The Kalman filter provides a finite dimensional solution when the signal and observation processes are linear and have Gaussian noise. In this paper the effect of a small non-linearity in the signal is discussed by considering stochastic flows for the signal and a Girsanov transformation for the observation. The result can be expressed in terms of Gaussian densities.
Filters With Small Non-Linearity

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ABSTRACT

The OBSERVATION process is taken to be of the form

\[ y_t = \int_0^t h_x(0,u)\,du + \omega_t. \]

As usual, we shall suppose \( x_0 \) is a Gaussian \( F_0 \) measurable random variable independent of \( \omega_t, B_t, t > 0 \).

Write \( \{Y_t\}, t \geq 0, \) for the right continuous complete filtration generated by the observations and

\[ \tilde{x}_t(x_s) = E[x_t \mid x_s, Y_t] \quad \text{for } t \geq s. \]

Then it is known that \( \tilde{x}_t(x_s) \) is a Gaussian random variable for \( t > s \) and

\[ \tilde{x}_t(x_s) = x_s + \int_s^t a_u \tilde{x}_u(x_s)\,du + \int_s^t P_{s,u}h_u(dy_u - h_u\tilde{x}_u(x_s))du \]

where

\[ P_{s,t} = E[x_t^2 \mid x_s, Y_t] - (E[x_t \mid x_s, Y_t])^2 \]

satisfies the deterministic equation

\[ \frac{dP_{s,t}}{dt} = -h_t^2 P_{s,t}^2 + 2a_t P_{s,t} + 1, \]

\[ P_{s,s} = 0. \]

Consequently, \( \tilde{x}_t(x_s) \) is Gaussian with conditional mean \( \tilde{x}_t(x_s) \) and variance \( P_{s,t} \).

Writing \( \hat{x}_t = E[x_t \mid Y_t] \) we see \( \hat{x}_t \) is Gaussian with mean and variance \( \hat{P}_t \) given by

\[ \hat{x}_t = E[x_0] + \int_0^t a_s \hat{x}_s ds + \int_0^t P_s h_s(dy_s - h_s\hat{x}_s ds) \]

\[ \frac{d\hat{P}_t}{dt} = -h_t^2 \hat{P}_t^2 + 2a_t \hat{P}_t + 1, \]

\[ \hat{P}_0 = E[x_0^2] - (E[x_0])^2. \]
The equations (1.5) and (1.6), or (1.7) and (1.8) are forms of the Kalmar filter. The innovation processes

\[ \beta_t(x_t) = y_t - \int_s^t h_u \tilde{z}_u(x_s) \, du, \quad t \geq s, \]

\[ \beta_t = y_t - \int_0^t h_u \tilde{z}_u \, du, \quad t \geq 0. \]

are \( \{Y_t\} \) Brownian motions. They generate the same filtration as \( \{y_t\} \).

The Gaussian measure on \( \mathbb{R} \) with mean \( m \) and variance \( P \)

\[ \text{it}(xs) = 0(s,t)z_s + \int_s^t \text{it}(s,u)\beta_u \, du + w_t \]

(1.9) will be denoted by \( \mu(m, P, dx) \). If \( g \) is a Borel measurable function on \( R \) we shall write

\[ \mathcal{I}(g, m, P) = \int_R g(x) \mu(m, P, dx). \]

If \( Z_t \) is an integrable process, \( t \geq 0 \), \( \Pi_t(Z) \) will denote the \( \{Y_t\} \)-predictable projection of \( Z \), so \( \Pi_t(Z) = E[Z_t | Y_t] \) a.s. For a function \( g(t, x) \) such that

\[ |g(t, x)| \leq K(1 + |x|^m) \]

for some \( K > 0 \), \( m > 0 \), we shall write

\[ \Pi_t(g) = \Pi_t(g(t, x_t)). \]

From [2] we quote the following results:

**Lemma 1.1.** a) Suppose \( 0 \leq s \leq t \). The conditional law of \( x_t \) given \( Y_t \) is

\[ \mu(m_s, P_s, dx) \]

where

\[ m_s = \dot{x}_t + \frac{P_s}{\gamma_s} \int_s^t \gamma_u h_u \, du \]

(1.9)

\[ P_s = P_s - \left( \frac{P_s}{\gamma_s} \right)^2 \int_s^t \gamma_u^2 \, du \]

(1.10)

and \( \gamma \) is the solution of

\[ \gamma_t = 1 + \int_0^t (a_s - P_s h_s^2) \gamma_s \, ds \]

(1.11)

so

\[ \gamma_t = \exp \left( \int_0^t (a_s - P_s h_s^2) \, ds \right). \]

b) Suppose \( g(t, x) \) and \( g_x(t, x) \) are Borel functions satisfying growth conditions as above. Then

\[ \Pi_t \left( \int_0^t g(s, x_s) \, ds \right) = \int_0^t \Pi_s(g) \, ds \]

\[ + \int_0^t \Pi_s \left( \int_0^s g_x(u, x_u) \frac{P_u}{\gamma_u} \, du \right) \gamma_s h_s \, ds. \]

(1.12)

From (1.3) we see the map

\[ x \to \xi_{s,t}(x) \]

is a diffeomorphism of \( R \) and

\[ \frac{\partial \xi_{s,t}(x)}{\partial x} = \Phi(s, t). \]

From (1.5) we can write

\[ \dot{x}_t(x_t) = \Phi(s, t) \left[ x_t + \int_s^t \Phi(s, u)^{-1} P_s u h_u d\beta_u(x_s) \right] \]

(1.13)

and

\[ \frac{\partial \dot{x}_t(x_s)}{\partial x_s} = \gamma_{s,t} \]

where

\[ \gamma_{s,t} = 1 + \int_s^t (a_u - P_u h_u^2) \gamma_u \, du \]

(1.14)

so

\[ \gamma_{s,t} = \exp \int_s^t (a_u - P_u h_u^2) \, du. \]

(1.15)

2. **Nonlinear Signal Equations**

For linear signal and observations the Kalmar filter provides a finite dimensional solution to the filtering problem. Consider a measurable function \( f(t, x) \) on \([0, \infty) \times \mathbb{R} \) which is twice differentiable in \( x \) and which satisfies the growth condition

\[ |f(t, x)| + |f_x(t, x)| \leq K(1 + |x|). \]

(2.1)

Let \( \varepsilon > 0 \) be a small positive number. Consider a signal process given by the non-linear equation

\[ \dot{x}_t = x_0 + \int_0^t (a_s \dot{x}_s + e(f(s, \dot{x}_s))) \, ds + \omega_t. \]

(2.2)

Consider the process \( z \) defined by

\[ z_t = x_0 + \int_0^t \Phi(0, s)^{-1} e(f(s, \xi_{0,s}(z_s))) \, ds \]

(2.3)

where \( \xi_{0,s}(\cdot) \) is the diffeomorphism defined by (1.1).

**Lemma 2.1.** The process \( \xi_{0,t}(z_t) \) is the solution of (2.2).

**Proof.** Substituting (2.3) in (1.3) we have

\[ \xi_{0,t}(z_t) = \Phi(0, t) \left[ x_0 + \int_0^t \Phi(0, s)^{-1} e(f(s, \xi_{0,s}(z_s))) \, ds \right. \]

\[ + \left. \int_0^t \Phi(0, s)^{-1} d\omega_s \right]. \]

(2.4)
Differentiating (2.4) in $t$ the result follows.

**REMARKS 2.2.** Because $f$ satisfies the linear growth condition (2.1) $\dot{z}_t = \xi_0, T(z_t)$ has finite moments of all orders.

If $Z_t$ is a process we shall write $Z_t = O(\epsilon^k)$ if 

$$E\left(\sup_{s \leq t} |Z_t|^p \right)^{1/p} = O(\epsilon^k)$$

for every $p \geq 1$.

**NOTATION 2.3.** Write

$$\Delta_{0,t} = \Phi(0,t) \int_0^t \Phi(0,s)^{-1} f(s,z_s) ds.$$ 

Using the mean value theorem we can quickly deduce

**PROPOSITION 2.4.** $\dot{z}_t - z_t = D_{0,t} = \epsilon \Delta_{0,t} + O(\epsilon^2)$.

**REMARKS 2.5.** To discuss the effect of the non-linear signal $\dot{z}_t = \xi_0, T(z_t)$ on the observations consider the measure $\bar{P}$ defined by

$$dP|_{F_t} = A_t^f$$

where

$$A_t^f = \exp \left( t \int_0^t h_s D_{0,s} dB_s - \frac{1}{2} \int_0^t h_s^2 D_{0,s}^2 ds \right).$$

Then under $\bar{P}$

$$\bar{B}_t = B_t - \int_0^t h_s D_{0,s} ds$$

is a Brownian motion, i.e.,

$$y_t = \int_0^t h_s \xi_0, T(z_s) ds + \bar{B}_t. \tag{2.5}$$

Therefore, under $\bar{P}$ the signal process is $\bar{z}$ and this now influences the observations as in (2.5). The non-linear filtering expression we wish to consider is

$$E[\xi_0, T(z_t) | Y_t].$$

By Baye's theorem this is

$$E[A_t^f \xi_0, T(z_t) | Y_t] \cdot (E[A_t^f | Y_t])^{-1}.$$ 

**LEMMA 2.6.** $A_t^f = 1 + \epsilon \int_0^t h_s \Delta_{0,s} dB_s + O(\epsilon^2)$.

**PROOF.** $A_t^f = 1 + \int_0^t \Lambda_t^f h_s D_{0,s} dB_s$ and the result follows by substituting for $A_t^f$ on the right and using Proposition 2.4.

From Proposition 3.3 of Picard [2] we have

**LEMMA 2.7.** $\Pi_t(\Lambda)^{-1} = 1 - \epsilon \Pi_t \left( \int_0^t h_s \Delta_{0,s} dB_s \right) + O(\epsilon^2)$.

The main result is the following theorem:

**THEOREM 2.8.** Writing $\dot{z}_t = \xi_0, T(z_t)$, $z_t = \xi_0, T(x_0)$

$$E[\dot{z}_t | Y_t] = E[z_t | Y_t] + \epsilon E \left[ \int_0^t h_s \Delta_{0,s} dB_s | Y_t \right]$$

$$+ \epsilon E[\Delta_{0,t} | Y_t] - \epsilon E[z_t | Y_t] E \left[ \int_0^t h_s \Delta_{0,s} dB_s | Y_t \right]$$

$$+ O(\epsilon^2). \tag{2.6}$$

**PROOF.**

$$E[\dot{z}_t | Y_t] = E[A_t^f \dot{z}_t | Y_t] \cdot E[A_t^f | Y_t]^{-1}$

$$= E[A_t^f (z_t + \epsilon \Delta_{0,t}) | Y_t]$$

$$\times E \left[ \left( 1 - \epsilon \int_0^t h_s \Delta_{0,s} dB_s \right) | Y_t \right] + O(\epsilon^2)$$

$$= E \left[ \left( 1 + \epsilon \int_0^t h_s \Delta_{0,s} dB_s \right) (z_t + \epsilon \Delta_{0,t}) | Y_t \right]$$

$$\times \left[ 1 - \epsilon E \left[ \int_0^t h_s \Delta_{0,s} dB_s | Y_t \right] \right] + O(\epsilon^2)$$

by Proposition 2.3 and Lemma 2.7.

**REMARKS 2.9.** These expectations are all expressible in terms of Gaussian measures because they are all expectations of functions of the original linear process $z_t$ under the original measure $P$. For example, $E[z_t | Y_t] = \bar{z}_t$ is given by the Kalmar filter. The remaining terms in (2.6) can be expressed in a recursive way; proofs can be found in [1]. For example, we have

**LEMMA 2.10.**

$$E[\Delta_{0,t} | Y_t] = \Phi(0,t) \left[ \int_0^t \Phi(0,s)^{-1} \Pi_s (f(s,z_s)) ds \right.$$

$$+ \int_0^t \Pi_s \left\{ \int_0^s f_2(u,u) \mu_u \gamma_u^{-1} du \right\} \Phi(0,s)^{-1} h_s \gamma_s dB_s \right].$$

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3. CONCLUSION

As in the paper of Picard [2] the first two terms in an expansion of the conditional mean in powers of $\epsilon$ have been determined. These coefficients have been expressed explicitly in terms of Gaussian measures by using stochastic flows.

REFERENCES
