Spectral Measurements of the Wall Shear Stress and Wall Pressure in a Turbulent Boundary Layer: Theory

William L. Keith
Submarine Sonar Department

Naval Underwater Systems Center
Newport, Rhode Island / New London, Connecticut

Approved for public release; distribution is unlimited.
PREFACE

This technical report was prepared under the NUSC IR/IED Program, "Modification of Turbulent Boundary Layer Pressure Fluctuations Through Surface Geometry Alterations," Program Element No. 61152N, Job Order No. A70260, Principal Investigator, Dr. W. L. Keith (Code 2141). Funding was also provided by the Office of Naval Research (Code 1125), "Fluctuating Wall Shear Stress," Program Element 61153N, Job Order No. A68400, Principal Investigators, Dr. H. P. Bakewell, Jr., (Code 2141) and Dr. W. L. Keith (Code 2141), ONR Program Manager, Marvin Blizzard (Code 1125A0).

The technical reviewer for this report was Dr. D. Hurdis (Code 2141).

The author is grateful to Professor W. Willmarth at the University of Michigan for his contribution to this effort, and also to Professor J. Bennett, Dr. W. Strawderman (Code 213), Dr. H. P. Bakewell, Jr., (Code 2141), and Dr. D. Hurdis (Code 2141) for many helpful discussions. The support of the Quiet Water Tunnel Experimental Facility by Dr. W. Von Winkle (formerly of NUSC, Code 10) is greatly appreciated.

REVIEWED AND APPROVED: 27 June 1990

F. J. KINGSBURY
HEAD: SUBMARINE SONAR DEPARTMENT
Two derivations are presented which are pertinent to the experimental measurement of turbulent wall shear stress, using flush mounted hot film sensors. The first, a steady state heat transfer analysis, is necessary in order to determine the thickness of the thermal boundary layer that developed over the hot film. The fundamental validity of the use of hot film sensors to measure the wall shear stress depends on the thickness of the thermal boundary layer relative to the thickness of the viscous sublayer in the turbulent boundary layer. The second derivation involves an unsteady heat transfer analysis, which is necessary in order to determine the frequency range over which the fluctuating wall shear stress may be accurately resolved.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>DERIVATION OF THE STEADY STATE RESPONSE OF THE HOT FILM SENSORS.</td>
<td>4</td>
</tr>
<tr>
<td>DERIVATION OF THE FREQUENCY RESPONSE OF THE HOT FILM SENSORS.</td>
<td>9</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>13</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>13</td>
</tr>
</tbody>
</table>
SPECTRAL MEASUREMENTS OF THE WALL SHEAR STRESS AND WALL PRESSURE IN A TURBULENT BOUNDARY LAYER: THEORY

INTRODUCTION

Turbulent boundary layer wall pressure fluctuations have received considerable attention over the past 30 years. Willmarth and Wooldridge [1,2] presented spectral and correlation measurements of the fluctuating wall pressure made in a thick turbulent boundary layer of air. The physical interpretations made by Willmarth and Wooldridge provided much of the framework on which subsequent models of the convected wall pressure field were developed. The issues of acoustic contamination (associated with the flow facility) at low frequencies and attenuations at high frequencies resulting from spatial averaging (associated with the finite size of the face of the pressure transducer), discussed by Willmarth and Wooldridge, continue to pose severe constraints on similar experimental research efforts. More recent efforts by Bull and Thomas [3] and Schewe [4] have been aimed at resolving the pressure fluctuations associated with the small scale (high wavenumber) structures in the near wall region.

Comparatively few investigations of wall pressure fluctuations performed in hydrodynamic facilities have been reported. Carey et al. [5] made wall pressure measurements in a turbulent pipe flow of water. The power spectra were shown to agree with the measurements of Bakewell et al. [6], which were made in a turbulent pipe flow of air. In comparing the spectra, an outer flow variable nondimensionalization was used. These measurements were also compared using the method proposed by Corcos [7] to correct for the effects of spatial averaging and were shown to agree very well. In principal, results from investigations performed in air may be scaled to predict the spectral levels existing for a particular hydrodynamic application, if corrections for spatial averaging are taken into account and changes in the turbulent structure with Reynolds numbers are negligible. To
investigate boundary layer control schemes, such as polymer additives, compliant coatings, or large eddy breakup devices, for reduction of flow noise, the use of water as a working fluid is necessary. The class of hydrodynamic boundary layers of interest have sufficiently small viscous length scales to cause spatial averaging to always pose a limitation in the highest frequencies (or smallest scales) which can be resolved with contemporary transducers. The correction method proposed by Corcos [7] is approximate in nature, as discussed by Willmarth and Roos [8]. The method does provide a systematic correction procedure that is useful for comparing results from various investigations. In addition, for a particular boundary layer and transducer, the method may be used to estimate the highest frequency for which an acceptable signal to noise ratio exists.

The related problem involving the fluctuating wall shear stress beneath turbulent boundary layers has received considerably less attention. A recent discussion of the current knowledge of wall shear fluctuations is given by Alfredsson et al. [9]. Measurements of the ratio $\tau_{rms}/\tau_0$ have varied drastically between various investigations. The spectra of the wall shear stress fluctuations have not been well established. The inherent difficulties associated with the various measurement methods are largely responsible. Alfredsson et al. concluded that flush mounted hot film sensors may be used to measure the mean and fluctuating wall shear stress in water applications where the heat loss from the hot film to the substrate is not significant. The validity of this measurement method requires that the thermal boundary layer developed over the hot film be entirely within the viscous sublayer of the turbulent boundary layer. In addition, the unsteady thermal response of the sensor poses a constraint on the highest frequency wall shear fluctuations which may be accurately resolved. This report contains two derivations necessary in order to determine the applicability of a particular hot film sensor for a particular application. The first derivation involves a steady state heat transfer analysis, from which the thermal boundary layer thickness is determined. The second derivation involves an unsteady heat transfer analysis, from which the one half power frequencies are determined.
The experimental results from this investigation are given by Keith and Bennett [10]. The goal was to resolve both the wall pressure and wall shear stress spectra in a fully developed hydrodynamic turbulent boundary layer. In addition to the frequency response of the hot film sensors, their absolute calibration posed an additional constraint on the accuracy of the measured spectra. This issue is discussed in detail in [10]. With regard to the results presented here, it is emphasized that for hydrodynamic applications, both the thermal boundary layer thickness and the frequency response associated with flush mounted hot films must be considered at the outset of any experiment. For many applications in air, the thermal boundary layer thickness is an order of magnitude less than the viscous sublayer thickness. For the hydrodynamic application here, the thermal boundary layer was approximately one half as thick as the viscous sublayer. Attenuations resulting from the thermal response limit the frequencies for which the spectral levels are accurately resolved.

Currently, it is not possible to estimate the effects of spatial averaging, due to the lack of measurements of the cross correlation of the wall shear stress field. For the boundary layer investigated here, the streamwise length of the hot film sensors was an order of magnitude smaller than the diameter of the pressure transducers. As a result, the size of the pressure transducers was the limiting factor with regard to the frequency range over which both the wall pressure and wall shear stress were accurately resolved. Significantly more experimental measurements of the wall shear stress field are needed in order for an understanding comparable to the wall pressure field to exist.
DERIVATION OF THE STEADY STATE RESPONSE OF THE HOT FILM SENSORS

Nomenclature

\( \delta(x) \) \hspace{1cm} \text{Turbulent boundary layer thickness (in.).}

\( U_\infty \) \hspace{1cm} \text{Free stream velocity (ft/sec).}

\( R_e \equiv U_\infty \Theta/\nu \) \hspace{1cm} \text{Momentum thickness Reynolds number.}

\( u(y) \) \hspace{1cm} \text{Mean streamwise velocity (ft/sec).}

\( \delta_t(x) \) \hspace{1cm} \text{Thermal boundary layer thickness (in.).}

where \( \delta_t(x_0) = 0 \), \( \delta_t(x_0 + L) = \delta_tL \).

\( T_\infty \) \hspace{1cm} \text{Fluid temperature (°F) for } y \geq \delta_t(x).

\( T(y) \) \hspace{1cm} \text{Mean fluid temperature (°F).}

\( T_0(y) \equiv T(y) - T_\infty \)

\( k \) \hspace{1cm} \text{Thermal conductivity (BTU/(sec-ft}^2 - °F/ft)).

\( C_p \) \hspace{1cm} \text{Constant pressure specific heat (BTU/(slug - °F)).}

\( \rho \) \hspace{1cm} \text{Fluid density (slug/ft}^3).\n
\( K \equiv k/\rho C_p \) \hspace{1cm} \text{Fluid thermal diffusivity (ft}^2/sec).\n
\( q_w(x) \) \hspace{1cm} \text{Heat transfer rate per unit area (BTU/(sec - ft}^2)).
\[ \frac{\partial}{\partial x} \int_0^{\delta_t(x)} u(y) T_0(y) dy = \frac{q_w(x)}{\rho C_p} = -K \frac{\partial T_0}{\partial y} \bigg|_{y=0}. \]  

From [11], a steady thermal boundary layer profile of the form

\[ T_0(n) = b(1-n)^3(1+n) = b(1-n)^2(1-n^2) \]

is used, where

\[ b = T(0) - T_{\infty} = \text{constant}, \]

and

\[ n = y/\delta_t(x). \]

Then

\[ \frac{\partial T_0}{\partial y} = \frac{\partial T_0}{\partial n} \frac{\partial n}{\partial y} = \frac{b}{\delta_t(x)} \left[ -2(1-n)(1-n^2) + (1-n)^2(-2n) \right] \]

and

\[ \frac{\partial T_0}{\partial y} \bigg|_{y=0} = \frac{-2b}{\delta_t(x)}. \]
Next, it is assumed that the thermal boundary layer thickness $\delta_t(x)$ is entirely within the viscous sublayer of the turbulent boundary layer, such that

$$u(y) = \tau_0 \frac{y}{\mu}.$$  

(3)

Combining Eqs. (2) and (3) with Eq. (1) results in

$$\frac{2bK}{\delta_t(x)} = \frac{\partial}{\partial x} \delta_t(x) \int_0^{\tau_0 y b (1-n)^2 (1-h^2)} dy$$

or

$$\frac{2K}{\delta_t(x)} = \frac{\partial}{\partial x} \frac{1}{\tau_0} \int_0^{\delta_t^2(x)(1-n)^2(1-g^2)\mu} d\mu.$$  

(4)

Assuming $\tau_0$ to be constant over the region $x_0 < x < x_0 + L$ gives

$$\frac{2K}{\delta_t(x)} = \frac{\tau_0}{\mu} \frac{\partial}{\partial x} \delta_t^2(x) \int_0^{(1-n)^2(1-g^2)} dn,$$  

which upon integration yields

$$\frac{2K}{\delta_t(x)} = \frac{\tau_0}{\mu} \frac{\partial}{\partial x} \left[ \delta_t^2(x)/15 \right]$$

or

$$\frac{2K}{\delta_t(x)} = \frac{\tau_0}{15\mu} \frac{\partial}{\partial x} \delta_t(x) \int_0^{\delta_t(x)} dx.$$  

(5)

Integrating Eq. (5) from $x_0$ to $x_0 + L$ yields

$$\int_{x_0}^{x_0 + L} K dx = \frac{\tau_0}{15\mu} \delta_t L \int_0^{\delta_t(x)} \delta_t^2(x) d\delta_t(x),$$

which results in

$$KL = \tau_0 \left( \delta_t L \right)^3/45\mu.$$
or

$$\left( \frac{\delta_{tL}}{L} \right)^3 = 45 \mu K / (\tau_0 L^2).$$

(6)

Defining \( L^+ \equiv L \frac{u^*}{v} \), Eq. (6) may be expressed in the form

$$\frac{\delta_{tL}}{L} = (45/\sigma)^{1/3} (L^+)^{-2/3}. \quad (7)$$

In this particular investigation [10], \( L = 0.005 \) (in.) and \( \sigma = 6.40 \) (corresponding to a fluid temperature of 72°F). The thermal boundary layer thickness was entirely within the viscous sublayer for each Reynolds number investigated, as shown in table 1. Here, the viscous sublayer thickness is assumed to be 10 viscous lengths.

**Table 1. Boundary Layer and Hot Film Parameters**

<table>
<thead>
<tr>
<th>( U_\infty ) (ft/sec)</th>
<th>( Re )</th>
<th>( \delta ) (in.)</th>
<th>( \nu/u^* ) (in.)</th>
<th>( Lu^*/\nu )</th>
<th>( \delta_t L u^*/\nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8,200</td>
<td>0.93</td>
<td>0.00034</td>
<td>15</td>
<td>4.65</td>
</tr>
<tr>
<td>20</td>
<td>13,400</td>
<td>0.74</td>
<td>0.00019</td>
<td>25</td>
<td>5.72</td>
</tr>
</tbody>
</table>

Having established that Eq. (3) holds throughout the thermal boundary layer, an expression relating the mean wall shear stress to the steady state mean heat transfer may now be derived. The steady state mean heat transfer rate (with respect to the streamwise element length \( L \)) per unit area is defined as

$$q_w = \frac{1}{L} \int_{x_0}^{x_0+L} q_w(x) dx.$$  

(8)

From Eqs. (1) and (2),

$$\frac{q_w(x)}{\rho C_p} = 2K b/\delta_t(x),$$
and from Eq. (5),

\[ \frac{q_w(x)}{\rho C_p} = \frac{2b}{15\mu} \frac{\tau_0}{\delta_t(x)} \frac{d\delta_t(x)}{dx}. \]  

(9)

Combining Eqs. (9) and (8) yields

\[ q_w = \frac{2\rho C_p \tau_0 b}{15\mu L} \frac{\delta_t(x)}{\tau_0} \delta_t(x) d\delta_t(x) = \frac{\rho C_p \tau_0 b}{15\mu L} \frac{\delta_t^2}{\tau_0}. \]

From Eq. (6),

\[ q_w = \frac{\rho C_p \tau_0 b}{15\mu L} \left[ \frac{45\mu KL}{\tau_0} \right]^{2/3}. \]  

(10)

Eq. (10) may be expressed in the form

\[ q_w = k b \left( \frac{3}{(5\mu L K)} \right)^{1/3} \tau_0^{1/3}, \]  

(11)

which is the desired relationship between \( q_w \) and \( \tau_0 \). Eq. (11) relates the steady state mean heat transfer rate \( q_w \) from the hot film to the mean wall shear stress \( \tau_0 \). In practice, it is not the mean heat transfer rate that is determined in order to calibrate the sensor, but rather the anemometer output voltage \( E \) necessary to control the hot film at a constant temperature. Sandborn [12] has shown that the relationship between the mean wall shear stress and anemometer output voltage is of the form

\[ \tau_0^{1/3} = AE^2 + B, \]  

(12)

where \( E \) is the voltage, and \( A \) and \( B \) are calibration constants. While the accurate determination of these constants for particular applications requires considerable effort, the result yields an equation relating the measurable quantity \( E \) to the desired quantity \( \tau_0 \). The particular method of calibration used in this investigation is fully discussed in [10].
DERIVATION OF THE FREQUENCY RESPONSE
OF THE HOT FILM SENSORS

To determine the dynamic response of the hot film sensors, the quantity $T_0(y) + T_f(y)e^{i\omega t}$ is combined with the unsteady form of the thermal energy integral equation to obtain

$$\frac{a}{a t} \int_0^a \delta_t(x) (T_0(y) + T_f(y)e^{i\omega t}) dy$$

$$+ \frac{a}{a x} \int_0^a \delta_t(x) u(y) (T_0(y) + T_f(y)e^{i\omega t}) dy$$

$$= -K \frac{a}{a y} (T_0(y) + T_f(y)e^{i\omega t}) \bigg|_{y=0}.$$  \hspace{1cm} (13)

The first term in Eq. (13) may be reduced to

$$i\omega e^{i\omega t} \int_0^a \delta_t(x) T_f(y) dy.$$ \hspace{1cm} (14)

Following [11], the fluctuating thermal boundary layer profile is approximated as

$$T_f(n) = -b[8(\omega)n(1-n)^2(1+2n)].$$ \hspace{1cm} (15)

where $n = y/\delta_t(x)$. The steady thermal profile is approximated as

$$T_0(n) = b(1-n)^2(1-n^2),$$ \hspace{1cm} (16)

and the fluctuating streamwise velocity $u(n)$ is expressed as
\[ u(\eta) = (\tau_0/\mu + \tau_1 e^{i\omega t}/\mu) \eta \delta_t(x). \] (17)

Combining Eqs. (15), (16), and (17) with Eq. (13) and performing the necessary integrations results in

\[ -i\omega B(\omega) \delta_t(x) (3/20) e^{i\omega t} \]

\[ + \left[ \frac{\tau_0}{\mu} + \tau_1 e^{i\omega t}/\mu \right] \left[ \frac{2}{15} \frac{\partial \delta_t(x)}{\partial x} (1 - B(\omega) e^{i\omega t}) \right] \]

\[ = \frac{K}{\delta_t(x)} (2 + B(\omega) e^{i\omega t}). \] (18)

Equating the constant terms in Eq. (18) to zero and integrating results in

\[ \delta_t(x)^3 = 45K\mu x/\tau_0. \] (19)

which is consistent with Eq. (6) from the steady state analysis.

Equating the coefficients of \( e^{i\omega t} \) in Eq. (18) to zero and neglecting higher order terms results in

\[ \frac{2}{15} \delta_t^2(x) \frac{\partial \delta_t(x)}{\partial x} \left[ \frac{\tau_1}{\mu} - \tau_0 B(\omega)/\mu \right] - \frac{3}{20} i\omega B(\omega) \delta_t^2(x) = KB(\omega). \] (20)

Eq. (20) may be integrated over the sensor length \( L \) (with respect to the streamwise direction \( x \)) to obtain

\[ B(\omega) = \frac{\tau_1}{\tau_0} \left[ \frac{3}{2} + \frac{3}{40} i\omega \delta_t^2/K \right]^{-1}. \] (21)

where the relationship

\[ \delta_t^2(x) \frac{\partial \delta_t(x)}{\partial x} = 15K\mu/\tau_0 \]
obtained in the steady analysis has been used, and

$$\overline{\delta_t^2} = \frac{1}{L} \int_0^L \delta_t^2(x)dx .$$

Combining Eqs. (21) and (15) and defining \( q_1(x) \) as

$$q_1(x) = -k \frac{\partial T_f}{\partial y} \bigg|_{y=0}$$

results in an expression for the fluctuating heat transfer given by

$$q_1(x) = \frac{kb}{\delta_t(x)} \frac{\tau_1}{\tau_0} \left[ \frac{3}{2} + \frac{3i\omega \overline{\delta_t^2}}{40K \delta_t^2} \right]^{-1} .$$

(22)

The mean value of \( q_1(x) \) defined as

$$\overline{q_1} = \frac{1}{L} \int_0^L q_1(x)dx$$

is obtained from Eq. (22) as

$$\overline{q_1} = \frac{kb}{L} \frac{\tau_1}{\tau_0} \left[ \frac{3}{2} + \frac{3i\omega \overline{\delta_t^2}}{40K \delta_t^2} \right]^{-1} \int_0^L \frac{dx}{\delta_t(x)} .$$

(23)

The mean value of the steady state heat transfer \( \overline{q_0} \) may be expressed as

$$\overline{q_0} = \frac{2bk}{L} \int_0^L \frac{dx}{\delta_t(x)} .$$

(24)

Combining Eqs. (24) with (23) yields

$$\frac{\overline{q_1}/\overline{q_0} = \frac{\tau_1}{2\tau_0} \left[ \frac{3}{2} + \frac{3i\omega \overline{\delta_t^2}}{40K \delta_t^2} \right]^{-1} .$$

(25)
Taking the magnitude of Eq. (24) yields

\[
\frac{q_1}{q_0} = \frac{\tau_1}{3 \tau_0} \left[ 1 + (\omega \frac{\delta_t^2}{20K})^2 \right]^{-1/2}.
\]  

(26)

From Eq. (19),

\[
\delta_t^2(x) = (K45\mu x/\tau_0)^{2/3}.
\]

Then

\[
\delta_t^2 = \frac{1}{L} \int_0^L (K45\mu x/\tau_0)^{2/3} dx = \frac{3}{5} (K45\mu L)^{2/3}.
\]  

(27)

From Eq. (26), the one-half power frequency \( \omega_{1/2} \) is expressed as

\[
\omega_{1/2} = \frac{\delta_t^2}{(20 K)} = 1.
\]  

(28)

Combining Eqs. (27) and (28) and rearranging the terms results in

\[
\omega_{1/2} = 4\nu \left[ \frac{\sigma^7}{54} (L\nu^2/\mu^2)^2 \right]^{-1/3}.
\]  

(29)

Eq. (29) defines the one-half power frequency for the hot film sensors. Table 2 presents both the dimensional and nondimensional one-half power frequencies for this particular investigation.

<table>
<thead>
<tr>
<th>( U_\infty ) (ft/sec)</th>
<th>( f_{1/2} ) (Hz)</th>
<th>( \omega_{1/2} \delta^*/U_\infty )</th>
<th>( \omega_{1/2} \nu/(\mu^2)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>468</td>
<td>3.18</td>
<td>0.24</td>
</tr>
<tr>
<td>20</td>
<td>1016</td>
<td>2.40</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 2. Hot Film Sensor One-Half Power Frequencies
CONCLUSIONS

The analysis pertinent for the use of flush mounted hot film sensors to measure the fluctuating wall shear stress has been presented. For the case where the thermal boundary layer thickness is of the order of or less than the fluid viscous sublayer thickness, the mean output voltage from the sensor is directly related to the mean wall shear stress. An approximate two-dimensional analysis of the unsteady heat transfer results in an expression that characterizes the frequency response of the sensor. Use of this equation for typical applications in water, i.e., those requiring a relatively low sensor temperature, shows that the attenuation at high frequencies due to the thermal response is significant. The results of this analysis show that the flush mounted hot film sensors used for the investigation of Keith and Bennett [10] were effective in resolving the lower frequencies of the fluctuating wall shear stress.

REFERENCES


<table>
<thead>
<tr>
<th>Addressee</th>
<th>No. of Copies</th>
</tr>
</thead>
<tbody>
<tr>
<td>ONR (Code 1125AO, Marvin Blizzard)</td>
<td>1</td>
</tr>
<tr>
<td>JCB Associates (John Bennett)</td>
<td>1</td>
</tr>
<tr>
<td>DTIC</td>
<td>2</td>
</tr>
</tbody>
</table>