An Analysis of Developing Turbulent Flow Between a Moving Cylinder and a Concentric Tube

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AN ANALYSIS OF DEVELOPING TURBULENT FLOW BETWEEN A MOVING CYLINDER AND A CONCENTRIC TUBE

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Abstract

A numerical analysis is made of developing and developed turbulent flow in the annular region between a cylinder moving at constant velocity within a fixed concentric tube. Turbulent shear is modeled by eddy viscosity, and a uniform velocity is assumed at the entrance to the annular region. The computations extend and modify the method of Sud and Chaddock (1981) to arbitrary Reynolds numbers and radius ratios. Approximate formulas are given for both developing and developed values of pressure drop, shear stress on the inner and outer walls, and total cylinder drag.

Nomenclature

\[ \begin{align*}
  a &= r_m / r_i \\
  b &= r_o / r_i \\
  C &= \text{parameter defined in Eq. (11)} \\
  Dh &= \text{hydraulic diameter, } 2(r_o - r_i) \\
  f &= \text{friction factor, Eq. (10)} \\
  F &= \text{drag force, Eq. (14)} \\
  L_e &= \text{entrance length, Fig. 7} \\
  \eta &= \text{eddy viscosity parameter, Table 1} \\
  p &= \text{pressure} \\
  r &= \text{radial coordinate} \\
  r_{mt} &= \text{radius of maximum velocity, Fig. 2} \\
  Re_v &= U_v r_i / v \\
  Re_b &= U_b r_i / v \\
  u &= \text{axial velocity} \\
  U &= \text{average annular velocity, Eq. (1)} \\
  U_v &= \text{inner cylinder or vehicle velocity} \\
  U_b &= \text{inviscid core velocity, Fig. 2} \\
  v_e &= \text{friction velocity, } (\tau_w / \rho)^{1/2} \\
  x &= \text{axial coordinate} \\
  y &= \text{coordinate normal to wall, Fig. 2} \\
  y_1 &= \text{eddy viscosity crossover point, Table 1} \\
  \delta &= \text{boundary layer thickness} \\
  \rho &= \text{density} \\
  \nu &= \text{kinematic viscosity} \\
  \tau &= \text{shear stress} \\
  \tau^* &= 2r_p U_v^2 \\
\end{align*} \]

\[ \text{Superscript} \]

\[ + \text{ = law-of-the-wall coordinates, Eq. (4)} \]

\[ \text{Subscripts} \]

\[ \text{i = inner wall} \]
\[ \text{o = outer wall} \]
\[ \text{w = either wall} \]
\[ \text{fd = fully developed} \]
\[ \text{a = inviscid core value} \]
\[ \text{E = in the entrance region} \]

Introduction

The problem of turbulent flow in a concentric annulus has been of interest for at least eight decades, beginning with the experimental results of Becker (1907). Numerous papers have treated the problem of pressure-driven flow between fixed cylinders. For fully developed flow, the eddy viscosity theory of Quarryby (1968) is in good agreement with his experimental results (Quarryby 1966). For both laminar and turbulent flow in a fixed annulus, the point of maximum axial velocity \( r_{mt} \) is closer to the inner radius (Lawn and Elliot 1972).

Developing annular flow is less well documented, but an integral analysis by Wilson and Medwell (1971) shows a development length for turbulent flow of about ten hydraulic diameters for Reynolds numbers from \( 10^4 \) to \( 3 \times 10^5 \) and radius ratios from 1.25 to 5.0. Their results agree reasonably well with data by Okiishi and Serov (1967).

The problem under study here is the annular flow driven by uniform axial motion of the inner cylinder, which simulates the motion caused by vehicles travelling in tubes or by the launching of torpedoes or cylindrical buoys. Experimental data on vehicles in tubes are summarized by Davidson (1974), and theoretical results are given in the book by Hammit (1973). Implicit in the analysis of the annular flow induced by a moving cylinder is the assumption that the volume swept out by the vehicle passes through the annulus to fill the void behind the vehicle, as in Figure 1. This simulates a very long outer tube, whereas in a short tube the inlet and outlet can serve as a source and sink of fluid, with less volume passing through the annulus. The only detailed analysis of the flow shown in Figure 1 is given by Sud and Chaddock (1981), for both developing and developed flows. After presenting a general analytical technique, Sud and Chaddock give computed results for only a single case, \( \rho_o = 8 \text{ ft (2.44 m)}, r_i = 6 \text{ ft (1.83 m)}, \text{ and } U_v = 200 \text{ ft/s (61 m/s)} \), at two air densities (Reynolds numbers).

It is the purpose of the present paper to extend the analysis of Sud and Chaddock to arbitrary radius ratios and Reynolds numbers, for both developing and developed flow. The developed flow analysis is identical to Sud and Chaddock. For developing flow, corrections are made to their analysis, as it is felt that Sud and Chaddock's results in the entrance region are somewhat inaccurate.
The turbulent core, \( y_L \leq y \leq y_M \), is modeled with Karman's similarity hypothesis for eddy viscosity:

\[
\frac{d^2 u^*}{dy^*} = \frac{\kappa (du^*/dy^*)^2}{(\nu \tau_w^* - du^*/dy^*)^{1/2}}, \quad \kappa = 0.36
\]

In both cases, the local shear stress is related to local radius through a momentum balance:

\[
\tau = \frac{r_w (r^2 - r_m^2)}{r_w} = \frac{r^2 - r_m^2}{r_m^2} \quad \text{or} \quad \frac{\tau}{r^2 - r_m^2} = \frac{r_w}{r_m^2}
\]

At high Reynolds numbers, Deissler's damping parameter \( n^2 \) equals 0.0154 with the cross-over point \( y_+^* = 15.0 \). At lower Reynolds numbers, values of \( n^2 \) and \( y_+^* \) were taken from a graph given by Quarmby (1968). Some typical values are given here in Table 1. These values are not thought to be especially accurate but are retained for comparison to the analyses of Quarmby (1968) and Sud and Chaddock (1981).

Equations (5,6,7) are a set of differential equations to solve for the velocity profiles \( u_i \) and \( u_o \) in the inner and outer regions. From Fig. 2, the boundary conditions are:

\[
y_1 = 0; \quad u_1 = 0
\]

\[
y_0 = 0; \quad u_0 = U_v
\]

The equations are solved by an iterative technique similar to Sud and Chaddock but modified by Kotlow (1985). One begins by guessing \( \tau_0 \) and \( a \) for \( r_m \), whence \( \tau_1 \) can be computed and the velocity profiles \( u_i \) and \( u_o \) computed by numerical (Runge-Kutta) integration of Eqs. (5,6,7). The value of \( \tau_0 \) is gradually adjusted and the integrations repeated until the core velocity \( U_v \) exceeds \( U_0 \). If the two profiles do not meet at \( r_m \) with \( U_{6i} = U_{6o} \) then a new value of \( a \) is computed from the relation

\[
a = \frac{(b + \alpha^2)}{b + \alpha^2} \quad \text{or} \quad \alpha = U_{6o}/U_{6i}
\]

The analysis follows Sud and Chaddock (1981). Both the inner and outer regions are divided into two computational regions as in Fig. 2: (1) the sublayer, and (2) the turbulent core. The flow is assumed steady, incompressible, and turbulent, with impermeable walls. The variables are non-dimensionalized in law-of-the-wall fashion:

\[
u^* = u/v; \quad y^* = y v^*/v; \quad \nu^* = (\nu \tau)^{1/2}
\]

The sublayer, \( 0 \leq y \leq y_s \), is modeled with Deissler's eddy viscosity formulation:
The error when compared to Fig. 3 is about ±3% in the range $Re_V = 10^4-10^6$.

Kotlow (1985) also tabulates values of the dimensionless inner and outer wall shear stresses. The inner stress is the higher and, for fully-developed flow, these stresses may be estimated by the following curve-fit formulas:

$$\tau_i = \frac{C}{(b+1)} \frac{b^2+1}{2(b-1)}$$

where $C = -b^2+1$ and $Re^* = \frac{b^2}{2(b-1)} Re_V$.

Equations (12) and (13) have an accuracy of about ±4% over the range $Re_V = 10^5-10^8$ and $b = 1.01-2.0$.

The present analysis follows Sud and Chaddock (1981) by assuming that the entrance velocity profile is uniform, $u = U_v$, as in Fig. 2. Boundary layers $\delta_1$ and $\delta_2$ grow on the walls until they meet at $x = L^*$, the entrance length. At any $x$, the core velocity $U_g(x)$ is assumed uniform.
At any given \( x \), variables \( u^i \) and \( y^i \) are scaled with local shears \( \tau_i(x) \) and \( \tau_0(x) \) and the boundary layer profiles \( u_i \) and \( u_0 \) computed by integration of Eqs. (5) and (6). Since \( \tau(y) \) is not known in the developing region, the linear profile assumption of Wilson and Medwell (1971) was adopted:

\[
\tau(y) = \tau_w(1 - y/\delta)
\]  

(15)

for both the inner and outer regions.

The study began by repeating the computations of Sud and Chaddock (1981) for the special case given in their Figures 3 and 4. Only qualitative agreement was obtained, and the computations indicated that local mass and momentum balances were not accurately satisfied. The discrepancy, if any, was difficult to analyze, for Sud and Chaddock used the transformation technique of Wilson and Medwell (1971), resulting in a double integral with extremely complex arguments. In any case, the Sud and Chaddock approach was abandoned and instead a local mass and momentum balance was used to compute incremental changes in velocity, pressure, and shear. Developing flow was thus computed for the ranges \( Re_y = 10^5-10^7 \) and \( b = 1.01-2.0 \).

A momentum balance over a short distance \( \Delta x \) of inner boundary layer gives Eq. (20) of Sud and Chaddock (1979):

\[
\frac{\partial \rho}{\partial x} \left( r^2 \frac{u^i}{r} \right) + \frac{\Delta x}{2} \left( \rho \frac{v^i}{r} \right) = \frac{\Delta x}{2} \left( \rho \frac{v_0^i}{r} \right) = \Delta I, \tag{17}
\]

where \( \Delta I \) is the average inner wall shear. An exactly similar relation holds for the outer boundary layer.

A direct comparison was made with the special case computed by Sud and Chaddock (1979, 1981): \( r_i = 1.83 \text{ m}, r_o = 2.44 \text{ m}, U_i = 61 \text{ m/s}, \) for air at \( 37^\circ \text{C} \) and \( 0.25 \text{ atm.} \) The results are shown in Fig. 5 for the local pressure drop. The two results are similar, but the present computations are smoother and perhaps slightly more accurate.
If the cylinder is of length $L_v$ greater than $L_s$, its overall drag coefficient is the sum of an entrance drag plus fully-developed drag on the aft portion ($L_v-L_s$):

$$C_D = \frac{L_s}{D_h} \left[ (4(b-1)r_i^* + r_{E}) \right]$$

$$+ \left( \frac{L_v-L_s}{D_h} (4(b-1)r_i^* + r_{fd}) \right)$$

(21)

Each of the parameters in this expression may be estimated from the previous curve-fit formulas (11) to (20). The overall accuracy for the drag is ±6%.

Although not shown here, the inner and outer velocity profiles, when plotted in law-of-the-wall coordinates, $u'$ versus $y'$ and $(u'-U)$ versus $y'$, were in excellent agreement with the traditional linear sublayer and logarithmic layer formulas. This was true for all $Re_v$ and $b$ computed, in both the developing and developed regions.

However, in their fixed-annulus experiment, Lawn and Elliot (1972) report that inner-wall velocity data failed to correlate with the law-of-the-wall for their largest $b = 11.36$. Such large radius ratios were not computed here.

Figure 6. Computed overall friction factors in the developing region.

Figure 6 shows the computed friction factors in the entrance region, defined as the dimensionless pressure drop between $x=0$ and $x=L_s$, the entrance length. These values everywhere exceed the fully-developed friction factors from Fig. 3. They may be curve-fit by the relation

$$f_E = f_{E_d} + 0.032 C^2 Re_v^{-0.2}$$

(18)

where $f_{E_d}$ is estimated from Fig. 3 or Eq. (11). The overall accuracy is ±3% in the range $Re_v = 10^5$-$10^7$ and $b = 1.01$-$2.0$.

Similarly, the integrated average inner-wall shear stress over the entrance length $L_s$ may be curve-fit by

$$\tau_i = \tau_{I_d} + 0.0068 C^2 Re_v^{-0.205}$$

(19)

where $\tau_i = 2\tau/p\mu_c^2$ and the fully-developed value is estimated from Eqs. (12) and (13). The overall accuracy of Eq. (19) is ±3.5% for $Re_v = 10^5$-$10^7$ and $b = 1.01$-$2.0$.

Finally, the computed entrance lengths $L_s$ are shown in Fig. 7 as a function of Reynolds number and radius ratio. These results are larger than the integral estimates for a fixed annulus flow by Wilson and Medwell (1971). They may be curve-fit by the formula

$$L_s/D_h = (4.17 \log_{10} Re_v - 7.875) b^{1.1}$$

(20)

with an overall accuracy of ±4%.

Conclusions

The use of straightforward eddy viscosity models and integral mass and momentum balances leads to a complete set of analytical results for the turbulent annular flow between a fixed outer cylinder and a moving concentric inner cylinder. The results are in good agreement with the two special cases computed earlier by Sud and Chaddock (1979, 1981) and with the data of Davidson (1974).
The pressure drop and wall shear stresses in both the developing and fully-developed regions are smoothly varying functions of Reynolds number and radius ratio. They may be fit to simple algebraic correlations which have an accuracy of about five percent. The friction factor in the fully-developed region may be directly related by scaling factors to the ordinary Moody-chart friction factor for circular pipe flow. Development lengths vary from ten to forty hydraulic diameters, increasing slowly with both Reynolds number and radius ratio.

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References


