4}{3}\left[\frac{\Delta t}{T4}\right]^3 - 3\right\}$$

$$= p4T4^2\frac{2}{3}\left[\frac{\Delta t}{T4}\right]^3$$

$$= \frac{2p4\Delta t^3}{3T4} \tag{36}$$

This will be a good approximation provided that the discretization interval $\Delta t$ is significantly less than the correlation time T4 (which it will be in PLANS).

Now the next term can also be simplified by similarly using the Maclaurin expansion and ignoring the higher order terms in $\Delta t/T4$:

$$d = b/c$$

$$= T4p4(1 - e^{-\Delta t/T4})^2/c \tag{37}$$

$$\therefore d \cong \frac{T4p4(1 - 1 + \Delta t/T4)^2}{\left(\frac{2p4\Delta t^3}{3T4}\right)}$$

$$= \frac{T4(\Delta t/T4)^2}{\left(\frac{2\Delta t^3}{3T4}\right)}$$

$$= \frac{3}{2\Delta t} \tag{38}$$

Finally the third term can also be simplified:

$$e = a - b^2/c$$

$$= p4(1-e^{-2\Delta t/T4}) - \frac{T4^2p4^2(1-e^{-2\Delta t/T4})^4}{p4T4^2\left(\frac{2\Delta t}{T4} + 4e^{-\Delta t/T4} - e^{-2\Delta t/T4} - 3\right)} \tag{39}$$

$$\cong p4\left(\frac{2\Delta t}{T4}\right) - \frac{T4^2p4^2\left(\frac{\Delta t}{T4}\right)^4}{\left(\frac{2p4\Delta t^3}{3T4}\right)}$$

$$= \frac{2p4\Delta t}{T4} - \frac{3T4^3p4\Delta t}{2T4^4}$$

$$= \frac{p4\Delta t}{T4}(2 - 3/2)$$

$$= \frac{p4\Delta t}{2T4} \tag{40}$$

In this case the exact solution of (34) is given by (35), (37) and (39), with a good approximation given by (36), (38) and (40):

$$\boxed{\begin{aligned} f &\cong \frac{2p4\Delta t^3}{3T4} \\ d &\cong \frac{3}{2\Delta t} \\ e &\cong \frac{p4\Delta t}{2T4} \end{aligned}} \tag{41}$$

## 4.2. EXACT DECOMPOSITION OF THE 3X3 BLOCK

The decomposition of the 3x3 block can be found in a similar way:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & g \end{bmatrix} = \begin{bmatrix} 1 & h & i \\ 0 & 1 & j \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ i & j & 1 \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} k & hL & im \\ 0 & L & jm \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ i & j & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (k+h^2L+i^2m) & (hL+ijm) & im \\ (hL+ijm) & (L+j^2m) & jm \\ im & jm & m \end{bmatrix} \tag{43}$$

Therefore we have:

$$m = g \tag{44}$$

$$jm = e$$

$$\Rightarrow \quad j = e/m$$

$$= e/g \tag{45}$$

$$im = c$$

$$\Rightarrow \quad i = c/m$$

$$= c/g \tag{46}$$

$$L+j^2m = d$$

$$\Rightarrow \quad L = d - e^2/m$$

$$= d - e^2/g \tag{47}$$

$$hL+ijm = b$$

$$\Rightarrow \quad h = (b - ijm)/L$$

$$= \frac{b - ce/m}{d - e^2g}$$

$$= \frac{b - ce/g}{d - e^2g} \tag{48}$$

$$k = a - h^2L - i^2m$$

$$= a - \frac{(b - ce/g)^2}{d - e^2g} - c^2/g \tag{49}$$

Now when the actual values for a, b, c, d, e and g are substituted from (27) and (42), we obtain the decomposition of the 3x3 block of the PLANS state vector driving noise covariance matrix, as follows:

$$m = \frac{2\Delta t^3}{3T6} S^2 \sin^2\theta\, p_6 \tag{50}$$

$$j = \cot\theta \tag{51}$$

$$i = \frac{2T6S\sin\theta\left[1 - e^{-\Delta t/T6}\left(1 - \frac{\Delta t}{T6}\right)\right] p6}{\frac{2\Delta t^3}{3T6} S^2 \sin\theta\cos\theta\, p6}$$

$$= \frac{3T6^2}{S\Delta t^3 \sin\theta}\left[1 - e^{-\Delta t/T6}\left(1 - \frac{\Delta t}{T6}\right)\right] \tag{52}$$

$$L = d - e\cdot\cot\theta$$

$$= 0 \tag{53}$$

$$h = 0 \tag{54}$$

$$k = p6\left(1 - e^{-2\Delta T/T6}\right) - \frac{\left[2T6S\sin\theta\left(1 - e^{-\Delta t/T6}\left(1 - \frac{\Delta t}{T6}\right)\right)p6\right]^2}{\frac{2\Delta t^3}{3T6} S^2 \sin^2\theta\, p6}$$

$$= p6\left(1 - e^{-2\Delta t/T6}\right) - 6\left[\frac{T6}{\Delta t}\right]^3 \left[1 - e^{-\Delta t/T6}\left(1 - \frac{\Delta t}{T6}\right)\right]^2 p6 \tag{55}$$

When these values are substituted into equation (42) we see that the exact decomposition of 3x3 block of $Q_k$ has the form:

$$Q_k = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & \cot\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & \cot\theta & 1 \end{bmatrix} \tag{56}$$

where:

$$
\begin{aligned}
m &= \frac{2\Delta t^3}{3T6} S^2 \sin^2\theta \, p_6 \\
i &= \frac{3T6^2}{S\Delta t^3 \sin\theta}\left[1- e^{-\Delta t/T6}\left(1-\frac{\Delta t}{T6}\right)\right] \\
k &= p6\left(1 - e^{-2\Delta t/T6}\right) - 6\left[\frac{T6}{\Delta t}\right]^3\left[1- e^{-\Delta t/T6}\left(1-\frac{\Delta t}{T6}\right)\right]^2 p6
\end{aligned}
\tag{57}
$$

Now unfortunately this solution does not satisfy the requirement that the diagonal elements (k, l and m) be non-negative. In particular it can be seen that k can be negative by substituting the model values for p6, T6 and $\Delta t$ into equation (57). This requirement is necessary in order to use the Modified Weighted Gram-Schmidt algorithm (described in reference [3]), which is used by PLANS to propagate the covariance matrix.

However, it is quite common to use a much rougher approximation for the discrete $Q_k$ matrix than is used here. In fact it is common to use $Q\Delta t$ in place of the integral of equation (56). This yields a $Q_k$ which is diagonal, and hence has a trivial $UDU^T$ decomposition. What has been done for PLANS however, is to find a decomposition which represents most elements of $Q_k$ exactly (including the diagonal terms) and approximates the others, as described in the next chapter. Although this is not entirely exact, it is much better than the usual approximation.

## 5. A GOOD APPROXIMATION FOR THE 3X3 DECOMPOSITION

First note that $Q_k$ is not positive definite. (The rank of the continuous 5x5 Q matrix, as shown in the bottom right corner of equation (3), is obviously only two.) Therefore the Cholesky decomposition of $Q_k$ does not necessarily exist (see for example reference [4]), as we have indeed discovered. Of course Q is positive semi-definite (since it is a covariance matrix), which is the more basic requirement for the Kalman filter equations. In order to use the more numerically stable algorithms however, a decomposible approximation to $Q_k$ must be found.

(After determining that the exact $UDU^T$ decomposition had a negative diagonal element, another decomposition was attempted: the $LDL^T$, which uses lower triangular rather than upper triangular matrices. This also (perhaps predictably?) produced a negative diagonal element.)

The following approximation was found by inspection:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b\cos\theta \\ 0 & 0 & b\sin\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b\cos\theta & b\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & bc\cos\theta \\ 0 & 0 & bc\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b\cos\theta & b\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} a^2c & abc\cdot\cos\theta & abc\cdot\sin\theta \\ abc\cdot\cos\theta & b^2c\cdot\cos^2\theta & b^2c\cdot\cos\theta\sin\theta \\ abc\cdot\sin\theta & b^2c\cdot\cos\theta\sin\theta & b^2c\cdot\sin^2\theta \end{bmatrix} \qquad (58)$$

By comparing this to equation (27), we see that this already has the correct $\theta$ dependence. In fact we would have an exact solution if we could find an a, b and c to satisfy the following:

$$a^2c = p6(1-e^{-2\Delta t/T6}) \qquad (59)$$

$$b^2c = p6\frac{2\Delta t^3}{3T6}s^2 \qquad (60)$$

$$\mathrm{abc} = 2p_6\, S\, T6 \left[ 1 - e^{-\Delta t/T6} \left(1 - \frac{\Delta t}{T6}\right) \right] \tag{61}$$

This does not generally (i.e. for arbitrary values of p6 T6 and $\Delta t$) have an exact solution, as can be seen by comparing (59)x(60) and $(61)^2$, which should both be equal to $(abc)^2$. However, by solving (59) and (60) exactly and approximating (61) we have:

$$\boxed{\begin{aligned} c &= p6 \\ a &= \sqrt{1 - e^{2\Delta t/T6}} \\ b &= S\Delta t \sqrt{\frac{2\Delta t}{3T6}} \end{aligned}} \tag{62}$$

This gives an exact solution on the diagonal and latitude/longitude cross terms (the (5,4) and (4,5) components of this 5x5 block of $Q_k$). The terms which are approximated (the (3,4) and (3,5) components) have the correct sign and the correct $\theta$ dependence. Further analysis indicates that the approximated terms are smaller than the true terms, provided only that the propagation interval is sufficiently short:

$$\Delta t < \sqrt{12}\ T6 \cong 200 \text{ minutes} \tag{63}$$

which will certainly be the case in PLANS. This can be seen by substituting (62) into (58) and comparing to (27). Thus this approximation for $Q_k$ is certainly better than the simpler $Q\Delta t$ approximation.

# REFERENCES

[1] "Design of an Optimally Integrated Primary Land Arctic Navigation System, Volume I: System Design," J.C. McMillan, DREO Report 946, Sept. 1986.

[2] "Design of an Optimally Integrated Primary Land Arctic Navigation System, Volume II: Failure Modes and Technical Appendices," J.C. McMillan, DREO Report 1003, March 1989.

[3] Bierman, G.J., *Factorization Methods for Discrete Sequential Estimation*, New York, Academic Press, 1977

[4] Stewart, G.W., *Introduction to Matrix Computations*, New York, Academic Press, 1973.

[5] Gelb, A., Editor, The Analytic Sciences Corporation, *Applied Optimal Estimation*, Cambridge, Massachusetts, The M.I.T. Press, 1974.

# APPENDIX A: BLOCK DIAGONAL FORM PRESERVATION UNDER DECOMPOSITION

In this section we shall prove that a block diagonal matrix can be decomposed (without loss of generality) by decomposing its diagonal blocks. This will be very useful for implementation of Kalman filters, since state models are often of block diagonal form, with a separate block for each independent sensor or subsystem. Deriving explicit decompositions for the corresponding driving noise covariance matrices $Q_k$ can then be greatly simplified.

We will first prove the result for a matrix with two blocks on its diagonal. The extension to the general case is a straightforward application of mathematical induction.

Consider the Cholesky $UDU^T$ decomposition of a positive definite square matrix M, which has two diagonal blocks:

$$M = \begin{bmatrix} M1 & 0 \\ 0 & M2 \end{bmatrix} \tag{A1}$$

$$= \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B^T & C^T \end{bmatrix} \tag{A2}$$

Where A and C are upper triangular matrices with one's on the diagonal, and D and E are diagonal matrices (with non-zero elements on the diagonal, since M is positive definite). Multiplying (A2) out we obtain:

$$M = \begin{bmatrix} AD & BE \\ 0 & CE \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B^T & C^T \end{bmatrix}$$

$$= \begin{bmatrix} (ADA^T + BEB^T) & BEC^T \\ CEB^T & CEC^T \end{bmatrix} \tag{A3}$$

Now for (A2) to be a decomposition of (A1), the off-diagonal blocks of (A3) must be zero. Thus:

$$CEB^T = 0 \tag{A4}$$

$$BEC^T = 0 \tag{A5}$$

Since C is upper triangular with 1's on the diagonal, and E is a diagonal matrix, then if we define:

$$F \equiv CE \tag{A6}$$

we can easily see that F is upper triangular with the (non-zero) elements of E on its diagonal. Then (A4) becomes:

$$FB^T = 0 \tag{A7}$$

$$\begin{bmatrix} e1 & x & \dots & x & x \\ 0 & e2 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e(n\text{-}1) & x \\ 0 & 0 & \dots & 0 & en \end{bmatrix} B^T = 0 \tag{A8}$$

Close examination of (A8) gives us the desired result: Starting with the last row of (A8), we can see that the bottom row of $B^T$ must be zero (since $e_n \neq 0$). Given that the bottom row of $B^T$ is zero, then examination of the second last row of (A8) shows that the second last row of $B^T$ also must be zero (since $e_{n-1} \neq 0$). This can be continued up the rows to show that all rows of $B^T$ are zero. From (A2) we can then see that the Cholesky decomposition of M is block diagonal, with the same block form as M.

Now this can easily be generalized to a matrix N with more than two blocks by separating one block at a time as follows. Let M1 in (A1) be the top block of N, so that M2 contains all the remaining blocks. The theorem as it stands proves that M2 can be decomposed separately from M1. Now simply apply the theorem again to M2 to see that its top diagonal block (the second block of N) can be decomposed separately from the rest (the third and remaining blocks of N). This can clearly be repeated until all the blocks have been separated.

# APPENDIX B: POSITION ERROR COVARIANCE PROPAGATION IN PLANS

The time dependence of the state vector error covariance matrix P, in the absence of measurement updates, is described by the covariance propagation equation (see for example reference [5]):

$$P_{t+\Delta t} \quad = \quad \Phi(t+\Delta t)\, P_t\, \Phi^T(t+\Delta t) \quad + \quad Q_k \qquad \text{(B1)}$$

Since the latitude and longitude error estimates are the last two elements of the state vector, the position error covariance is described by the last two diagonal elements of the covariance matrix P. Thus we will examine the propagation of these last two elements of P. From (B1) we can see that this involves only the bottom two rows of $\Phi$ and their transpose (the last two columns of $\Phi^T$). From equation (5) we can see that the first four columns of the last two rows of $\Phi$ are zero, and can therefore be ignored.

Equation (27) can be used to obtain the relevant elements of $Q_k$. Here the $p_6$ refers to the steady state covariance of the sixth state (the odometer scale factor error), as explained by equation (11). This is a constant which comes from the error model, and has a value of about 0.001 (dimensionless).

If we assign the Markov process covariances, P(5,5) and P(6,6), to their steady state values, then we would have a gyro heading error P(5,5) of about (0.1 radian)$^2$ and an odometer scale factor error P(6,6) of about (1%)$^2$. This is what would be expected in the absence of measurements, and its reasonableness has been verified by simulation. The relevant portion of (B1) can then be written as follows, using (5) for the form of $\Phi$ and (30) for the form of $Q_k$ :

$$P_{t+\Delta t} \cong$$

$$\begin{bmatrix} a & b & 1 & 0 \\ b & -a & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.01 & 0 & d & e \\ 0 & 0.0001 & f & g \\ d & f & h & i \\ e & g & i & j \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} k\cos^2\theta & k\sin\theta\cos\theta \\ k\sin\theta\cos\theta & k\sin^2\theta \end{bmatrix} \qquad \text{(B2)}$$

where the position error covariance before propagation is:

$$h = P(7,7)_t \tag{B3}$$

$$j = P(8,8)_t \tag{B4}$$

The elements a and b of the propagation matrix $\Phi$ are shown in (5) to be:

$$a = -\Delta t \, S \sin\theta \tag{B5}$$

$$b = \Delta t \, S \cos\theta \tag{B6}$$

Assuming a propagation interval of $\Delta t = 1$ second, the relevant elements of the driving noise covariance $Q_k$ can be found from equation (30), which implies that :

$$k = 0.0002 \, p_6 \, S^2 \tag{B7}$$

Multiplying (B2) out we obtain:

$$P_{t+\Delta t} \cong$$

$$\begin{bmatrix} (0.01a + d) & (0.0001b + f) & (ad + bf + h) & (ae + bg + i) \\ (0.01b + e) & (-0.0001a + g) & (bd - af + i) & (be - ag + j) \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + Q_k \tag{B8}$$

$$=$$

$$\begin{bmatrix} (0.01a^2+ad+0.0001b^2+bf+ad+bf+h) & x \\ x & (0.01b^2+be+0.0001a^2-ag+be-ag+j) \end{bmatrix} + Q_k$$

Therefore, the position error covariance terms are:

$$P(7,7)_{t+\Delta t} \cong (0.01a^2 + 0.0001b^2 + 2(ad + bf) + h) + k\cos^2\theta \tag{B9}$$

$$= P(7,7)_t + 0.01a^2 + 0.0001b^2 + k\cos^2\theta + 2(ad + bf)$$

$$\cong P(7,7)_t + S^2(0.01\sin^2\theta + 0.0001\cos^2\theta + 0.0000002\cos^2\theta) + 2S(\underline{-\sin\theta \, d + \cos\theta \, f}) \tag{B10}$$

$$P(8,8)_{t+\Delta t} \cong (0.01b^2 + 0.0001a^2 + 2(be - ag) + j)) \quad + \quad k\sin^2\theta \qquad \text{(B11)}$$

$$= P(8,8)_t + 0.01b^2 + 0.0001a^2 + k\sin^2\theta + 2(be - ag)$$

$$\cong P(8,8)_t + S^2( 0.01\cos^2\theta + 0.0001\sin^2\theta + 0.0000002\sin^2\theta) + 2S\underline{( \cos\theta e + \sin\theta g)} \qquad \text{(B12)}$$

Thus the position covariance *can* decrease in the absence of measurements, if the underlined terms in equations (B10) and (B12) are large enough in the negative sense. This will happen for certain values of heading $\theta$ and speed S, provided the d, e, f and g terms are not too small. Simulations have shown that these terms can be large enough to cause P(7,7) and P(8,8) to decrease, particularly in the absence of position measurements. The physical interpretation is that while the vehicle is heading back towards its point of origin the heading and speed errors start to cancel the "outbound" errors. In the absence of position update measurements (from GPS or Transit) the outbound position errors will of course be caused entirely by the heading and speed errors, and will therefore be statistically correlated to them through the cross covariances d, e, f and g.

## DOCUMENT CONTROL DATA

(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)

1. ORIGINATOR (the name and address of the organization preparing the document. Organizations for whom the document was prepared, e.g. Establishment sponsoring a contractor's report, or tasking agency, are entered in section 8.)
National Defence Headquarters
Ottawa, Ontario
K1A 0Z4

2. SECURITY CLASSIFICATION (overall security classification of the document including special warning terms if applicable)
UNCLASSIFIED

3. TITLE (the complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S,C or U) in parentheses after the title.)

FACTORIZATION OF THE DISCRETE NOISE COVARIANCE MATRIX FOR PLANS (U)

4. AUTHORS (Last name, first name, middle initial)

MCMILLAN, J.C.

5. DATE OF PUBLICATION (month and year of publication of document)
FEBRUARY 1991

6a. NO. OF PAGES (total containing information. Include Annexes, Appendices, etc.)
35

6b. NO. OF REFS (total cited in document)
5

7. DESCRIPTIVE NOTES (the category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

DREO REPORT

8. SPONSORING ACTIVITY (the name of the department project office or laboratory sponsoring the research and development. Include the address.)
Defence Research Establishment Ottawa
Ottawa, Ontario
K1A 0Z4

9a. PROJECT OR GRANT NO. (if appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant)

041LJ

9b. CONTRACT NO. (if appropriate, the applicable number under which the document was written)

10a. ORIGINATOR'S DOCUMENT NUMBER (the official document number by which the document is identified by the originating activity. This number must be unique to this document.)

DREO REPORT 1071

10b. OTHER DOCUMENT NOS. (Any other numbers which may be assigned this document either by the originator or by the sponsor)

11. DOCUMENT AVAILABILITY (any limitations on further dissemination of the document, other than those imposed by security classification)

(X) Unlimited distribution
( ) Distribution limited to defence departments and defence contractors; further distribution only as approved
( ) Distribution limited to defence departments and Canadian defence contractors; further distribution only as approved
( ) Distribution limited to government departments and agencies; further distribution only as approved
( ) Distribution limited to defence departments; further distribution only as approved
( ) Other (please specify):

12. DOCUMENT ANNOUNCEMENT (any limitation to the bibliographic announcement of this document. This will normally correspond to the Document Availabilty (11). However, where further distribution (beyond the audience specified in 11) is possible, a wider announcement audience may be selected.)
UNLIMITED

13. ABSTRACT ( a brief and factual summary of the document. It may also appear elsewhere in the body of the document itself. It is highly desirable that the abstract of classified documents be unclassified. Each paragraph of the abstract shall begin with an indication of the security classification of the information in the paragraph (unless the document itself is unclassified) represented as (S), (C), or (U). It is not necessary to include here abstracts in both offical languages unless the text is bilingual).

The PLANS (Primary Land Arctic Navigation System), developed at DREO, optimally integrates a directional gyro/gyrocompass, an odometer, a 3-axis strapdown magnetometer, a GPS receiver, a Transit receiver, a baroaltimeter and a digital terrain elevation map, for the purpose of navigating a land vehicle in the Canadian Arctic under potentially adverse conditions. This report derives the exact form of the discrete driving noise covariance matrix $Q_k$ which is needed to propagate the covariance matrix in the Kalman filter used by PLANS. It is shown that the exact $Q_k$ does not have a Cholesky $UDU^T$ decomposition. However, a good approximation is shown to have the necessary decomposition for use in the Biermann-Agee-Turner formulation of the Kalman filter. This approximate decomposition is then found. A general result on the preservation of block diagonal form under $UDU^T$ decomposition is also proven.

14. KEYWORDS, DESCRIPTORS or IDENTIFIERS (technically meaningful terms or short phrases that characterize a document and could be helpful in cataloguing the document. They should be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location may also be included. If possible keywords should be selected from a published thesaurus. e.g. Thesaurus of Engineering and Scientific Terms (TEST) and that thesaurus-identified. If it is not possible to select indexing terms which are Unclassified, the classification of each should be indicated as with the title.)

KALMAN FILTER
MATRIX DECOMPOSITION
NAVIGATION
PLANS