A Procedure for Numerical Evaluation of the Performance of a $TM_{01}$ Circular to $TE_{10}$ Rectangular Waveguide Mode Converter

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A Procedure for Numerical Evaluation of the Performance of a \( TM_{01} \) Circular to \( TE_{10} \) Rectangular Waveguide Mode Converter

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Two identical rectangular waveguides are fed by means of two symmetrically placed apertures in a circular waveguide. Power is transmitted from the \( TM_{01} \) mode in the circular waveguide to the \( TE_{10} \) dominant mode in each rectangular waveguide. A previously described method of evaluating this transmitted power is refined to obtain an algorithm for which a computer program was written. Several plots of transmitted power and magnitude of electric field in one of the apertures are presented. The computer program that was used to obtain the numerical data for these plots will be described and listed in a subsequent report.

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Chapter 1
Introduction

There is, as shown in Fig. 1.1, a circular waveguide which is closed at one end. Two symmetrically placed apertures in the lateral wall of this waveguide are backed by rectangular waveguides of identical dimensions. The dimensions of the waveguides are such that only the $TE_{11}$ and $TM_{01}$ modes can propagate in the circular waveguide and that only the $TE_{10}$ mode can propagate in the rectangular waveguides. The problem is, as stated in [1, Chapter 1], to find out how much of the power of an incident $TM_{01}$ wave in the circular waveguide is reflected in the circular waveguide and how much of this power is transmitted into the rectangular waveguides.

In this report, the analytical results of [1] for this problem are manipulated into expressions suitable for evaluation by means of a digital computer. These analytical results are not derived here; they are merely referred to. For this reason, the reader of this report should obtain a copy of reference [1].

A computer program was written in FORTRAN. Some numerical results obtained by using this computer program are presented. The computer program will be described and listed in a forthcoming report.

As is shown in Fig. 1.2, the interiors of the left-hand rectangular waveguide, the right-hand rectangular waveguide, and the circular waveguide are called regions 1, 2, and 3, respectively. The electromagnetic field in region 1 is radiated by $\mathcal{M}^{(1)}$. The field in region 2 is radiated by $\mathcal{M}^{(2)}$. The field in region 3 is radiated by the combination of $\mathcal{J}^{imp}$, $-\mathcal{M}^{(1)}$, and $-\mathcal{M}^{(2)}$. The magnetic currents in Fig. 1.2 are supposed to be located right on (infinitesimal distances from either side) of the closing conductors. The finite displacement of these magnetic currents from the closing conductors in Fig. 1.2 is only for
Fig. 1.1. Top and side views of the $TM_{01}$ to $TE_{10}$ mode converter.
Fig. 1.2. Top and side views of the situation equivalent to that of Fig. 1.1.
the purpose of illustration. The magnetic currents $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are given by [1, eqs. (2.11) and (2.12)] in which the $V$'s are the elements of the column vector on the left-hand side of [1, eq. (2.22)]. In [1, eq. (2.22)], $Y^1$, $Y^2$, and $Y^3$ are the admittance matrices for regions 1, 2, and 3, respectively. The column vector on the right-hand side of [1, eq. (2.22)] is called the excitation vector. The matrices $Y^1$ and $Y^2$ are treated in Chapter 2, the matrix $Y^3$ is treated in Chapter 3, and the excitation vector is treated in Chapter 4.

After solving [1, eq. (2.22)] for the $V$'s which determine $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ according to [1, eqs. (2.11) and (2.12)], we find the electromagnetic fields in regions 1, 2, and 3. Expressions for the fields in regions 1 and 2 are obtained in Chapter 5. The field in region 3 due to the combination of $-\mathcal{M}^{(1)}$ and $-\mathcal{M}^{(2)}$ is considered in Chapter 6. An expression is obtained for this field below the apertures where $z < -c/2$. Expressions are also obtained for this field in the apertures. In Chapter 7, numerical results are presented for the power transmitted into the rectangular waveguides, the power reflected back into the circular waveguide, and the magnitudes of the $\phi$- and $z$-components of the electric field in one of the apertures when a $TE_{01}$ wave is incident in the circular waveguide and when the loads $Z_1$ and $Z_2$ that terminate the rectangular waveguides are both matched loads.

In Appendix A, the expansion functions $\{\mathcal{M}_m^{1TM}, \mathcal{M}_m^{1TE}, \mathcal{M}_m^{2TM}, \mathcal{M}_m^{2TE}\}$ are ordered so that each one of them can be identified by means of a single positive integer rather than the attached combination of subscripts and superscripts. In Appendix B, a numerical procedure for obtaining roots of Bessel functions and their derivatives is described. Heretofore, the excitation has simply been a $z$-traveling $TM_{01}$ wave in the circular waveguide. In Appendix C, the response due to this excitation is used to find the response due to excitation by a transverse sheet of $TM_{01}$ electric current between two impedance loads as shown in Fig. C.1.
Chapter 2

The Admittance Matrices for the Rectangular Waveguides

The admittance matrix for region 1, the left-hand rectangular waveguide in Fig. 1.2, is $Y^1$ given by [1, eq. (2.25)] where the $Y$'s on the right-hand side of [1, eq. (2.25)] are approximated by $\hat{Y}$'s given by [1, eqs. (3.44)-(3.47)]. The superscripts on the $Y$'s are the same as those on the $\hat{Y}$'s.

For convenience, [1, eqs. (3.45) and (3.46)] are repeated:

\[
\hat{Y}^{1,1TE,1TM}_{ij} = 0 \quad (2.1)
\]
\[
\hat{Y}^{1,1TM,1TE}_{ij} = 0. \quad (2.2)
\]

In [1, eqs. (3.44) and (3.47)], we have [1, eq. (A.12)]

\[
\gamma_{pq} = \begin{cases} 
  j\beta_{pq}, & k_{pq} < k \\
  \sqrt{k_{pq}^2 - k^2}, & k_{pq} \geq k
\end{cases} \quad (2.3)
\]

where $k = \omega \sqrt{\mu \varepsilon}$ in which $\omega$ is the angular frequency. Moreover, $\mu$ and $\varepsilon$ are, respectively, the permeability and permittivity of the medium in regions 1, 2, and 3. In (2.3),

\[
k_{pq} = \sqrt{\left(\frac{p\pi}{b}\right)^2 + \left(\frac{q\pi}{c}\right)^2} \quad (2.4)
\]
\[
\beta_{pq} = \sqrt{k^2 - k_{pq}^2}. \quad (2.5)
\]
There is a correspondence between each pair of integers \((p, q)\) used in [1, eqs. (3.44) and (3.47)] and the subscript \(j\) in [1, eqs. (3.44) and (3.47)]. This correspondence is described in Appendix A. If \((p, q) = (1, 0)\), then, because the \(TE_{10}\) mode propagates in the rectangular waveguide, \(k > k_{10}\) and substitution of (2.3) into [1, eq. (3.47)] and subsequent multiplication by \(-j\eta\) where \(\eta = \sqrt{\mu/\varepsilon}\) gives

\[-j\eta Y_{ij}^{1,1TE,1TE} = -j \beta_{10} (\cos \beta_{10} x_1 + j Z_1 Y_{10}^{TE} \sin \beta_{10} x_1) \delta_{ij},\]

\((p, q) = (1, 0)\) \hspace{1cm} (2.6)

where

\[x_1 = L_1 - x_0\] \hspace{1cm} (2.7)

\[x_0 = a \sin \phi_0\] \hspace{1cm} (2.8)

\[\phi_0 = \sin^{-1} \frac{b}{2a}\] \hspace{1cm} (2.9)

\(Y_{10}^{TE}\) is the characteristic admittance of the \(TE_{10}\) mode in the rectangular waveguide [1, eq. (A.25)]

\[Y_{10}^{TE} = \frac{\beta_{10}}{k\eta},\] \hspace{1cm} (2.10)

and \(\delta_{ij}\) is the Kronecker delta function given by

\[\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.\] \hspace{1cm} (2.11)

The identities [2, formulas 6.5.6 and 654.7] were used to obtain (2.6). In (2.6), the subscript \(j\) is not to be confused with the other \(j\)'s. Each of these other \(j\)'s is \(\sqrt{-1}\). Because the \(TE_{pq}\) mode does not propagate in the rectangular waveguide when \((p, q) \neq (1, 0)\), \(\gamma_{pq}\) is real when \((p, q) \neq (1, 0)\) and, from [1, eq. (3.47)],

\[-j\eta Y_{ij}^{1,1TE,1TE} = -\frac{\gamma_{pq}}{k} \delta_{ij}, \quad (p, q) \neq (1, 0).\] \hspace{1cm} (2.12)

The factor \(-j\) on the left-hand side of (2.12) has rendered the right-hand side of (2.12) real. The factor \(\eta\) on the left-hand sides of (2.6) and (2.12) has
rendered the right-hand sides of (2.6) and (2.12) independent of \( \eta \). Multiplication of [1, eq. (3.44)] by \(-j\eta\) gives

\[
-j\eta Y_{ij}^{1,1TM,1TM} = \frac{k}{\eta_{pq}} \delta_{ij}.
\] (2.13)

Because all the \( TM \) modes in the rectangular waveguide are evanescent, \( \gamma_{pq} \) is real in (2.13) so that the right-hand side of (2.13) is real.

The admittance matrix for region 2, the right-hand rectangular waveguide in Fig. 1.2, is \( Y_2 \) given by [1, eq. (2.27)] where the \( Y \)'s on the right-hand side are approximated by \( \tilde{Y} \)'s given by [1, eqs. (3.49)-(3.52)]. Similar to (2.1), (2.2), (2.6), (2.7), (2.12), and (2.13), we have

\[
\tilde{Y}_{ij}^{2,2TE,2TM} = 0
\] (2.14)

\[
\tilde{Y}_{ij}^{2,2TM,2TE} = 0
\] (2.15)

\[
-j\eta \tilde{Y}_{ij}^{2,2TE,2TE} = -j \beta_{10} \left( \cos \beta_{10} x_2 + Z_2 Y_{10}^{TE} \sin \beta_{10} x_2 \right) \frac{k}{j \sin \beta_{10} x_2 + Z_2 Y_{10}^{TE} \cos \beta_{10} x_2} \delta_{ij},
\] \((p, q) = (1, 0)\) (2.16)

\[
x_2 = L_2 - x_o
\] (2.17)

\[
-j\eta \tilde{Y}_{ij}^{2,2TE,2TE} = -\frac{\gamma_{pq}}{k} \delta_{ij}, \quad (p, q) \neq (1, 0)
\] (2.18)

\[
-j\eta \tilde{Y}_{ij}^{2,2TM,2TM} = \frac{k}{\gamma_{pq}} \delta_{ij}.
\] (2.19)
Chapter 3

The Admittance Matrix for the Circular Waveguide

The admittance matrix for region 3, the circular waveguide in Fig. 1.2, is $Y^3$ given by [1, eq. (2.29)] where the Y's on the right-hand side of [1, eq. (2.29)] are given by [1, eqs. (4.93), (4.111), (4.112), and (4.113)] in which $T$, $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$ are given by [1, eqs. (4.94)–(4.99)]. The previously mentioned equations for the Y's are recast as

$$-j\eta Y_{ij}^{3,\alpha TM,\gamma TM} = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \hat{T}\{W_8\hat{S}_1 + W_9\hat{S}_3 - W_{10}\hat{S}_4 - W_{11}\hat{S}_5\}$$

$$-j\eta Y_{ij}^{3,\alpha TE,\gamma TM} = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \hat{T}\{W_9\hat{S}_1 - W_8\hat{S}_3 - W_{11}\hat{S}_4 + W_{10}\hat{S}_5\}$$

$$-j\eta Y_{ij}^{3,\alpha TM,\gamma TE} = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \hat{T}\{W_{10}\hat{S}_1 + W_{11}\hat{S}_3 + W_8\hat{S}_4 + W_9\hat{S}_5\}$$

$$-j\eta Y_{ij}^{3,\alpha TE,\gamma TE} = \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \hat{T}\{W_{11}\hat{S}_1 - W_{10}\hat{S}_3 + W_9\hat{S}_4 - W_8\hat{S}_5\}$$

where

$$\hat{T} = \frac{2\pi}{k\alpha(k_x b)(k_y b)}\sqrt{\epsilon_m\epsilon_n\epsilon_p\epsilon_q}$$

$$W_8 = \frac{nq b}{c}$$

$$W_9 = mq$$
\[ W_{10} = np \quad (3.8) \]
\[ W_{11} = \frac{mpc}{b} \quad (3.9) \]
\[ \dot{S}_1 = (z_1 - z_2)\dot{\phi}^{\alpha \gamma 2} \quad (3.10) \]
\[ \dot{S}_3 = z_3\dot{\phi}^{\alpha \gamma 1} \quad (3.11) \]
\[ \dot{S}_4 = z_4\dot{\phi}^{\alpha \gamma 3} \quad (3.12) \]
\[ \dot{S}_5 = z_5\dot{\phi}^{\alpha \gamma 4} \quad (3.13) \]
\[ \dot{\phi}^{\alpha \gamma 1} = (-1)^{\alpha} \dot{\phi}^{\alpha \gamma 1} \quad (3.14) \]
\[ \dot{\phi}^{\alpha \gamma 2} = \dot{\phi}^{\alpha \gamma 2} \quad (3.15) \]
\[ \dot{\phi}^{\alpha \gamma 3} = (-1)^{\gamma} \dot{\phi}^{\alpha \gamma 3} \quad (3.16) \]
\[ \dot{\phi}^{\alpha \gamma 4} = (-1)^{\alpha + \gamma} \dot{\phi}^{\alpha \gamma 4} \quad (3.17) \]
\[ z_1 = \frac{(ka)^2(k_{rs}^{TM}a)^2J_r^r(k_{rs}^{TM}a)\dot{z}^{(1)}(r)}{x_{rs}^2J_{r+1}^r(x_{rs})} \frac{\epsilon_r}{2} \quad (3.18) \]
\[ z_2 = \frac{r^2J_r^r(k_{rs}^{TE}a)\dot{z}^{(2)}(r)}{(x_{rs}^2 - r^2)J_r^r(x_{rs})} \quad (3.19) \]
\[ z_3 = \frac{(k_{rs}^{TE}a)2rJ_r^r(k_{rs}^{TE}a)\dot{z}^{(3)}(r)}{(x_{rs}^2 - r^2)J_r^r(x_{rs})} \frac{\sin \phi_o}{\phi_o} \quad (3.20) \]
\[ z_4 = \frac{(k_{rs}^{TE}a)2rJ_r^r(k_{rs}^{TE}a)\dot{z}^{(4)}(r)}{(x_{rs}^2 - r^2)J_r^r(x_{rs})} \frac{\sin \phi_o}{\phi_o} \left(\frac{\epsilon_r}{2}\right) \quad (3.21) \]
\[ z_5 = \frac{(k_{rs}^{TE}a)4J_r^r(k_{rs}^{TE}a)\dot{z}^{(5)}(r)}{(x_{rs}^2 - r^2)J_r^r(x_{rs})} \frac{\sin \phi_o}{\phi_o} \left(\frac{\epsilon_r}{2}\right) \quad (3.22) \]
\[ z^{(1)} = \frac{4z^{(1)}}{c^2 \gamma_{rs}^{TM} a} \quad (3.23) \]
\[ z^{(2)} = \frac{4z^{(2)} \gamma_{rs}^{TE} a}{c^2} \quad (3.24) \]
\[ z^{(3)} = \frac{4z^{(3)}}{c^2} \quad (3.25) \]
\[ z^{(4)} = \frac{4z^{(4)}}{c^2} \quad (3.26) \]
\[ z^{(5)} = \frac{4z^{(5)}}{c^2 \gamma_{rs}^{TE} a} \quad (3.27) \]
In (3.5)-(3.9), $j$ determines $p$ and $q$ in the manner described in Appendix A; $i$ determines $m$ and $n$ in the same manner. In obtaining (3.1)-(3.4), we started the index $r$ of summation in [1, eqs. (4.96) and (4.97)] at 0 instead of 1. This was possible because the $r = 0$ terms so introduced are zero. We multiplied [1, eq. (4.93)] by $\sqrt{\epsilon_m \epsilon_n \epsilon_q / 16}$, [1, eq. (4.111)] by $\sqrt{\epsilon_p \epsilon_q / 4}$, and [1, eq. (4.112)] by $\sqrt{\epsilon_m \epsilon_n / 4}$. This was possible because none of the indices $m$, $n$, $p$, and $q$ which appear in these multipliers is ever zero.

### 3.1 Evaluation of the $\hat{\phi}$'s in (3.10)-(3.13)

The $\hat{\phi}$'s in (3.10)-(3.13) are given by (3.14)-(3.17). Substitution of [1, eqs. (E.31)-(E.34)] into (3.14)-(3.17) gives

\[
\hat{\phi}^{\alpha \gamma_1} = (-1)^{\alpha + \gamma} \left\{ \phi_p^{(2)} \phi^{\alpha 2 \gamma_1} - \phi_p^{(1)} \phi^{\alpha 2 \gamma_2} \right\} 
\]

(3.28)

\[
\hat{\phi}^{\alpha \gamma_2} = \phi_p^{(2)} \phi^{\alpha 1 \gamma_2} + \phi_p^{(1)} \phi^{\alpha 1 \gamma_1} 
\]

(3.29)

\[
\hat{\phi}^{\alpha \gamma_3} = \phi_p^{(4)} \phi^{\alpha 1 \gamma_1} - \phi_p^{(3)} \phi^{\alpha 1 \gamma_2} 
\]

(3.30)

\[
\hat{\phi}^{\alpha \gamma_4} = (-1)^{\alpha + \gamma} \left\{ \phi_p^{(4)} \phi^{\alpha 2 \gamma_2} + \phi_p^{(3)} \phi^{\alpha 2 \gamma_1} \right\} 
\]

(3.31)

For $\gamma = \alpha$, we have [1, eqs. (E.46)-(E.49)]

\[
\hat{\phi}^{\alpha 1 \gamma_1} = \phi_m^{(1)}, \quad \gamma = \alpha 
\]

(3.32)

\[
\hat{\phi}^{\alpha 2 \gamma_1} = \phi_m^{(3)}, \quad \gamma = \alpha 
\]

(3.33)

\[
\hat{\phi}^{\alpha 1 \gamma_2} = \phi_m^{(2)}, \quad \gamma = \alpha 
\]

(3.34)

\[
\hat{\phi}^{\alpha 2 \gamma_2} = \phi_m^{(4)}, \quad \gamma = \alpha. 
\]

(3.35)

For $\gamma \neq \alpha$, we have [1, eqs. (E.53)-(E.56)]

\[
\hat{\phi}^{\alpha 1 \gamma_1} = (-1)^{\gamma} \left\{ \phi_m^{(2)} \sin \frac{rb}{x_0} - \phi_m^{(1)} \cos \frac{rb}{x_0} \right\}, \quad \gamma \neq \alpha 
\]

(3.36)

\[
\hat{\phi}^{\alpha 2 \gamma_1} = (-1)^{\gamma} \left\{ \phi_m^{(4)} \sin \frac{rb}{x_0} - \phi_m^{(3)} \cos \frac{rb}{x_0} \right\}, \quad \gamma \neq \alpha 
\]

(3.37)

\[
\hat{\phi}^{\alpha 1 \gamma_2} = (-1)^{\gamma} \left\{ \phi_m^{(2)} \cos \frac{rb}{x_0} + \phi_m^{(1)} \sin \frac{rb}{x_0} \right\}, \quad \gamma \neq \alpha 
\]

(3.38)

\[
\hat{\phi}^{\alpha 2 \gamma_2} = (-1)^{\gamma} \left\{ \phi_m^{(4)} \cos \frac{rb}{x_0} + \phi_m^{(3)} \sin \frac{rb}{x_0} \right\}, \quad \gamma \neq \alpha. 
\]

(3.39)
In (3.28)-(3.31), the $\phi_p$'s are given by [1, eqs. (E.23)-(E.26)]

$$\phi_p^{(1)} = \frac{b}{2x_o} \left\{ \frac{\sin A^-}{A^-} - \frac{\sin A^+}{A^+} \right\}$$  \hspace{1cm} (3.40)

$$\phi_p^{(2)} = \frac{b}{2x_o} \left\{ \frac{\sin^2(A^-/2)}{(A^-/2)} + \frac{\sin^2(A^+/2)}{(A^+/2)} \right\}$$  \hspace{1cm} (3.41)

$$\phi_p^{(3)} = \frac{b}{2x_o} \left\{ \frac{-\sin^2(A^-/2)}{(A^-/2)} + \frac{\sin^2(A^+/2)}{(A^+/2)} \right\}$$  \hspace{1cm} (3.42)

$$\phi_p^{(4)} = \frac{b}{2x_o} \left\{ \frac{\sin A^-}{A^-} + \frac{\sin A^+}{A^+} \right\}$$  \hspace{1cm} (3.43)

where

$$A^+ = p\pi + \frac{rb}{x_0}$$  \hspace{1cm} (3.44)

$$A^- = p\pi - \frac{rb}{x_0}.$$  \hspace{1cm} (3.45)

The $\phi$'s in (3.36)-(3.39) are given by the right-hand sides of (3.40)-(3.43) with $A^+$ and $A^-$ replaced by the right-hand sides of (3.44) and (3.45) with $p$ replaced by $m$.

### 3.2 Evaluation of the $z$'s in (3.10)-(3.13)

The $z$'s in (3.10)-(3.13) are given by (3.18)-(3.22). In (3.18)-(3.22), $\epsilon_r$ is Neumann's number given by [1, eq. (B.9)]

$$\epsilon_r = \begin{cases} 1, & r = 0 \\ 2, & r = 1, 2, \ldots \end{cases}$$  \hspace{1cm} (3.46)

$J_r$ is the Bessel function of the first kind of order $r$, $x_{rs}$ is the $s$th root of $J_r$, $J'_r$ is the derivative of $J_r$ with respect to its argument, and $x'_{rs}$ is the $s$th root of $J'_r$. The roots \{${x_{rs}, r = s = 1, 2, \ldots}$\} and \{${x'_{rs}, r = 1, 2, \ldots}$\} are ordered such that

$$0 < x_{r1} < x_{r2} < x_{r3} \cdots$$  \hspace{1cm} (3.47)

$$0 < x'_{r1} < x'_{r2} < x'_{r3} \cdots$$  \hspace{1cm} (3.48)
Still in (3.18–3.22), we have [1, eqs. (B.7) and (B.41)]

\[ k_{rs}^{TM} = \frac{x_{rs}}{a} \quad (3.49) \]

\[ k_{rs}^{TE} = \frac{x_{rs}'}{a}. \quad (3.50) \]

Since \( J_r(x_{rs}) = 0 \), we have [3, formula 9.1.27]

\[ J_r'(x_{rs}) = -J_{r+1}(x_{rs}). \quad (3.51) \]

Substitution of (3.49)–(3.51) into (3.18)–(3.22) yields

\[ z_1 = (ka)^2 \left( \frac{\epsilon_r}{2} \right) z^{(1)} \quad (3.52) \]

\[ z_2 = \frac{x_{rs}^2 z^{(2)}}{x_{rs}^2 - r^2} \quad (3.53) \]

\[ z_3 = \frac{r x_{rs}' z^{(3)}}{x_{rs}^2 - r^2} \left( \frac{\sin \phi_0}{\phi_0} \right) \quad (3.54) \]

\[ z_4 = \frac{r x_{rs}' z^{(4)}}{x_{rs}^2 - r^2} \left( \frac{\epsilon_r}{2} \right) \left( \frac{\sin \phi_0}{\phi_0} \right) \quad (3.55) \]

\[ z_5 = \frac{x_{rs}^2 z^{(5)}}{x_{rs}^2 - r^2} \left( \frac{\epsilon_r}{2} \right) \left( \frac{\sin \phi_0}{\phi_0} \right)^2. \quad (3.56) \]

The \( z \)'s in (3.52)–(3.56) are given by (3.23)–(3.27) where [1, eqs. (B.24) and (B.53)]

\[ \gamma_{rs}^{TM} a = \begin{cases} 
   j \beta_{rs}^{TM} a, & x_{rs} < ka \\
   \sqrt{x_{rs}^2 - (ka)^2}, & x_{rs} \geq ka 
\end{cases} \quad (3.57) \]

\[ \gamma_{rs}^{TE} a = \begin{cases} 
   j \beta_{rs}^{TE} a, & x_{rs}' < ka \\
   \sqrt{x_{rs}'^2 - (ka)^2}, & x_{rs}' \geq ka 
\end{cases} \quad (3.58) \]

where

\[ \beta_{rs}^{TM} a = \sqrt{(ka)^2 - x_{rs}^2} \quad (3.59) \]

\[ \beta_{rs}^{TE} a = \sqrt{(ka)^2 - x_{rs}'^2}. \quad (3.60) \]
From [1, eqs. (F.60), (F.61), (F.76), (F.81), (F.118), and (F.119)],

\[ z^{(1)} = \frac{j}{4} \left( D^{TM} G^{TM} + c^2 F^{TM} \right) \]  \hspace{1cm} (3.61)  \\
\[ z^{(2)} = \frac{j}{4} \left( D^{TE} G^{TE} + c^2 F^{TE} \right) \]  \hspace{1cm} (3.62)  \\
\[ z^{(3)} = \frac{1}{4} \left( D^{(3)} G^{(3)} + c^2 F^{(3)} \right) \]  \hspace{1cm} (3.63)  \\
\[ z^{(4)} = -\frac{1}{4} \left( D^{TE} G^{(4)} + c^2 F^{(4)} \right) \]  \hspace{1cm} (3.64)  \\
\[ z^{(5)} = \frac{a_{s s}^{TE} Z^{TE}}{(k^{TE})^2} + \frac{j}{4} \left( D^{(3)} G^{(4)} + c^2 F^{(5)} \right) \]  \hspace{1cm} (3.65)  \\

where the D's, the G's, the F's, and \( z_{ss}^{TE} \) are dealt with in [1, Appendix F]. In view of [1, eq. (F.83)] and (3.50), substitution of (3.61)–(3.65) into (3.23)–(3.27) gives

\[ \hat{z}^{(1)} = j \left( \frac{F^{TM} + \hat{G}^{TM}_q \hat{D}^{TM}_n}{\gamma^{TM}_{rs} a} \right) \]  \hspace{1cm} (3.66)  \\
\[ \hat{z}^{(2)} = j \left( F^{TE} + \hat{G}^{TE}_q \hat{D}^{TE}_n \right) \gamma^{TE}_{rs} a \]  \hspace{1cm} (3.67)  \\
\[ \hat{z}^{(3)} = F^{(3)} + \hat{G}^{TE}_q \hat{D}^{(3)}_n \]  \hspace{1cm} (3.68)  \\
\[ \hat{z}^{(4)} = -\left( F^{(4)} + \hat{G}^{TE}_q \hat{D}^{(4)}_n \right) \]  \hspace{1cm} (3.69)  \\
\[ \hat{z}^{(5)} = \frac{4a_{s s}^{TE}}{c^2 x_{rs}^2} + j \left( F^{(5)} + \hat{G}^{(4)}_q \hat{D}^{(3)}_n \right) \frac{1}{\gamma^{TE}_{rs} a} \]  \hspace{1cm} (3.70)  \\

where

\[ \hat{G}^{\delta}_q = \frac{G^{\delta}}{c}, \quad \delta = TM, TE \]  \hspace{1cm} (3.71)  \\
\[ \hat{D}^{\delta}_n = \frac{D^{\delta}}{c}, \quad \delta = TM, TE \]  \hspace{1cm} (3.72)  \\
\[ \hat{D}^{(3)}_n = \frac{D^{(3)}}{c} \]  \hspace{1cm} (3.73)  \\
\[ \hat{G}^{(4)}_q = \frac{G^{(4)}}{c}. \]  \hspace{1cm} (3.74)
3.2.1 The $T.M$ Quantity $\hat{z}^{(1)}$ for $x_{rs} < ka$

With $x_{rs} < ka$, substitution of (3.57) into (3.66) gives

$$\hat{z}^{(1)} = \frac{F_{T.M} + \hat{G}_{q}^{T.M} \hat{D}_{n}^{T.M}}{\beta_{rs}^{T.M} a}. \quad (3.75)$$

Substituting [1, eq. (F.87)] into (3.71), we obtain

$$\hat{G}_{q}^{\delta} = \frac{\sin(q^{\delta}-c) \cos(\beta_{rs}^{\delta} L_{3}^{\delta}) - 2 \sin^{2} \left(\frac{q^{\delta}-c}{2}\right) \sin(\beta_{rs}^{\delta} L_{3}^{\delta})}{q^{\delta}-c} \sin(q^{\delta}+c) \cos(\beta_{rs}^{\delta} L_{3}^{\delta}) + 2 \sin^{2} \left(\frac{q^{\delta}+c}{2}\right) \sin(\beta_{rs}^{\delta} L_{3}^{\delta})}{q^{\delta}+c} \quad (3.76)$$

where $\delta$ may be either $T.M$ or $T.E$ and where [1, eqs. (F.11), (F.23), and (F.24)]

$$L_{3}^{\delta} = L_{3} + \frac{c}{2} \quad (3.77)$$

$$q^{\delta}-c = q\pi - \beta_{rs}^{\delta} c \quad (3.78)$$

$$q^{\delta}+c = q\pi + \beta_{rs}^{\delta} c. \quad (3.79)$$

Using (3.77)-(3.79) and [2, formulas 403.02, 401.03, and 401.04], we reduce (3.76) to

$$\hat{G}_{q}^{\delta} = \begin{cases} \frac{\sin \left(\frac{q^{\delta}+c}{2}\right)}{q^{\delta}+c} + (-1)^{q} \frac{\sin \left(\frac{q^{\delta}-c}{2}\right)}{q^{\delta}-c} \cos \left(\frac{\beta_{rs}^{\delta} L_{3} - q\pi}{2}\right) \end{cases} \quad (3.80)$$

Substituting [1, eq. (F.85)] into (3.72), we obtain

$$\hat{D}_{n}^{\delta} = \begin{cases} \frac{-j \sin(n^{\delta}-c) - 2 \sin^{2} \left(\frac{n^{\delta}-c}{2}\right)}{n^{\delta}-c} + \frac{-j \sin(n^{\delta}+c) + 2 \sin^{2} \left(\frac{n^{\delta}+c}{2}\right)}{n^{\delta}+c} \end{cases} e^{-j\beta_{rs}^{\delta} L_{3}^{\delta}} \quad (3.81)$$
where \( n^{-c} \) and \( n^{+c} \) are given by (3.78) and (3.79) with \( q \) replaced by \( n \). In the same manner as we reduced (3.76) to (3.80), we can reduce (3.81) to

\[
\hat{D}_n^\delta = - \left\{ \frac{\sin\left(\frac{n^{+c}}{2}\right)}{n^{+c}} + (-1)^n \frac{\sin\left(\frac{n^{-c}}{2}\right)}{n^{-c}} \right\} 
\cdot \left\{ \sin\left(\beta_{rs}^\delta L_3 - \frac{n\pi}{2}\right) + j \cos\left(\beta_{rs}^\delta L_3 - \frac{n\pi}{2}\right) \right\}
\] (3.82)

As for the quantity \( F^TM \) in (3.75), we have [1, eq. (F.79)]

\[
F^\delta = -f(n^{-c}, -q^{-c}) + f(n^{+c}, q^{-c}) - f(n^{+c}, q^{+c}) + f(n^{-c}, -q^{+c})
\] (3.83)

where \( \delta \) may be either \( TM \) or \( TE \) and [1, eq. (F.97)]

\[
f(x, y) = \begin{cases} 
-\sin x, & \frac{y}{x} \neq 0, |y| \leq \frac{\pi}{2} \\
(-1)^I \sin y, & \frac{y}{x} \neq 0, |y| \leq \frac{\pi}{2} \\
y - \sin y, & \frac{y}{x} \neq 0, |y| > 0.1 \\
\frac{y - y^3 + y^5}{3! - 5! + 7!}, & \frac{y}{x} \neq 0, |y| \leq 0.1
\end{cases}
\] (3.84)

where \( I \) is the integer that satisfies

\[
x + y = I\pi.
\] (3.85)

When \( x_{rs} < ka \), the \( TM \) quantity \( \tilde{\varepsilon}^{(1)} \) is now given by (3.75) in which \( \beta_{rs}^{TM}, \hat{G}_q^{TM}, \hat{D}_n^{TM}, \) and \( F^TM \) are given by (3.59), (3.80), (3.82) and (3.83), respectively.

### 3.2.2 The TM Quantity \( \tilde{\varepsilon}^{(1)} \) for \( x_{rs} > ka \)

When \( x_{rs} > ka \), we proceed to evaluate expression (3.66) for \( \tilde{\varepsilon}^{(1)} \), which contains the quantities \( \hat{G}_q^{TM}, \hat{D}_n^{TM}, \) and \( F^TM \). These quantities with the superscript \( TM \) replaced by \( \delta \) are, according to (3.71), (3.72), and [1, eqs. (F.104),...
(F.100), (F.101), and (F.107)-(F.109)], given by

\[ \hat{G}_q^\delta = c_q \left\{ \sinh(gL_3^+) - (-1)^q \sinh \left( g(L_3^+ - c) \right) \right\} \]  
\[ \hat{D}_n^\delta = 2j c_n e^{-g(L_3^+ - \frac{\delta}{2})} \left\{ \begin{array}{l} - \sinh \left( \frac{2q}{2} \right), \quad n \text{ even} \\ \cosh \left( \frac{2q}{2} \right), \quad n \text{ odd} \end{array} \right\} \]  
\[ F^\delta = j \left\{ \begin{array}{l} (-1)^n c_n c_q \sinh(gc), \quad q \neq n \\ (-1)^n c_n c_q \sinh(gc) - c_q, \quad q = n \neq 0 \\ \frac{4(\sinh(gc) - gc)}{(gc)^2}, \quad q = n = 0 \end{array} \right\} \]

where

\[ g = \gamma_{rs}^\delta \]  
\[ c_n = \frac{2\gamma_{rs}^\delta c}{(n\pi)^2 + (\gamma_{rs}^\delta c)^2}. \]

Furthermore, \( c_q \) is the right-hand side of (3.90) with \( n \) replaced by \( q \). The truncated series approximation inherent in [1, eq. (F.110)] is introduced later in this section. The case where \( x_{rs} = ka \) is not allowed because, if \( x_{ra} = ka \), then, according to (3.57), \( \gamma_{rs}^{TMa} \) would be zero so that division by \( \gamma_{rs}^{TMa} \) in (3.66) would be impossible.

Substituting (3.77) into (3.86) and using [2, formulas 651.06 and 651.07], we obtain

\[ \hat{G}_q^\delta = 2c_q \left\{ \begin{array}{l} \sinh(gc/2) \cosh(gL_3) \quad q \text{ even} \\ \sinh(gL_3) \cosh(gc/2) \quad q \text{ odd} \end{array} \right\}. \]

Substitution of (3.77) into (3.87) gives

\[ \hat{D}_n^\delta = 2j c_n e^{-gL_3} \left\{ \begin{array}{l} - \sinh(gc/2), \quad n \text{ even} \\ \cosh(gc/2), \quad n \text{ odd} \end{array} \right\}. \]

Combining (3.88), (3.91), and (3.92) and using [2, formulas 652.12 and 654.5] to simplify the result, we obtain

\[ j \left( F^\delta + \hat{G}_q^\delta \hat{D}_n^\delta \right) = c_{nq} + c_n \left\{ \begin{array}{l} 1, \quad q = n \neq 0 \\ 0, \quad \text{otherwise} \end{array} \right\} \]
where

\[
c_{nq} = \begin{cases} 
  c_n c_q z_{ee}, & \text{n even} \\
  q \text{ even} \\
  q \neq n \\
  c_n c_q z_{oe}, & \text{n even} \\
  q = n \neq 0 \\
  c_n c_q z_o, & q = n = 0 \\
  c_n c_q z_{oo}, & \text{n odd} \\
  q \text{ even} \\
  q \text{ odd} \\
  \text{n odd} \\
  q \text{ odd} 
\end{cases}
\]  

(3.94)

In (3.94),

\[
z_{ee} = 2 \left\{ e^{-2gL_3} \sinh \frac{gc}{2} - e^{-\frac{2g}{2}} \right\} \sinh \frac{gc}{2} 
\]  

(3.95)

\[
z_o = z_{ee} + gc 
\]  

(3.96)

\[
z_{oe} = -e^{-2gL_3} \sinh gc 
\]  

(3.97)

\[
z_{oo} = 2 \left\{ e^{-2gL_3} \cosh \frac{gc}{2} - e^{-\frac{2g}{2}} \right\} \cosh \frac{gc}{2} 
\]  

(3.98)

The value of \(\sinh (gc/2)\) is excessively large when \(gc/2\) is only moderately large. Moreover, \(z_o\) approaches zero more rapidly than \(gc\) as \(gc\) approaches zero. To avoid computational difficulties, we replace (3.95)-(3.98) by

\[
z_{ee} = \begin{cases} 
  e^{-gc} - e^{-2gL_3} - 1 + \frac{1}{2} \left\{ e^{-g(2L_3-c)} + e^{-g(2L_3+c)} \right\}, & gc \geq 1 \\
  2 \left\{ e^{-2gL_3} \sinh \frac{2g}{2} - e^{-\frac{2g}{2}} \right\} \sinh \frac{2g}{2}, & gc < 1 \\
  e^{-gc} - e^{-2gL_3} - 1 + \frac{1}{2} \left\{ e^{-g(2L_3-c)} + e^{-g(2L_3+c)} \right\} + gc, & gc \geq 1 \\
  2 \left\{ e^{-2gL_3} \sinh \frac{2g}{2} - e^{-\frac{2g}{2}} \right\} \sinh \frac{2g}{2} + gc 
\end{cases} 
\]  

(3.99)

\[
z_o = \begin{cases} 
  2 \left\{ e^{-\frac{2g}{2}} \sinh \frac{2g}{2} + e^{-2gL_3} \sinh \frac{2g}{2} \right\} \sinh \frac{2g}{2} \\
  - \left( \frac{(2g)^3}{3!} + \frac{(2g)^5}{5!} + \frac{(2g)^7}{7!} \right) 
\end{cases} 
\]  

(3.100)

\[\begin{array}{c}
\text{, } gc < 0.01
\end{array}\]
\[ z_{oe} = \begin{cases} \frac{1}{2} \left\{ e^{-2(2L_3+c)} - e^{-2(2L_3-c)} \right\}, & gc \geq 1 \\ -e^{-2L_3} \sinh gc, & gc < 1 \end{cases} \] (3.101)

\[ z_{oo} = \begin{cases} e^{-2L_3} - e^{-2c} - 1 + \frac{1}{2} \left\{ e^{-2(2L_3-c)} + e^{-2(2L_3+c)} \right\}, & gc \geq 1 \\ 2 \left\{ e^{-2L_3} \cosh \frac{2c}{2} - e^{-2c} \right\} \cosh \frac{2c}{2}, & gc < 1 \end{cases} \] (3.102)

We used the approximation [2, formula 657.1]

\[ \sinh x - x = \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \] (3.103)

to obtain \( z_0 \) of (3.100) for \( gc < 0.01 \). From Fig. 1.2, \( L_3 \geq c/2 \) so that all the arguments of the exponentials in (3.99)-(3.102) are less than or equal to zero. The exponentials in (3.99)-(3.102) will be excessively small when \( gL_3 \) is only moderately large. However, this will not cause any difficulty if we use a computing system which treats an underflow by setting the number equal to zero and proceeding without an error message. The right-hand side of (3.94) cannot be evaluated when \( gc = 0 \) and when \( n = 0 \) or \( q = 0 \) because \( c_n = 2/(gc) \) when \( n = 0 \) and \( c_q = 2/(gc) \) when \( q = 0 \). However, the right-hand side of (3.94) remains finite as \( gc \) approaches zero when \( n = 0 \) or \( q = 0 \).

When \( x_{rs} > ka \), the \( TM \) quantity \( z^{(1)} \) is now given by (3.66) in which \( \gamma_{rs}^{TM} a \) and \( j(F^{TM} + \hat{G}_q^{TM} \hat{D}_n^{TM}) \) are given by (3.57) and (3.93), respectively. In (3.93), \( c_n \) and \( \gamma_{nq} \) are given by (3.90) and (3.94), respectively. In (3.94), \( z_{cc}, z_{o}, z_{oe}, \) and \( z_{oo} \) are given by (3.99)-(3.102) where \( g \) is given by (3.89).

### 3.2.3 The \( TE \) Quantities \( z^{(2)}-z^{(5)} \) for \( x_{rs} < ka \)

When \( x_{rs} < ka \), \( \gamma_{rs}^{TE} a \) is given by (3.58) so that expressions (3.67)-(3.70) for \( z^{(2)}-z^{(5)} \) become

\[ z^{(2)} = - \left( F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE} \right) \beta_{rs}^{TE} a \] (3.104)

\[ z^{(3)} = F^{(3)} + \hat{G}_q^{(3)} \hat{D}_n^{(3)} \] (3.105)

\[ z^{(4)} = - \left( F^{(4)} + \hat{G}_q^{(4)} \hat{D}_n^{(4)} \right) \beta_{rs}^{TE} a \] (3.106)

\[ z^{(5)} = \frac{F^{(5)} + \hat{G}_q^{(4)} \hat{D}_n^{(3)}}{\beta_{rs}^{TE} a} + \begin{cases} \frac{2a}{c_{x_{rs}}^2}, & n = q \neq 0 \\ 0, & \text{otherwise} \end{cases} \] (3.107)
We obtained (3.107) by substituting [1, eq. (F.121)] for $z_{rr}^{TE}$.

In (3.104), $\beta_{rr}^{TE}$, $\hat{G}_q^{TE}$, $\hat{D}_n^{TE}$, and $F^{TE}$ are given by (3.60), (3.80), (3.82), and (3.83), respectively. As for $\hat{G}_q^{(4)}$ in (3.106) and (3.107), substitution of [1, eq. (F.122)] into (3.74) gives

$$\hat{G}_q^{(4)} = \frac{\sin(q^{TE}-c)\cos(\beta_{rr}^{TE} L_3^+) - 2 \sin^2\left(\frac{q^{TE}-c}{2}\right)\sin(\beta_{rr}^{TE} L_3^+)}{q^{TE}-c} \frac{\sin(q^{TE}+c)\cos(\beta_{rr}^{TE} L_3^+) + 2 \sin^2\left(\frac{q^{TE}+c}{2}\right)\sin(\beta_{rr}^{TE} L_3^+)}{q^{TE}+c}.$$ (3.108)

Note that the right-hand side of (3.108) is that of (3.76) with $\delta$ replaced by $TE$ and with the sign of the coefficient of $1/(q^s+c)$ changed. If we recall also that (3.76) reduced to (3.80), we see that (3.108) reduces to

$$\hat{G}_q^{(4)} = \left\{ \begin{array}{c} \sin\left(\frac{q^{TE}+c}{2}\right) \sin\left(\frac{q^{TE}-c}{2}\right) \\ \frac{q^{TE}+c}{2} \end{array} + (-1)^n \frac{q^{TE}+c}{2} \right\} \cos(\beta_{rr}^{TE} L_3 - \frac{q\pi}{2}).$$ (3.109)

As for $\hat{D}_n^{(3)}$ in (3.105) and (3.107), substitution of [1, eq. (F.86)] into (3.73) gives

$$\hat{D}_n^{(3)} = \left\{ \begin{array}{c} j \sin(n^{TE}-c) + 2 \sin^2\left(\frac{n^{TE}-c}{2}\right) \\ n^{TE}-c \end{array} + \frac{-j \sin(n^{TE}+c) + 2 \sin^2\left(\frac{n^{TE}+c}{2}\right)}{n^{TE}+c} \right\} e^{-j\beta_{rr}^{TE} L_3^+}.$$ (3.110)

The right-hand side of (3.110) is that of (3.81) with $\delta$ replaced by $TE$ and with the sign of the coefficient of $1/(n^s-c)$ changed; recalling that (3.81) reduced to (3.82), we see that (3.110) reduces to

$$\hat{D}_n^{(3)} = \left\{ \sin\left(\frac{n^{TE}+c}{2}\right) - \sin\left(\frac{n^{TE}-c}{2}\right) \right\} e^{-j\beta_{rr}^{TE} L_3^+}.$$ (3.110)
\[ \cdot \left\{ \sin \left( \beta_{r,s}^{TE} L_3 - \frac{n\pi}{2} \right) + j \cos \left( \beta_{r,s}^{TE} L_3 - \frac{n\pi}{2} \right) \right\}. \]  

(3.111)

The \( F \)'s in (3.105)-(3.107) are given by [1, eqs. (F.84), (F.123), and (F.124)]

\[
F^{(3)} = f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\
+ f(n^{TE-}c, q^{TE+}c) + f(n^{TE+}c, -q^{TE+}c) 
\]  

(3.112)

\[
F^{(4)} = -f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\
+ f(n^{TE-}c, q^{TE+}c) - f(n^{TE+}c, -q^{TE+}c) 
\]  

(3.113)

\[
F^{(5)} = f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\
- f(n^{TE-}c, q^{TE+}c) - f(n^{TE+}c, -q^{TE+}c) 
\]  

(3.114)

where \( f \) is given by (3.84).

### 3.2.4 The \( TE \) Quantities \( \mathcal{Z}^{(2)}-\mathcal{Z}^{(5)} \) for \( x'_{rs} > ka \)

When \( x'_{rs} > ka \), we proceed to evaluate expressions (3.67)-(3.70) for \( \mathcal{Z}^{(2)} - \mathcal{Z}^{(5)} \). Expression (3.67) is

\[
\mathcal{Z}^{(2)} = j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE}) \gamma_{rs}^{TE} a 
\]  

(3.115)

where \( \gamma_{rs}^{TE} a \) is given by (3.58). In (3.115), \( j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE}) \) is given by the right-hand side of (3.93) with \( \delta \) replaced by \( TE \), that is, with \( g \) (which appears in the \( z \)'s of (3.95)-(3.98)) given by

\[
g = \gamma_{rs}^{TE}, 
\]  

(3.116)

with \( c_n \) given by

\[
c_n = \frac{2\gamma_{rs}^{TE} c}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2}, 
\]  

(3.117)

and with \( c_q \) given by the right-hand side of (3.117) with \( n \) replaced by \( q \). Throughout Section 3.2.4, \( g \) and \( c_n \) are given by (3.116) and (3.117). This \( g \) and this \( c_n \) are not to be confused with the \( g \) and the \( c_n \) given by (3.89) and (3.90) of Section 3.2.2.

Expression (3.68) is

\[
\mathcal{Z}^{(3)} = F^{(3)} + \hat{G}_q^{TE} \hat{D}_n^{(3)}. 
\]  

(3.118)
In (3.118), $F^{(3)}$ is given by [1, eqs. (F.111)-(F.113)]

$$
F^{(3)} = \frac{n \pi}{g c} \left\{ \begin{array}{ll}
(-1)^n c_n c_q \sinh(g c), & q \neq n \\
(-1)^n c_n c_q \sinh(g c) - c_q, & q = n \neq 0 \\
\frac{4 \{\sinh(g c) - g c\}}{(g c)^2}, & q = n = 0
\end{array} \right\}. 
$$  \hfill (3.119)

As given by (3.119), $F^{(3)} = 0$ when $q = n = 0$. The right-hand side of (3.119) was written so as to be similar to the right-hand side of (3.88). In (3.118), $\hat{G}^{TE}_q$ is given by the right-hand side of (3.91) with $g$ given by (3.116) rather than by (3.89). As for $\hat{D}^{(3)}$, in view of (3.77), substitution of [1, eqs. (F.102) and (F.103)] into (3.73) gives

$$
\hat{D}^{(3)} = 2n \pi c_n e^{-g l_3} \left\{ \begin{array}{ll}
- \sinh(g c / 2), & n \text{ even} \\
\cosh(g c / 2), & n \text{ odd}
\end{array} \right\}. 
$$  \hfill (3.120)

Comparing the quantities $F^{(3)}$, $\hat{G}^{TE}_q$, and $\hat{D}^{(3)}$ of this paragraph with the quantities $F^6$ of (3.88), $\hat{G}^6_q$ of (3.91), and $\hat{D}^6$ of (3.92) and noting that the latter quantities combined to give (3.93), we obtain

$$
F^{(3)} + \hat{G}^{TE}_q \hat{D}^{(3)} = -\frac{n \pi}{g c} \left\{ j(F^{TE} + \hat{G}^{TE} \hat{D}^{TE}) \right\} 
$$  \hfill (3.121)

where $j(F^{TE} + \hat{G}^{TE} \hat{D}^{TE})$ is given by the right-hand side of (3.93) with $\delta$ replaced by $TE$ as described in the third sentence of Section 3.2.4. Substitution of (3.121) into (3.118) gives

$$
z^{(3)} = -\frac{n \pi}{g c} \left\{ j(F^{TE} + \hat{G}^{TE} \hat{D}^{TE}) \right\}. 
$$  \hfill (3.122)

Expression (3.69) is

$$
z^{(4)} = -(F^{(4)} + \hat{G}^{(4)} \hat{D}^{TE}). 
$$  \hfill (3.123)

In (3.123), $F^{(4)}$ is given by [1, eqs. (F.129)-(F.131)]

$$
F^{(4)} = \frac{-q \pi}{g c} \left\{ \begin{array}{ll}
(-1)^n c_n c_q \sinh(g c), & q \neq n \\
(-1)^n c_n c_q \sinh(g c) - c_q, & q = n \neq 0 \\
\frac{4 \{\sinh(g c) - g c\}}{(g c)^2}, & q = n = 0
\end{array} \right\}. 
$$  \hfill (3.124)
As given by (3.124), \( F^{(4)} = 0 \) when \( q = n = 0 \). The right-hand side of (3.124) was written so as to be similar to the right-hand side of (3.88). As for \( \hat{G}_q^{(4)} \) in (3.123), substitution of [1, eq. (F.126)] into (3.74) gives

\[
\hat{G}_q^{(4)} = j \frac{q\pi c_q}{gc} \left\{ \sinh(gL_3^+) - (-1)^q \sinh(gL_3^- - c) \right\}. \tag{3.125}
\]

Substituting (3.77) into (3.125) and using [2, formulas 651.06 and 651.07], we obtain

\[
\hat{G}_q^{(4)} = j \frac{2q\pi c_q}{gc} \left\{ \begin{array}{ll}
\sinh(gc/2) \cosh(gL_3), & q \text{ even} \\
\sinh(gL_3) \cosh(gc/2), & q \text{ odd}
\end{array} \right\}. \tag{3.126}
\]

In (3.123), \( \hat{D}_n^{TE} \) is given by (3.92) with \( \delta \) replaced by \( TE \). Thus,

\[
\hat{D}_n^{TE} = 2j c_n e^{-gL_3} \left\{ \begin{array}{ll}
-\sinh(gc/2), & n \text{ even} \\
\cosh(gc/2), & n \text{ odd}
\end{array} \right\}. \tag{3.127}
\]

where \( g \) and \( c_n \) are given by (3.116) and (3.117). Comparing the quantities \( F^{(4)} \) of (3.124), \( \hat{G}_q^{(4)} \) of (3.126), and \( \hat{D}_n^{TE} \) of (3.127) with the quantities \( F^\delta \) of (3.88), \( \hat{G}_q^\delta \) of (3.91), and \( \hat{D}_n^\delta \) of (3.92) and noting that the latter quantities combined to give (3.93), we obtain

\[
F^{(4)} + \hat{G}_q^{(4)} \hat{D}_n^{TE} = \frac{q\pi}{gc} \left\{ j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE}) \right\} \tag{3.128}
\]

where \( j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE}) \) is as in (3.121). Substitution of (3.128) into (3.123) gives

\[
\hat{z}^{(4)} = -\frac{q\pi}{gc} \left\{ j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE}) \right\}. \tag{3.129}
\]

Substitution of (3.116) and [1, eq. (F.121)] into (3.70) gives

\[
\hat{z}^{(5)} = \frac{2a}{cx^2} \left\{ \begin{array}{ll}
1, & n = q \neq 0 \\
0, & \text{otherwise}
\end{array} \right\} + \frac{j(F^{(5)} + \hat{G}_q^{(4)} \hat{D}_n^{(3)})}{ga} \tag{3.130}
\]

where \( F^{(5)} \) is given by [1, eqs. (F.132)-(F.134)]

\[
F^{(5)} = j \frac{nq\pi^2}{(gc)^2} \left\{ \begin{array}{ll}
(-1)^n c_n c_q \sinh(gc), & q \neq n \\
(-1)^n c_n c_q \sinh(gc), & q = n \neq 0 \\
4\{\sinh(gc) - gc\} \left(\frac{1}{gc} - 1\right), & q = n = 0
\end{array} \right\} + j c_q \left\{ \begin{array}{ll}
1, & q = n \neq 0 \\
0, & \text{otherwise}
\end{array} \right\}. \tag{3.131}
\]
Note that [1, eq. (F.132)] is not correct. Please correct [1, eq. (F.132)] by multiplying the denominator of the right-hand side by \((n\pi)^2 + (\gamma_{rs}^{TE})^2\). As given by (3.131), \(F^{(5)} = 0\) when \(q = n = 0\). The right-hand side of (3.131) was written so as to be as much as possible like the right-hand side of (3.88). In (3.130), \(\hat{G}^{(4)}_q\) and \(\hat{D}^{(3)}_n\) are given by (3.126) and (3.120), respectively. Comparing the quantities \(F^{(5)}\) of (3.131), \(\hat{G}^{(4)}_q\) of (3.126), and \(\hat{D}^{(3)}_n\) of (3.120) with the quantities \(F^{(6)}\) of (3.88), \(\hat{G}^{(6)}_q\) of (3.91), and \(\hat{D}^{(6)}_n\) of (3.92) and noting that the latter quantities combined to give (3.93), we obtain

\[
j(F^{(5)} + \hat{G}^{(4)}_q \hat{D}^{(3)}_n) = \frac{nq\pi^2 c_n q}{(gc)^2} - c_n \left\{ \begin{array}{ll} 1, & q = n \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}. \tag{3.132}
\]

where \(c_n\) is given by (3.117) and \(c_{nq}\) is given by (3.94) in which the \(z\)'s are given by (3.95)–(3.98) with \(g\) given by (3.116). Substitution of (3.132) into (3.130) gives

\[
\hat{z}^{(5)} = \frac{nq\pi^2 c_n q}{(ga)(gc)^2} + \left( \frac{2a}{cx^2_{rs}} - \frac{c_n}{ga} \right) \left\{ \begin{array}{ll} 1, & q = n \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}. \tag{3.133}
\]
Chapter 4

The Excitation Vector

The elements of the excitation vector are given by [1, eqs. (5.15) and (5.16)]

\[
I^{TM}_t = \frac{8\phi_0 n}{k_{mn}a} \sqrt{\frac{a^2 \pi}{2}} \frac{y_{sm} \zeta_{c}\alpha_{T} \eta_{L}}{4bc} e^{-i\beta_{01}^{TM} L_3}
\]

(4.1)

\[
I^{TE}_t = \frac{8\phi_0 m}{k_{mn}a} \frac{c}{b} \sqrt{\frac{a^2 \pi \epsilon_m \epsilon_n}{4bc}} y_{sm} \zeta_{c}\alpha_{T} \eta_{L} e^{-i\beta_{01}^{TM} L_3}
\]

(4.2)

where [1, eq. (5.21)]

\[
y_{sm} = \begin{cases} 
0 & \text{m even} \\
\frac{2}{m\pi} & \text{m odd}
\end{cases}
\]

(4.3)

and [1, eq. (5.24)]

\[
z_{c}\alpha_{T} = \frac{\sin(n\pi - \beta_{01}^{TM} c) \cos(\beta_{01}^{TM} L_3^\alpha) - 2\sin^2\left(\frac{n\pi - \beta_{01}^{TM} c}{2}\right) \sin(\beta_{01}^{TM} L_3^\alpha)}{2(n\pi - \beta_{01}^{TM} c)}
\]

\[
+ \frac{\sin(n\pi + \beta_{01}^{TM} c) \cos(\beta_{01}^{TM} L_3^\alpha) + 2\sin^2\left(\frac{n\pi + \beta_{01}^{TM} c}{2}\right) \sin(\beta_{01}^{TM} L_3^\alpha)}{2(n\pi + \beta_{01}^{TM} c)}
\]

(4.4)

In (4.1) and (4.2), \( \alpha \) is either 1 or 2, and \( i \) determines \( m \) and \( n \) as described in Appendix A. Comparing the right-hand side of (4.4) with that of (3.76), we see that

\[
z_{c}\alpha_{T} = \frac{1}{2} [G_{n}^{TM}]_{01}
\]

(4.5)
where \([\hat{G}^{TM}_{n}]_{01}\) is \(\hat{G}^{TM}_{n}\) when \(r = 0\) and \(s = 1\). Substitution of (4.3) and (4.5) into (4.1) and (4.2) gives

\[
I^{TM}_{i} = \frac{8\phi_{0}n}{mk_{mn}b} \sqrt{\frac{b}{\pi c}} [\hat{G}^{TM}_{n}]_{01} e^{-j\beta^{TM}_{01}L_{3}} \begin{cases} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{cases}
\]

(4.6)

\[
I^{TE}_{i} = \frac{8\phi_{0}}{k_{mn}b} \sqrt{\frac{\epsilon_{m}\epsilon_{n}c}{4\pi b}} [\hat{G}^{TM}_{n}]_{01} e^{-j\beta_{01}^{TM}L_{3}} \begin{cases} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{cases}
\]

(4.7)

Multiplying both (4.6) and (4.7) by \(-je^{j\beta_{01}^{TM}L_{3}}\) and noting that \(\epsilon_{m}/2 = 1\) whenever \(m\) is odd, we obtain

\[
-jI^{TM}_{i} e^{j\beta_{01}^{TM}L_{3}} = -j \frac{8\phi_{0}n}{mk_{mn}b} \sqrt{\frac{b}{\pi c}} [\hat{G}^{TM}_{n}]_{01} \begin{cases} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{cases}
\]

(4.8)

\[
-jI^{TE}_{i} e^{j\beta_{01}^{TM}L_{3}} = -j \frac{8\phi_{0}}{k_{mn}b} \sqrt{\frac{\epsilon_{n}c}{2\pi b}} [\hat{G}^{TM}_{n}]_{01} \begin{cases} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{cases}
\]

(4.9)
Chapter 5

The Electric Field in the Rectangular Waveguides

In this chapter, modal expansions are found for the electric fields in the rectangular waveguides. Afterwards, the time-average powers of the TE_{10} modes in the rectangular waveguides are obtained.

5.1 Expansions in Terms of the Fields of the Magnetic Currents

The electric field $\mathbf{E}^{(1)}$ in the left-hand rectangular waveguide (region 1) in Fig. 1.2 is due to $\mathcal{M}^{(1)}$ and is given by \cite[eqs. (2.3) and (2.11)]{1}

$$\mathbf{E}^{(1)} = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} V_{pq}^{TM} \mathbf{E}^{(1)}(0, \mathcal{L}_{pq}^{TM}) + \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} V_{pq}^{TE} \mathbf{E}^{(1)}(0, \mathcal{L}_{pq}^{TE}). \quad (5.1)$$

In obtaining (5.1), the upper limits on $p$ and $q$ in \cite[eq. (2.11)]{1} were suppressed. We truncated the double summations in (5.1) by retaining only terms for which both $p$ and $q$ are so small that, according to (A.2),

$$\sqrt{(p\pi)^2 + \left(\frac{q\pi b}{c}\right)^2} \leq BKM. \quad (5.2)$$
The electric field $E^{(2)}$ in the right-hand rectangular waveguide (region 2) in Fig. 1.2] is similarly given by
\[
E^{(2)} = \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} V_{pq}^{2TM} E^{(2)}(0, M_{pq}^{2TM}) + \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} V_{pq}^{2TE} E^{(2)}(0, M_{pq}^{2TE}).
\] (5.3)

We approximate the $M$'s on the right-hand sides of (5.1) and (5.3) by the $M$'s given by [1, eqs. (3.3) and 3.13)]. In Sections 5.2 to 5.5, we express the resulting approximate $E$'s on the right-hand sides of (5.1) and (5.3) in terms of the modes of the rectangular waveguides.

5.2 The Electric Field of the Magnetic Current $\hat{M}_{pq}^{1TM}$

The electric field $E^{(1)}(0, M_{pq}^{1TM})$ due to $M_{pq}^{1TM}$ in region 1 of Fig. 1.2 is given by [1, eq. (3.30)]
\[
E^{(1)}(0, M_{pq}^{1TM}) = \{ \psi_{pq}^{TM}(y^+, z^+) - \frac{k_{pq}^2 \psi_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}} \} e^{\gamma_{pq}(x-x_0)}. \] (5.4)

From [1, eqs. (A.3) and (A.13)], the mode field $E_{pq}^{TM-}$ is given by
\[
E_{pq}^{TM-} = -Z_{pq}^{TM} \left\{ \frac{\psi_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}} \right\}_e^{\gamma_{pq}(x-x_0)}. \] (5.5)

In view of (5.5), we recast (5.4) as
\[
E^{(1)}(0, M_{pq}^{1TM}) = \left( -\frac{1}{Z_{pq}^{TM}} E_{pq}^{TM-} \right) e^{\gamma_{pq}(x-x_0)}. \] (5.6)

5.3 The Electric Field of the Magnetic Current $\hat{M}_{pq}^{2TM}$

The electric field $E^{(2)}(0, M_{pq}^{2TM})$ due to $M_{pq}^{2TM}$ in region 2 of Fig. 1.2 is given by [1, eq. (3.36)]
\[
E^{(2)}(0, M_{pq}^{2TM}) = \{ \psi_{pq}^{TM}(y^+, z^+) + \frac{k_{pq}^2 \psi_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}} \} e^{-\gamma_{pq}(x-x_0)}. \] (5.7)
From [1, eqs. (A.2) and (A.3)], the mode field $E_{pq}^{TM+}$ is given by

$$E_{pq}^{TM+} = Z_{pq}^{TM} \left\{ \xi_{pq}^{TM}(y^+, z^+) + \frac{k_{pq}^{-1} v_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}} \right\} e^{-\gamma_{pq} x}. \quad (5.8)$$

In view of (5.8), we recast (5.7) as

$$E^{(2)}(0, \hat{M}_{pq}^{2TM}) = \left( \frac{1}{Z_{pq}^{TM}} E_{pq}^{TM+} \right) e^{\gamma_{pq} x}. \quad (5.9)$$

## 5.4 The Electric field of the Magnetic Current $\hat{M}_{pq}^{1TE}$

The electric field $E^{(1)}(0, \hat{M}_{pq}^{1TE})$ due to $\hat{M}_{pq}^{1TE}$ in region 1 of Fig. 1.2 is given by [1, eqs. (3.32) and (3.34)]

$$E^{(1)}(0, \hat{M}_{pq}^{1TE}) = \epsilon_{p0}^{TE}(y^+, z^+) \frac{j \sin(\beta_{10}(L_1 + x)) + Z_1 Y_{10}^{TE} \cos(\beta_{10}(L_1 + x))}{j \sin(\beta_{10}x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10}x_1)} \quad (5.10)$$

$$E^{(1)}(0, \hat{M}_{pq}^{1TE}) = \epsilon_{p0}^{TE}(y^+, z^+) e^{\gamma_{pq}(x_1 + x)}, \quad (p, q) \neq (1, 0) \quad (5.11)$$

where

$$x_1 = L_1 - x_0. \quad (5.12)$$

In obtaining (5.10), we substituted $j\beta_{10}$ for $\gamma_{10}$ and used [2, formulas 654.6 and 654.7]. Equation (5.10) is recast as

$$E^{(1)}(0, \hat{M}_{10}^{1TE}) = \frac{(Z_1 Y_{10}^{TE} + 1)e^{j\beta_{10}(L_1 + x)} + (Z_1 Y_{10}^{TE} - 1)e^{-j\beta_{10}(L_1 + x)}}{2(j \sin(\beta_{10}x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10}x_1))} \epsilon_{10}^{TE}(y^+, z^+). \quad (5.13)$$

From [1, eqs. (A.14) and (A.15)], the mode fields $E_{10}^{TE+}$ and $E_{10}^{TE-}$ are given by

$$E_{10}^{TE+} = \epsilon_{10}^{TE}(y^+, z^+) e^{-j\beta_{10}x} \quad (5.14)$$

$$E_{10}^{TE-} = \epsilon_{10}^{TE}(y^+, z^+) e^{j\beta_{10}x}, \quad (p, q) \neq (1, 0). \quad (5.15)$$
In obtaining (5.14) and (5.15), we substituted $j\beta_1$ for $\gamma_{10}$. In view of (5.14) and (5.15), we recast (5.13) as
\[
E^{(1)}(0, \dot{M}_1^{1TE}) = \frac{(Z_1 Y_{10}^{TE} + 1)E_1^{TE-} e^{j\beta_1 L_1} + (Z_1 Y_{10}^{TE} - 1)E_1^{TE+} e^{-j\beta_1 L_1}}{2(j \sin(\beta_{10} x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10} x_1))}.
\]
(5.16)

From [1, eq. (A.15)], the mode field $E_{pq}^{TE-}$ is given by
\[
E_{pq}^{TE-} = e_{pq}^{TE} (y^+, z^+) e^{\gamma_{pq} x}.
\]
(5.17)

In view of (5.17), we recast (5.11) as
\[
E^{(1)}(0, \dot{M}_1^{1TE}) = E_{pq}^{TE-} e^{\gamma_{pq} x}, \quad (p, q) \neq (1, 0).
\]
(5.18)

### 5.5 The Electric Field of the Magnetic Current $\dot{M}_{pq}^{2TE}$

The electric field $E^{(2)}(0, \dot{M}_{pq}^{2TE})$ due to $\dot{M}_{pq}^{2TE}$ in region 2 of Fig. 1.2 is given by [1, eqs. (3.38) and (3.40)]
\[
E^{(2)}(0, \dot{M}_{pq}^{2TE}) = \frac{j \sin(\beta_{10}(L_2 - x)) + Z_2 Y_{10}^{TE} \cos(\beta_{10}(L_2 - x))}{j \sin(\beta_{10} x_2) + Z_2 Y_{10}^{TE} \cos(\beta_{10} x_2)} e^{TE} (y^+, z^+).
\]
(5.19)

\[
E^{(2)}(0, \dot{M}_{pq}^{2TE}) = e_{pq}^{TE} (y^+, z^+) e^{-\gamma_{pq} (x-x_2)}, \quad (p, q) \neq (1, 0)
\]
(5.20)

where
\[
x_2 = L_2 - x_0.
\]
(5.21)

In obtaining (5.19), we substituted $j\beta_{10}$ for $\gamma_{10}$ and used [2, formulas 654.6 and 654.7]. Equation (5.19) is recast as
\[
E^{(2)}(0, \dot{M}_{10}^{2TE}) = \frac{(Z_2 Y_{10}^{TE} + 1)e^{-j\beta_1 (L_2 - x)} + (Z_2 Y_{10}^{TE} - 1)e^{-j\beta_1 (L_2 - x)}}{2(j \sin(\beta_{10} x_2) + Z_2 Y_{10}^{TE} \cos(\beta_{10} x_2))}.
\]
(5.22)
In view of (5.14) and (5.15), we recast (5.22) as
\[
E^{(2)}(0, \hat{M}_{10}^{2TE}) = \frac{(Z_2 Y_{10}^{TE} + 1) E_{10}^{TE+} e^{j \beta_{10} L_2} + (Z_2 Y_{10}^{TE} - 1) E_{10}^{TE-} e^{-j \beta_{10} L_2}}{2(j \sin(\beta_{10} x_2) + Z_2 Y_{10}^{TE} \cos(\beta_{10} x_2))}.
\] (5.23)

From [1, eq. (A.14)], the mode field \( E_{pq}^{TE+} \) is given by
\[
E_{pq}^{TE+} = E_{pq}^{TE}(y^+, z^+) e^{-\gamma_{pq} x}.
\] (5.24)

In view of (5.24), we recast (5.20) as
\[
E^{(2)}(0, \hat{M}_{pq}^{2TE}) = E_{pq}^{TE+} e^{\gamma_{pq} x_0}, \quad (p, q) \neq (1, 0).
\] (5.25)

### 5.6 Expansions in Terms of Waveguide Modes

Substitution of (5.6), (5.16) and (5.18) into (5.1) with the \( M \)'s replaced by \( \hat{M} \)'s gives
\[
E^{(1)} = \frac{(Z_1 Y_{10}^{TE} + 1) E_{10}^{TE-} e^{j \beta_{10} L_1} + (Z_1 Y_{10}^{TE} - 1) E_{10}^{TE+} e^{-j \beta_{10} L_1}}{2(j \sin(\beta_{10} x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10} x_1))} V_{10}^{1TE} + \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} V_{pq}^{1TM} \left( \frac{-1}{Z_{pq}^{TM}} E_{pq}^{TM-} \right) e^{\gamma_{pq} x_0}
\] + \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} V_{pq}^{1TE} E_{pq}^{TE-} e^{\gamma_{pq} x_0}.
\] (5.26)

Substitution of (5.9), (5.23) and (5.25) into (5.3) with the \( M \)'s replaced by \( \hat{M} \)'s gives
\[
E^{(2)} = \frac{(Z_2 Y_{10}^{TE} + 1) E_{10}^{TE+} e^{j \beta_{10} L_2} + (Z_2 Y_{10}^{TE} - 1) E_{10}^{TE-} e^{-j \beta_{10} L_2}}{2(j \sin(\beta_{10} x_2) + Z_2 Y_{10}^{TE} \cos(\beta_{10} x_2))} V_{10}^{2TE} + \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} V_{pq}^{2TM} \left( \frac{1}{Z_{pq}^{TM}} E_{pq}^{TM+} \right) e^{\gamma_{pq} x_0}
\] + \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} V_{pq}^{2TE} E_{pq}^{TE+} e^{\gamma_{pq} x_0}.
\] (5.27)
5.6.1 Normalization of $E^{(1)}$ and $E^{(2)}$

The quantities $E^{(1)}$ and $E^{(2)}$ are due to the $z$ traveling wave whose electric field in the circular waveguide is $E_{01}^{TM}$ given by

$$E_{01}^{TM} = Z_{01}^{TM} e^{j\beta_{01} z} \psi_{01}^{TM} (\rho, \phi) e^{-j\beta_{01} z} + \frac{(k_{01}^{TM})^2 \psi_{01}^{TM} (\rho, \phi) e^{-j\beta_{01} z}}{j\omega e}$$  \hspace{1cm} (5.28)

where

$$Z_{01}^{TM} = \frac{\eta Z_{01}}{k}.$$  \hspace{1cm} (5.29)

Equation (5.28) was obtained by substituting (3.57) into [1, eq. (B.1)]. Equation (5.29) was obtained by substituting (3.57) into [1, eq. (B.25)]. In this subsection, suitable expressions are found for the quantities $E^{(1)} e^{j\beta_{01} L_3} / \sqrt{Z_{01}^{TM}}$ and $E^{(2)} e^{j\beta_{01} L_3} / \sqrt{Z_{01}^{TM}}$. These quantities are, according to (5.28), due to the $z$-traveling wave whose transverse electric field is $\sqrt{Z_{01}^{TM}} e^{j\beta_{01} L_3} (\rho, \phi)$ at $z = L_3$ in the circular waveguide. In Section 6.3, it will be shown that the $z$-directed time-average power associated with this field is unity.

Multiplying both sides of (5.26) and (5.27) by $e^{j\beta_{01} L_3} / \sqrt{Z_{01}^{TM}}$ and using (5.29), we obtain

$$\frac{(e^{j\beta_{01} L_3})}{\sqrt{Z_{01}^{TM}}} E^{(1)} = C_{10}^{TE} e^{-j\beta_{01} x_0} E_{10}^{TE} + C_{10}^{TE} e^{j\beta_{01} x_0} E_{10}^{TE} - \sum_{q=1} \sum_{p=1} C_{pq}^{TM} e^{-\gamma_{pq} x_0} E_{pq}^{TM} - \sum_{q=0} \sum_{p=0} C_{pq}^{TM} e^{\gamma_{pq} x_0} E_{pq}^{TM}$$  \hspace{1cm} (5.30)

$$\frac{(e^{j\beta_{01} L_3})}{\sqrt{Z_{01}^{TM}}} E^{(2)} = C_{10}^{TE} e^{-j\beta_{01} x_0} E_{10}^{TE} + C_{10}^{TE} e^{j\beta_{01} x_0} E_{10}^{TE} + \sum_{q=1} \sum_{p=1} C_{pq}^{TM} e^{-\gamma_{pq} x_0} E_{pq}^{TM} + \sum_{q=0} \sum_{p=0} C_{pq}^{TM} e^{\gamma_{pq} x_0} E_{pq}^{TM}$$  \hspace{1cm} (5.31)
What is the x-directed time-average power associated with an arbitrary electromagnetic field \((E, H)\) in a source-free region of a rectangular waveguide? This field can be expressed as

\[
E = \sum_{p,q} \left( C_{p,q}^{TM+} E_{p,q}^{TM+} + C_{p,q}^{TM-} E_{p,q}^{TM-} + C_{p,q}^{TE+} E_{p,q}^{TE+} + C_{p,q}^{TE-} E_{p,q}^{TE-} \right) 
\]

\[
H = \sum_{p,q} \left( C_{p,q}^{TM+} H_{p,q}^{TM+} + C_{p,q}^{TM-} H_{p,q}^{TM-} + C_{p,q}^{TE+} H_{p,q}^{TE+} + C_{p,q}^{TE-} H_{p,q}^{TE-} \right). 
\]

On the right-hand sides of (5.40) and (5.41), the \(C\)'s are constants, and the \(E\)'s and the \(H\)'s are the mode fields defined by [1, eqs. (A.2), (A.3), (A.14), and (A.15)]. The \(x\)-directed time-average power \(P\) associated with \((E, H)\) is given by [4, eqs. (1-57) and (1-58)]

\[
P = \int_0^b dy^+ \int_0^c dz^+ \text{Re}(E \times H^*) \cdot \mathbf{u}_x 
\]

where "*" denotes complex conjugate and "Re" denotes real part. Substituting (5.40) and (5.41) into (5.42) and using the previously mentioned definitions of the mode fields and the last of the orthogonality relations [1, eq. (A.26)], we obtain

\[
P = \sum_{p,q} \left( |C_{p,q}^{TM+}|^2 e^{-2\text{Re}(\gamma_{pq})z} - |C_{p,q}^{TM-}|^2 e^{2\text{Re}(\gamma_{pq})z} \right) \text{Re}(Z_{p,q}^{TM}) 
\]

\[
-2 \sum_{p,q} \text{Imag}(C_{p,q}^{TM+} (C_{p,q}^{TM-})^* e^{-2j\text{Imag}(\gamma_{pq})z}) \text{Imag}(Z_{p,q}^{TM}) 
\]

5.7 Time-Average Power
In (5.30),

\[
C_{10}^{1TE-} = \left( \sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}} \right) \left\{ \frac{(Z_{1}Y_{10}^{TE} + 1)e^{j\beta_{10}x_{1}}}{2(Z_{1}Y_{10}^{TE} \cos(\beta_{10}x_{1}) + j \sin(\beta_{10}x_{1}))} \right\} \cdot \left( \frac{V_{10}^{1TE} e^{j\beta_{01}^{TM}L_{3}}}{\eta} \right) \quad (5.32)
\]

\[
C_{10}^{1TE+} = \left( \sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}} \right) \left\{ \frac{(Z_{1}Y_{10}^{TE} - 1)e^{-j\beta_{10}x_{1}}}{2(Z_{1}Y_{10}^{TE} \cos(\beta_{10}x_{1}) + j \sin(\beta_{10}x_{1}))} \right\} \cdot \left( \frac{V_{10}^{1TE} e^{j\beta_{01}^{TM}L_{3}}}{\eta} \right) \quad (5.33)
\]

\[
C_{pq}^{1TM-} = \frac{jk}{\sqrt{\gamma_{pq}^{TM}}} \left( \frac{V_{pq}^{1TM} e^{j\beta_{01}^{TM}L_{3}}}{\eta} \right) \quad (5.34)
\]

\[
C_{pq}^{1TE-} = \left( \sqrt{\frac{\gamma_{pq}}{\beta_{01}^{TM}}} \right) \left( \frac{V_{pq}^{1TE} e^{j\beta_{01}^{TM}L_{3}}}{\eta} \right) \quad (5.35)
\]

In (5.31),

\[
C_{10}^{2TE+} = \left( \sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}} \right) \left\{ \frac{(Z_{2}Y_{10}^{TE} + 1)e^{j\beta_{10}x_{2}}}{2(Z_{2}Y_{10}^{TE} \cos(\beta_{10}x_{2}) + j \sin(\beta_{10}x_{2}))} \right\} \cdot \left( \frac{V_{10}^{2TE} e^{j\beta_{01}^{TM}L_{3}}}{\eta} \right) \quad (5.36)
\]

\[
C_{10}^{2TE+} = \left( \sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}} \right) \left\{ \frac{(Z_{2}Y_{10}^{TE} - 1)e^{-j\beta_{10}x_{2}}}{2(Z_{2}Y_{10}^{TE} \cos(\beta_{10}x_{2}) + j \sin(\beta_{10}x_{2}))} \right\}
\]
\[ + \sum_{p,q} \{|C_{pq}^{TE+}|^2 e^{-2 \text{Re}(\gamma_{pq}) \cdot z} - |C_{pq}^{TE-}|^2 e^{2 \text{Re}(\gamma_{pq}) \cdot z}\} \text{Re}(Y_{pq}^{TE}) \]

\[-2 \sum_{p,q} \text{Imag}\{C_{pq}^{TE+}(C_{pq}^{TE-})^* e^{-2j\text{Imag}(\gamma_{pq}) \cdot z}\} \text{Imag}(Y_{pq}^{TE}) \] (5.43)

where "Imag" denotes imaginary part. In (5.43), we have [1, eqs. (A.13) and (A.25)]

\[ Z_{pq}^{TM} = -j \frac{\eta \gamma_{pq}}{k} \] (5.44)

\[ Y_{pq}^{TE} = -j \frac{\gamma_{pq}}{k\eta} \] (5.45)

where \( \gamma_{pq} = -j \beta_{pq} \) if the mode propagates, and \( \gamma_{pq} \) is purely real if the mode does not propagate. Since only the \( TE_{10} \) mode propagates, (5.43) reduces to

\[ P = \frac{2\eta}{k} \left\{ \sum_{p,q} \gamma_{pq} \text{Imag}\{C_{pq}^{TM+}(C_{pq}^{TM-})^*\} \right\} + \frac{\beta_{10}}{k\eta} \left\{ |C_{10}^{TE+}|^2 - |C_{10}^{TE-}|^2 \right\} \]

\[ + \frac{2}{k\eta} \left\{ \sum_{p,q} \gamma_{pq} \text{Imag}\{C_{pq}^{TE+}(C_{pq}^{TE-})^*\} \right\} . \] (5.46)

### 5.7.1 Time-Average Power in the Rectangular Waveguides

The normalized electric fields \( E^{(1)} e^{j\beta_{01}^{TM} L_3}/\sqrt{Z_{01}^{TM\text{eo}}} \) and \( E^{(2)} e^{j\beta_{01}^{TM} L_3}/\sqrt{Z_{01}^{TM\text{eo}}} \) of (5.37) and (5.38) are due to the \( z \)-traveling wave whose transverse electric field is \( E_{01}^{TM+} e^{-j\beta_{01}^{TM}(z-L_3)}/\sqrt{Z_{01}^{TM\text{eo}}} \) in the circular waveguide. The \( z \)-directed time-average power of this field in the circular waveguide is, as given by an expression very similar to (5.31), equal to unity. The \( -x \)-directed time-average power of \( E^{(1)} e^{j\beta_{01}^{TM} L_3}/\sqrt{Z_{01}^{TM\text{eo}}} \) is \( P^{(1)}/Z_{01}^{TM\text{eo}} \) given by (5.31) as

\[ \frac{P^{(1)}}{Z_{01}^{TM\text{eo}}} = |C_{10}^{TE+}|^2 - |C_{10}^{TE-}|^2. \] (5.47)
The $x$- directed time-average power of $P^{(2)}$, $L_3/\sqrt{Z_{01}}^{TE}$ given by (5.31) as
\[ P^{(2)} = \frac{IC_0}{Z_{01}} I T - C_2 + Z_{01}^{TE} = \frac{C_1^{2TE+}}{C_1^{2TE-}} + 2. \]

When the incident time-average power in the circular waveguide is unity, the time-average power $P_t$ transmitted into the rectangular waveguide is the sum of (5.47) and (5.48):
\[ P_t = |C_{10}^{1TE-}|^2 - |C_{10}^{1TE+}|^2 + |C_{10}^{2TE+}|^2 - |C_{10}^{2TE-}|^2. \]
Chapter 6

The Electric Field in the Circular Waveguide

In this chapter, a modal expansion is found for the electric field $E^{(3)}$ in the part of the circular waveguide for which $z_s < z < -c/2$ where $z_s$ is such that the impressed source $J^{\text{imp}}$ lies in the region for which $z < z_s$. See Fig. 1.2 where the entire region inside the circular waveguide is called Region 3. As stated in Chapter 1, only the $TM_{01}$ and $TE_{11}$ modes propagate in Region 3. The coefficients of the $TM_{01}$ and $TE_{11}$ modes in the modal expansion for $E^{(3)}$ are then expressed in forms suitable for computation. Finally, the time-average powers of the $TM_{01}$ and $TE_{11}$ modal contributions to $E^{(3)}$ are obtained.

The electric field $E^{(3)}$ in the circular waveguide is given by [1, eq. (2.7)]

$$E^{(3)} = E^{(3)}(0, -M^{(1)} - M^{(2)}) + E^{(3)}(J^{\text{imp}}, 0)$$  \hspace{1cm} (6.1)

where $E^{(3)}(J^{\text{imp}}, 0)$ is the electric field due to $J^{\text{imp}}$, and $E^{(3)}(0, -M^{(1)} - M^{(2)})$ is the electric field due to $-M^{(1)} - M^{(2)}$ where $M^{(1)} - M^{(2)}$ is the combination of $-M^{(1)}$ and $-M^{(2)}$. Each of the sources $J^{\text{imp}}, -M^{(1)}$, and $-M^{(2)}$ radiates in the circular waveguide with the apertures closed, with the short at $z = L_3$, and with a matched load at the other end where $z << 0$. 

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6.1 The Electric Field $E^{(3)}(0, -M^{(1)} - M^{(2)})$

Since $M^{(1)}$ and $M^{(2)}$ are given by [1, eqs. (2.11) and (2.12)], we have

$$E^{(3)}(0, -M^{(1)} - M^{(2)}) = -\sum_{\gamma,q,p} V_{pq}^{TM} E^{(3)}(0, M_{pq}^{TM})$$

$$-\sum_{\gamma,q,p} V_{pq}^{TE} E^{(3)}(0, M_{pq}^{TE}).$$

(6.2)

The first $\sum_{\gamma,q,p}$ on the right-hand side of (6.2) stands for $\sum_{p=1}^{2} \sum_{q=0}^{2} \sum_{p=0}^{2}$. The second $\sum_{\gamma,q,p}$ stands for $\sum_{p=1}^{2} \sum_{q=0}^{2} \sum_{p=0}^{2}$ Here, no upper limits are placed on the indices $p$ and $q$. Terms are retained only for which $p$ and $q$ are so small that (5.2) is true. The $E^{(3)}$'s on the right-hand side of (6.2) are given by [1, eqs. (4.47) and (4.61)]:

$$E^{(3)}(0, M_{pq}^{TM}) = -\frac{4a}{k_{pq} \sqrt{bc}} \left\{ \left( \frac{q}{b} \right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{e_{r} (k_{rs}^{TM})^{2} J_{r} (k_{rs}^{TM} a) E_{r}}{2 x_{rs}^{2} J_{r+1}^{2} (x_{rs})} \right. $$

$$+ \left( \frac{q}{c} \right) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TM})^{2} r J_{r} (k_{rs}^{TM} a) E_{r}}{2 x_{rs}^{2} J_{r+1}^{2} (x_{rs})} \right\}.$$  

(6.3)

$$E^{(3)}(0, M_{pq}^{TE}) = -\frac{4a}{k_{pq} \sqrt{ep_{2}^{2}/4bc}} \left\{ \left( \frac{p}{b} \right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{e_{r} (k_{rs}^{TE})^{2} J_{r} (k_{rs}^{TE} a) E_{r}}{2 x_{rs}^{2} J_{r+1}^{2} (x_{rs})} \right. $$

$$+ \left( \frac{q}{c} \right) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TE})^{2} r J_{r} (k_{rs}^{TE} a) E_{r}}{2 x_{rs}^{2} J_{r+1}^{2} (x_{rs})} \right\}.$$  

(6.4)

In (6.3) and (6.4), $\gamma_{rs}^{TE}$ is given by (3.58). Furthermore, $E_{r}^{TM}$, $E_{r}^{TE}$, and $E_{r}^{TEz}$ are given by [1, eqs. (4.74)-(4.76)]:

$$E_{r}^{TM} = \left\{ -z^{TM1} \sinh (\gamma_{rs}^{TM} (L_{3} - z)) + z^{TM2} e^{-\gamma_{rs}^{TM} (L_{3} - z)} \right\}$$

$$\cdot \left\{ \frac{\phi_{r}^{TM2} J_{r} (k_{rs}^{TM} \rho)}{k_{rs}^{TM}} - \frac{\phi_{r}^{TM1} J_{r} (k_{rs}^{TM} \rho)}{k_{rs}^{TM}} \right\}.$$
+ \left\{ z^{TM_1} \cosh(\gamma_{\tau s}^{TM}(L_3-z)) + z^{TM_2} e^{-\gamma_{\tau s}^{TM}(L_3-z)} \right\} \frac{\phi^{\gamma_2} k_{rs}^{TM} J_r(k_{rs}^{TM})}{\gamma_{\tau s}^{TM}} (6.5)

\begin{align*}
E^{\gamma TE_\phi} &= \left\{ -z^{TE_1} \sinh(\gamma_{\tau s}^{TE}(L_3-z)) + z^{TE_2} e^{-\gamma_{\tau s}^{TE}(L_3-z)} \right\} \\
& \cdot \left\{ \mathcal{U}_{\rho} \frac{\phi^{\gamma_2} r J_r(k_{rs}^{TE})}{k_{rs}^{TE} \rho} - \mathcal{U}_{\phi} \phi^{\gamma_1} J'_r(k_{rs}^{TE} \rho) \right\} (6.6)

E^{\gamma TE_z} &= \left\{ z^{TE_3} \sinh(\gamma_{\tau s}^{TE}(L_3-z)) + z^{TE_4} e^{-\gamma_{\tau s}^{TE}(L_3-z)} \right\} \\
& \cdot (-1)^{\gamma} \left\{ \mathcal{U}_{\rho} \frac{\phi^{\gamma_3} r J_r(k_{rs}^{TE})}{k_{rs}^{TE} \rho} + \mathcal{U}_{\phi} \phi^{\gamma_4} J'_r(k_{rs}^{TE} \rho) \right\}. (6.7)

\end{align*}

The \( \phi \)'s in (6.5)-(6.7) are given by [1, eqs. (E.10)-(E.13)]:

\begin{align*}
\phi^{\gamma_1} &= -(-1)^{\gamma} \left\{ \phi^{(1)}_p \cos \left( \frac{ry^{\gamma_1}}{x_o} \right) - \phi^{(2)}_p \sin \left( \frac{ry^{\gamma_1}}{x_o} \right) \right\} (6.8) \\
\phi^{\gamma_2} &= \phi^{(2)}_p \cos \left( \frac{ry^{\gamma_1}}{x_o} \right) + \phi^{(1)}_p \sin \left( \frac{ry^{\gamma_1}}{x_o} \right) (6.9) \\
\phi^{\gamma_3} &= -(-1)^{\gamma} \left\{ \phi^{(3)}_p \cos \left( \frac{ry^{\gamma_1}}{x_o} \right) - \phi^{(4)}_p \sin \left( \frac{ry^{\gamma_1}}{x_o} \right) \right\} (6.10) \\
\phi^{\gamma_4} &= \phi^{(4)}_p \cos \left( \frac{ry^{\gamma_1}}{x_o} \right) + \phi^{(3)}_p \sin \left( \frac{ry^{\gamma_1}}{x_o} \right). (6.11)
\end{align*}

The \( y \)'s on the right-hand sides of (6.8)-(6.11) are given by [1, eqs. (2.15) and (2.16)]:

\begin{align*}
y^{1+} &= (\pi - \phi)x_o + \frac{b}{2} (6.12) \\
y^{2+} &= \phi x_o + \frac{b}{2}. (6.13)
\end{align*}

The \( \phi \)'s on the right-hand sides of (6.8)-(6.11) are given by (3.40)-(3.43). For simplicity, we assume that \( z < -c/2 \) so that [1, eqs. (4.77) and (4.83)]

\begin{align*}
z^{TM_1} = z^{TE_1} = z^{TE_2} = 0. (6.14)
\end{align*}

Since \( z < -c/2 \), the remaining superscripted \( z \)'s in (6.5)-(6.7) (namely \( z^{TM_2} \), \( z^{TE_1} \), and \( z^{TE_4} \)) are, as stated in [1, page 34], given by their expressions in
[1, Appendix F] with \( z^+ \) replaced by zero. For \( x, \), \( x < ka \), we have [1, eqs. (F.26) and (F.35)]

\[
\frac{z^{\delta_2}}{c} = \left\{ \sin(q^{\delta_-}c) \cos(\beta^{\delta_+}_r L_3^-) - 2 \sin^2\left(\frac{q^{\delta_-}c}{2}\right) \sin(\beta^{\delta_+}_r L_3^-) \right\} / (2q^{\delta_-}c) \\
+ \left\{ \sin(q^{\delta_+}c) \cos(\beta^{\delta+}_r L_3^+) + 2 \sin^2\left(\frac{q^{\delta_+}c}{2}\right) \sin(\beta^{\delta+}_r L_3^+) \right\} / (2q^{\delta_+}c) \\
(6.15)
\]

\[
\frac{z^{TE4}}{c} = -j \\
\left\{ \sin(q^{TE-}c) \cos(\beta^{TE}_r L_3^-) - 2 \sin^2\left(\frac{q^{TE-}c}{2}\right) \sin(\beta^{TE}_r L_3^-) \right\} / (2q^{TE-}c) \\
+ j \left\{ \sin(q^{TE+}c) \cos(\beta^{TE}_r L_3^+) + 2 \sin^2\left(\frac{q^{TE+}c}{2}\right) \sin(\beta^{TE}_r L_3^+) \right\} / (2q^{TE+}c) .
(6.16)
\]

For \( x, \), \( x \geq ka \), we have [1, eqs. (F.33) and (F.42)]

\[
\frac{z^{\delta_2}}{c} = \left\{ \gamma_r^\delta c \sinh(\gamma_r^\delta L_3^+) - (-1)^q \gamma_r^\delta c \sinh(\gamma_r^\delta (L_3^+ - c)) \right\} / \left\{ (q\pi)^2 + (\gamma_r^\delta c)^2 \right\} \\
(6.17)
\]

\[
\frac{z^{TE4}}{c} = q\pi \left\{ \sinh(\gamma_r^{TE} L_3^+) - (-1)^q \sinh(\gamma_r^{TE} (L_3^+ - c)) \right\} / \left\{ (q\pi)^2 + (\gamma_r^{TE} c)^2 \right\} .
(6.18)
\]

In (6.15) and (6.17), \( \delta \) is either \( TM \) or \( TE \).

### 6.1.1 The Quantities \( E^{TM\phi}_\gamma \), \( E^{TE\phi}_\gamma \), and \( E^{TE\zeta}_\gamma \)

In this section, expressions (6.5)-(6.7) for \( E^{TM\phi}_\gamma \), \( E^{TE\phi}_\gamma \), and \( E^{TE\zeta}_\gamma \) are first reduced by means of (6.14), and then expanded by means of (6.8)-(6.13). Next, the \(-z\)-traveling modes of the circular waveguide are introduced. Finally, the expressions for \( E^{TM\phi}_\gamma \), \( E^{TE\phi}_\gamma \), and \( E^{TE\zeta}_\gamma \) that were obtained by means of (6.8)-(6.14) are recast in terms of the \(-z\)-traveling modes of the circular waveguide.
Reduction of (6.5)-(6.7) by means of (6.14)

Substitution of (6.14) into (6.5)-(6.7) gives

\[ E^{\gamma TM} = z^{TM} e^{-\gamma r s (L_3 - z)} \left\{ u_o \phi^{\gamma 2} J_r(k_{rs}^{TM} \rho) - u_e \phi^{\gamma 1} J_r(k_{rs}^{TM} \rho) + u_z \frac{\phi^{\gamma 2} J_r(k_{rs}^{TM} \rho)}{\gamma_r^{TM}} \right\} \]  \tag{6.19}

\[ E^{\gamma TE} = z^{TE} e^{-\gamma r s (L_3 - z)} \left\{ u_o \phi^{\gamma 2} J_r(k_{rs}^{TE} \rho) - u_e \phi^{\gamma 1} J_r(k_{rs}^{TE} \rho) \right\} \]  \tag{6.20}

\[ E^{\gamma TE} = z^{TE} e^{-\gamma r s (L_3 - z)} (-1)^{\gamma} \left\{ u_o \phi^{\gamma 3} J_r(k_{rs}^{TE} \rho) + u_e \phi^{\gamma 4} J_r(k_{rs}^{TE} \rho) \right\} \]  \tag{6.21}

Expansion of (6.19)-(6.21) by means of (6.8)-(6.13)

Substituting (6.12) and (6.13) into the arguments of the trigonometric functions in (6.8)-(6.11) and using [2, formulas 401.01-401.04], we obtain

\[ \cos \left( \frac{ry_{r s}}{x_0} \right) = (-1)^{\gamma} \left\{ \cos \left( \frac{rb}{2x_0} \right) \cos(r \phi) - (-1)^{\gamma} \sin \left( \frac{rb}{2x_0} \right) \sin(r \phi) \right\} \]  \tag{6.22}

\[ \sin \left( \frac{ry_{r s}}{x_0} \right) = (-1)^{\gamma} \left\{ \sin \left( \frac{rb}{2x_0} \right) \cos(r \phi) + (-1)^{\gamma} \cos \left( \frac{rb}{2x_0} \right) \sin(r \phi) \right\} \]  \tag{6.23}

Substitution of (6.22) and (6.23) into (6.8)-(6.11) gives

\[ \phi^{p 1} = (-1)^{\gamma} \left\{ (-1)^{\gamma+1} \phi^{b 1}_p \cos(r \phi) + \phi^{b 2}_p \sin(r \phi) \right\} \]  \tag{6.24}

\[ \phi^{p 2} = (-1)^{\gamma} \left\{ \phi^{b 2}_p \cos(r \phi) + (-1)^{\gamma} \phi^{b 1}_p \sin(r \phi) \right\} \]  \tag{6.25}

\[ \phi^{p 3} = (-1)^{\gamma} \left\{ (-1)^{\gamma+1} \phi^{b 3}_p \cos(r \phi) + \phi^{b 4}_p \sin(r \phi) \right\} \]  \tag{6.26}

\[ \phi^{p 4} = (-1)^{\gamma} \left\{ \phi^{b 4}_p \cos(r \phi) + (-1)^{\gamma} \phi^{b 3}_p \sin(r \phi) \right\} \]  \tag{6.27}

where

\[ \phi^{b 1}_p = \phi_p^{(1)} \cos \left( \frac{rb}{2x_0} \right) - \phi_p^{(2)} \sin \left( \frac{rb}{2x_0} \right) \]  \tag{6.28}

\[ \phi^{b 2}_p = \phi_p^{(2)} \cos \left( \frac{rb}{2x_0} \right) + \phi_p^{(1)} \sin \left( \frac{rb}{2x_0} \right) \]  \tag{6.29}
\[ \phi_p^{b3} = \phi_p^{(3)} \cos \left( \frac{rb}{2x_0} \right) - \phi_p^{(4)} \sin \left( \frac{rb}{2x_0} \right) \]  
(6.30)

\[ \phi_p^{b4} = \phi_p^{(4)} \cos \left( \frac{rb}{2x_0} \right) + \phi_p^{(3)} \sin \left( \frac{rb}{2x_0} \right) \]  
(6.31)

Substituting (6.24)-(6.27) into (6.19)-(6.21), we obtain

\[ E_{\gamma TM} = (-1)^{\gamma} \zeta_{TM} e^{-\gamma L_3 z} \left\{ (-1)^{\gamma} \phi_p^{b1} \right. \]

\[ + \left. \{ u_p J'(k_{rs} \rho) \sin(r\phi) + u_{\phi} r J_r(k_{rs} \rho) \cos(r\phi) \right\} + \phi_p^{b2} \{ u_p J'(k_{rs} \rho) \cos(r\phi) \}
\]

\[ - \left. u_{\phi} \frac{r J_r(k_{rs} \rho) \sin(r\phi)}{k_{rs} \rho} \} \right\} \]  
(6.32)

\[ E_{\gamma TE} = (-1)^{\gamma} \zeta_{TE} e^{-\gamma L_3 z} \left\{ \right. \]

\[ (-1)^{\gamma} \phi_p^{b1} \{ u_p \frac{r J_r(k_{rs} \rho) \sin(r\phi)}{k_{rs} \rho} + u_{\phi} J'_r(k_{rs} \rho) \cos(r\phi) \}
\]

\[ + \phi_p^{b2} \{ u_p J'_r(k_{rs} \rho) \cos(r\phi) - u_{\phi} J'_r(k_{rs} \rho) \sin(r\phi) \} \} \]  
(6.33)

\[ E_{\gamma TEx} = (-1)^{\gamma} \zeta_{TE} e^{-\gamma L_3 z} \left\{ \right. \]

\[ - \phi_p^{b3} \{ u_p \frac{r J_r(k_{rs} \rho) \cos(r\phi)}{k_{rs} \rho} - u_{\phi} J'_r(k_{rs} \rho) \sin(r\phi) \} + (-1)^{\gamma} \]

\[ \phi_p^{b4} \{ u_p \frac{r J_r(k_{rs} \rho) \sin(r\phi)}{k_{rs} \rho} + u_{\phi} J'_r(k_{rs} \rho) \cos(r\phi) \} \}. \]  
(6.34)

The \(-z\)-Traveling Modes of the Circular Waveguide

The modes of the circular waveguide that travel in the \(-z\)-direction are \(E_{rs}^{-TM}, E_{rs}^{-TM}, E_{rs}^{-TE}, \) and \(E_{rs}^{-TE}\) given by \([1, eqs. (B.2), (B.27), (B.36)], \)
and (B.56)]

\[
E_{rs}^{TM_{e-}} = \left\{ -Z_{rs}^{TM_{e0}} \varepsilon_{rs}^{TM_{e}}(\rho, \phi) + \frac{\mu_z (k_{rs}^{TM})^2 \psi_{rs}^{TM_{e}}(\rho, \phi)}{j \omega} \right\} e^{j \gamma_{rs}^{TM_{e}}} \tag{6.35}
\]

\[
E_{rs}^{TM_{o-}} = \left\{ -Z_{rs}^{TM_{o0}} \varepsilon_{rs}^{TM_{o}}(\rho, \phi) + \frac{\mu_z (k_{rs}^{TM})^2 \psi_{rs}^{TM_{o}}(\rho, \phi)}{j \omega} \right\} e^{j \gamma_{rs}^{TM_{o}}} \tag{6.36}
\]

\[
E_{rs}^{TE_{e-}} = \varepsilon_{rs}^{TE_{e}}(\rho, \phi) e^{j \gamma_{rs}^{TE_{e}}} \tag{6.37}
\]

\[
E_{rs}^{TE_{o-}} = \varepsilon_{rs}^{TE_{o}}(\rho, \phi) e^{j \gamma_{rs}^{TE_{o}}} \tag{6.38}
\]

where [1, eqs. (B.25), (B.7), (B.22), (B.30), (B.33), (B.51), and (B.62)]

\[
Z_{rs}^{TM_{e0}} = \frac{\gamma_{rs}^{TM}}{j \omega} \tag{6.39}
\]

\[
\psi_{rs}^{TM_{e}}(\rho, \phi) = \sqrt{\frac{\varepsilon_r}{\pi}} J_r(k_{rs}^{TM} \rho) \cos(\tau \phi) \tag{6.40}
\]

\[
\varepsilon_{rs}^{TM_{e}}(\rho, \phi) = \varepsilon_{rs}^{TM_{e}}(\rho, \phi) \tag{6.41}
\]

\[
\psi_{rs}^{TM_{o}}(\rho, \phi) = \sqrt{\frac{2}{\pi}} J_n(k_{np}^{TM} \rho) \sin(n \phi) \tag{6.42}
\]

\[
\varepsilon_{rs}^{TM_{o}}(\rho, \phi) = -\sqrt{\frac{2}{\pi}} \left( \frac{1}{a J_{r+1}(x_{rs})} \right) \tag{6.43}
\]

\[
V_{rs}^{TE_{e}}(\rho, \phi) = \varepsilon_{rs}^{TE_{e}}(\rho, \phi) \tag{6.44}
\]

\[
V_{rs}^{TE_{o}}(\rho, \phi) = -\varepsilon_{rs}^{TE_{o}}(\rho, \phi) \tag{6.45}
\]
In view of (6.39) and (3.49), substitution of (6.40) and (6.41) into (6.35) and (6.36) gives

\[
E_{rs}^{TM_{\phi}} = \sqrt{\frac{\varepsilon_r}{\pi}} \left( \frac{Z_{TM_{\phi}}}{a_{J_r+1}(x_{rs})} \right) e^{\gamma_{rs}^{TM_{\phi}}} \left\{ \frac{u_{\phi}}{k_{rs}^{TE}} J_{r}(k_{rs}^{TE} \rho) \cos(\gamma \phi) + \frac{k_{rs}^{TM} J_{r}(k_{rs}^{TM} \rho) \cos(\gamma \phi)}{\gamma_{rs}^{TM}} \right\} \right. \quad (6.46)
\]

\[
E_{rs}^{TM_{\theta}} = \sqrt{\frac{2}{\pi}} \left( \frac{Z_{TM_{\theta}}}{a_{J_r+1}(x_{rs})} \right) e^{\gamma_{rs}^{TM_{\theta}}} \left\{ \frac{u_{\phi}}{k_{rs}^{TE}} J_{r}(k_{rs}^{TE} \rho) \sin(\gamma \phi) + \frac{k_{rs}^{TM} J_{r}(k_{rs}^{TM} \rho) \cos(\gamma \phi)}{\gamma_{rs}^{TM}} \right\} \right. \quad (6.47)
\]

Substituting (6.44) and (6.45) into (6.37) and (6.38), we obtain

\[
E_{rs}^{TE_{\phi}} = \sqrt{\frac{\varepsilon_r}{\pi(x_{rs}^2 - r^2)}} \left( \frac{k_{rs}^{TE}}{J_{r}(x_{rs}^2)} \right) e^{\gamma_{rs}^{TE_{\phi}}} \left\{ \frac{u_{\phi}}{k_{rs}^{TE}} J_{r}(k_{rs}^{TE} \rho) \sin(\gamma \phi) + \frac{k_{rs}^{TM} J_{r}(k_{rs}^{TM} \rho) \cos(\gamma \phi)}{\gamma_{rs}^{TM}} \right\} \right. \quad (6.48)
\]

\[
E_{rs}^{TE_{\theta}} = -\sqrt{\frac{2}{\pi(x_{rs}^2 - r^2)}} \left( \frac{k_{rs}^{TE}}{J_{r}(x_{rs}^2)} \right) e^{\gamma_{rs}^{TE_{\theta}}} \left\{ \frac{u_{\phi}}{k_{rs}^{TE}} J_{r}(k_{rs}^{TE} \rho) \cos(\gamma \phi) + \frac{k_{rs}^{TM} J_{r}(k_{rs}^{TM} \rho) \sin(\gamma \phi)}{\gamma_{rs}^{TM}} \right\} \right. \quad (6.49)
\]

Expressions for $E_{\gamma TM_{\phi}}$, $E_{\gamma TE_{\phi}}$, and $E_{\gamma TE_{\theta}}$ in Terms of Waveguide Modes

Equations (6.46) and (6.47) reduce (6.32) to

\[
E_{\gamma TM_{\phi}} = (-1)^{\gamma} \sqrt{\frac{\pi}{\varepsilon_r}} \left( \frac{Z_{TM_{\phi}}}{a_{J_r+1}(x_{rs})} \right) \frac{J_{r+1}(x_{rs})}{Z_{TM_{\phi}}} \phantom{a} \{ \phi_{p}^{\rho 2} E_{rs}^{TM_{\phi}} + (-1)^{\gamma} \phi_{p}^{\rho 1} E_{rs}^{TM_{\phi}} \} e^{-\gamma L_{3}}. \quad (6.50)
\]
Equations (6.48) and (6.49) reduce (6.33) and (6.34) to

\[
E^{nTE\phi} = (-1)^{\gamma r} \sqrt{\frac{\pi(x_{rs}^2 - r^2)}{\epsilon_r}} \left( \frac{z^{TE2}J_r(x_{rs}^2)}{k_{rs}} \right) \cdot \left\{ (-1)^{\gamma} \phi_p^{b1} E_{rs}^{TEe-} - \phi_p^{b2} E_{rs}^{TEe0-} \right\} e^{-\gamma_{rs}TE L_3}
\]

(6.51)

\[
E^{nTEz} = (-1)^{\gamma r} \sqrt{\frac{\pi(x_{rs}^2 - r^2)}{\epsilon_r}} \left( \frac{z^{TE4}J_r(x_{rs}^2)}{k_{rs}} \right) \cdot \left\{ (-1)^{\gamma} \phi_p^{b4} E_{rs}^{TEe-} + \phi_p^{b3} E_{rs}^{TEe0-} \right\} e^{-\gamma_{rs}TE L_3}.
\]

(6.52)

### 6.1.2 Expression for \( E^{(3)}(0, -\mathcal{M}^{(1)} - \mathcal{M}^{(2)}) \) in Terms of Waveguide Modes

In this section, expressions (6.50)-(6.52) for \( E^{TM\phi}, E^{TEm\phi}, \) and \( E^{TEz} \) are substituted into expressions (6.3) and (6.4) for \( E^{(3)}(0, \mathcal{M}_{pq}^{TM}) \) and \( E^{(3)}(0, \mathcal{M}_{pq}^{TE}) \). The resulting expressions for \( E^{(3)}(0, \mathcal{M}_{pq}^{TM}) \) and \( E^{(3)}(0, \mathcal{M}_{pq}^{TE}) \) are then substituted into expression (6.2) for \( E^{(3)}(0, -\mathcal{M}^{(1)} - \mathcal{M}^{(2)}) \).

Substituting (6.50)-(6.52) into (6.3) and (6.4) and using (3.49) (3.51), we obtain

\[
E^{(3)}(0, \mathcal{M}_{pq}^{TM}) = \frac{2}{k_{pq} b} \sqrt{\frac{2\pi b}{c}} \left\{ \sum_{r=0}^{\infty} (-1)^{\gamma r} \sqrt{\frac{\epsilon_r}{2}} \sum_{s=1}^{\infty} \frac{z^{TM2}}{cZ_{TMco}} 
\cdot \left( \phi_p^{b2} E_{rs}^{TMe} + (-1)^{\gamma} \phi_p^{b1} E_{rs}^{TMo-} \right) e^{-\gamma_{rs}TM L_3} - q \sum_{r=1}^{\infty} (-1)^{\gamma r} \sum_{s=1}^{\infty} \frac{z^{TE2}}{c \sqrt{x_{rs}^2 - r^2}} 
\cdot \left( -1 \right)^{\gamma} \phi_p^{b1} E_{rs}^{TEe-} + \phi_p^{b4} E_{rs}^{TEe0-} \right\} e^{-\gamma_{rs}TE L_3} \right\}
\]

(6.53)

\[
E^{(3)}(0, \mathcal{M}_{pq}^{TE}) = \frac{2}{k_{pq} b} \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_p \epsilon_q}{4}} \left\{ \frac{pc}{b} \sum_{r=0}^{\infty} (-1)^{\gamma r} \sqrt{\frac{\epsilon_r}{2}} \sum_{s=1}^{\infty} \frac{z^{TM2}}{cZ_{TMco}} 
\cdot \left( \phi_p^{b2} E_{rs}^{TMe} + (-1)^{\gamma} \phi_p^{b1} E_{rs}^{TMo-} \right) e^{-\gamma_{rs}TM L_3} - \frac{pc}{b} \sum_{r=1}^{\infty} (-1)^{\gamma r} \sum_{s=1}^{\infty} \frac{z^{TE2}}{c \sqrt{x_{rs}^2 - r^2}} 
\cdot \left( -1 \right)^{\gamma} \phi_p^{b1} E_{rs}^{TEe-} + \phi_p^{b3} E_{rs}^{TEe0-} \right\} e^{-\gamma_{rs}TE L_3} \right\}
\]

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\[
\cdot \left( (-1)^\gamma \phi_p^{b_1} E_{rs}^{TE_e} - \phi_p^{b_2} E_{rs}^{TE_o} \right) e^{-\gamma_{rs} T L_3} - q \left( \frac{\sin \phi_o}{\phi_o} \right) \sum_{r=0}^{\infty} (-1)^r \sqrt{\frac{c}{2}} \sum_{s=1}^{\infty} \frac{x_{rs}^{TE_4}}{\gamma_{rs}^{TE} ac \sqrt{x_{rs}^{TE} - r^2}} \left( (-1)^\gamma \phi_p^{b_4} E_{rs}^{TE_e} + \phi_p^{b_3} E_{rs}^{TE_o} \right) e^{-\gamma_{rs} T L_3} \right}.
\]

Substitution of (6.53) and (6.54) into (6.2) gives

\[
E^{(3)}(\Omega, -M^{(1)} - M^{(2)}) = 2 \sqrt{\frac{2 \pi b}{c}} \sum_{r=0}^{\infty} \sqrt{\frac{c}{2}} \sum_{s=1}^{\infty} \left\{ \frac{1}{Z_{TM}^{Meo}} \cdot \left( S_{TM e}^{TM} E_{rs}^{TM e} - S_{TM o}^{TM} E_{rs}^{TM o} \right) e^{-\gamma_{rs} T L_3}
\]

where

\[
S_{TM e}^{TM} = - \sum_{\gamma=1}^{2} \sum_{q=0}^{\infty} \left( \frac{z^{TM e}}{c} \right) \sum_{p=0}^{\infty} \epsilon_{pq} C_{1} \phi_p^{b_2}
\]

\[
S_{TM o}^{TM} = - \sum_{\gamma=1}^{2} (-1)^\gamma \sum_{q=0}^{\infty} \left( \frac{z^{TM o}}{c} \right) \sum_{p=0}^{\infty} \epsilon_{pq} C_{1} \phi_p^{b_1}
\]

\[
S_{TE e}^{TM e} = \frac{1}{\sqrt{x_{rs}^{TE e} - r^2}} \sum_{\gamma=1}^{2} (-1)^\gamma \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \epsilon_{pq}
\]

\[
\cdot \left\{ r \left( \frac{z^{TE e}}{c} \right) C_{1} \phi_p^{b_1} - \left( \frac{\sin \phi_c}{\phi_o} \right) \left( \frac{z^{TE e}}{c} \right) \frac{x_{rs}^{TE} C_{2} \phi_p^{b_4}}{\gamma_{rs}^{TE o}} \right\}
\]

\[
S_{TE o}^{TM o} = \frac{-1}{\sqrt{x_{rs}^{TE o} - r^2}} \sum_{\gamma=1}^{2} \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \epsilon_{pq}
\]

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\[
\begin{align*}
\cdot \left\{ r \left( \frac{z^{TE2}}{c} \right) C_1 \phi_p^b + \left( \frac{\sin \phi_o}{\phi_o} \right) \left( \frac{z^{TE1}}{c} \right) \frac{x_{r_2}^b C_2 \phi_p^b}{\gamma_{TE}^{r_3}} \right\} & \quad (6.59) \\
\epsilon_{pq} = \frac{1}{k_{pq} b} \sqrt{\frac{\epsilon_p \epsilon_q}{4}} & \quad (6.60) \\
C_1 = (-1)^\gamma \left( q V^{TM}_{pq} + \left( \frac{pc}{b} \right) V^{TE}_{pq} \right) & \quad (6.61) \\
C_2 = (-1)^\gamma \left( \frac{pc}{b} V^{TM}_{pq} - q V^{TE}_{pq} \right). & \quad (6.62)
\end{align*}
\]

In (6.61) and (6.62), \( V^{TM}_{pq} = 0 \) when \( p = 0 \) or \( q = 0 \).

### 6.2 The Electric Field \( \mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) \)

Since \( x_{01} < ka \), we have, from (3.57),
\[
\gamma_{01}^{TM} = j \beta_{01}^{TM}. & \quad (6.63)
\]

Substitution of (6.63) into [1, eq. (5.10)] gives
\[
\mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) = \frac{2 \epsilon e^{-j \beta_{01}^{TM} L_3}}{\sqrt{\pi \alpha \omega \epsilon J_1(x_{01})}} \left\{ u_{\rho} \beta_{01}^{TM} J_1(k_{01} \rho) \sin (\beta_{01}^{TM} (L_3 - z)) \\
- u_{z} k_{01}^{TM} J_0(k_{01} \rho) \cos (\beta_{01}^{TM} (L_3 - z)) \right\} & \quad (6.64)
\]
where \( \beta_{01}^{TM} \) is given by (3.59).

### 6.2.1 Expression for \( \mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) \) in Terms of Waveguide Modes

Since, as stated in [1, page 37], the only \( z \)-traveling wave contained in \( \mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) \) is the unit amplitude \( z \) traveling \( TM_{01} \) wave, we suspect that \( \mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) \) contains only \( TM_{01} \) modes. Expression (6.64) for \( \mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) \) is recast as
\[
\mathbf{E}^{(3)}(\mathbf{J}^{\text{imp}}, \mathbf{Q}) = \frac{1}{\sqrt{\pi a J_1(x_{01})}} \left\{ \frac{u_{\rho} \beta_{01}^{TM} J_1(k_{01} \rho)}{\omega \epsilon} + \frac{u_{z} k_{01}^{TM} J_0(k_{01} \rho)}{j \omega \epsilon} \right\} e^{-j \beta_{01}^{TM} z} 
\]
\[ + \frac{e^{-2j\beta_{01}T^ML_3}}{\sqrt{\pi}aJ_1(x_{01})} \left\{ - \frac{\beta_{01}^TMJ_1(k_{01}^TM\rho)}{\omega_\epsilon} + u_\epsilon \frac{k_{01}^TMJ_0(k_{01}^TM\rho)}{j\omega_\epsilon} \right\} e^{j\beta_{01}T^Mz}. \] (6.65)

From (6.40) and (6.41), we have, in view of (3.46),

\[ \psi_{01}T^M(\rho, \phi) = \frac{J_0(k_{01}^M\rho)}{\sqrt{\pi}x_{01}J_1(x_{01})} \] (6.66)

\[ e_{01}T^M(\rho, \phi) = -u_\epsilon \frac{J_0'(k_{01}^M\rho)}{\sqrt{\pi}aJ_1(x_{01})} \] (6.67)

so that (6.65) becomes, with the help of (3.51) and (3.49),

\[ E^{(3)}(j^{imp}, \Omega) = \left\{ \frac{\beta_{01}^TM}{\omega_\epsilon} e_{01}T^M(\rho, \phi) + u_\epsilon \frac{(k_{01}^M)^2\psi_{01}T^M(\rho, \phi)}{j\omega_\epsilon} \right\} e^{-j\beta_{01}T^Mz} + e^{-2j\beta_{01}T^ML_3} \left\{ - \frac{\beta_{01}^TM}{\omega_\epsilon} e_{01}T^M(\rho, \phi) + u_\epsilon \frac{(k_{01}^M)^2\psi_{01}T^M(\rho, \phi)}{j\omega_\epsilon} \right\} e^{j\beta_{01}T^Mz}. \] (6.68)

Substitution of (6.63) into (6.39) gives

\[ \frac{\beta_{01}^TM}{\omega_\epsilon} = \gamma_{01}T^Me^o. \] (6.69)

Substituting (6.69) into (6.68) and recalling (6.63), we obtain

\[ E^{(3)}(j^{imp}, \Omega) = E_{01}T^M+ + e^{-2j\beta_{01}T^ML_3} E_{01}T^M- \] (6.70)

where \( E_{01}T^M+ \) and \( E_{01}T^M- \) are given by [1, eqs. (B.1) and (B.2)].

### 6.3 The Electric Field \( E^{(3)} \)

Since (6.47) gives

\[ E_{01}T^M- = 0 \] (6.71)

and since (3.57) gives (6.63) and

\[ \gamma_{11}TE = j\beta_{11}TE, \] (6.72)
substitution of (6.55) and (6.70) into (6.1) yields
\[
E^{(3)} = E_{01}^{TM+} + \left(2\sqrt{\frac{\pi b}{c}} \frac{S_{11}^{TM} e^{-j\beta^{TM}_{01} L_3}}{Z_{01}^{TM\omega}} + e^{-2j\beta^{TM}_{01} L_3} \right) E_{01}^{TM-} \\
+ 2\sqrt{\frac{2\pi b}{c}} \left(S_{11}^{TE} E_{11}^{TE+} + S_{11}^{TE} E_{11}^{TE-} \right) e^{-j\beta^{TE}_{11} L_3} + 2\sqrt{\frac{2\pi b}{c}} \sum_{r=0}^{\infty} \sqrt{\frac{e_r}{2}} \\
\left\{ \sum_{s=1}^{(r,s)\neq (0,1)} \frac{(S_{1s}^{TM} E_{1s}^{TM+} + S_{1s}^{TM} E_{1s}^{TM-}) e^{-\gamma^{TM}_{1s} L_3}}{Z_{1s}^{TM\omega}} \\
+ \sum_{s=1}^{(r,s)\neq (1,1)} (S_{1s}^{TE} E_{1s}^{TE+} + S_{1s}^{TE} E_{1s}^{TE-}) e^{-\gamma^{TE}_{1s} L_3} \right\} \quad (6.73)
\]

The only \(z\)-traveling wave on the right-hand side of (6.73) is the incident wave whose electric field \(E_{01}^{TM+}\) is given by [1, eq. (B.1)]

\[
E_{01}^{TM+} = \left\{ \frac{Z_{01}^{TM\omega} e^{TM}}{\omega_0 e^{TM}} (\rho, \phi) + \sqrt{2} \frac{(k^{TM}_{01})^2 \phi_{01}^{TM}(\rho, \phi)}{j\omega} \right\} e^{-j\beta^{TM}_{01} z} \quad (6.74)
\]

The time-average \(z\)-directed power associated with \(E_{01}^{TM+}\) is, as given by a formula similar to (5.31), \(Z_{01}^{TM\omega}\). Moreover, \(E_{01}^{TM+}\) has the phase factor \(e^{-j\beta^{TM}_{01} L_3}\) when \(z = L_3\). However, the normalized electric field \(E_{01}^{TM+} e^{j\beta^{TM}_{01} L_3}/\sqrt{Z_{01}^{TM\omega}}\) has unit power and no phase factor when \(z = L_3\). Thus, the \(z\)-traveling wave part of \(E^{(3)} e^{j\beta^{TM}_{01} L_3}/\sqrt{Z_{01}^{TM\omega}}\) has unit power and no phase factor when \(z = L_3\). Multiplying both sides of (6.73) by \(e^{j\beta^{TM}_{01} L_3}/\sqrt{Z_{01}^{TM\omega}}\), we obtain the normalized electric field \(E^{(3)} e^{j\beta^{TM}_{01} L_3}/\sqrt{Z_{01}^{TM\omega}}\) given by

\[
\left( \frac{e^{j\beta^{TM}_{01} L_3}}{\sqrt{Z_{01}^{TM\omega}}} \right) E^{(3)} = \left( \frac{e^{j\beta^{TM}_{01} L_3}}{\sqrt{Z_{01}^{TM\omega}}} \right) E_{01}^{TM+} - \left( \frac{C^{TM} e^{-j\beta^{TM}_{01} L_3}}{\sqrt{Z_{01}^{TM\omega}}} \right) E_{01}^{TM-} \\
+ \left( \frac{e^{-j\beta^{TE}_{11} L_3}}{\sqrt{V_{TE}}} \right) \left( C_{11}^{TE} E_{11}^{TE+} + C_{11}^{TE} E_{11}^{TE-} \right) \\
- \sum_{r=0}^{\infty} \sum_{s=1}^{(r,s)\neq (0,1)} \left\{ \frac{e^{-\gamma^{TM}_{1s} L_3}}{\sqrt{Z_{1s}^{TM\omega}}} \right\} \left( C_{rs}^{TM} E_{rs}^{TM+} + C_{rs}^{TM} E_{rs}^{TM-} \right) \\
\]

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\[
\sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \left\{ \left( \frac{e^{-\gamma_{r,s} L_3}}{|Y_{r,s}^{TEo}|} \right) \left( C_{r,s}^{TEe} E_{r,s}^{TEe} + C_{r,s}^{TEo} E_{r,s}^{TEo} \right) \right\} (6.75)
\]

where \(Z_{r,s}^{TMeo}\) is given by (6.39) and [1, eq. (B.54)]

\[
Y_{r,s}^{TEo} = \frac{\gamma_{r,s}^{TE}}{j\omega \mu}.
\]

In (6.39), \(\gamma_{01}^{TM}\) is to be replaced by \(j\beta_{01}^{TM}\). In (6.76), \(\gamma_{11}^{TE}\) is to be replaced by \(j\beta_{11}^{TE}\). The \(C's\) in (6.7) are given by

\[
C_{01}^{TM} = -1 - 2 \sqrt{\frac{\pi b}{c}} \left( \frac{S_{11}^{TM} e^{j\beta_{01}^{TM} L_3}}{Z_{01}^{TMeo}} \right)
\]

\[
C_{11}^{TEe} = 2 \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{Y_{11}^{TEo}}{Z_{01}^{TMeo} S_{11}^{TEe}}} e^{j\beta_{01}^{TM} L_3}
\]

\[
C_{11}^{TEo} = 2 \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{Y_{11}^{TEo}}{Z_{01}^{TMeo} S_{11}^{TEo}}} e^{j\beta_{01}^{TM} L_3}
\]

\[
C_{r,s}^{TM} = -2 \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Z_{r,s}^{TMeo}|}{Z_{01}^{TMeo}}} \left( \frac{S_{r,s}^{TM}}{Z_{r,s}^{TMeo}} \right) e^{j\beta_{01}^{TM} L_3}
\]

\[
C_{r,s}^{TMo} = -2 \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Z_{r,s}^{TMeo}|}{Z_{01}^{TMeo}}} \left( \frac{S_{r,s}^{TMo}}{Z_{r,s}^{TMeo}} \right) e^{j\beta_{01}^{TM} L_3}
\]

\[
C_{r,s}^{TEe} = 2 \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Y_{r,s}^{TEo}|}{Z_{01}^{TMeo}}} S_{r,s}^{TEe} e^{j\beta_{01}^{TM} L_3}
\]

\[
C_{r,s}^{TEo} = 2 \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Y_{r,s}^{TEo}|}{Z_{01}^{TMeo}}} S_{r,s}^{TEo} e^{j\beta_{01}^{TM} L_3}
\]

The magnitudes of the squares of the constants \(C_{01}^{TM}, C_{11}^{TEe}, \) and \(C_{11}^{TEo}\) are time-average powers (see Section 6.3.1). The minus sign which precedes the \(E_{01}^{TM-}\) term on the right-hand side of (6.75) makes the phase of \(C_{01}^{TM}\) equal to the phase of the coefficient of \(E_{01}^{TM}(\rho, \phi)\). See (6.35). The \(E_{r,s}^{TM-}, E_{r,s}^{TE-}\),
and \( E_{TE}\) terms in (6.75) were patterned after the \( E_{01}^{TM+} \), \( E_{11}^{TE+} \), and \( E_{11}^{TE-} \) terms, respectively.

### 6.3.1 Time-Average Power

The time-average power of the normalized electric field \( E_0^{(3)} e^{j \beta_0^{TM} L_3} / \sqrt{Z_{01}^{TM\phi}} \) of (6.75) is given by a formula similar to (5.31). This power consists only of the time-average powers associated with the \( E_{01}^{TM+} \), \( E_{01}^{TM-} \), \( E_{11}^{TE+} \), and \( E_{11}^{TE-} \) terms on the right-hand side of (6.75). The time-average \( z\)-directed power associated with the \( E_{01}^{TM+} \) term in (6.75) is 1 W. The time-average \( -z\)-directed powers associated with the \( E_{01}^{TM-} \), \( E_{11}^{TE+} \), and \( E_{11}^{TE-} \) terms in (6.75) are \(|C_{01}^{TM}|^2\), \(|C_{11}^{TE+}|^2\), and \(|C_{11}^{TE-}|^2\), respectively.

### 6.3.2 The Coefficient \( C_{01}^{TM} \)

In this subsection, \( C_{01}^{TM} \) of (6.77) is expressed in a form suitable for calculation. In (6.77), \( S_{01}^{TM\phi} \) is given by (6.56) in which the quantities \( z^{TM2}/c \) and \( \phi_{p}^{k2} \) appear. Comparing (6.15) with (3.76), we see that, since \( x_0 < ka \),

\[
\frac{z^2}{c} = \frac{1}{2} \left[ \hat{G}_q^\delta \right]_{01} \tag{6.84}
\]

where \( \left[ \hat{G}_q^\delta \right]_{01} \) is \( \hat{G}_q^\delta \) when \( r = 0 \) and \( s = 1 \). Here, \( \delta \) is either TM or TE. Setting \( r = 0 \) in (6.29), we obtain

\[
\phi_{p}^{k2} = \phi_{p}^{(2)}. \tag{6.85}
\]

Substitution of (6.61), (6.84), and (6.85) into (6.56) gives

\[
S_{01}^{TM\phi} = \sum_{\gamma=1}^{\gamma} \sum_{\nu=0}^{\nu} \left[ \hat{G}_q^{TM} \right]_{01} \sum_{p=0}^{p} \epsilon_{pq} \phi_{p}^{(2)} \left( q V_{pq}^{TM} + \left( \frac{p c}{b} \right) V_{pq}^{TM} \right). \tag{6.86}
\]

The quantity \( Z_{01}^{TM\phi} \) in (6.77) is given by (6.69):

\[
Z_{01}^{TM\phi} = \frac{\eta_{01}^{TM}}{k} \tag{6.87}
\]
Substituting (6.86) and (6.87) into (6.77), we obtain

\[ C_{01}^{TM_e} = -1 + \sum_{\gamma=1}^{2} \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \sum_{p+q \neq 0} \left\{ C_{01,pq}^{TM_e,\gamma TM} \left( \frac{V_{\gamma TM} e^{j\beta_{01}^{TM} L_3}}{\eta} \right) \right\} + C_{01,pq}^{TM_e,\gamma TE} \left( \frac{V_{\gamma TE} e^{j\beta_{01}^{TM} L_3}}{\eta} \right) \}

where

\[ C_{01,pq}^{TM_e,\gamma TM} = \sqrt{\frac{\pi b}{c}} \left( \frac{k}{\beta_{01}^{TM}} \right) \epsilon_{pq} \left[ \hat{G}_{11}^{TM} \right]_{01} \phi_p^{(2)} \]

\[ C_{01,pq}^{TM_e,\gamma TE} = \sqrt{\frac{\pi b}{c}} \left( \frac{k}{\beta_{01}^{TM}} \right) \left( \frac{p}{b} \right) \epsilon_{pq} \left[ \hat{G}_{11}^{TM} \right]_{01} \phi_p^{(2)}. \]

### 6.3.3 The Coefficients \( C_{11}^{TE_e} \) and \( C_{11}^{TE_0} \)

In this section, \( C_{11}^{TE_e} \) of (6.78) and \( C_{11}^{TE_0} \) of (6.79) are expressed in forms suitable for calculation. In (6.78) and (6.79), \( S_{11}^{TE_e} \) and \( S_{11}^{TE_0} \) are given by (6.58) and (6.59), respectively. The quantities \( z_{TE_4}/c \) and \( \gamma_{11}^{TE} a \) appear in both (6.58) and (6.59). Comparing (6.16) with (3.108), we see that, since \( x_{11}' < ka \),

\[ \frac{z_{TE_4}}{c} = \frac{j}{2} \left[ \hat{G}_4^{(4)} \right]_{11} \]

where \( \left[ \hat{G}_4^{(4)} \right]_{11} \) is \( \hat{G}_4^{(4)} \) when \( r = 1 \) and \( s = 1 \). From (3.58), we have

\[ \gamma_{11}^{TE} = j\beta_{11}^{TE}. \]

Substitution of (6.61), (6.62), (6.84), (6.91), and (6.92) into (6.58) and (6.59) gives

\[ S_{11}^{TE_e} = \frac{1}{2\sqrt{x_{11}'^2 - 1}} \sum_{\gamma=1}^{2} \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \epsilon_{pq} \left\{ \left[ \hat{G}_q^{TE} \right]_{11} \phi_p^{(1)} \left( qV_{pq}^{TM} + \left( \frac{p}{b} \right) V_{pq}^{TE} \right) \right\}. \]
\[
\begin{align*}
S_{11}^{\text{TE}o} &= \frac{-1}{2\sqrt{\lambda^2_{11} - 1}} \sum_{\gamma=1}^{2} \sum_{q=0}^{\gamma} \sum_{p=0}^{\gamma} \epsilon_{pq} \left\{ \frac{\hat{G}^{\text{TE}}_q}{\beta_{11} a} \left( \tilde{V}_{pq} + \frac{pc}{b} \tilde{V}_{pq} \right) \right\} \\
& \quad - \left( \frac{\sin \phi_0}{\phi_0} \right) \frac{\hat{G}^{(4)}_q}{\beta_{11} a} \left( \tilde{V}_{pq} \right) \end{align*}
\]

The admittance \( Y_{11}^{\text{TE}oo} \) in (6.78) and (6.79) is obtained by substituting (6.92) into [1, eq. (B.54)]:
\[
Y_{11}^{\text{TE}oo} = \frac{\beta_{11}}{k\eta}. 
\]

Substituting (6.87), (6.95), and (6.93) into (6.78), we obtain
\[
C_{11}^{\text{TE}e} = \sum_{\gamma=1}^{2} \sum_{q=0}^{\gamma} \sum_{p=0}^{\gamma} \left\{ C_{11,pq}^{\text{TE},\gamma TM} \left( \frac{\tilde{V}_{pq} e^{j\theta_{01}^{TM} L_3}}{\eta} \right) \right\} 
\]

where
\[
\begin{align*}
C_{11}^{\text{TE},\gamma TM} &= \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\beta_{11}^{\text{TE}}}{\beta_{01}^{TM}}} \frac{\epsilon_{pq}}{\sqrt{x_{11}^2 - 1}} \left\{ q(\hat{G}^{\text{TE}}_q)_{11} \phi_{51}^{b1} \right. \\
& \quad \left. + \left( \frac{pc}{b} \right) \left( \frac{\sin \phi_0}{\phi_0} \right) \frac{x_{11}^{\prime 2}(\hat{G}^{(4)}_q)_{11} \phi_{51}^{b1}}{\beta_{11}^{\text{TE} a}} \right\} 
\end{align*}
\]

\[
\begin{align*}
C_{11}^{\text{TE},\gamma TE} &= \sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\beta_{11}^{\text{TE}}}{\beta_{01}^{TM}}} \frac{\epsilon_{pq}}{\sqrt{x_{11}^2 - 1}} \left\{ (\frac{pc}{b})(\hat{G}^{\text{TE}}_q)_{11} \phi_{51}^{b1} \right. \\
& \quad \left. - q \left( \frac{\sin \phi_0}{\phi_0} \right) \frac{x_{11}^{\prime 2}(\hat{G}^{(4)}_q)_{11} \phi_{51}^{b1}}{\beta_{11}^{\text{TE} a}} \right\}.
\end{align*}
\]
Substituting (6.87), (6.95), and (6.94) into (6.79), we obtain

\[
C_{11}^{TE} = \sum_{\gamma=1}^{2} \sum_{q=0}^{\infty} \sum_{p=0 \atop p+q \neq 0} \left\{ C_{11}^{TE,\gamma TM} \left( \frac{V_{pq} e^{i \beta_{pq}^{TM} L_3}}{\eta} \right) \right. \\
+ C_{11, pq}^{TE,\gamma TE} \left( \frac{V_{pq} e^{i \beta_{pq}^{TM} L_3}}{\eta} \right) \left\} \right. 
\]

(6.99)

where

\[
C_{11}^{TE,\gamma TM} = -\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\beta_{11}^{TE}}{\beta_{01}^{TM}}} \frac{(-1)^{\gamma} \epsilon_{pq}}{\sqrt{x_{11}^2 - 1}} \left\{ q (\hat{G}_{q}^{TE})_{11} \phi_p^{b2} \right. \\
- \left( \frac{pc}{b} \right) \left( \sin \phi_o \right) \frac{x_{11}^2 (\hat{G}_{q}^{(4)})_{11} \phi_p^{b3}}{\beta_{11}^{TE} a} \left\} 
\]

(6.100)

\[
C_{11}^{TE,\gamma TE} = -\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\beta_{11}^{TE}}{\beta_{01}^{TM}}} \frac{(-1)^{\gamma} \epsilon_{pq}}{\sqrt{x_{11}^2 - 1}} \left\{ \left( \frac{pc}{b} \right) (\hat{G}_{q}^{TE})_{11} \phi_p^{b2} \\
+ q \left( \sin \phi_o \right) \frac{x_{11}^2 (\hat{G}_{q}^{(4)})_{11} \phi_p^{b3}}{\beta_{11}^{TE} a} \right\}. 
\]

(6.101)

The \( \phi_p \)'s in (6.97), (6.98), (6.100), and (6.101) are obtained by setting \( r = 1 \) in (6.28)–(6.31). From (2.8) and (2.9), we have

\[
\frac{\phi}{2x_o} = \phi_o 
\]

(6.102)

so that

\[
\sin \left( \frac{b}{2x_o} \right) = \frac{b}{2a}. 
\]

(6.103)

Assuming that \( \phi_o \leq \pi/2 \), we have

\[
\cos \left( \frac{b}{2x_o} \right) = \sqrt{1 - \left( \frac{b}{2a} \right)^2}. 
\]

(6.104)
Substitution of (6.103) and (6.104) into (6.28)–(6.31) yields

\[
\phi_{o1} = \sqrt{1 - \left( \frac{b}{2a} \right)^2 \phi_p^{(1)} - \left( \frac{b}{2a} \right) \phi_p^{(2)}}
\]  
(6.105)

\[
\phi_{o2} = \sqrt{1 - \left( \frac{b}{2a} \right)^2 \phi_p^{(2)} + \left( \frac{b}{2a} \right) \phi_p^{(1)}}
\]  
(6.106)

\[
\phi_{o3} = \sqrt{1 - \left( \frac{b}{2a} \right)^2 \phi_p^{(3)} - \left( \frac{b}{2a} \right) \phi_p^{(4)}}
\]  
(6.107)

\[
\phi_{o4} = \sqrt{1 - \left( \frac{b}{2a} \right)^2 \phi_p^{(3)} + \left( \frac{b}{2a} \right) \phi_p^{(3)}}
\]  
(6.108)
Chapter 7

The Tangential Electric Field in the Apertures

The tangential part of the electric field in the left-hand aperture \( A_1 \) is called \( \mathbb{E}_t^{(A_1)}(\phi, z) \). The tangential part of the electric field in the right-hand aperture \( A_2 \) is called \( \mathbb{E}_t^{(A_2)}(\phi, z) \). These tangential parts are given by

\[
\mathbb{E}_t^{(A\gamma)}(\phi, z) = \mathbb{u}_\rho \times \mathbb{M}^{(\gamma)}, \quad \gamma = 1, 2
\]

(7.1)

where \( \mathbb{u}_\rho \) is the unit vector in the \( \rho \)-direction and \( \mathbb{M}^{(\gamma)} \) is given by [1, eqs. (2.11) and (2.12)]. Substituting [1, eqs. (2.11) and (2.12)] into (7.1), we obtain

\[
\mathbb{E}_t^{(A\gamma)}(\phi, z) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left\{ V_{pq}^{\gamma TM} (\mathbb{u}_\rho \times \mathbb{M}_{pq}^{\gamma TM}(\phi, z)) \right. \\
+ \left. V_{pq}^{\gamma TE} (\mathbb{u}_\rho \times \mathbb{M}_{pq}^{\gamma TE}(\phi, z)) \right\}
\]

(7.2)

where the double summation is truncated as in (5.1). In (7.2),

\[
V_{pq}^{\gamma TM} = 0, \quad p = 0 \text{ or } q = 0.
\]

(7.3)

Substitution of [1, eqs. (2.13) and (2.14)] into (7.2) gives

\[
\mathbb{E}_t^{(A\gamma)}(\phi, z) = \mathbb{u}_\phi (-1)^\gamma \left( \frac{\sin \phi_0}{\phi_0} \right)
\]

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\[
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left\{ V_{\gamma TM} e_{yq}^{TM}(y^{\gamma^+}, z^+) + V_{\gamma TE} e_{zq}^{TE}(y^{\gamma^+}, z^+) \right\} + M_z \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left\{ V_{p q}^{TM} e_{yq}^{TM}(y^{\gamma^+}, z^+) + V_{p q}^{TE} e_{zq}^{TE}(y^{\gamma^+}, z^+) \right\} \tag{7.4}
\]

where \(y^{\gamma^+}\) and \(z^+\) are given by [1, eqs. (2.15)-(2.17)]

\[
y^{\gamma^+} = (2 - \gamma) \pi x_o + (-1)^\gamma \phi x_o + \frac{b}{2} \tag{7.5}
\]

\[
z^+ = z + \frac{c}{2} \tag{7.6}
\]

where, from (6.102),

\[
x_o = \frac{b}{2\phi_o} \tag{7.7}
\]

Now, \(e_{yq}^{TM}\) and \(e_{zq}^{TM}\) are the \(y\)- and \(z\)-components of \(e_{pq}^{TM}\) given by (A.10), and \(e_{yq}^{TE}\) and \(e_{zq}^{TE}\) are the \(y\)- and \(z\)-components of \(e_{pq}^{TE}\) given by (A.23) so that (7.4) becomes

\[
E_t^{(A\gamma)}(\phi, z) = \frac{1}{c} \left( \frac{\sin \phi_o}{\phi_o} \right) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sqrt{\frac{\varepsilon_p \varepsilon_q}{4}} \left( \frac{1}{k_{pq} b} \right) \left( -\frac{p}{b} V_{\gamma TM}^{\gamma} + \frac{q}{c} V_{\gamma TE}^{\gamma} \right) \cos \frac{p\pi y^+}{b} \sin \frac{q\pi z^+}{c}
\]

\[
-2\pi \sqrt{\frac{b}{c}} \sum_{p+q \neq 0} \left( \frac{1}{k_{pq} b} \right) \left( \frac{q}{c} V_{\gamma TM}^{\gamma} + \frac{p}{b} V_{\gamma TE}^{\gamma} \right) \sin \frac{p\pi y^+}{b} \cos \frac{q\pi z^+}{c} \tag{7.8}
\]

We want to normalize \(E_t^{(A\gamma)}(\phi, z)\) of (7.8) by dividing by \(-|E_{01}^{TM+}|_{\text{rms}}\) where the subscript "rms" denotes the root mean square value of the transverse part over the waveguide cross section at \(z = 0\). Recall that \(E_{01}^{TM+}\) is the electric field of the \(z\) traveling wave in the circular waveguide. We choose to divide by \(-|E_{01}^{TM+}|_{\text{rms}}\) rather than \(|E_{01}^{TM+}|_{\text{rms}}\) because the \(z\)-directed electric current associated with \(E_{01}^{TM+}\) is negative at \((\rho, z) = (a, 0)\). This electric
current is given by \(-u_x \times H_{01}^{TM+}\) where \(H_{01}^{TM+}\) is given by [1, eqs. (5.2) and (5.9)].

According to (6.74), the transverse part of \(E_{01}^{TM+}\) is \(Z_{01}^{TM\alpha} e_0^{TM\alpha}(\rho, \phi) e^{-j\beta_{01}^{TM}z}\) so that

\[
|E_{01}^{TM+}|_{\text{rms}} = |Z_{01}^{TM\alpha}| \left\{ \frac{1}{\pi a^2} \int_{\rho}^{a} \int_{0}^{2\pi} d\rho d\phi (e_0^{TM\alpha} \cdot e_0^{TM\alpha}) \right\}^{1/2}. \tag{7.9}
\]

One of the orthogonality relationships in [1, eq. (B.64)] and

\[
Z_{01}^{TM\alpha} = \frac{\eta \beta_{01}^{TM}}{k} \tag{7.10}
\]

reduce (7.9) to

\[
|E_{01}^{TM+}|_{\text{rms}} = \frac{\eta \beta_{01}^{TM}}{\sqrt{\pi ka}}. \tag{7.11}
\]

Equation (7.10) was obtained by replacing \(1/(\omega c)\) by \(\eta/k\) in (6.69). Substitution of (7.11) and (7.5)-(7.7) into (7.8) gives

\[
\frac{E_{\alpha}^{(A\gamma)}(\phi, z)}{|E_{01}^{TM+}|_{\text{rms}}} = \frac{\pi}{k} \frac{2}{c} \left( \frac{k}{\beta_{01}^{TM}} \right) \left( \frac{\sin \phi_0}{\phi_0} \right) e^{-j\beta_{01}^{TM} \rho} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\epsilon_p \epsilon_q}{4} \left( \frac{\rho a}{b} \right) \left( \frac{\eta}{\eta} \right) \left( \frac{\eta}{\eta} \right)
\]

\[
\times \cos \left\{ \left( \frac{p\pi}{2\phi_0} \right) ((-1)^\gamma \phi + (2 - \gamma)\pi) + \frac{p\pi}{2} \right\} \sin \left( \frac{q\pi z}{c} + \frac{q\pi}{2} \right)
\]

\[
+ \frac{\pi}{2} \frac{2}{c} \left( \frac{k}{\beta_{01}^{TM}} \right) e^{-j\beta_{01}^{TM} \rho} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\epsilon_p \epsilon_q}{4} \left( \frac{\rho a}{b} \right) \left( \frac{\eta}{\eta} \right) \left( \frac{\eta}{\eta} \right)
\]

\[
\times \sin \left\{ \left( \frac{p\pi}{2\phi_0} \right) ((-1)^\gamma \phi + (2 - \gamma)\pi) + \frac{p\pi}{2} \right\} \cos \left( \frac{q\pi z}{c} + \frac{q\pi}{2} \right). \tag{7.12}
\]

When \(z = 0\), the \(\phi\)-component of (7.12) is \(E_{\phi}^{(A\gamma)}(\phi, 0)/|E_{01}^{TM+}|_{\text{rms}}\) given
by

\[
\frac{E^{(A\gamma)}(\phi, 0)}{|E_{01}^{TM\phi+}|_{\text{rms}}} = (-1)^{2}\sqrt{\pi} \frac{2}{c} \left( \frac{k}{\beta_{01}^{TM}} \right) \left( \sin \frac{\phi}{\gamma} \right) e^{-j\beta_{01}^{TM}L_3} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sqrt{\epsilon_p \epsilon_q} \frac{1}{4} \\
\cdot \left( \frac{1}{k_{pq} b} \right) \left\{ \left( \frac{p \alpha}{b} \right) \left( \frac{V \nu^{TM} e^{j\beta_{01}^{TM}L_3}}{\eta} \right) - \left( \frac{q \alpha}{c} \right) \left( \frac{V \nu^{TE} e^{j\beta_{01}^{TM}L_3}}{\eta} \right) \right\} \\
\cdot \cos \left\{ \left( \frac{p \pi}{2} \right) \left( \frac{\gamma \phi + (2 - \gamma)\pi}{\phi} \right) + \frac{p \pi}{2} \right\} \sin \frac{q \pi}{2}.
\]

(7.13)

When \( \phi = (2 - \gamma)\pi \), the z-component of (7.12) is \( \frac{E^{(A\gamma)}((2 - \gamma)\pi, z)}{|E_{01}^{TM\phi+}|_{\text{rms}}} \) given by

\[
\frac{E^{(A\gamma)}((2 - \gamma)\pi, z)}{|E_{01}^{TM\phi+}|_{\text{rms}}} = 2\sqrt{\pi} \frac{b}{c} \left( \frac{k}{\beta_{01}^{TM}} \right) e^{-j\beta_{01}^{TM}L_3} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sqrt{\epsilon_p \epsilon_q} \frac{1}{4} \\
\cdot \left( \frac{1}{k_{pq} b} \right) \left\{ \left( \frac{q \alpha}{c} \right) \left( \frac{V \nu^{TM} e^{j\beta_{01}^{TM}L_3}}{\eta} \right) + \left( \frac{p \alpha}{b} \right) \left( \frac{V \nu^{TE} e^{j\beta_{01}^{TM}L_3}}{\eta} \right) \right\} \\
\cdot \sin \frac{p \pi}{2} \cos \left( \frac{q \pi z}{c} + \frac{q \pi}{2} \right).
\]

(7.14)
Chapter 8

Numerical Results

A computer program was written to calculate the time-average power transmitted into the rectangular waveguides, the time-average power reflected in the circular waveguide, and the $\phi$ and $z$ components of the electric field in the apertures. This computer program will be described and listed in a subsequent report. Some numerical results obtained by means of this computer program are presented in this chapter.

When the time-average incident power in the circular waveguide is unity, the time-average power transmitted into the rectangular waveguides is $P_t$ given by (5.49) and the time-average power reflected in the circular waveguide is called $P_r$. According to the discussion in Section 6.3.1,

$$P_r = |C_{01}^{TM}|^2 + |C_{11}^{TE}|^2 + |C_{11}^{TE_0}|^2. \quad (8.1)$$

Figures 8.1 to 8.5 show plots of $P_t$ and $P_r$ versus $ka$ for various values of $L_3$ when

$$\begin{align*}
\frac{b}{a} &= 1.1 \\
\frac{c}{a} &= 0.5
\end{align*} \quad (8.2)$$

The plots of Figs. 8.1 to 8.5 are for $L_3/[\lambda_{01}^{TM}]_{ka=2.95} = 0.35, 0.40, 0.45, 0.50,$ and 0.55, respectively. Here, $[\lambda_{01}^{TM}]_{ka=2.95}$ is the wavelength of the $TM_{01}$ mode in the circular waveguide when $ka = 2.95$:

$$[\lambda_{01}^{TM}]_{ka=2.95} = \frac{2\pi a}{\sqrt{(2.95)^2 - x_{01}^2}}. \quad (8.3)$$
Fig. 8.1. Plots of the ratio $P_t$ of the transmitted power to the input power and the ratio $P_r$ of the reflected power to the input power when $L_3 = 0.35 \left[ \frac{\lambda_{TM}}{2a} \right]_{ka=2.95}$. The input power is the power of the incident $TM_{01}$ wave in the circular waveguide. $P_t + P_r = 1$.

Fig. 8.2. Plots of the ratio $P_t$ of the transmitted power to the input power and the ratio $P_r$ of the reflected power to the input power when $L_3 = 0.40 \left[ \frac{\lambda_{TM}}{2a} \right]_{ka=2.95}$. The input power is the power of the incident $TM_{01}$ wave in the circular waveguide. $P_t + P_r = 1$. 

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Fig. 8.3. Plots of the ratio $P_t$ of the transmitted power to the input power and the ratio $P_r$ of the reflected power to the input power when $L_3 = 0.45 \left[ \lambda_{01}^{TM} \right]_{ka=2.95}$. The input power is the power of the incident $TM_{01}$ wave in the circular waveguide. $P_t + P_r = 1$.

Fig. 8.4. Plots of the ratio $P_t$ of the transmitted power to the input power and the ratio $P_r$ of the reflected power to the input power when $L_3 = 0.50 \left[ \lambda_{01}^{TM} \right]_{ka=2.95}$. The input power is the power of the incident $TM_{01}$ wave in the circular waveguide. $P_t + P_r = 1$.  

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Fig. 8.5. Plots of the ratio $P_t$ of the transmitted power to the input power and the ratio $P_r$ of the reflected power to the input power when $L_3 = 0.55 \left[ \lambda_{01}^{TM} \right]_{ka=2.95}$. The input power is the power of the incident $TM_{01}$ wave in the circular waveguide. $P_t + P_r = 1$.

With the value of $x_{01}$ given in [5, page 2], (8.3) becomes

$$[\lambda_{01}^{TM}]_{ka=2.95} = \frac{2\pi a}{\sqrt{(2.95)^2 - (2.40482556)^2}} = 3.67738806a. \quad (8.4)$$

The value $ka = 2.95$ was chosen because it is fairly central to the range of values of $ka$ in Figs. 8.1 to 8.5 (see the next paragraph). The values of $L_1$, and $L_2$ do not matter because the loads $Z_1$ and $Z_2$ were chosen to be matched loads, that is,

$$Z_1 = Z_2 = Z_{10}^{TE}. \quad (8.5)$$

The curves of Figs. 8.1 to 8.5 are plotted for the entire range of $ka$ such that only the $TE_{10}$ mode propagates in the rectangular waveguides and only the $TM_{01}$ and $TE_{11}$ modes propagate in the circular waveguide. Since only the $TE_{10}$ mode propagates in the rectangular waveguides,

$$\pi < kb < \min \left( 2\pi, \frac{\pi b}{c} \right) \quad (8.6)$$

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where “min” denotes the minimum of the values in parentheses. Since only the $TM_{01}$ and $TE_{11}$ modes propagate in the circular waveguide,

$$x_{01} < ka < x'_{21}. \quad (8.7)$$

Substituting (8.2) and (8.3) into (8.6), we obtain

$$2.85599332 < ka < 5.7119866. \quad (8.8)$$

Taking the values of $x_{01}$ and $x'_{21}$ given in [5, pages 2 and 32], (8.7) becomes

$$2.40482556 < ka < 3.05423693. \quad (8.9)$$

Combining (8.8) and (8.9), we have

$$2.85599332 < ka < 3.05423693. \quad (8.10)$$

In Figs. 8.1 to 8.5, $P_t$ approaches zero as $ka$ approaches 2.8559932. This is expected because the $TE_{10}$ mode, which carries the transmitted power, ceases to propagate when $ka$ becomes less than 2.8559932.

The numerical data of Figs. 8.1 to 8.5 were computed with

$$BKM = 15 \quad (8.11)$$

$$XM = 40. \quad (8.12)$$

The parameters $BKM$ and $XM$ are not written with italicized letters because they are input variables for our computer program. The parameter $BKM$ is introduced in (A.1) and used in (5.2). The constraint (5.2) on the values of $p$ and $q$ determines the order of the moment matrix $[Y^1 + Y^2 + Y^3]$ which appears in [1, eq. (2.22)]. When $BKM = 15$, the order of the moment matrix is 32. The parameter $XM$ is introduced in (B.4). The effect of $XM$ is to truncate the doubly infinite sum $\sum_{r=0}^{\infty} \sum_{s=1}^{\infty}$ that appears in (3.1)-(3.4). The truncated sum is $\sum_{r=0}^{r_{\text{max}}} \sum_{s=1}^{s_{\text{max}}}$ where $s_{\text{max}}$, which depends on $r$, is the largest value of $s$ such that

$$j_{0,s} \leq XM, \quad r = 0$$

$$j_{r,s} \leq XM, \quad r = 1, 2, \ldots$$

Assuming that $XM > j_{0,1}$, $r_{\text{max}}$ is the largest value of $r$ such that $j_{r,1} \leq XM$. 63
Some of the time-average incident power is transmitted into the rectangular waveguides. The rest of it is reflected in the circular waveguide. Therefore,

\[ P_t + P_r = 1. \]  

(8.14)
The plots of \( P_t \) and \( P_r \) shown in Figs. 8.1 to 8.5 do indeed satisfy (8.14). Because the values of \( P_t \) and \( P_r \) computed separately from (5.49) and (8.1) satisfy (8.12) we have some confidence in their accuracy.

The magnetic field due to the impressed source \( \mathbf{J}^{\text{imp}} \) radiating in the circular waveguide with the apertures closed is \( H^{(3)}(\mathbf{J}^{\text{imp}}, 0) \) given by [1, eq. (5.11)]. The electric current at \( z = 0 \) on the wall of the circular waveguide associated with this magnetic field is \( \mathbf{J} \) given by

\[ \mathbf{J} = -\mathbf{j} \frac{2e^{-j\beta_{01}^TM L_3}}{\sqrt{\pi a}} \cos(\beta_{01}^T M I_3). \]  

(8.15)
The magnitude of \( \mathbf{J} \) of (8.15) is maximum when \( L_3 = 0.5\lambda_{01}^T M \) where \( \lambda_{01}^T M \) is the wavelength of the \( TM_{01} \) mode in the circular waveguide. If an aperture is put where the electric current would otherwise be maximum, the tangential electric field will be large in this aperture. A large aperture field gives a large transmitted power \( P_t \). Thus, we expect \( P_t \) to be large at \( ka = 2.95 \) in Fig. 8.4 because \( L_3 \) is then equal to \( 0.5\lambda_{01}^T M \). Actually, \( P_t \) is even larger at \( ka = 2.95 \) in Figs. 8.2 and 8.3 where \( L_3 = 0.4\lambda_{01}^T M \) and \( 0.45\lambda_{01}^T M \), respectively. When \( ka \) is held at 2.95, the curve of \( P_t \) versus \( L_3/\lambda_{01}^T M \) shown in Fig. 8.6 attains a maximum at a value of \( L_3/\lambda_{01}^T M \) somewhat less than 0.5. This phenomenon may be due to the finite extent of the aperture in the \( z \)-direction. Note that \( P_t = 0 \) at \( L_3/\lambda_{01}^T M = 0.25 \) in Fig. 8.6. This is expected because the magnitude of \( \mathbf{J} \) of (8.15) vanishes when \( L_3/\lambda_{01}^T M = 0.25 \). In this case, the aperture has little effect because there is no flow of electric current to stop at \( z = 0 \).

The data for the plot of \( P_t \) versus \( L_3/\lambda_{01}^T M \) of Fig. 8.6 were computed with

\[ \begin{align*}
\frac{b}{a} &= 1.1 \\
\frac{c}{a} &= 0.5 \\
ka &= 2.95 \\
\text{BKM} &= 15 \\
\text{XM} &= 40 \\
Z_1 &= Z_2 = Z_{01}^{TE}
\end{align*} \]  

(8.16)
Fig. 8.6. Plot of the ratio $P_t$ of the transmitted power to the input power versus $L_3/\lambda_{01}^{TM}$ when $ka = 2.95$. The input power is the power of the incident $TM_{01}$ wave in the circular waveguide.

The curve in Fig. 8.6 was terminated at $L_3/\lambda_{01}^{TM} = 1.5$. As $L_3/\lambda_{01}^{TM}$ becomes larger and larger, $P_t$ versus $L_3/\lambda_{01}^{TM}$ becomes more and more periodic with period one. Adding one to the value of $L_3/\lambda_{01}^{TM}$ does not change the reflection of the $TM_{01}$ wave from the short at $z = L_3$. It only changes the reflections of the even and odd $TE_{11}$ modes and all the nonpropagating modes. Now, with the parameters of (8.16), our solution for the electric field in the circular waveguide did not contain any $TE_{11}$ modes. The computed values of the constants $C_{11}^{TEe}$ and $C_{11}^{TEo}$ in (6.75) were zero. When $L_3/\lambda_{01}^{TM}$ is large, there is not much reflection of nonpropagating modes from the short at $z = L_3$ because any nonpropagating mode suffers attenuation on its journey from the aperture to the short at $z = L_3$. Furthermore, its reflection suffers the same amount of attenuation in going from the short at $z = L_3$ back to the aperture.

In Fig. 8.6, $P_t$ could not be plotted for $L_3/\lambda_{01}^{TM} < 0.067983034$ because $L_3$ cannot be less than $c/2$. The approximate value 0.067983034 is obtained
by writing

\[ \frac{L_3}{\lambda_{01}^{TM}} = \frac{c}{2\lambda_{01}^{TM}} \]  

(8.17)

and substituting (8.3) and (8.5) into the right-hand side of (8.17) to obtain

\[ \frac{L_3}{\lambda_{01}^{TM}} = \sqrt{\frac{(2.95)^2 - (2.40482556)^2}{8\pi}} = 0.067983034. \]  

(8.18)

Figures 8.7 to 8.13 show plots of \(|E_{(A2)}^{(\phi, 0)}|/|E_{01}^{TM+}|_{\text{rms}}\) of (7.13) versus \(\phi/\phi_0\) and \(|E_{(A2)}^{(0, z)}|/|E_{01}^{TM+}|_{\text{rms}}\) versus \(z/c\) for \(ka = 2.855994, 2.86, 2.90, 2.95, 3.00, 3.05,\) and 3.054236 when \(L_3/[\lambda_{01}^{TM}]_{ka=2.95} = 0.5\). As given by (7.13), \(E_{(A2)}^{(\phi, 0)}|/|E_{01}^{TM+}|_{\text{rms}}\) is the \(\phi\)-component of the normalized electric field at \(z = 0\) in the right-hand aperture \(A_2\). As given by (7.14), \(E_{(A2)}^{(0, z)}|/|E_{01}^{TM+}|_{\text{rms}}\) is the \(z\)-component of the normalized electric field at \(\phi = 0\) in \(A_2\). The values 2.855994 and 3.054236 were purposely chosen close to the lower and upper bounds in (8.10). The curves in Figs. 8.7 to 8.13 are not smooth because they were obtained by drawing straight lines between points spaced 0.025 apart in \(\phi/\phi_0\) and 0.05 apart in \(z/z_0\). In Figs. 8.7 to 8.13, \(|E_{(A2)}^{(0, z)}|\) is generally much larger than \(|E_{(A2)}^{(\phi, 0)}|\). This is expected because the aperture \(A_2\) stops only the \(z\)-directed electric current \(J\) of (8.15). There is no \(\phi\)-directed electric current to stop.

The data for the plots in Figs. 8.7 to 8.13 were computed with

\[
\begin{align*}
\frac{b}{a} &= 1.1 \\
\frac{c}{a} &= 0.5 \\
L_3 &= 0.5[\lambda_{01}^{TM}]_{ka=2.95} \\
\text{BKM} &= 33 \\
\text{XM} &= 100 \\
Z_1 &= Z_2 = Z_{01}^{TE}
\end{align*}
\]

(8.19)

where \([\lambda_{01}^{TM}]_{ka=2.95}\) is given by (8.5). For the data in (8.19), the computed values of \(V_{pq}^{TM}\) and \(V_{pq}^{TE}\) satisfy

\[
\begin{align*}
V_{pq}^{1TM} &= V_{pq}^{2TM} \\
V_{pq}^{1TE} &= V_{pq}^{2TE}
\end{align*}
\]

(8.20)
Fig. 8.7. The ratios (a) $|E_\phi^{(A2)}|/|E_{01}^{TM+}|_\text{rms}$ and (b) $|E_z^{(A2)}|/|E_{01}^{TM+}|_\text{rms}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 2.855994$ and $L_3 = 0.5 \lambda_{01}^{TM}|_{k_a=2.95}$. 
Fig. 8.8. The ratios (a) $|E_{\phi}^{(A2)}|/|E_{01}^{TM_01+}|_{\text{rms}}$ and (b) $|E_z^{(A2)}|/|E_{01}^{TM_01+}|_{\text{rms}}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 2.86$ and $L_3 = 0.5 \left[ \frac{\lambda_{01}}{k} \right]_{ka=2.95}$. 

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Fig. 8.9. The ratios (a) $|E^{(A2)}_\phi|/|E^{TM_{e+}}_{01}|_{\text{rms}}$ and (b) $|E^{(A2)}_z|/|E^{TM_{e+}}_{01}|_{\text{rms}}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 2.90$ and $L_3 = 0.5 \left[ \lambda^{TM}_{01} \right]_{ka=2.95}$. 

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Fig. 8.10. The ratios (a) $|E_{\phi}^{(A2)}|/|E_{01}^{TM\epsilon+}|_{\text{rms}}$ and (b) $|E_{z}^{(A2)}|/|E_{01}^{TM\epsilon+}|_{\text{rms}}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 2.95$ and $L_3 = 0.5 \left[\lambda_{01}^M\right]_{ka=2.95}$. 

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Fig. 8.11. The ratios (a) $|E_{\phi}^{(A2)}|/|E_{01}^{TM+}|_{\text{rms}}$ and (b) $|E_{z}^{(A2)}|/|E_{01}^{TM+}|_{\text{rms}}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 3.00$ and $L_3 = 0.5 \lambda_{01}^{TM}$.
Fig. 8.12. The ratios (a) $|E_{\phi}^{(A2)}|/|E_{01}^{TM} + |_{rms}$ and (b) $|E_{z}^{(A2)}|/|E_{01}^{TM} + |_{rms}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 3.05$ and $L_3 = 0.5 \left[ \lambda_0^{TM} \right]_{ka=2.95}$. 

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Fig. 8.13. The ratios (a) $\frac{|E^{342}_\phi|}{|E_{01}^{TM_{e+}}|_{\text{rms}}}$ and (b) $\frac{|E^{342}_z|}{|E_{01}^{TM_{e+}}|_{\text{rms}}}$ of the magnitudes of the $\phi$- and $z$-directed electric fields in the aperture $A_2$ to the root mean square value of the electric field of the incident $TM_{01}$ wave in the circular cylinder when $ka = 3.054236$ and $L_3 = 0.5 \left[ \lambda_{01}^{TM} \right]_{ka=2.95}$. 

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the angle $\pi - \phi$ in $A_2$. Furthermore, the $z$ component of the electric field at the angle $\phi$ in $A_1$ is equal to the $z$ component of the electric field at the angle $\pi - \phi$ in $A_2$.

Note that the values of BKM and XM in (8.19) are larger than those in (8.16). Larger values of BKM and XM were needed for accurate calculation of the tangential electric field in $A_2$ than for accurate calculation of $P_i$. Larger values were needed because the convergence of the electric field in $A_2$ with increasing BKM and XM was slower than the convergence of $P_i$. The electric field in $A_2$ converges slowly because its $\phi$-component tends toward infinity as $1/(1-(\phi/\phi_o)^2)^\nu$ when $\phi/\phi_o$ approaches $\pm 1$ where $\nu = (1-2\phi_o/\pi)/(3-2\phi_o/\pi)$, and its $z$-component tends toward infinity as $1/(1-(2z/c)^2)^{1/3}$ when $2z/c$ approaches $\pm 1$ [6, page 387].
Appendix A

Ordering the Expansion Functions

The expansion function $M_{mn}^{1\delta}(\phi, z)$ for the equivalent magnetic current $M^{(1)}$ is, according to [1, eq. (2.13)], defined in terms of $\xi_{mn}^\delta$. Here, $\delta$ may be either $TM$ or $TE$, $\xi_{mn}^{TM}$ is the electric-type mode vector of the $TM_{mn}$ rectangular waveguide mode, and $\xi_{mn}^{TE}$ is the electric-type mode vector of the $TE_{mn}$ rectangular waveguide mode. The expansion function $M_{mn}^{2\delta}(\phi, z)$ is, according to [1, eq. (2.14)], also defined in terms of $\xi_{mn}^\delta$.

The expansion functions are arranged in the following order:

1. $\{M_{m1}^{1TM}, m = 1, 2, \cdots, MM(2)\}$, $\{M_{m2}^{1TM}, m = 1, 2, \cdots, MM(3)\}$,
   $\{M_{m3}^{1TM}, m = 1, 2, \cdots, MM(4)\}$, ...
2. $\{M_{m0}^{1TE}, m = 1, 2, \cdots, MM(1)\}$, $\{M_{m1}^{1TE}, m = 0, 1, \cdots, MM(2)\}$,
   $\{M_{m2}^{1TE}, m = 0, 1, \cdots, MM(3)\}$, ...
3. $\{M_{m1}^{2TM}, m = 1, 2, \cdots, MM(2)\}$, $\{M_{m2}^{2TM}, m = 1, 2, \cdots, MM(3)\}$,
   $\{M_{m3}^{2TM}, m = 1, 2, \cdots, MM(4)\}$, ...
4. $\{M_{m0}^{2TE}, m = 1, 2, \cdots, MM(1)\}$, $\{M_{m1}^{2TE}, m = 0, 1, \cdots, MM(2)\}$,
   $\{M_{m2}^{2TE}, m = 0, 1, \cdots, MM(3)\}$, ...

As listed above, items 3 and 4 are, respectively, items 1 and 2 with the
The MM's in items 1, 2, 3, and 4 are variables that are calculated in our computer program. The value of $\text{MM}(n+1)$ is determined in the following manner. The sequences of expansion functions $\{_{\text{TM}}M_{mn}^{1}\}$, $\{_{\text{TM}}M_{mn}^{2}\}$, and $\{_{\text{TE}}M_{mn}^{2}\}$ are terminated by requiring that the nonnegative integers $m$ and $n$ be small enough so that

$$k_{mn}b \leq \text{BKM} \quad (A.1)$$

where $k_{mn}$ is the mode cutoff wavenumber given by (2.4), and BKM is a specified constant. The parameter BKM is not written with italicized letters because it is an input variable for our computer program. Substitution of (2.4) into (A.1) gives

$$\sqrt{(m\pi)^2 + \left(\frac{n\pi b}{c}\right)^2} \leq \text{BKM}. \quad (A.2)$$

According to (A.2), $\text{MM}(n+1)$ is the largest integer such that

$$\sqrt{(\text{MM}(n+1)\pi)^2 + \left(\frac{n\pi b}{c}\right)^2} \leq \text{BKM}. \quad (A.3)$$

All values of $n$ so large that no integer $\text{MM}(n+1)$ satisfies (A.3) are disallowed.

If we define $M_{i1}^{\text{TM}}$ to be the $i^{\text{th}}$ $M_{11}^{\text{TM}}$ in item 1 for $i = 1, 2, \ldots$, then there will be a pair of integers $m$ and $n$ such that $M_{i1}^{\text{TM}} = M_{mn}^{1}$). Thus, there will, for the $M_{11}^{\text{TM}}$'s, be a correspondence between each pair of integers $m$ and $n$ in use and each single integer $i$. Replacing $m$, $n$, and $i$ by $p$, $q$, and $j$, respectively, there will, for the $M_{11}^{\text{TM}}$'s, be a correspondence between each pair of integers $p$ and $q$ in use and each single integer $j$. This is what is called in the sentence following (2.5) the correspondence between each pair of integers $(p, q)$ used in [1, eq. (3.44)] and the subscript $j$ in [1, eq. (3.44)].

If we define $M_{i1}^{\text{TE}}$ to be the $i^{\text{th}}$ $M_{11}^{\text{TE}}$ in item 2 for $i = 1, 2, \ldots$, then there will be a pair of integers $m$ and $n$ such that $M_{i1}^{\text{TE}} = M_{mn}^{1}$. Thus, there will, for the $M_{11}^{\text{TE}}$'s, be a correspondence between each pair of integers $m$ and $n$ in use and each single integer $i$. Replacing $m$, $n$, and $i$ by $p$, $q$, and $j$, respectively, there will, for the $M_{11}^{\text{TE}}$'s, be a correspondence between each pair of integers $p$ and $q$ in use and each single integer $j$. This is what is called in the sentence following (2.5) the correspondence between each pair of integers $(p, q)$ used in [1, eq. (3.47)] and the subscript $j$ in [1, eq. (3.47)].
Appendix B

Roots of Bessel Functions and Their Derivatives

This appendix describes how numerical values of \{j_{ns}, s = 1, 2, \ldots, s_{\text{max}}; n = 0, 1, \ldots\} and \{j'_{ns}, s = 1, 2, \ldots, s_{\text{max}}; n = 0, 1, \ldots\} are obtained. Here, \(j_{ns}\) is the \(s^{th}\) root of \(J_n\):

\[
J_n(j_{ns}) = 0. \tag{B.1}
\]

Furthermore, \(j'_{ns}\) is the \(s^{th}\) root of \(J'_n\):

\[
J'_n(j'_{ns}) = 0. \tag{B.2}
\]

In (B.1), \(J_n\) is the Bessel function of the first kind of order \(n\). In (B.2), \(J'_n\) is the derivative of \(J_n\) with respect to its argument. Now, the roots \(\{j_{ns}\}\) and \(\{j'_{ns}\}\) are ordered so that

\[
\begin{align*}
0 < j_{0,1} &< j'_{0,1} < j_{0,2} \ldots < j_{0,s_{\text{max}}} < j'_{0,s_{\text{max}}}, \\
n < j'_{n,1} &< j_{n,1} < j'_{n,2} \ldots < j_{n,s_{\text{max}}} < j'_{n,s_{\text{max}}}, \quad n = 1, 2, \ldots
\end{align*} \tag{B.3}
\]

Here, \(s_{\text{max}}\) depends on \(n\). Given \(n\), \(s_{\text{max}}\) is the largest value of \(s\) such that

\[
\begin{align*}
j_{0,s} &\leq XM, \quad n = 0 \\
j'_{n,s} &\leq XM, \quad n = 1, 2, \ldots
\end{align*} \tag{B.4}
\]

where XM is not written with italicized letters because it is an input variable for our computer program. Assuming that \(XM > j_{0,1}\), all values of \(n\) so large that \(j'_{n,1} > XM\) are disallowed. In (B.1), XM is a constant that controls the
number of roots that are calculated. If one wants to calculate more roots, one should choose a larger value of XM.

Our numerical values of \{j_{ns}, s = 1, 2, \ldots, 49; n = 0, 1, \ldots, 19\} and \{j'_{ns}, s = 1, 2, \ldots, 49; n = 0, 1, \ldots, 19\} are taken directly from [5, Tables 1 and 2]. For other values of n and s, we calculate \(j_{ns}\) and \(j'_{ns}\) by means of formulas given in [5]. In the body of the present report, the \(s^{th}\) roots of \(J_n\) and \(J'_n\) were called \(x_{ns}\) and \(x'_{ns}\) to coincide with the notation in [1]. Here in Appendix B, these roots are, more in harmony with the notation in [5], called \(j_{ns}\) and \(j'_{ns}\). However, our notation in Appendix B is slightly different from that of [5]. Our \(j_{ns}\) is what Olver calls \(j_{n,s}\). Our \(j'_{0,s}\) is what Olver calls \(j'_{0,s+1}\). Our \(j'_{ns}\) for \(n \geq 1\) is what Olver calls \(j'_{n,s}\). Our "inner fringe" calculated values of \{\(j_{20s}, s = 1, 2, \ldots, 50\)\}, \{\(j_{50n}, n = 0, 1, 2, \ldots, 19\)\}, \{\(j'_{20s}, s = 1, 2, \ldots, 50\)\}, and \{\(j'_{50n}, n = 0, 1, 2, \ldots, 19\)\} agree well with the "outer fringe" tabulated values in [5, Tables 1 and 2]. There is no entry in [5, Table 2] which corresponds to our \(j_{0,50}\). However, according to [3, formula 9.1.28], \(j'_{0,50} = j_{1,50}\), and there is an entry in [5, Table 1] which corresponds to our \(j_{1,50}\). Although, as stated earlier in this paragraph, our notation does not normally place a comma between the subscripts of \(j\) and \(j'\), we had to use a comma in the previous two sentences in order to separate the "0" from the "50" and the "1" from the "50".

### B.1 Evaluation of Roots of Bessel Functions of Large Order

For \(n \geq 20\) and \(s \geq 1\), we approximate \(j_{ns}\) by [5, eq. (9.01)]

\[
j_{ns} = nz + \frac{p_1}{n} + \frac{p_2}{n^3} \quad (B.5)
\]

where \(z\), \(p_1\), and \(p_2\) are tabulated functions of \(-\zeta\) where

\[
-\zeta = -n^{-2/3}a_s. \quad (B.6)
\]

In (B.6), \(a_s\) is the \(s^{th}\) negative root of the Airy function \(Ai\):

\[
Ai(a_s) = 0, \quad s = 1, 2, \ldots \quad (B.7)
\]

Here,

\[
0 < -a_1 < -a_2 < -a_3 < \cdots \quad (B.8)
\]
The roots \( \{a_s, s = 1, 2, \ldots, 50\} \) are tabulated [5, Table V, page 78]. For \( s > 50 \) [5, eq. (9.07)],
\[
a_s = -\lambda^{2/3} \left( 1 + \frac{5}{48\lambda^2} \right)
\]
where [5, eq. (9.09)],
\[
\lambda = \frac{3\pi}{8}(4s - 1).
\]
Actually, \( z, p_1, \) and \( p_2 \) are tabulated functions of \(-\zeta\) only for \( (0 \leq -\zeta \leq 7.5) \) [5, Table IV, pages 72 and 74]. If \(-\zeta > 7.5\), then \( z = 2/(3\zeta^3) \) and \( p_1 \) are tabulated functions of \( \xi \) [5, Table IV, page 74] where
\[
\xi = \frac{1}{\sqrt{-\zeta}}.
\]
If \(-\zeta > 7.5\), then \( p_2 = 0 \).

The modified interpolation formula of Everett [5, eq. (9.04)], [7, page 57],
\[
f_p = (1 - p)f_0 + p f_1 + E_2 \delta_{m0}^2 + F_2 \delta_{m1}^2 + M_4 \gamma_0^4 + N_4 \gamma_1^4,
\]
is used to obtain accurate values of \( z, p_1, \) and \( p_2 \). In (B.12), the \( f \)'s are values of the function being interpolated, the \( \delta_m^2 \)'s are modified second-order differences, and the \( \gamma^4 \)'s are modified fourth-order differences. In particular, \( f_p \) is the interpolated value of \( f \) at the actual value of the argument; \( f_0, \delta_{m0}^2, \) and \( \gamma_0^4 \) are the tabulated values of \( f, \delta_m^2, \) and \( \gamma^4 \) at the nearest smaller tabulated value of the argument; and \( f_1, \delta_{m1}^2, \) and \( \gamma_1^4 \) are the tabulated values of \( f, \delta_m^2, \) and \( \gamma^4 \) at the nearest larger tabulated value of the argument. The argument is either \(-\zeta\) or \( \xi \). Numerically, \( p \) is the ratio of the difference between the actual value and the nearest smaller tabulated value of the argument to the difference between the nearest larger and nearest smaller tabulated values of the argument. Thus, the actual value of the argument is a fraction \( p \) of the way from the nearest lower tabulated value to the nearest upper tabulated value. In (B.12), \( E_2, F_2, M_4, \) and \( N_4 \) are given by [7, pages 56 and 57]
\[
E_2 = -\frac{p(p-1)(p-2)}{6}
\]
\[
F_2 = \frac{(p+1)p(p-1)}{6}
\]
\[
M_4 = 1000E_2 \left\{ \frac{(p+1)(p-3)}{20} + 0.184 \right\}
\]
\[ N_4 = 1000F_2 \left\{ \frac{(p + 2)(p - 2)}{20} + 0.184 \right\}. \quad (B.16) \]

**B.2 Evaluation of Roots of Derivatives of Bessel Functions of Large Order**

For \( n \geq 20 \) and \( s \geq 1 \), we approximate \( j'_n \) by \([5, \text{eq. (9.02)}] \)

\[ j'_{ns} = nz + \frac{q_1}{n} + \frac{q_2}{n^3} + \frac{q_3}{n^5} \quad (B.17) \]

where \( z, q_1, q_2, \) and \( q_3 \) are tabulated functions of \( -\zeta \) \([5, \text{Table IV, pages 72-75}] \) where

\[ -\zeta = -n^{-2/3}a'_s. \quad (B.18) \]

In (B.18), \( a'_s \) is the \( s^{th} \) negative root of \( \text{Ai}' \), the derivative of the Airy function \( \text{Ai} \):

\[ \text{Ai}'(a'_s) = 0, \ s = 1, 2, \ldots. \quad (B.19) \]

Here,

\[ 0 < -a'_1 < -a'_2 < -a'_3 < \ldots. \quad (B.20) \]

The roots \( \{a'_s, \ s = 1, 2, \ldots, 50\} \) are tabulated \([5, \text{Table V, page 78}] \). For \( s > 50 \) \([5, \text{eq. (9.08)}] \),

\[ a'_s = -\mu^{2/3} \left( 1 - \frac{7}{48\mu^2} \right) \quad (B.21) \]

where \([5, \text{eq. (9.09)}] \)

\[ \mu = \frac{3\pi}{8}(4s - 3). \quad (B.22) \]

The tabulation of \( z \) was described in the two sentences following (B.10). Actually, \( q_1, q_2, \) and \( q_3 \) are tabulated functions of \( -\zeta \) only for \( 0 \leq \zeta \leq 7.5 \) \([5, \text{Table IV, pages 73 and 75}] \). If \( -\zeta > 7.5 \), then \( q_2 = q_3 = 0 \) and \( q_1 \) is a tabulated function of \( \xi \) \([5, \text{Table IV, page 75}] \) where \( \xi \) is given by (B.11).

The interpolation formula (B.12) is used to obtain accurate values of \( z, q_1, q_2, \) and \( q_3 \).
B.3 Evaluation of Large Roots of Bessel Functions

For $0 \leq n \leq 19$ and $s \geq 50$, we approximate $j_n \beta_n$ and $j'_n \beta'_n$ by the truncated McMahon expansions [5, eqs. (1.10) and (1.12)]

\[
j_n = \beta - \sum_{r=1}^{4} \frac{A_{2r-1}}{(2r-1)!23^r \beta^{2r-1}} \quad (B.23)
\]
\[
j'_n = \beta' - \sum_{r=1}^{4} \frac{A'_{2r-1}}{(2r-1)!23^r \beta'^{(2r-1)}} \quad (B.24)
\]

In (B.23),

\[
\beta = (2n + 4s - 1) \frac{\pi}{4} \quad (B.25)
\]

and [5, eq. (1.11)]

\[
A_1 = \mu - 1 \quad (B.26)
\]
\[
A_3 = (\mu - 1)(7\mu - 31) \quad (B.27)
\]
\[
A_5 = 4(\mu - 1)(83\mu^2 - 982\mu + 3779) \quad (B.28)
\]
\[
A_7 = 6(\mu - 1)(6949\mu^3 - 153855\mu^2 + 1585743\mu - 6277237) \quad (B.29)
\]

where

\[
\mu = 4n^2. \quad (B.30)
\]

In (B.24),

\[
\beta' = (2n + 4s - 3) \frac{\pi}{4} \quad (B.31)
\]

and [5, eq. (1.13)]

\[
A'_1 = \mu + 3 \quad (B.32)
\]
\[
A'_3 = 7\mu^2 + 82\mu - 9 \quad (B.33)
\]
\[
A'_5 = 4(83\mu^3 + 2075\mu^2 - 3039\mu + 3537) \quad (B.34)
\]
\[
A'_7 = 6(6949\mu^4 + 296492\mu^3 - 1248002\mu^2 + 7414380\mu - 5853627) \quad (B.35)
\]

where $\mu$ is given by (B.30). Expressions (B.23) and (B.24) expand to

\[
j_n = \beta - \frac{A_1}{8\beta} - \frac{A_3}{384\beta^3} - \frac{1}{15360\beta^5} \left( \frac{A_5}{4} \right) - \frac{1}{3440640\beta^7} \left( \frac{A_7}{6} \right) \quad (B.36)
\]
\[
j'_n = \beta' - \frac{A'_1}{8\beta'} - \frac{A'_3}{384\beta'^3} - \frac{1}{15360\beta'^5} \left( \frac{A'_5}{4} \right) - \frac{1}{3440640\beta'^7} \left( \frac{A'_7}{6} \right). \quad (B.37)
\]
Appendix C

The Effect of Loads on the \( TM_{01} \) Source in the Circular Waveguide

In the body of this report, we find the electric fields in the waveguide mode converter due to the excitation of a unit-amplitude \( z \)-traveling \( TM_{01} \) wave. The source of this wave is taken to be the electric current source \( J_{\text{imp}} \) whose \(-z\)-traveling waves see a matched load. Specializing further, we take \( J_{\text{imp}} \) to be the transverse electric surface current density located at \( z = -L_5 \) that launches the \( z \)-traveling wave whose electromagnetic field is \( (E_{\text{Te},+}^{TM_{01}}, H_{\text{Te},+}^{TM_{01}}) \) in the region for which \( z > -L_5 \). Here,

\[
\begin{align*}
E_{\text{Te},+}^{TM_{01}} &= Z_{\text{Te},0}^{TM_{01}} \varphi_{\psi_{\text{Te},0}^{TM_{01}}} (\rho, \phi) e^{-j\beta_{01}^{TM_{01}} z} + \frac{y_{\psi_{\text{Te},0}^{TM_{01}}} (\rho, \phi) e^{-j\beta_{01}^{TM_{01}} z}}{j\omega \varepsilon} \\
H_{\text{Te},+}^{TM_{01}} &= \frac{\psi_{\text{Te},0}^{TM_{01}} (\rho, \phi) e^{-j\beta_{01}^{TM_{01}} z}}{j\omega} \left\{ \right. \\
\end{align*}
\]  \( (C.1) \)

Equation (C.1) was obtained by substituting (6.63) into [1, eq. (B.1)]. The electric and magnetic fields defined by (C.1) are those of the \( z \)-traveling \( TM_{01} \) mode. We assume that \( J_{\text{imp}} \) also launches the \(-z\)-traveling wave whose electromagnetic field is \( C(E_{\text{Te},-}^{TM_{01}}, H_{\text{Te},-}^{TM_{01}}) \) in the region for which \( z < -L_5 \). Here, \( C \) is an unknown constant and

\[
\begin{align*}
E_{\text{Te},-}^{TM_{01}} &= -Z_{\text{Te},0}^{TM_{01}} \varphi_{\psi_{\text{Te},0}^{TM_{01}}} (\rho, \phi) e^{j\beta_{01}^{TM_{01}} z} + \frac{y_{\psi_{\text{Te},0}^{TM_{01}}} (\rho, \phi) e^{j\beta_{01}^{TM_{01}} z}}{j\omega \varepsilon} \\
H_{\text{Te},-}^{TM_{01}} &= \frac{\psi_{\text{Te},0}^{TM_{01}} (\rho, \phi) e^{j\beta_{01}^{TM_{01}} z}}{j\omega} \left\{ \right. \\
\end{align*}
\]  \( (C.2) \)
Equation (C.2) was obtained by substituting (6.63) into [1, eq. (B.2)]. The electric and magnetic fields defined by (C.2) are those of the $-z$-traveling $TM_{01}$ mode. Requiring the transverse electric field to be continuous at $z = -L_5$, we obtain

$$C = -e^{j2\beta_{01}TML_5}.$$  \hspace{1cm} (C.3)

The $TM_{01}$ waves radiated by $J_{\text{imp}}$ in the circular waveguide are shown in Fig. C.1 where “1” is the coefficient of the $z$-traveling $TM_{01}$ mode in the region for which $-L_5 < z < -c/2$. “Γ” is the coefficient of the $-z$-traveling $TM_{01}$ mode in the region for which $-L_5 < z < -c/2$. “$C + \Gamma$” is the coefficient of the $-z$-traveling $TM_{01}$ mode in the region for which $-\infty < z < -L_5$. The $TM_{01}$ waves shown in Fig. C.1 are those dealt with in the body of this report. Thus, from (6.75), we have

$$\Gamma = -C_{01}TM e^{-j2\beta_{01}TM L_5}.$$  \hspace{1cm} (C.4)

Here, $\Gamma$ is a reflection coefficient because $\gamma$ is the ratio of the coefficient of the $-z$-traveling mode field (C.2) to the coefficient of the $z$-traveling mode field (C.1) in the region for which $-L_5 < z < -c/2$. Since the magnetic fields of the mode fields are $h_{01}^{TM}(\rho, \phi)e^{\mp j\beta_{01}TM z}$ rather than $\pm h_{01}^{TM}(\rho, \phi)e^{\mp j\beta_{01}TM z}$, $\gamma$ is a reflection coefficient for the current rather than for the voltage.

The loads mentioned in the title of this appendix are taken to be the $TM_{01}$ loads $Z_{L4}$ at $z = -L_4$ and $Z_{L6}$ at $z = -L_6$. See Fig. C.2. A $TM_{01}$ load is a load that acts on the voltage and current of only the $TM_{01}$ waves. When the loads $Z_{L4}$ and $Z_{L6}$ are in place, the $TM_{01}$ electric and magnetic fields $E$ and $H$ in the circular waveguide are, as indicated in Fig. C.2, assumed to be given by

$$\begin{align*}
E &= C_4^+ E_{01}^{TM+} + C_4^- E_{01}^{TM-} & - L_4 < z < L_5 \\
H &= C_4^+ H_{01}^{TM+} + C_4^- H_{01}^{TM-}
\end{align*}$$  \hspace{1cm} (C.5)

$$\begin{align*}
E &= C_5^+ E_{01}^{TM+} + C_5^- E_{01}^{TM-} & - L_5 < z < -L_6 \\
H &= C_5^+ H_{01}^{TM+} + C_5^- H_{01}^{TM-}
\end{align*}$$  \hspace{1cm} (C.6)

$$\begin{align*}
E &= C_6^+ E_{01}^{TM+} + C_6^- E_{01}^{TM-} & - L_6 < z < -c/2 \\
H &= C_6^+ H_{01}^{TM+} + C_6^- H_{01}^{TM-}
\end{align*}$$  \hspace{1cm} (C.7)
Fig. C.1. The $TM_{01}$ waves radiated by $J^{\text{imp}}$ in the circular waveguide. The situation in Fig. C.1 is the same as that in the body of this report. There are no additional loads.

Fig. C.2. The $TM_{01}$ waves radiated by $J^{\text{imp}}$ when the additional loads $Z_{L4}$ and $Z_{L6}$ are present in the circular waveguide.
where the $C$'s are unknown constants.

The effect of the loads $Z_{L4}$ and $Z_{L6}$ is to change the electric field incident on the aperture-perforated section of circular waveguide from $E_{01}^{TM_{Me}+}$ in Fig. C.1 to $C_6^+ E_{01}^{TM_{Me}+}$ in Fig. C.2. Viewing this incident electric field as the excitation of the aperture-perforated section of circular waveguide, we deduce that the fields in this section of waveguide in Fig. C.2 are those in Fig. C.1 multiplied by $C_6^+$. Therefore,

$$C_6^- = \Gamma C_6^+. \quad (C.8)$$

Equation (C.8) is one simultaneous equation in the variables $C_4^+,$ $C_4^-$, $C_5^+,$ $C_5^-$, $C_6^+$, and $C_6^-$. In the following five paragraphs, we obtain five more simultaneous equations in these six variables.

Because the waves launched by $Z_{\text{imp}}$ in Fig. C.2 are the same as those in Fig. C.1, we have

$$C_5^+ = C_4^+ + 1 \quad (C.9)$$

$$C_4^- = C_5^- + C. \quad (C.10)$$

Recall that the loads $Z_{L4}$ and $Z_{L6}$ act on the voltages and currents of the $TM_{01}$ waves. These voltages and currents are called the $TM_{01}$ voltages and currents. Seeking to define the $TM_{01}$ voltages and currents, we substitute (C.1) and (C.2) into (C.5)-(C.7) and take only the transverse part of $E$, which is called $E_t$:

$$E_t = \left( C_4^+ e^{-j\beta_{01} T M_z} - C_4^- e^{j\beta_{01} T M_z} \right) Z_{01}^{TM_{Me}} e_{01}^{TM_{Me}}(\rho, \phi) \right) \{ -L_4 < z < -L_5 \} \quad (C.11)$$

$$H = \left( C_4^+ e^{-j\beta_{01} T M_z} + C_4^- e^{j\beta_{01} T M_z} \right) h_{01}^{TM_{Me}}(\rho, \phi) \right) \} \{ -L_4 < z < -L_5 \} \quad (C.12)$$

$$E_t = \left( C_5^+ e^{-j\beta_{01} T M_z} - C_5^- e^{j\beta_{01} T M_z} \right) Z_{01}^{TM_{Me}} e_{01}^{TM_{Me}}(\rho, \phi) \right) \{ -L_5 < z < -L_6 \} \quad (C.13)$$

$$H = \left( C_5^+ e^{-j\beta_{01} T M_z} + C_5^- e^{j\beta_{01} T M_z} \right) h_{01}^{TM_{Me}}(\rho, \phi) \right) \} \{ -L_5 < z < -\frac{c}{2} \} \quad (C.14)$$

The $TM_{01}$ voltages are defined to be the coefficients of $e_{01}^{TM_{Me}}(\rho, \phi)$ in (C.11)-(C.13). The $TM_{01}$ currents are defined to be the coefficients of $h_{01}^{TM_{Me}}(\rho, \phi)$ in
(C.11)-(C.13). Viewing Fig. C.2 as a circuit, the \( TM_{01} \) voltage is the voltage of the upper line with respect to the lower one, and the \( TM_{01} \) current is the \( z \)-directed electric current on the upper line.

The presence of \( Z_{L4} \) at \( z = -L_4 \) requires that

\[
\left[ \frac{V}{I} \right]_{z=-L_4^+} = -Z_{L4} \tag{C.14}
\]

where \( V \) is the \( TM_{01} \) voltage and \( I \) is the \( TM_{01} \) current. The subscript \( "z = -L_4^+" \) denotes the limit as \( z \) approaches \( -L_4 \) from above. “From above" means through values which are greater than \( -L_4 \). Extracting \( V \) and \( I \) from (C.11) and substituting them into (C.14), we obtain

\[
\frac{C_4^+ e^{jl_4} - C_4^- e^{-jl_4}}{C_4^+ e^{jl_4} + C_4^- e^{-jl_4}} = -\frac{Z_{L4}}{Z_{TM_{01}e_0}} \tag{C.15}
\]

where

\[
l_4 = \beta_{01}^{TM} L_4. \tag{C.16}
\]

Solving (C.15) for \( C_4^+ \) in terms of \( C_4^- \), we arrive at

\[
C_4^+ = \Gamma_4 C_4^- \tag{C.17}
\]

where

\[
\Gamma_4 = \frac{Z_{TM_{01}e_0} - Z_{L4}}{Z_{TM_{01}e_0} + Z_{L4}} e^{-jl_4}. \tag{C.18}
\]

Since there is no series load at \( z = -L_6 \),

\[
[V]_{z=-L_6^-} = [V]_{z=-L_6^+} \tag{C.19}
\]

where the subscript \( "z = -L_6^-" \) denotes the the limit as \( z \) approaches \( -L_6 \) from below and \( "z = -L_6^+" \) denotes the limit as \( z \) approaches \( -L_6 \) from above. Extracting the \( V \)'s from (C.12) and (C.13) and substituting them into (C.19), we obtain

\[
C_5^+ e^{jl_6} - C_5^- e^{-jl_6} = C_6^+ e^{jl_6} - C_6^- e^{-jl_6} \tag{C.20}
\]

where

\[
l_6 = \beta_{01}^{TM} L_6. \tag{C.21}
\]
At $z = -L_6$, the $TM_{01}$ current $V/Z_{L6}$ flows from the upper terminal of $Z_{L6}$ to its lower terminal so that

$$[I]_{z=-L_6} = \frac{1}{Z_{L6}}[V]_{z=-L_6} + [I]_{z=-L_6^+}. \quad (C.22)$$

Extracting $V$ and $I$ from (C.12) and (C.13) and substituting them into (C.22), we obtain

$$C^+_5 e^{j\ell_6} + C^-_5 e^{-j\ell_6} = C^+_6 e^{j\ell_6} + C^-_6 e^{-j\ell_6} + \frac{C^+_5 e^{j\ell_6} - C^-_5 e^{-j\ell_6}}{Z} \quad (C.23)$$

where

$$Z = \frac{Z_{L6}}{Z_{TM_{01}}}. \quad (C.24)$$

Equation (C.23) becomes

$$C^+_5 (1 - Z)e^{j\ell_6} - C^-_5 (1 + Z)e^{-j\ell_6} + C^+_6 Ze^{j\ell_6} + C^-_6 Ze^{-j\ell_6} = 0. \quad (C.25)$$

Equations (C.17), (C.10), (C.9), (C.20), (C.25), and (C.8), ordered as cited, are written in matrix form as

$$\begin{bmatrix} 1 & -\Gamma_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -e^{j\ell_6} & e^{-j\ell_6} & e^{j\ell_6} & -e^{-j\ell_6} \\ 0 & 0 & (1 - Z)e^{j\ell_6} & -(1 + Z)e^{-j\ell_6} & Z e^{j\ell_6} & Z e^{-j\ell_6} \\ 0 & 0 & 0 & 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C^+_4 \\ C^-_4 \\ C^+_5 \\ C^-_5 \\ C^+_6 \\ C^-_6 \end{bmatrix} = \begin{bmatrix} 0 \\ C \\ 1 \\ 0 \\ 0 \end{bmatrix}. \quad (C.26)$$

We proceed to solve the matrix equation (C.26) for $C^+_4$, $C^-_4$, $C^+_5$, $C^-_5$, $C^+_6$, and $C^-_6$.
Adding the first row of (C.26) and the product of $\Gamma_4$ with the second row to the third row, we obtain, in view of (C.3),

$$
\begin{bmatrix}
1 & -\Gamma_4 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -\Gamma_4 & 0 \\
0 & 0 & -e^{j\ell_6} & e^{-j\ell_6} & e^{j\ell_6} - e^{-j\ell_6} \\
0 & 0 & (1 - Z)e^{j\ell_6} & -(1 + Z)e^{-j\ell_6} & Z e^{j\ell_6} Z e^{-j\ell_6} \\
0 & 0 & 0 & 0 & -\Gamma
\end{bmatrix}
\begin{bmatrix}
C_4^+ \\
C_4^- \\
C_5^+ \\
C_5^- \\
C_6^+ \\
C_6^-
\end{bmatrix} =
\begin{bmatrix}
0 \\
C \\
D \\
0 \\
0 \\
0
\end{bmatrix}
$$

(C.27)

where

$$D = 1 - \Gamma_4 e^{j2\vartheta_0^M L_5}. \quad \text{(C.28)}$$

The last four rows of (C.27) are

$$
\begin{bmatrix}
1 & -e^{j\ell_6} & 0 & 0 \\
-e^{-j\ell_6} & e^{-j\ell_6} & e^{j\ell_6} - e^{-j\ell_6} \\
(1 - Z)e^{j\ell_6} & -(1 + Z)e^{-j\ell_6} & Z e^{j\ell_6} Z e^{-j\ell_6} \\
0 & 0 & 0 & -\Gamma
\end{bmatrix}
\begin{bmatrix}
C_5^+ \\
C_5^- \\
C_6^+ \\
C_6^-
\end{bmatrix} =
\begin{bmatrix}
D \\
0 \\
0 \\
0
\end{bmatrix}. \quad \text{(C.29)}
$$

Adding the product of $e^{j\ell_6}$ with the first row to the second row of (C.29) and adding the product of $-(1 - Z)e^{j\ell_6}$ with the first row to the third row, we obtain

$$
\begin{bmatrix}
1 & -\Gamma_4 \\
0 & e^{-j\ell_6} - \Gamma_4 e^{j\ell_6} \\
0 & -(1 + Z)e^{-j\ell_6} + (1 - Z)\Gamma_4 e^{j\ell_6} \\
0 & 0 & -\Gamma
\end{bmatrix}
\begin{bmatrix}
C_5^+ \\
C_5^- \\
C_6^+ \\
C_6^-
\end{bmatrix} =
\begin{bmatrix}
D \\
D e^{j\ell_6} \\
-(1 - Z)D e^{j\ell_6} \\
0
\end{bmatrix}. \quad \text{(C.30)}
$$
Discarding the first row of (C.30) and multiplying the second and third rows by $e^{-j\theta_6}$, we obtain
\[
\begin{bmatrix}
    e^{-j\theta_6} - \Gamma_4 \\
    -(1 + Z)e^{-j\theta_6} + (1 - Z)\Gamma_4 \\
    0
\end{bmatrix}
\begin{bmatrix}
    1 \\
    Z \\
    -\Gamma
\end{bmatrix}
\begin{bmatrix}
    C_5^- \\
    C_6^+ \\
    C_6^-
\end{bmatrix} = 
\begin{bmatrix}
    D \\
    -(1 - Z)D \\
    0
\end{bmatrix}.
\] (C.31)

Adding the product of the first row with \{(1 + Z)e^{-j\theta_6} - (1 - Z)\Gamma_4\}/\{e^{-j\theta_6} - \Gamma_4\} to the second row of (C.31), we obtain
\[
\begin{bmatrix}
    e^{-j\theta_6} - \Gamma_4 \\
    0 \\
    0
\end{bmatrix}
\begin{bmatrix}
    1 \\
    \frac{(1 + 2Z)e^{-j\theta_6} - \Gamma_4}{e^{-j\theta_6} - \Gamma_4} \\
    -\Gamma
\end{bmatrix}
\begin{bmatrix}
    C_5^- \\
    C_6^+ \\
    C_6^-
\end{bmatrix} = 
\begin{bmatrix}
    D \\
    \frac{2ZDe^{-j\theta_6}}{e^{-j\theta_6} - \Gamma_4} \\
    0
\end{bmatrix}.
\] (C.32)

Discarding the first row of (C.32) and multiplying the second row by $e^{-j\theta_6} - \Gamma_4$, we obtain
\[
\begin{bmatrix}
    (1 + 2Z)e^{-j\theta_6} - \Gamma_4 \\
    -\Gamma
\end{bmatrix}
\begin{bmatrix}
    -e^{-j\theta_6} \\
    1
\end{bmatrix}
\begin{bmatrix}
    C_5^- \\
    C_6^+
\end{bmatrix} = 
\begin{bmatrix}
    2ZDe^{-j\theta_6} \\
    0
\end{bmatrix}.
\] (C.33)

Adding the product of the first row of (C.33) with \(\Gamma/\{(1 + 2Z)e^{-j\theta_6} - \Gamma_4\}\) to the second row, we obtain
\[
\left\{1 - \Gamma\frac{e^{-j\theta_6} + (2Z - 1)\Gamma_4e^{-j\theta_6}}{(1 + 2Z)e^{-j\theta_6} - \Gamma_4}\right\} C_6^- = \frac{2ZD\Gamma e^{-j\theta_6}}{(1 + 2Z)e^{-j\theta_6} - \Gamma_4}.
\] (C.34)
Solving (C.34) for $C_6^-$, we obtain

$$C_6^- = \frac{2ZD\Gamma}{\Delta} \quad (C.35)$$

where

$$\Delta = 1 + 2Z - (2Z - 1)\Gamma_4 - \Gamma_4 e^{j2\theta} - \Gamma e^{-j2\theta}. \quad (C.36)$$

Substitution of (C.35) into the second row of (C.33) gives

$$C_6^+ = \frac{2ZD}{\Delta}. \quad (C.37)$$

Substituting (C.35) and (C.36) into the first row of (C.32) and solving for $C_5^-$, we arrive at

$$C_5^- = \frac{\{(2Z - 1)\Gamma + e^{j2\theta}\}D}{\Delta}. \quad (C.38)$$

Substitution of (C.38) into the first row of (C.30) gives

$$C_5^+ = \frac{\{(2Z + 1 - \Gamma e^{-j2\theta})D}{\Delta}. \quad (C.39)$$

Next, $C_4^-$ is given by (C.10). Finally, $C_4^+$ is given by (C.17). It can be verified that the $C$'s given by (C.35), (C.37), (C.38), (C.39), (C.10), and (C.17) do indeed satisfy (C.26).
References


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