This report describes the results obtained during the two years of funding extending from 1/88 to 1/90 in support of our project entitled "The problem of Robust Compensation for Systems with Unmodeled Dynamics." The project has consisted of several lines of research which are quite distinct, but which show great promise toward combining them into a single comprehensive theory. Our goal has been to explore the problem of designing feedback control systems that are insensitive to the presence of high-frequency dynamics not accounted for explicitly in the mathematical model of the plant. Some of our previous work suggests that it is possible to design controllers which simultaneously stabilize a given nominal system as well as a large class of small singular perturbations of the system. Attached are six papers summarizing work which has been supported all or in part by the present grant and which either have appeared, have been accepted, or have been submitted for publication.
Abstract

This report describes the results obtained during the two years of funding extending from 1/88 to 1/90 in support of our project entitled "The problem of Robust Compensation for Systems with Unmodeled Dynamics." The project has consisted of several lines of research which are quite distinct, but which show great promise toward combining them into a single comprehensive theory. Our goal has been to explore the problem of designing feedback control systems that are insensitive to the presence of high-frequency dynamics not accounted for explicitly in the mathematical model of the plant. Some of our previous work suggests that it is possible to design controllers which simultaneously stabilize a given nominal system as well as a large class of small singular perturbations of the system. Attached are six papers summarizing work which has been supported all or in part by the present grant and which either have appeared, have been accepted, or have been submitted for publication.
This report describes the results obtained during the two years of funding extending from 1/88 to 1/90 with regard to our project entitled "The Problem of Robust Compensation for Systems with Unmodeled Dynamics." The project has consisted of several lines of research which are quite distinct, but which show great promise toward combining them into a single comprehensive theory. Our goal has been to explore the problem of designing feedback control systems that are insensitive to the presence of high-frequency dynamics not accounted for explicitly in the mathematical model of the plant. Some of our previous work suggests that it is possible to design controllers which simultaneously stabilize a given nominal system as well as a large class of small singular perturbations of the system. We say such a controller is "robust" with respect to the certain class of unmodeled dynamics.

Attached are six papers summarizing work which has been supported all or in part by the present grant and which either have appeared, have been accepted, or have been submitted for publication. Two have already appeared in the *IEEE Transactions on Automatic Control*, one is scheduled to appear in the same journal in May, one has appeared in the *Proceedings of the 27th IEEE Conference on Decision and Control*, and one is presently under review. In addition, a summary page is included describing a paper presented at the *SIAM Conference on Control in the 90's* (5/89, San Francisco). We also are in the process of writing a journal article describing the work carried out in the final stage of the project. The manuscript should be available within the next two months. Part of the research effort over the past two years has been carried out by my graduate student Mingde Tan as part of his Ph.D. research. His dissertation should be available sometime this spring.
Reference [1] contains work which was partially carried out during the funding period and which was instrumental in establishing the direction outlined in the original grant proposal. Our basic idea was to explore the role of parasitics in the performance of automatic control systems, without having to resort to explicit representations of specific parasitic effects. It was our desire to develop a comprehensive theory of compensator design which would guarantee performance in the presence of a large class of possible parasitic effects. Reference [1] contains preliminary results directed at this goal.

Along somewhat different theoretical lines, but with a similar class of engineering problems in mind, references [2]-[4] constitute initial attempts, carried out with the aid of a colleague here at Wisconsin, Professor Chris DeMarco, to characterize the geometric structure of the class of system perturbations under which a control system retains stability and perhaps other performance characteristics. Contained in [2]-[4] is a thorough treatment of the case where system order is constant (the nonsingular case). Our more recent work may be viewed as extending these results to the case where parameter variations can cause changes in system order (the singular case).

Reference [5] summarizes the work carried out by my graduate student, Mingde Tan, and me over the initial phase of his dissertation research (the first year of the funding period). Our principal idea was to study the effects of small system perturbations on internal closed-loop stability. Ultimately, we wish to examine many of the recent robust control theories of Vidyasagar, Zames and Francis, Stein and Doyle, and others in terms of internal closed-loop system behavior. For example, suppose a plant model is given and a corresponding compensator is designed using some methodology such that the closed-loop configuration is input-output stable. The design methods
of the researchers just mentioned typically guarantee that, for a certain class of perturbations of the plant (and sometimes the compensator), the corresponding perturbed closed-loop system is also stable in an input-output sense. It would be highly desirable to know whether the perturbed closed-loop system is internally stable as well. From our perspective, it is especially crucial that the compensator be insensitive to perturbations corresponding to unmodelled high-frequency dynamics.

In order to formulate this problem precisely, one must first develop an understanding of how internal system structure is affected by perturbations of the transfer function. To this end we have developed a "perturbational" analogue of the standard state-space realization theory for rational matrices. In our theory, families of rational matrices are considered (either convergent sequences or continuous parametrizations); it is desired to find corresponding (convergent) families of state equations which realize the given transfer matrices. We have obtained results that show, for example, that every such family of transfer matrices has a realization and that "minimality" of a realization can be related to both the degrees of the given rational functions as well as to controllability and observability of the realizations themselves. In short, we have succeeded in developing the perturbational analogues of the standard results concerning realization of a fixed rational matrix. This body of results is among our main accomplishments in the past two years.

Based on our understanding of the fundamental issues surrounding internal realizations of perturbed transfer functions, we have spent a large portion of the past year exploring the relationships between robust input-output stability and robust internal stability. We have succeeded in establishing simple conditions under which internal as well as input-output stability is
robust to parasitic model uncertainty. These results form the second half of Mingde Tan's dissertation and are presently being organized for journal publication.

In addition to the work with Mingde Tan, I personally have continued the effort I began in 1986 which addresses questions similar to those described above related to internal behavior of robust closed-loop systems. The culmination of this effort so far is the paper [6] which shows that robust design methodologies necessarily must incorporate some internal system information; the exact form of the "minimal" internal information is as yet unknown. A recent breakthrough has been summarized in [7] which was presented in May 1989 at the SIAM Conference on Control in the 90's in San Francisco. (There were no published Proceedings for this conference.) For a certain large class of linear systems, it is now possible to precisely characterize the family of singular perturbations under which closed-loop stability is retained.

Overall, our two-year research effort has been fruitful, answering many important questions and leading to new ones. In the final analysis, the entire body of results we have generated is not as coherent as we original hoped for; however, we feel that, given the time-limit set for the project, the progress made in each research direction more than compensate for this fact.
Toward a Theory of Robust Compensation for Systems with Unknown Parasitics

J. DANIEL COBB, MEMBER, IEEE

Abstract—We consider the problem of designing a robust compensator based on a plant model with order uncertainty. The uncertainty is characterized mathematically as a class of generalized singular perturbations of the plant. This paper considers the case of static compensation. A necessary and sufficient condition is established under which actual closed-loop behavior is close to that predicted by the plant model under sufficiently small singular perturbations. The condition is shown to be generic.

I. INTRODUCTION

The problem of robust compensation may be roughly stated as that of designing a good controller for a given physical system on the basis of a model which contains less than complete information about that system. The resulting closed-loop configuration should exhibit reasonable performance in spite of the uncertain aspects of the system. In the strictest sense, every model contains uncertainty; hence, any good controller design should address the issue of robustness.

Among the many types of robust control theories appearing in the literature is the asymptotic approach. Typical results in this area guarantee reasonable closed-loop performance under sufficiently small perturbations of a nominal model (e.g., variations in the coefficients of a single differential equation). Although only local in nature, such results are often a first step in developing a global theory where an explicit characterization is attained for classes of systems which can be simultaneously compensated. The results of this paper fall into the asymptotic category.

It is possible to view most asymptotic robustness theories within a common mathematical framework. Let \( \mathcal{P}, \mathcal{Q}, \) and \( \mathcal{J} \) be topological spaces, and let \( \mathcal{R} \subseteq \mathcal{P} \times \mathcal{Q} \) inherit set subset topology. \( \mathcal{P}, \mathcal{Q}, \) and \( \mathcal{J} \) correspond to the sets of all possible models of plants, compensators, and closed-loop systems, respectively. The topologies on \( \mathcal{P} \) and \( \mathcal{Q} \) are chosen so that small perturbations characterize measurement error inherent in developing each model; small perturbations in the topology of \( \mathcal{J} \) reflect tolerable closed-loop performance error. If \( \mathcal{R} \) is interpreted as the class of all plant–compensator pairs which lead to closed-loop systems that are well-defined and which satisfy any additional constraints present in the design problem, we may naturally define the loop-closing map \( \mathcal{C} : \mathcal{R} \rightarrow \mathcal{J} \) which takes each plant and compensator into their corresponding closed-loop configuration. Many robustness questions then reduce to that of finding the points of continuity of \( \mathcal{C} \). In other words, we wish to characterize the class of all plant–compensator pairs such that small perturbations of each pair result in small perturbations in the closed-loop system.

We now examine various existing theories which lie within the asymptotic framework. The most obvious body of such results centers around the well-known fact that, for state-space models, the parameters of the closed-loop system are continuous functions of the open-loop plant and compensator parameters. For example, if we let \( \mathcal{P} \) be the set of all matrix triples \( \xi = (A, B, C) \) and \( \mathcal{Q} \) consist of all feedback matrices \( K \), and if we combine \( \xi \) and \( K \) in a standard way, then \( \mathcal{R} = \mathcal{P} \times \mathcal{Q} \) and \( \mathcal{J} \) consists of triples \( \xi, K \) \( = (A + BK, B, C) \). Adopting Euclidean topology on \( \mathcal{P}, \mathcal{Q}, \) and \( \mathcal{J} \), it follows that \( \mathcal{C} \) is continuous everywhere, i.e., every compensator is robust relative to every plant. One immediate consequence of this observation is that closed-loop eigenvalues are continuous functions of plant and compensator parameters; hence, every stable closed-loop configuration remains stable under sufficiently small parameter variations. These facts are used routinely in many control system analyses without explicit mention. It should be noted, however, that the perturbations considered here do not alter either plant or compensator order. Therefore, this approach alone is inadequate when dealing with order uncertainty.

The main body of existing results that does deal with order uncertainty in an asymptotic setting can be broadly termed singular perturbation theory (see [1]–[3]). Here a typical analysis treats a parametrized system of the form

\[
\begin{bmatrix}
I & 0 \\
0 & \varepsilon I
\end{bmatrix} \dot{x} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} x+
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u
\]

\[ y = [C_1 \ C_2]x \tag{1} \]

with \( A_{22} \) stable and seeks to achieve some closed-loop performance criteria for all sufficiently small \( \varepsilon \geq 0 \). (In this case, we might take \( \mathcal{P} = [0, \infty) \).) A major drawback with this approach is that explicit knowledge of the parasitic structure giving rise to order uncertainty is assumed. If more than one perturbation (1) need to be considered, serious problems may develop. For example, the system

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \varepsilon & 0 \\
0 & 0 & \varepsilon
\end{bmatrix} \dot{x} =
\begin{bmatrix}
1 & 1 & 0 \\
0 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix} x+
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} u
\]

\[ y = [-1 \ 1 \ 0]x \tag{2} \]

is nominally \( \varepsilon = 0 \) unstable, but can be stabilized with the static compensator \( u = 2y \). The perturbed system \( \varepsilon > 0 \) is also stabilized by the same compensator for sufficiently small \( \varepsilon \). Setting \( \varepsilon = 0 \), premultiplication of (2) by the matrix

\[
M =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\]

yields an equivalent system equation which may in turn be
perturbed according to
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \epsilon & 0 \\
0 & 0 & \epsilon \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 0 \\
2 & -1 & -2 \\
1 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}

y = (-1 1 0)x.
\]

In this case the compensator \( u = 2y \) yields a perturbed closed-loop system having a pair of eigenvalues \( \lambda_1 \) and \( \lambda_2 \), with \( \text{Re} \lambda_1, \lambda_2 \to +\infty \) as \( \epsilon \to 0^+ \). Such divergent behavior does not coincide with any reasonable definition of small perturbations in 3. We may therefore conclude that examination of a single parasitic effect is in general not sufficient to guarantee robustness of a compensator with respect to other order uncertainties.

Additional singular perturbation results include the multiple time-scale extensions [7] and [8] and the robust compensation theorems of [9]. Multiple time-scale techniques suffer from the same drawback as single time-scale analyses based on (1) in that they assume an explicit knowledge of parasitic structure. Also, much less is known about the \( \epsilon \)-dependence of the time response of multiple time-scale systems than in the single time-scale case.

In [5] it is shown that any compensator having a strictly proper transfer function matrix, which stabilized (1) with \( \epsilon = 0 \), also stabilizes (1) when \( \epsilon > 0 \) is sufficiently small. Furthermore, it is shown that the corresponding family of closed-loop transfer matrices converges uniformly on compact subsets of the right-half complex plane as \( \epsilon \to 0^+ \). These results thus provide a means for robustly compensating a system in the presence of a large class of possible perturbations. One drawback to this theory is that only single time-scale systems (1) are treated. In practice, a much larger class of perturbations may be required to model all relevant effects. Additional problems are that the results of [5] do not take into account uncertainties in the compensator model and that uniform convergence on compact sets in \( \theta \) is difficult to relate to time-domain performance of the system.

Another notable asymptotic robustness theory is that of [6] where the graph topology is introduced. Let \( \mathcal{G} \) and \( \mathcal{Q} \) each be the space of all rational matrices, \( \mathcal{G} \) the space of strictly proper and stable rational matrices, equipped with the \( H_\infty \) norm, and \( \mathcal{Q} = \mathbb{C}^{-1}(\mathcal{G}) \). The graph topology is the weakest topology on \( \mathcal{G} \) and \( \mathcal{Q} \) under which \( \mathcal{E} \) is continuous. We have shown in [9], however, that singularly perturbed systems generically do not converge in the graph topology; hence, in this sense, robust compensation in the presence of order uncertainty is unattainable.

In view of the shortcomings of the existing asymptotic techniques, we wish to propose a framework as well as some preliminary results for an alternative robustness theory which will be taken into account: 1) multirate and other relatively unexplored classes of singular perturbations; 2) the necessity of dealing simultaneously with a large class of system perturbations, each corresponding to a possible higher order model; and 3) time-domain behavior of the closed-loop system. Although treatment of 1) and 2) seems on the surface to be a formidable task, we will see that it is possible to approach the problem in a roundabout way, thus avoiding having to explicitly characterize all possible parasitic phenomena. We feel that the inclusion of 3) is a desirable feature for any good robustness theory, since the goal of system design must ultimately be satisfactory closed-loop time response. In view of this fact, a time-domain approach has certain advantages over frequency domain techniques, since the relationship between time response and frequency-domain behavior can be rather complex.

Before becoming too engrossed in technicalities, we will briefly describe (in rough terms) the problem we wish to address. Consider the system

\[ E\dot{x} = Ax + Bu \]

where \( E, A, B, \) are real matrices with \( E \) and \( A \) square. We assume that (4) exhibits existence and uniqueness of solutions for each initial condition \( x_0 \) and each input function \( u \); from [17] we know that this is equivalent to

\[ \|E - A\| \neq 0. \]

Such systems have been studied extensively (e.g., see [14]-[16]), and are referred to as singular when \( E \) is singular and regular otherwise. The polynomial

\[ \Delta(s) = \|E - A\| \]

may be considered the characteristic polynomial of (4) and its roots the eigenvalues of \( \xi \). An important property of singular systems is that small perturbations in the entries of \( E \) and \( A \) can change the system order; one example of this phenomenon is (1).

Suppose we wish to find a compensator of the form \( u = KY - v \) which is robust with respect to perturbations in \( E, A, B \) and \( C \). Since we are inevitably interested in time response, we might ask which compensators result in a closed-loop system whose time response varies continuously with \( E, A, B, \) and \( C \), regardless of the perturbation. Unfortunately, it is easy to show that for any \( K \) there exist perturbations in the system matrices that yield divergent behavior in the closed-loop system trajectories for some initial conditions. A more meaningful problem can be formulated by first observing that not necessarily all perturbations in the matrix entries of (4) are physically realistic. For example, a simple \( RC \) circuit consisting of a single resistor, capacitor, and voltage source may be modeled as

\[ \begin{align*}
\dot{x} &= -x + u \\
y &= x
\end{align*} \]

where \( x \) is the capacitor voltage, \( R = 1 \), and \( C = \epsilon \). Positive \( \epsilon \) makes perfect physical sense, and it seems reasonable to try to design a compensator based on the low-order model corresponding to \( \epsilon = 0 \). On the other hand, if \( \epsilon \) is negative, the system engineer could not expect to produce a robust compensator without first being aware of the negative capacitance and then using an appropriate higher order (in this case, first-order) model.

A simple way to characterize physically meaningful perturbations in the plant is to look at their effect on plant trajectories for various inputs and initial conditions. For example, in (6) an initial condition \( x_0 = 1 \) yields \( x(t) = e^{-\epsilon t} \) which converges on compact subintervals of \( (0, \infty) \) as \( \epsilon \to 0^+ \), but diverges as \( \epsilon \to 0^- \). Strictly speaking, we are really not saying as much about perturbations of (4) which can occur in the physical world as we are about those perturbations which are consistent with the measurements taken while formulating our plant model; a system model is good only if it is capable of predicting the behavior of the actual physical system.

We may now state our definition of asymptotic robustness more precisely. For a given plant of the form (4), a compensator is robust if all perturbations in both the plant and compensator, which bring about only small variations in the trajectories of each system individually under all inputs and initial conditions, result in only small variations in the closed-loop system trajectories. The meaning of the phrase small variations will be precisely defined in Section III. In the same section we will see that our approach implicitly incorporates the idea that small system variations should correspond to only small changes in system parameters.

II. PRELIMINARIES

In this section we summarize the constructions of [10], [11], [13], and [14] which are pertinent to subsequent developments. Let

\[ \Sigma(n, m, p) = \{(E, A, B, C) \in \mathbb{R}^{(n+m+p)} | \|E - A\| \neq 0\} \]
and let \( \mathcal{L}(n, m, p) \) be the corresponding quotient manifold (see [18]) determined by the equivalence

\[
(E_1, A_1, B_1, C_1) = (E_2, A_2, B_2, C_2) \quad \text{iff} \quad C_1 = C_2 \quad \text{and} \quad M \text{ nonsingular}
\]

(7)

(The arguments \( n, m, \) and \( p \) will be dropped when clear from context.) We choose the equivalence relation (7) because pre-multiplication by \( M \) has no significant effect on the system representation. Indeed, pre-multiplication by \( M \) merely performs elementary row operations on the system of scalar equations (7). Hence, we are merely identifying systems formed from each other by reshuffling the equations. We do not wish to identify systems which are related by a coordinate change on the state variable \( x \), since this would reduce the system space to one consisting of input-output descriptions. Our intention is to produce results which exploit internal information.

The equivalence class containing \( \sigma = (E, A, B, C) \) is denoted \( \xi = [E, A, B, C] \). In this case, we say \( \sigma \) represents \( \xi \). Let

\[
\sigma = [E, A, B, C] \quad \text{ord} \sigma = \text{ord} \xi = \deg \Delta
\]

where \( \Delta \) is the characteristic polynomial (5) of \( \xi \), and note that a unique matrix \( C \) is determined by each \( \xi \in \mathcal{L} \). A sequence \( \xi_k \in \mathcal{L} \) converges weakly to \( \xi \in \mathcal{L} \) if \( \xi_k \to \xi \) in manifold topology. Since \( \mathcal{L} \) is a quotient manifold, the natural projection \( \pi : (E, A, B, C) \to [E, A, B, C] \in \mathcal{L} \) is continuous with respect to weak convergence. Conversely, we have shown in [10] that, for each convergent sequence \( \xi_k \to \xi \) in \( \mathcal{L} \), there exists a sequence \( (E_k, A_k, B_k, C) \to [E, A, B, C] \) in \( \Sigma \) such that \( [E, A, B, C] = \xi \) and \( [E_k, A_k, B_k, C_k] = \xi_k \) for every \( k \).

Let \( \xi = [E, A, B, C] \) with \( E \) singular. In [11] is shown that there exist nonsingular matrix sequences \( M_k \to M \) and \( N_k \to N \) such that

\[
M_k E_k N_k = \begin{bmatrix} I & 0 \\ 0 & A_k \end{bmatrix}, \quad M_k A_k N_k = \begin{bmatrix} A_k & 0 \\ 0 & I_{n-r} \end{bmatrix}
\]

(8)

where \( r = \text{ord} \xi, A_k \to A, \) and \( A_k \to A_j \) with \( A_j \) nilpotent. For sufficiently large \( k \), the matrices \( A_k \) and \( A_j \) are unique up to a similarity transformation. For a constant sequence, the decomposition (8) reduces to the Weierstrass decomposition for matrix pencils (see [17])

\[
\begin{bmatrix} I & 0 \\ 0 & A_j \end{bmatrix}, \quad \text{MEN} = [I, 0, A_j], \quad \text{MAN} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I_{n-r} \end{bmatrix}
\]

(9)

The matrices \( M \) and \( N \) may also be used to decompose (4), yielding

\[
MB = \begin{bmatrix} B_k \\ B_j \end{bmatrix}, \quad CN = [C_k, C_j].
\]

Referring to [14], we say that (4) is slow controllable if and only if

\[
\text{rank} \{E - A, B\} = n
\]

(10)

for every \( \lambda \in \sigma \) and fast controllable if and only if

\[
\text{rank} \{E, B\} = n.
\]

(11)

The system is controllable if and only if both (10) and (11) hold. In addition, we say that (4) is impulse controllable if and only if

\[
\text{Im} A_k + \text{Ker} A_k + \text{Im} B_k = \mathbb{R}^{n-r}.
\]

(All four system properties can also be defined directly in terms of the solutions of the differential equation (4), but we find the linear algebraic characterizations more useful in the context of this paper.) Controllability and observability imply impulse controllability and impulse observability, respectively. The corresponding definitions for observability are dual to (10), (11), and (12) (see [14]). Since each of these definitions is invariant under the equivalence transformation (7), we may also consider the subsets \( \mathcal{L}_{<r}, \mathcal{L}_{<e}, \mathcal{L}_{<r}, \mathcal{L}_{<e}, \mathcal{L}_{<e}, \mathcal{L}_{<e}, \mathcal{L}_{<e}, \mathcal{L}_{<e} \subset \mathcal{L} \) determined by (10), (11), (12), and their duals, as well as the controllable and observable systems \( \mathcal{L}_{<r} \neq \mathcal{L}_{<e} \). Various properties of these spaces are studied in [13]; for example, \( \mathcal{L}_{<e} \) and \( \mathcal{L}_{<e} \) are open, and \( \mathcal{L}_{<r} \) and \( \mathcal{L}_{<e} \) are dense in \( \mathcal{L} \).

Other important subsets of \( \mathcal{L} \) are the singular subspace \( \mathcal{L}^s \), consisting of all points \( [E, A, B] \) with \( E \) singular, the regular subspace \( \mathcal{L}^n = \mathcal{L} - \mathcal{L}^s \), and the subspace of unit index systems

\[
\mathcal{L}_1 = \{ \xi \in \mathcal{L} | \deg sE - A = \text{rank} E \}.
\]

In [10] it is shown that \( \mathcal{L}^s \) is open and dense in \( \mathcal{L} \); from [13], \( \mathcal{L}_{<e} \cap \mathcal{L}^s \) is dense in \( \mathcal{L}^s \).

Let \( \mathcal{L} \) be the set of all \( C^\infty \) functions \( f : \mathbb{R} \to \mathbb{R} \) with compact support and let \( \mathcal{D}_+ \) be the space of distributions with support in \( [0, \infty) \) (see [19]). To define convergence in \( \mathcal{D}_+ \), we adopt the weak* topology: A sequence \( f_k \in \mathcal{D}_+ \) is said to converge to \( f \) if \( (f, \phi) \to 0 \) for every \( \phi \in \mathcal{D}_+ \). A defined at the point \( \phi \).

Associated with each initial condition \( x_0 \in \mathbb{R}^n \) and each piecewise continuous input \( u \) there exists a unique solution \( \Psi_{xu}(\xi) = (\mathbb{R}^n, x_0) \subseteq \mathcal{D}_+ \) of the system \( \xi \) (see [17]). From linearity it follows that the solution can be decomposed into natural and forced response

\[
\Psi_{xu}(\xi) = \Psi_{x0}(\xi) + \Psi_{0u}(\xi).
\]

Letting

\[
\begin{bmatrix} B_k \\ B_j \end{bmatrix} = [A_k, C_k], \quad [C_k, C_j] = C_k N_k, \quad \begin{bmatrix} x_{0k} \\ x_{0j} \end{bmatrix} = N_k^{-1} x_0
\]

we have from [17] that

\[
\Psi_{x0}(\xi) = \exp (A_k) x_0 + \exp (A_k) \ast B_k u
\]

(14)

where \( \exp (A) \in \mathcal{D}_+ \) is defined by

\[
\exp (A)(t) = e^{A t} \quad \text{and} \quad "\ast" \quad \text{denotes convolution. Each} \quad \Psi_{xu}(\xi) \quad \text{satisfies several properties of continuity. Indeed, convergence of} \quad \mathcal{D}_+ \quad \text{guarantees uniform convergence of} \quad \exp (A_k) \quad \text{on compact intervals and, hence, weak* convergence. Continuity of convolution with respect to both types of convergence assures that each sequence} \quad \Psi_{x0}(\xi) \quad \text{converges weak* and uniformly on compact intervals whenever} \quad \xi_k \to \xi. \quad \text{Furthermore, since} \quad \Psi_{x0}(\xi) = \Psi_{x0}(\xi) \quad \text{for} \quad \xi \in \mathcal{L}^s, \quad \Psi_{x0}(\xi) \quad \text{satisfies the same properties when restricted to} \quad \mathcal{L}^n.\]

To aid in writing a general expression for \( \Psi_{x0}(\xi) \), we note that there exists a nonsingular matrix sequence \( T_k \) (not necessarily convergent) such that

\[
T_k^{-1} A_k T_k = \begin{bmatrix} A_k & 0 \\ 0 & A_k \end{bmatrix}
\]

(15)

where \( A_k \) is nonsingular and \( A_k \) is nilpotent. Then from [17],

\[
\Psi_{x0}(\xi) = \begin{bmatrix} \exp (A_k) & 0 \\ 0 & \sum_{i=0}^{o_k-1} (A_k)^i B_k u \end{bmatrix} T_k^{-1} x_{0k} + T_k \begin{bmatrix} \exp (A_k) + A_k^{-1} B_k u \\ - \sum_{i=0}^{o_k-1} A_k^i B_k u \end{bmatrix}
\]

(16)
where
\[
\begin{bmatrix}
\beta_{kr} \\
\beta_{r}\end{bmatrix} = T_k^{-1}B_k
\]
and \(\beta_i^t\) and \(\beta^t\) denote the \(i\)th distribution derivatives.

### III. Problem Formulation

We are now in a position to precisely state the basic problem under consideration. For two reasons we are forced to select a rather abstract mathematical framework for our constructions. First, since perturbations leading to changes in order require the continuity of the loop-closing map \(G\) and \(K\), rather abstract mathematical framework for our constructions, we restrict attention to system (4) is fast cyclic if, in the Weierstrass decomposition (8), we are interested in choosing a compensator such that the resulting closed-loop system perturbations should lead to only small changes in the problem of interest is to characterize the class of all robust plant-compensator pairs of \((G, K) \in \mathcal{G}\) for any given values of \(n, m,\) and \(p\). Equivalently, we seek to describe the class of all compensators \(K\) which are robust with respect to a given plant model \(G\).

### IV. The Class of Asymptotically Robust Compensators

Before starting our main result on robustness, we need to consider one more algebraic system property of (4). We say that a system (4) is fast cyclic if, in the Weierstrass decomposition (8), the nilpotent matrix \(A_f\) is cyclic. If \(A_f\) is in Jordan form, fast cyclicity is equivalent to
\[
\begin{bmatrix}
0 & 1 \\
& & & \ddots \\
& & & & & & & & 1 \\
& & & & & & & & 0
\end{bmatrix}
\]
Hence, from (9), a system (4) is fast cyclic if and only if rank \(E = n - 1\). Note that fast cyclicity is independent of the choice of representation for \(G\).

In order to prove that a certain algebraic condition on the compensator \(K\) is well defined, we next present a pair of lemmas. It will eventually be proven that this condition is necessary and sufficient for robustness.

**Lemma 4.1:** Let \(N\) and \(T\) be any \(n \times n\) matrices with \(N\) nilpotent and having index \(q\). Then \(\text{Ker} \, N^q = N^{q-1}T\)-invariant.

**Proof:** Since \(N(N^{q-1}T) = N^qT = 0,\) Im \(N^{q-1}T \subset \text{Ker} \, N\). Hence,
\[
(N^{q-1}T) \text{Ker} \, N \subset \text{Ker} \, N.
\]

Note that, if \(N\) is cyclic, \(\text{Ker} \, N\) is one-dimensional.

**Lemma 4.2:** If \(G \in \mathcal{G}\) is fast cyclic, impulse controllable, and impulse observable, and \((A_f, B_f, C_f)\) is obtained from the...
Weierstrass decomposition of any representation of $\xi$, then
\[
A_{r-r-1}^{-1}(I+A_{r}K_{C}) | \text{Ker} A_{r} > 0 \quad (20)
\]
determines a nonempty open affine half-space in $\mathbb{R}^{m}$ which is independent of the representation. (The vertical bar denotes the restriction of the linear operator to the subspace Ker $A_{r}$.)

**Proof:** In view of Lemma 4.1, (20) is well-defined. For the case $r = n - 1$, $A_{r} = 0$ and
\[
A_{r-r-1}^{-1}(I + B_{r}K_{C}) = 1 + B_{r}K_{C} \quad (21)
\]
For $r < n - 1$, choose a nonsingular $T$ so that $T^{-1}A_{r}T$ is in Jordan form. Letting
\[
\begin{bmatrix}
    b_1 \\
    \vdots \\
    b_{n-r}
\end{bmatrix} = T^{-1}B_{r}, [c_1 \cdots c_{n-r}] = C_{r}T \quad (22)
\]
we know from [14] that impulse controllability and impulse observability guarantee $b_{n-r} \neq 0$ and $c_1 \neq 0$. Also,
\[
A_{r-r-1}^{-1}(I_{n-r}+B_{r}K_{C}) \quad (23)
\]
Setting (21) and (23) positive determines nonempty open affine half-spaces.

From [11], $(A_{r}, B_{r}, C_{r})$ is unique up to similarity transformation for a given $\xi$. Clearly, similarity transformation does not alter (21), so the resulting half-space is unchanged. To see how (22) is affected by similarity transformation, note that (23) means
\[
A_{r-r-1}^{-1}(I_{n-r}+B_{r}K_{C}) = b_{n-r}, K_{C} x
\]
for any $x \in \text{Ker} A_{r}$. Let $z = T^{-1}x$. Then
\[
T^{-1}A_{r-r-1}^{-1}(I_{n-r}+B_{r}K_{C}) Tz = b_{n-r}, K_{C} z
\]
so
\[
(T^{-1}A_{r}T)^{n-r-1}(I_{n-r}+(T^{-1}B_{r})) \quad (24)
\]
But $T^{-1}\text{Ker} A_{r} = \text{Ker} (T^{-1}A_{r}T)$ so the resulting half-space is again unchanged.

A final technical lemma is needed to prove our main robustness theorem.

**Lemma 4.3:** Let $a_{k}, i = 0, \ldots, \mu$ be convergent sequences in $\mathbb{R}$ with a $a_{k} \neq 0$ for every $k$, and let $f_{z,k} : \mathbb{R} \rightarrow \mathbb{R}$; $i = 1, \ldots, \nu = 1$; $k = 1, 2, \ldots$ be continuous at the origin and satisfy $f_{z,k}(0) = 0$, where $\nu > \mu$. Then there exists a sequence $e_{k}$ in $\mathbb{R}$ such that for every $k$:
1) $0 < |e_{k}| < 1/k$
2) $\text{sgn} e_{k} = -\text{sgn} a_{k}$
3) the polynomial $e_{k} s^{2} + f_{z,k-1}(e_{k}) s^{r-1} + \cdots + f_{z,k+1}(e_{k}) s^{r-1} + (a_{k} + f_{z,k}(e_{k})) s^{r} + \cdots + (a_{k} + f_{z,k}(e_{k})) s^{r} + a_{k}$ has at least one real root $\lambda_{k}$ with $\lambda_{k} > 0$.

**Proof:** Fix $k$, let $a_{j} = -1/j \text{ sgn} a_{k}$, and consider the sequence (in $j$)
\[
p_{j}(s) = a_{j} s^{r} + f_{z,k-1}(a_{j}) s^{r-1} + \cdots + f_{z,k+1}(a_{j}) s^{r-1} + (a_{k} + f_{z,k}(a_{j})) s^{r} + \cdots + (a_{k} + f_{z,k}(a_{j})) s^{r} + a_{k} \quad (25)
\]
From [12, Lemma 4.3], $p_{j}$ can be factored as
\[
p_{j}(s) = \phi_{j}(s^{r} + b_{n-1}, s^{r-1} + \cdots + b_{0}) \prod_{l=1}^{r} (a_{l} s^{l} - 1) \quad (26)
\]
where each $a_{j} \rightarrow 0$ and $\phi_{j}$, $b_{i}$; $i = 0, \ldots, \mu - 1$ all converge. Equating the coefficients in (24) and (25) of $s^{r}$ and $s^{r-1}$ yields
\[
\begin{align*}
    &a_{j} = \phi_{j}, \prod_{l=1}^{r} a_{l} \\
    &a_{k} = (-1)^{n-r} \lim \phi_{j}.
\end{align*}
\]
For sufficiently large $j$ it follows that
\[
\text{sgn} \prod_{l=1}^{r} a_{l} = \text{sgn} a_{j} \text{ sgn} \phi_{j} = - \text{sgn} a_{k} \text{ sgn} \lim \phi_{j} = (-1)^{n-r+1}.
\]
Hence, for each sufficiently large $j$ there must exist an $i$ such that
\[
a_{j} \in \mathbb{R}, a_{j} > 0.
\]
Since $a_{j} \rightarrow 0$, there exists $j > k$ such that $1/a_{j} > k$. Set $\lambda_{k} = 1/\alpha_{j}$ and $e_{k} = a_{j}$.

Our main result, Theorem 4.4 completely characterizes the robust static compensator gains $K$.

**Theorem 4.4:** Let $\xi \in \mathbb{R}^{n}$.
1) A robust $K \in \mathbb{R}^{m}$ exists iff $\xi$ is fast cyclic, impulse controllable, and impulse observable.
2) Under the conditions of part 1), $K$ is robust iff
\[
A_{r-r-1}^{-1}(I_{n-r}+B_{r}K_{C}) | \text{Ker} A_{r} > 0. \quad (27)
\]

**Proof:**

1) (Necessary): Let $r = \text{ord} \xi$. We need only consider the case $r < n - 1$, since $r = n - 1$ implies $\xi$ is fast cyclic, impulse controllable, and impulse observable (see [14]). Suppose $r < n - 1$ and choose a representation $(E_{r}, A_{r}, B_{r}, C_{r})$ for $\xi$. Invoke the Weierstrass decomposition (9), select a similarity transformation to put $A_{r}$ in Jordan form, use the notation (22), and let
\[
\begin{bmatrix}
    0 & \gamma_{1} \\
    \vdots \\
    0 & \gamma_{n-r-1}
\end{bmatrix} = T^{-1}A_{r}T. \quad (28)
\]
(Each $\gamma_{i}$ is either 0 or 1.) If $\xi$ is not fast cyclic, impulse controllable, or impulse observable, then either $\gamma_{i} = 0$ for some $i$, $b_{n-r} = 0$, or $c_{r} = 0$ (see [14]). Choose nonzero sequences $\gamma_{i} \rightarrow \gamma_{i}, b_{n-r} \rightarrow b_{n-r}, c_{1} \rightarrow c_{1},$ and $K_{r} \rightarrow K$ such that
\[
b_{n-r,k} K_{r} c_{1} \prod_{i=1}^{n-r-1} \gamma_{i} < 0
\]
for every $k$, and define
\[
B_{r} = T \begin{bmatrix}
    b_{1} \\
    \vdots \\
    b_{n-r-1} \\
    b_{n-r,k}
\end{bmatrix}, \quad C_{r} = \begin{bmatrix}
    c_{1} c_{2} \cdots c_{n-r-1}
\end{bmatrix} T^{-1} \quad (29)
\]
COBB: ROBUST COMPENSATION

\[
A_R(x) = T \begin{bmatrix} x & \gamma_1 k \\ \vdots & \vdots \\ x_{n-r-1,k} & \end{bmatrix} T^{-1}, \tag{30}
\]

Now, we may uniquely define sequences \(a_{n,i} = 0, \ldots, n-1\) and polynomials \(p_{n,k} = 1, \ldots, n-1\), with \(p_{n,0} = 0\), according to

\[
x^{n-r} + (a_{n-1,k} + p_{n-1,k}(x)) x^{n-1} + \cdots + (a_{1,k} + p_{1,k}(x)) x + a_{0,k}
= \begin{vmatrix}
\gamma_1 k \\
\vdots \\
\gamma_{n-r-1,k}
\end{vmatrix}
\]

By elementary matrix arguments,

\[
\alpha_{n-1,k} = (-1)^{n-r} a_{n-1,k} k \gamma_{1,k} \prod_{i=1}^{n-r} \gamma_{i,k} \tag{31}
\]

Letting \(f_k(x) = p_{n,k}(x^{1/(n-r)})\), \(n = n\), and \(\mu = n - 1\), we may select a sequence \(e_k\) satisfying the properties in Lemma 4.3 and define

\[
\alpha_k = -\left(\frac{1}{\epsilon_k}\right)^{1/(n-r)}.
\]

Since \(\epsilon_k \to 0\) and \(\text{sgn } \epsilon_k = -\text{sgn } a_{n-1,k}\), (28) and (31) guarantee that \(\alpha_k \to +\infty\). If we set

\[
E_k = M^{-1} \begin{bmatrix} I_{n-r} & 0 \\ 0 & A_R \left(\frac{1}{\alpha_k}\right) \end{bmatrix} N^{-1}, A_k = A
\]

\[
B_k = M^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}, C_k = [C_1, C_2] N^{-1} \tag{32}
\]

we have

\[
\det(s E_k - (A_k + B_k K_k C_k))
= \beta(e_k s^n + (a_{n-1,k} + f_{n-1,k}(e_k)) s^{n-1} + \cdots + (a_{1,k} + f_1(e_k)) s + a_{0,k})
\]

for some constant \(\beta\). From Lemma 4.3, (33) has at least one real root \(\lambda_k > k\) for every \(k\). Since (33) is just the characteristic polynomial of the closed-loop system \(C(\xi, k)\), (15) shows that \(\lambda_k\) must be an eigenvalue of the closed-loop \(A_k^{1/2}\) for sufficiently large \(k\). Thus, \(\exp \left( A_k^{1/2} \right) \) cannot converge uniformly on compact subintervals of \((0, \infty)\), since this would imply uniform convergence of its eigenvalue exp (\(\lambda_k\)). Letting \(u = 0\), it follows from (13) and (16) that \(\Psi_{f_k}(C(\xi, k))\) does not converge for every \(x_0\).

In order to prove that \(K\) is not robust, we have only left to show that \(\Psi_{f_k}(\xi) \to \Psi_{f_k}(\xi)\) in the weak* sense, where \(\xi_k = [E_k, A_k, B_k, C_k]\). To do so, we note that

\[
\Psi_{f_k}(\xi) = \exp \left( A_R \left(\frac{1}{\alpha_k}\right)^{-1} \right) x_0 + A_R \left(\frac{1}{\alpha_k}\right)^{-1} x_0 A_R \left(\frac{1}{\alpha_k}\right)^{-1} x_0 + \cdots + A_R \left(\frac{1}{\alpha_k}\right)^{-1} x_0 A_R \left(\frac{1}{\alpha_k}\right)^{-1} x_0 + A_R u
\]

and

\[
A_R \left(\frac{1}{\alpha_k}\right)^{-1} \exp \left( A_R \left(\frac{1}{\alpha_k}\right)^{-1} \right) = \sum_{i=0}^{n-1} \frac{d^i}{ds^i} e^{i\xi u} \bigg|_{s = -1/\alpha_k} \tag{30}
\]

Consider the matrix

\[
\Sigma_k = \begin{bmatrix} -\frac{1}{\alpha_k} & 1 & & & \\
\vdots & \vdots & \ddots & \ddots & \\
\end{bmatrix}.
\]

A routine calculation shows that the \((i, j)\)th entry of \(\Sigma_k^{-1} \exp(\tau \Sigma_k^{-1})\) is

\[
o_{ij} = \frac{1}{(j-i)!} \frac{d^{j-i}}{ds^{j-i}} \left( \frac{1}{s} e^{\xi u} \right) \bigg|_{s = -1/\alpha_k}, \quad j \geq l.
\]

It was shown in the proof of [10, Theorem 4] that

\[
\Sigma_k^{-1} \exp(\tau \Sigma_k^{-1}) \to - \begin{bmatrix} 1 & \delta & \cdots & \delta^{n-r-1} \\
\vdots & \ddots & \ddots & \ddots \\
\delta & \cdots & 1
\end{bmatrix}
\]

Therefore, \(o_{ij} \to -\delta^{j-i}\) for each \(j \geq i\) and \(\Psi_{f_k}(\xi) \to \Psi_{f_k}(\xi)\) for any \(x_0u\). It follows from (13), weak* continuity of convolution, and continuity of \(\Psi_{f_k}\) that \(\Psi_{f_k}(\xi_k) \to \Psi_{f_k}(\xi)\).

(Sufficient): Let \(\xi_k\) be any sequence in \(E\) such that \(\Psi_{f_k}(\xi_k) \to \Psi_{f_k}(\xi)\) for every \(x_0u\) with and \(C_k \to C\). Then, from [10], \(\xi_k \to \xi\) and the decomposition (8) may be invoked. It follows that

\[
M_k \mid s E_k - A_k \mid N_k = \mid s E_k - A_k \mid s A_{j_0} \mid s A_{j_0} - I_{n-r}\]

for any convergent representation \((E_k, A_k, B_k, C_k)\). Suppose \((-1)^{n-r}[A_k] < 0\) for infinitely many \(k\). Since \([A_{j_0}]\) is just the product of the eigenvalues of \(A_{j_0}\), there exists a subsequence \(A_{j_0}\), with at least one real, positive eigenvalue for each \(k_j\). It follows that \(A_{j_0}^{-1}\) has an eigenvalue \(\lambda_j \to \infty\). Let \(u = 0\) and observe that

\[
\langle \Psi_{f_k}(\xi_k), \phi \rangle = \Gamma_j x_0 \delta_{i,j}
\]
Applying the decomposition (8) to the closed-loop system yields nonsingular transformations $M_k$ and $N_k$ such that

$$
\begin{align*}
M_k E_k \dot{A}_k &= \begin{bmatrix}
I_{n-1} & 0 \\
0 & \alpha_k
\end{bmatrix}, \\
M_k (A_k + B_k K_k C_k) \hat{N}_k &= \begin{bmatrix}
\hat{A}_{nk} & 0 \\
0 & 1
\end{bmatrix}
\end{align*}
$$

where all sequences converge. The decomposition (13) may also be applied to the closed-loop system yielding

$$
\Psi_{x_{0k}}(\mathcal{E}(\xi_k)) = \hat{N}_k \begin{bmatrix}
\Psi_{x_{0k}}(\xi_k) \\
\hat{N}_k \mathcal{E}(\xi_k)
\end{bmatrix}.
$$

From Lemma 4.3, $\Psi_{x_{0k}}(\xi_k) \to \Psi_{x_{0k}}(\xi)$, and

$$
\Psi_{x_{0k}}'(\xi_k) = \begin{cases}
\exp \left( \frac{1}{\alpha_k} \right) x_{0k} + \frac{1}{\alpha_k} \exp \left( \frac{1}{\alpha_k} \right) B_k u & \text{if } \alpha_k < 0 \\
- B_k u & \text{if } \alpha_k > 0
\end{cases}
$$

for every $k$. Now define $a_{1k}, \ldots, a_{n-1,k}, \beta_{1k}, \ldots, \beta_{n-1,k}$ according to

$$
x s^n + (a_{n-1,k} + \beta_{n-1,k} s) s^{n-1} + \cdots + (a_{1,k} + \beta_{1,k}) s + a_{0,k}
$$

$$
= s^{n-1} - (A_k + B_k K_k C_k) + B_k K_k C_k x s - (1 + B_k K_k C_k).
$$

Then

$$
a_{n-1,k} = -(1 + B_k K_k C_k).$$

Letting $f_k(x) = \beta_k x$, we can find a sequence $\xi_k$ satisfying the properties in Lemma 4.3; define $\alpha_k = -1/\varepsilon_k$. Since $\varepsilon_k \to 0$ and $\sgn \varepsilon_k = - \sgn a_{n-1,k}$, (39) and (40) guarantee that $\alpha_k \to +\infty$. If we set

$$
E_k = M^{-1} \begin{bmatrix}
I_{n-1} & 0 \\
0 & -\frac{1}{\alpha_k}
\end{bmatrix} N^{-1}, \quad A_k = A
$$

$$
B_k = B, \quad C_k = C
$$

we have that det $(s E_k - (A_k + B_k K_k C_k))$ has at least one real root $\lambda_k > k$ for each $k$. As in the sufficiency proof of part 1, $\Psi_{x_{0k}}(\mathcal{E}(\xi_k, K))$ does not converge for some $x_{0k}$, $u_k$. Since

$$
\Psi_{x_{0k}}'(\xi_k) = \exp (-\alpha_k) x_{0k} - \alpha_k \exp (-\alpha_k) B_k u
$$

(41)

in the open-loop system, $\Psi_{x_{0k}}'(\xi_k) \to \Psi_{x_{0k}}'(\xi)$ and $K$ is not robust.

If (26) fails and $r = n - 1$, we may adopt the notation (29) and (30) and observe that $b_{n-1,k} C_k \leq 0$. Since fast cyclability, impulse controllability, and impulse observability guarantee that $b_{n-1,k} \neq 0$ and $c_{1,k} \neq 0$, a sequence $K_\xi \to K$ may be chosen so that $b_{n-1,k} C_k < 0$ for every $K$. The remaining arguments are the same as in the necessity part of 1) with $b_{n-1,k} C_k = -c_{1,k}$ and $\gamma_k = 1$.

Theorem 4.4 is somewhat pessimistic in that, in the strictest...
theoretical sense, robustness can only be guaranteed when at most one degree of singularity is present in the plant (4) (rank \( E = n - 1 \)). In physical terms this can be interpreted as meaning that a static compensator can handle only a first-order unmodeled dynamic element. In our opinion, this indicates that some basic assumptions which are as yet not well understood are conventionally placed on system models in engineering practice.

For a mathematical explanation of how nonrobust compensators may fail to stabilize a system, consider the matrix condition in part 2) of Theorem 4.4. This condition determines an open affine half-space in the set of compensation gains \( K \). Examination of the proof of Theorem 4.4 reveals that, for systems which are fast cyclic, impulse controllable, and impulse observable, a static compensator results in positive feedback either for all admissible perturbations simultaneously or for none at all. The half-space of robust feedback gains is simply the set of all \( K \) with the appropriate sign to guarantee negative feedback for all perturbations (4) of the system (4). The system (6) illustrates this point. The robust gains are simply those satisfying \( r < 1 \). On the other hand, part 1) of Theorem 4.4 maintains that unless the plant is fast cyclic, impulse controllable, and impulse observable, the class of admissible perturbations is so broad that any compensator results in positive feedback with respect to some perturbation; hence, no compensator is robust. This is illustrated by (2) and (3).

Another important point to note at this stage is that, although all definitions and technical arguments until now have been couched in terms of sequences, each statement applies equally well to nets in the various topological spaces. This observation is important, since the space of distributions \( \mathbb{D} \), does not satisfy the first axiom of countability (see [20]).

To conclude this section we compare our results to those of [5]. Specifically, [5, Theorem 1] shows that, for any system

\[
\dot{x} = Ax + Bu
\]

\[](42)

and any compensation matrix \( K \), there exists a singular perturbation of (42) of the form (1) which destabilizes the closed-loop system. (The result of [5] is somewhat more general in that it applies to all dynamic compensators which are proper but not strictly proper.) According to Theorem 4.4, if we take such a perturbation and set \( \epsilon = 0 \), we obtain a nominal system

\[
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u
\]

\[](43)

which must either fail to be fast cyclic, impulse controllable, or impulse observable. For example, setting \( \epsilon = 0 \) in (3) yields a system of the form (43) which can be shown to be not fast cyclic. While the result of [5] illustrates that a specialized class of parasitics can lead to closed-loop destabilization, our results characterize the same phenomenon but in the context of a broader class of perturbations and a larger family of nominal systems. For example, our Theorem 4.4 applies to systems of the form (43) with \( A_{22} \) singular (as long as \( \text{det} E - A \neq 0 \) is satisfied), while [5] considers only the case of \( A_{22} \) nonsingular. Our result also shows when destabilization can occur as a result of perturbations to a given order; the perturbed order required to destabilize the closed-loop system in [5] is not specified.

V. GENERICITY

We now consider the class of systems (4) for which there exists a robust compensator \( K \). The sets of impulse controllable and impulse observable systems were shown in [11] to be dense in the system space \( \mathcal{L} \). The next result characterizes those systems which are also fast cyclic.

Proposition:

1. \( \xi \) is fast cyclic iff \( \xi \in \mathcal{L}^n \cup \mathcal{L}^{n-1} \).
2. \((\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_c \cap \mathcal{L}_o \) is open in \( \mathcal{L} \).
3. \( \mathcal{L}^{n-1} \cap \mathcal{L}_c \cap \mathcal{L}_o \) is dense in \( \mathcal{L} \).

Proof:

1. Let \((E, A, B, C)\) be any representative of \( \xi \). If \( r = \text{ord} \xi \), the Weierstrass decomposition (9) shows that \( E = r + \text{rank} A_f \). But \( \xi \) is fast cyclic if and only if either \( r = n \) or rank \( A_f = n - r \). Hence, \( \xi \) is fast cyclic iff rank \( E = n \) or rank \( E = n - 1 \).

2. Let \( \xi = [E, A, B, C] \in \Omega = (\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_c \cap \mathcal{L}_o \)

and apply the decomposition (9). Then \( A_f \) is cyclic, and \( \text{ker} A_f \subset \text{ker} A_f \). Since \( \xi \) is impulse controllable and impulse observable.

\[
\text{Im} A_f^* \cap \text{Im} A_f^* = \text{Im} A_f + \text{ker} A_f \cap \text{Im} A_f \cap \text{ker} C_f = 0
\]

so \( \xi \in \mathcal{L}_c \cap \mathcal{L}_o \) (see [14]). It follows that

\[
\Omega = (\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_c \cap \mathcal{L}_o.
\]

VI. DISCUSSION AND CONCLUSIONS

In this section we discuss some of the implications of our theory and use these to suggest further research. Theorem 4.5 shows that a generic class of systems can be robustly compensated using static compensators \( K \). This does not mean, however, that the complement of the open and dense subset \((\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_c \cap \mathcal{L}_o \) does not contain interesting systems. On the contrary, it is easy to show that all systems of the form (4) with \( r < n - 1 \) and \( A_{22} \) nonsingular lie outside the generic class described by Theorem 4.5. Another interesting observation is that even a system which does lie in the generic set can be trivially augmented so that it sits outside the generic set in a higher dimensional system space. For example, the dimension of (4) may be increased simply by defining a new (scalar-valued) state variable \( z = 0 \) and noting that

\[
\begin{bmatrix}
E & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} + \begin{bmatrix}
B \\
0
\end{bmatrix} u
\]

\[
y = [C \ 0] \begin{bmatrix}
x \\
z
\end{bmatrix}.
\]

System (44) is a member of \( \mathcal{L}(n + 1, m, p) \). It is easy to show that (44) is not fast cyclic and, hence, cannot be robustly compensated. The latter point can be countered by arguing that only variables of interest should be included in a well-devised state-space model; therefore, the variable \( z \) would never be present.

There are at least a couple of avenues of research which might eventually resolve these issues. Dynamic compensation is still relatively unexplored in the context of singular perturbations. One promising result is [5, Theorem 2] which suggests that, when parasitics are present, strictly proper compensators are more robust than nonstrictly proper ones. Since [5] treats only the single time-scale case, more work needs to be done to see whether this
result stands up to a larger class of perturbations. As pointed out in Section 1, the issue of which class of perturbations is meaningful in a given system analysis is of fundamental importance. Our main results can in fact be proven under a somewhat more general definition of system perturbation than the one provided here (strong convergence). However, preliminary work suggests that even such a generalization might be too restrictive to allow a coherent robustness theory to be developed. We intend to explore these issues more fully in the future.

REFERENCES

THE MINIMAL DIMENSION OF STABLE FACES REQUIRED
TO GUARANTEE STABILITY OF A MATRIX POLYTOPE

J. Daniel Cobb
Christopher L. DeMarco

Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

We consider the problem of determining whether each point in a polytope of n×n matrices is stable. Our approach is to check stability of certain faces of the polytope. For n≥3, we show that stability of each point in every (2n-4)-dimensional face guarantees stability of the entire polytope. Furthermore, we prove that, for any k≤n^2, there exists a k-dimensional polytope containing a strictly unstable point and such that all its subpolytopes of dimension min(k-1,2n-5) are stable.

This work was supported in part by NSF Grants ECS-8612948 and ECS-8611728 and in part by AFOSR Grant No. 88-0087.
1. Background and Introduction

In this paper we consider the problem of ascertaining whether certain subsets of $\mathbb{R}^{n \times n}$ consist entirely of stable matrices. (Here we take stability of a matrix to mean that all its eigenvalues are in the open left half plane.) First we need some definitions. A (convex) polytope $\mathcal{P}$ in a vector space $V$ is the convex hull of any nonempty finite subset of $V$. The dimension of $\mathcal{P}$ is the dimension of the affine hull $\text{aff}(\mathcal{P})$ of $\mathcal{P}$. The relative boundary of $\mathcal{P}$ is the boundary of $\mathcal{P}$ as a subset of the topological space $\text{aff}(\mathcal{P})$. A face of $\mathcal{P}$ is any set of the form $\Pi \mathcal{P}$, where $\Pi$ is a supporting hyperplane of $\mathcal{P}$. A vertex of $\mathcal{P}$ is a 0-dimensional face. An edge of $\mathcal{P}$ is a 1-dimensional face. A subpolytope of $\mathcal{P}$ is the convex hull of any set of vertices of $\mathcal{P}$. Finally, a $k$-dimensional half-plane in $V$ is any nonempty set of the form $\mathcal{N}=\mathbb{R}S$, where $\mathcal{N}$ is a closed half-space, $S$ is a $k$-dimensional affine subspace, and $S \subseteq \mathbb{R}$. (Note that this implies that the affine hull of $\mathcal{N}$ is simply $S$.)

In the robust control literature, considerable interest has been generated by the problem of determining whether stability of a polytope in either $\mathbb{R}^n$ or $\mathbb{R}^{n \times n}$ can be guaranteed simply by checking stability of low-dimensional faces. (Stability of a vector $x \in \mathbb{R}^n$ means simply that the polynomial $s^n + x_n s^{n-1} + \ldots + x_1$ is Hurwitz.) We first note that the cases $n=0$ and $n=1$ are trivial; stability of the vertices always guarantees stability of the polytope. Several recent papers consider the case $n \geq 2$. For example, polynomial polytopes of a particularly simple structure ("interval polynomials") were addressed by Kharitonov [1]; he showed that only four specially constructed vertices need be checked. A more recent result of Bartlett, Hollot, and Lin [2] demonstrates that, for an arbitrary polynomial polytope checking all edges is sufficient to guarantee stability of $\mathcal{P}$. With respect to polytopes in $\mathbb{R}^{n \times n}$, Fu and Barmish [3] have shown that stability of
all 1-dimensional subpolytopes is insufficient to guarantee stability of $\mathcal{P}$.

DeMarco [4] has shown that, for $n \geq 3$, $(n-2)$-dimensional faces are insufficient, but $2n$-dimensional faces are sufficient.

In this paper we refine the bounds of [4] and show that stability of all $m$-dimensional faces is sufficient to guarantee stability of $\mathcal{P}$, where

$$m(n) = \begin{cases} 1, & n=2 \\ 2n-4, & n>2 \end{cases}$$

Furthermore, we show that for any $n$ and $k \leq n^2$ there exists a polytope of dimension $k$, containing a strictly unstable point (a matrix with an eigenvalue $\lambda$ satisfying $\Re \lambda > 0$), and such that all its $\min(k-1,m-1)$-dimensional subpolytopes are stable; hence, in this sense, $m$ is minimal.

2. Sufficiency of $m$

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map $f : \mathbb{R}^k \to \mathbb{R}^{n^2}$ determines an affine isomorphism between $\mathbb{R}^k$ and $f(\mathbb{R}^k)$. Among other things, this implies that, for any polytope $\mathcal{P} \subset \mathbb{R}^k$, $f(\mathcal{P})$ is also a polytope of the same dimension as $\mathcal{P}$; furthermore, $f$ sets up a one-to-one correspondence between $q$-dimensional faces of $\mathcal{P}$ and $q$-dimensional faces of $f(\mathcal{P})$. In addition, $f$ maps each $k$-dimensional half-plane in $\mathbb{R}^k$ into another $k$-dimensional half-plane (e.g., see [5]). Finally, we note that every polytope is compact and that the set $\{x \in \mathbb{R}^k | |x|_\infty \leq 1\}$ is a polytope whose $q$-dimensional faces are generated by fixing $k-q$ entries of $x$ at either $\pm 1$ and letting the remaining $q$ entries vary independently over $[-1,1]$.

With these observations in mind, we prove a result characterizing the affine structure of the set of unstable points in $\mathbb{R}^{n \times n}$.
Lemma 2.1 For each unstable \( A \in \mathbb{R}^{n \times n} \) there exists an \( (n^2 - m) \)-dimensional half-plane \( \mathcal{H} \subset \mathbb{R}^{n \times n} \) such that 1) \( A \in \mathcal{H} \) and 2) \( B \in \mathcal{H} \) implies \( B \) is unstable.

Proof Case I -- \( A \) has a real eigenvalue \( \lambda \neq 0 \): Let \( T = [v \; w] \), where \( v \) is an eigenvector corresponding to \( \lambda \) and \( w \) is chosen to make \( T \) nonsingular. Clearly, the map \( f: \mathbb{R}^{n^2-n-1} \to \mathbb{R}^{n \times n} \) determined by

\[
f(\lambda, y, z) = T \begin{bmatrix} \lambda & y \\ 0 & z \end{bmatrix} T^{-1}
\]

is affine and one-to-one. Let \( \mathcal{H} \) be the \((n^2-n+1)\)-dimensional half-plane

\[
\mathcal{H} = \{ f(\lambda, y, z) \mid \lambda \geq \lambda_0, y \in \mathbb{R}^{1 \times n-1}, z \in \mathbb{R}^{n-1 \times 1} \}
\]

Then \( A \in \mathcal{H} \) and every matrix in \( \mathcal{H} \) is unstable. Since \( n^2-n+1 \geq n^2 - m \), we need only select any \((n^2-m)\)-dimensional half-plane \( \mathcal{H} \) satisfying \( A \in \mathcal{H} \).

Case II -- \( A \) has a complex eigenvalue pair \( \alpha_0 \pm i \beta_0 \) with \( \alpha_0 > 0 \): Let \( T = [v \; w \; X] \), where \( v+iw \) is an eigenvector corresponding to \( \alpha_0 + i \beta_0 \) and \( X \) is chosen to make \( T \) nonsingular. Let \( \overline{\mathcal{H}} \) be the \((n^2-2n+4)\)-dimensional half-plane

\[
\overline{\mathcal{H}} = \{ T \begin{bmatrix} U & Y \\ 0 & Z \end{bmatrix} T^{-1} \mid \text{tr} U \geq 2\alpha_0, Y \in \mathbb{R}^{2 \times n-2}, Z \in \mathbb{R}^{n-2 \times n-2} \}
\]

(\( \text{tr} U \geq 2\alpha_0 \) describes a 4-dimensional half-plane, since \( \text{tr} U = \langle U, I \rangle \).) \( \overline{\mathcal{H}} \) contains only unstable points, since \( \text{tr} U \geq 2\alpha_0 \) implies \( U \) has at least one eigenvalue \( \lambda \) with \( \text{Re} \lambda \geq \alpha_0 \). Also, \( A \in \overline{\mathcal{H}} \), since our choice of \( T \) guarantees that \( A \) has
Finally, \( n^2 - 2n + 4 \geq n^2 - m \), so the desired \( \mathcal{H} \cap \mathcal{R} \) exists.

Next we prove an easy result concerning the intersection of affine sets.

**Lemma 2.2** Let \( V \) be a \( p \)-dimensional Euclidean space, \( \mathcal{H} \subseteq V \) a \( k \)-dimensional half-plane, and \( \Gamma \) a \( q \)-dimensional affine subspace with \( k+q>p \). Consider any vector \( x_0 \in \mathcal{H} \cap \Gamma \). There exists a \((k+q-p)\)-dimensional half-plane \( \mathcal{H} \) such that \( x_0 \in \mathcal{H} \cap \Gamma \).

**Proof** By definition, \( \mathcal{H} = R \cap S \), where \( R \) is a closed half-space and \( S \) is a \( k \)-dimensional affine subspace satisfying \( S \subseteq R \). There exists an affine subspace \( \tilde{S} \subseteq \mathcal{H} \) with \( \dim \tilde{S} = k+q-p \) and \( x_0 \in \tilde{S} \). If \( \tilde{S} \subseteq R \), let \( \mathcal{H} \cap \tilde{S} \) be any \((k+q-p)\)-dimensional half-space containing \( x_0 \). Then \( \mathcal{H} \cap \tilde{S} \supseteq \mathcal{H} \cap \Gamma \). If \( \tilde{S} \not\subseteq R \), let \( \mathcal{H} = \mathcal{H} \cap \tilde{S} \). Then \( x_0 \in \mathcal{H} \), since \( x_0 \in \mathcal{H} \cap \Gamma \). Also, \( \dim \mathcal{H} = k+q-p \), since \( \mathcal{H} \) is nonempty.

We are now in a position to prove our first main result.

**Theorem 2.3** Stability of every matrix in every \( m \)-dimensional face of \( \mathcal{F} \) guarantees stability of every matrix in \( \mathcal{F} \).

**Proof** Our arguments here are similar to those used in [2, Lemma 1]. Suppose \( \mathcal{P}_k \) is an unstable polytope of dimension \( k>m \). Then there exists an unstable matrix \( A_1 \in \mathcal{P}_k \). From Lemma 2.1, there is a \((n^2-m)\)-dimensional half plane \( \mathcal{H}_1 \), consisting entirely of unstable points and containing \( A_1 \). Since \( \mathcal{H}_1 \) is unbounded, there exists an \( A_2 \in \mathcal{H}_1 \) lying on the boundary of \( \mathcal{P}_k \) and, hence, in
one of its $(k-1)$-dimensional faces $P_{k-1}$. From Lemma 2.2, the intersection $H_1 \cap \text{aff}(P_{k-1})$ contains a $(k-m-1)$-dimensional half-plane $H_2$ such that $A_2 \subseteq H_2$. Proceeding inductively, we find that there exists an $m$-dimensional face $P_m$ and a point $A_{k-m} \in P_m$ such that $A_{k-m}$ is unstable.

3. Minimality of $m$

Our next task is to show that $m$ is the smallest integer such that stability of all $m$-dimensional faces of $P$ guarantees stability of $P$.

Theorem 3.1 For each integer $n \geq 2$ there exists an $m$-dimensional polytope $P \subset \mathbb{R}^{n \times n}$ containing an unstable point and such that all its $(m-1)$-dimensional faces are stable.

Proof Case I -- $n=2$: Consider the affine, one-to-one map

$$f(x) = \begin{bmatrix} 0 & x \\ -x & -1 \end{bmatrix}$$

and the corresponding 1-dimensional polytope $P = \{f(x) \mid x \leq 1\}$. Each matrix in $P$ has characteristic polynomial $\Delta(s) = s^2 + s + x^2$; hence, each vertex of $P$ ($x = \pm 1$) is stable, but the point corresponding to $x = 0$ is unstable.

Case II -- $n=3$: The 2-dimensional polytope

$$P = \left\{ \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & -y \\ x & y & -1 \end{bmatrix} \mid \begin{bmatrix} x \\ y \end{bmatrix} \leq 1 \right\}$$

has characteristic polynomial $\Delta(s) = s^3 + s^2 + (1-x^2+y^2)s + 1$. Each coefficient of
(\text{s}) is positive, and the corresponding $2 \times 2$ Hurwitz matrix has a leading principal second-order minor equal to $M_2(x,y)=x^2-y^2$. Thus, each edge is stable, but the matrix corresponding to $x=y=0$ is unstable.

Case III -- $n \geq 4$: Consider the $(2n-4)$-dimensional polytope

$$P = \begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix} \left\{ \begin{bmatrix} X \\ y \end{bmatrix} : \begin{bmatrix} X \\ y \end{bmatrix} \leq 1 \right\}$$

A routine calculation shows that $P$ has characteristic polynomial $\tilde{p}(s)=(s+1)^{n-4} \Delta(s)$, where

$$\Delta(s) = s^4 + 2s^3 + (2 + x^T x + y^T y)s^2 + (2 + x^T x + y^T y)s + 1 + x^T x y^T y - (x^T y)^2$$

From the Schwartz inequality, it is clear that all coefficients of $\Delta$ are strictly positive. The corresponding $4 \times 4$ Hurwitz matrix has its leading principal third-order minor equal to

$$M_3(x,y) = 4x^T x + 4y^T y + 4(x^T y)^2 + (x^T x - y^T y)^2$$

Clearly, $M_3 \geq 0$ with equality if and only if $x=y=0$. Thus the $(2n-5)$-dimensional faces of $P$ are stable, but the point corresponding to $x=y=0$ is unstable.

It is interesting to note that Theorem 3.1 also implies that the half-planes considered in Lemma 2.1 are maximal in the sense that there exists an unstable matrix $A$ in $\mathbb{R}^{n \times n}$ such that every half-plane of dimension greater than $n^2-m$ containing $A$ must also contain a stable matrix. Indeed, if this
were not the case, the arguments in Theorem 2.3 could be used to prove that $m$ is not minimal.

4. A Stronger Version of the Minimality Theorem

The construction in the proof of Theorem 3.1 is weak in three respects:

1) The polytope $P$ contains only a single marginally unstable matrix (i.e., a matrix having all eigenvalues $\lambda$ satisfying $\Re \lambda \leq 0$ and at least one with $\Re \lambda = 0$). 
2) The construction yields only a polytope of dimension $m$. 
3) Arbitrary subpolytopes are not considered; thus it is not clear that checking all subpolytopes of dimension, say, $m-1$ would not guarantee stability. The minimality proof would be more convincing if it could be extended to give a family of polytopes, each 1) containing a strictly unstable point (and, hence, infinitely many unstable points), 2) having arbitrary dimension $k$, and 3) having all $\min(k-1,m-1)$-dimensional subpolytopes stable.

Theorem 4.2 shows that such improvements over Theorem 3.1 can be made. The proof requires a simple lemma. For any normed linear space $V$, subset $\Omega \subset V$, and point $\gamma \in V$ consider the distance function

$$d(\gamma, \Omega) = \inf_{\omega \in \Omega} |\gamma - \omega|$$

Let $\text{conv}(\Omega)$ denote the convex hull of $\Omega$. 

-7-
Lemma 1: Suppose \( \Omega \subseteq V \) is convex, \( \varepsilon > 0 \), and \( V \subseteq V \) is any set such that \( d(\eta, \Omega) \leq \varepsilon \) for every \( \eta \in \Gamma \). Then \( d(\eta, \Omega) < \varepsilon \) for every \( \eta \in \text{conv}(\Omega \cup \Gamma) \).

**Proof** Each \( \eta \in \text{conv}(\Omega \cup \Gamma) \) is of the form \( \eta = a\eta_1 + (1-a)\eta_2 \), where \( a \in [0, 1] \) and \( \eta_1, \eta_2 \in \Omega \cup \Gamma \). There exist \( \omega_1, \omega_2 \in \Omega \) such that \( \|\eta_1 - \omega_1\| < \varepsilon \) and \( \|\eta_2 - \omega_2\| < \varepsilon \). Let \( \omega = a\omega_1 + (1-a)\omega_2 \). Then \( \omega \in \Omega \), and

\[
\begin{align*}
    d(\eta, \Omega) &\leq \|\eta - \omega\| \\
    &= \|a(\eta_1 - \omega_1) + (1-a)(\eta_2 - \omega_2)\| \\
    &\leq a\|\eta_1 - \omega_1\| + (1-a)\|\eta_2 - \omega_2\| \\
    &< \varepsilon
\end{align*}
\]

Theorem 4.2 For any integers \( n, k \) with \( n \geq 2 \) and \( 1 \leq k \leq n^2 \), there exists a polytope \( P_k \) of dimension \( k \) containing a strictly unstable point and such that each \( \min(k-1, m-1) \)-dimensional subpolytope is stable.

**Proof** Suppose a marginally unstable polytope \( P_k \) of dimension \( k \) is constructed such that all its \( \min(k-1, m-1) \) subpolytopes are stable. Then, since the set of stable points in \( \mathbb{R}^{n \times n} \) is open and the union of all \( \min(k-1, m-1) \)-dimensional subpolytopes of \( P_k \) is compact, the subpolytopes of \( P_k \) of the same dimension are stable for sufficiently small \( \varepsilon \), but \( P_k \) is strictly unstable. Thus, it suffices to construct any \( k \)-dimensional unstable \( P_k \) with stable subpolytopes.
If $n=2$, let $f(x, y) = \begin{bmatrix} 0 & x \\ -x & -1 \end{bmatrix}$, where $x, y$ range over $\mathbb{R}$; otherwise, let

$$f(x, y) = \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & -y \\ x & y & -1 \end{bmatrix}$$

where $x, y \in \mathbb{R}^{n-2}$. We consider two cases: first, assume $k < m$. Define $f_k: \mathbb{R}^k \rightarrow \mathbb{R}^{n \times n}$ according to $f_k(\begin{bmatrix} w \\ 0 \end{bmatrix}, \begin{bmatrix} z \\ 0 \end{bmatrix})$, where the vectors $x$ and $y$ are partitioned in any way such that $\begin{bmatrix} w \\ z \end{bmatrix} \in \mathbb{R}^k$. Since each $f_k$ is affine and one-to-one, the set

$$\mathcal{Q}_k = \{ f_k(w, z) \mid \|w\|_\infty \leq 1 \}$$

is a $k$-dimensional polytope. As in the proof of Theorem 3.1, each matrix in $\mathcal{Q}_k$ is stable except for the point corresponding to $w=z=0$. The union of the $(k-1)$-dimensional subpolytopes of $\mathcal{Q}_k$ is compact and nowhere dense in $f(\mathbb{R}^k)$; hence, there exist vectors $w_0, z_0 \in f(\mathbb{R}^k)$ such that

$$\mathcal{P}_k = \{ f_k(w+w_0, z+z_0) \mid \|w\|_\infty \leq 1 \}$$

is unstable, but has all its $(k-1)$-dimensional subpolytopes stable.

Next consider the case $k > m$. The union of the $(m-1)$-dimensional subpolytopes of the $m$-dimensional polytope $\mathcal{P}$ (defined in Theorem 3.1) are compact and nowhere dense in $f(\mathbb{R}^m)$; hence, there exist $x_0, y_0$ such that
\[ C = \{ f(x-x_0, y-y_0) \mid \|x\|_\infty \leq 1 \} \]

has all its \((m-1)\)-dimensional subpolytopes stable, but \(Q\) is unstable. If \(n=2\), let
\[ g(x, y, z) = f(x, y) + \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} \]
Otherwise, define
\[ g(x, y, z) = f(x, y) + \begin{bmatrix} z_1 & z_2 & \cdots & z_n \\ z_{n+1} & z_{n+2} & \cdots & z_{2n} \\ 0 & 0 & \cdots & z_{2n+1} \end{bmatrix} \]

In each case, \(z \in \mathbb{R}^{n^2-m}\). Also let \(g_k(x, y, w) = g(x, y, \begin{bmatrix} \frac{\varepsilon}{2}w \\ 0 \end{bmatrix})\), where \(w \in \mathbb{R}^{k-2n+4}\) and \(\varepsilon > 0\). Note that each \(g_k\) is affine and one-to-one.

Now consider the \(k\)-dimensional polytope
\[ P_k = \{ g_k(x, y, w) \mid \|x\|_\infty \leq 1 \} \]

If we choose the matrix norm \(\|M\| = \max |m_{ij}|\), it follows that for every vertex \(A\) of \(P_k\) there exists a vertex \(\bar{A}\) of \(Q\) such that \(\|A - \bar{A}\| \leq \varepsilon\). Furthermore, every \((m-1)\)-dimensional subpolytope of \(P_k\) can be expressed as a disjoint finite union \(\cup \Lambda_j\), where each \(\Lambda_j\) is the convex hull of \(m-1\) vertices \(A_1, \ldots, A_{m-1}\) of \(P_k\). Suppose \(A_1, \ldots, A_q \in Q\) and \(A_{q+1}, \ldots, A_{m-1} \in \overline{Q}\); let
\( \Omega = \text{conv}(A_1, \ldots, A_q, \tilde{A}_q, \ldots, \tilde{A}_{m-1}) \), where each \( \tilde{A}_i \) is a vertex of \( \Omega \) satisfying \( A_i - \tilde{A}_i < \varepsilon \), and let \( \Gamma = (A_{q-1}, \ldots, A_{m-1}) \). From Lemma 4.1, every \( B \in \text{conv}(A_1, \ldots, A_{m-1}) \subseteq \text{conv}(\Omega \cup \Gamma) \) satisfies \( d(B, \Omega) < \varepsilon \). Hence, for sufficiently small \( \varepsilon \), each \((m-1)\)-dimensional subpolytope of \( \mathcal{P}_K \) is stable.

5. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [2] can be extended to the case of \( n \times n \) matrices. We have shown that, without further information describing the particular structure of a polytope, \((2n-4)\)-dimensional faces must be checked for stability. Since testing even one such face can be a formidable task when \( n \) is large, and since the number of \((2n-4)\)-dimensional faces grows exponentially with \( n \), more work needs to be done before a computationally tractable algorithm can be devised for checking stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.
REFERENCES


THE MINIMAL DIMENSION OF STABLE FACES REQUIRED TO GUARANTEE
STABILITY OF A MATRIX POLYTOPE: D-STABILITY

J. Daniel Cobb
Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

We consider the problem of determining whether a polytope \( P \) of \( \mathbb{R}^n \) matrices is D-stable -- i.e., whether each point in \( P \) has all its eigenvalues in a given nonempty, open, convex, conjugate-symmetric subset \( D \) of the complex plane. Our approach is to check D-stability of certain faces of \( P \). In particular, for each \( D \) and \( n \) we determine the smallest integer \( m \) such that \( D \)-stability of every \( m \)-dimensional face guarantees \( D \)-stability of \( P \).

1. Introduction

Let \( D \subset \mathbb{C} \) be nonempty, open, convex, and conjugate-symmetric (symmetric about the real axis), and define an \( n \times n \) real matrix \( M \) to be D-stable if each eigenvalue \( \lambda \) of \( M \) satisfies \( \lambda \in \mathbb{D} \); otherwise, \( M \) is D-unstable. We consider the problem of determining whether certain subsets of \( \mathbb{R}^{n \times n} \) consist entirely of D-stable matrices. To facilitate discussion we begin with some definitions.

A (convex) polytope \( \mathcal{P} \) in a vector space \( V \) is the convex hull \( \text{conv}(\mathcal{P}) \) of any nonempty finite subset \( \mathcal{P} \subset V \). The dimension of \( \mathcal{P} \) is the dimension of the affine hull \( \text{aff}(\mathcal{P}) \) of \( \mathcal{P} \). A face of \( \mathcal{P} \) is any set of the form \( \mathcal{P} \cap \Pi \), where \( \Pi \) is a supporting hyperplane of \( \mathcal{P} \). Finally, a \( k \)-dimensional half-plane in \( V \) is any nonempty set of the form \( \Pi \cap \mathcal{S} \), where \( \Pi \) is a closed half-space, \( S \) is a \( k \)-dimensional affine subspace, and \( \Pi \cap \mathcal{S} \subset S \). (Note that this implies that \( \text{aff}(\Pi) \) is simply \( S \).

In the robust control literature, considerable interest has been generated by the problem of determining whether a family of linear systems can be shown to consist entirely of D-stable systems by checking D-stability of certain representative members of that family. In many cases, such problems can be reduced to that of determining whether a polytope or other subset of \( \mathbb{R}^n \) or \( \mathbb{R}^{n \times n} \) consists entirely of D-stable points \([1,2]\). (D-stability of a vector \( x \in \mathbb{R}^n \) means simply that the polynomial \( s^n - x_1 s^{n-1} - \cdots - x_n \) has all its roots in \( D \).) We are primarily interested in the technique of checking D-stability of lower dimensional faces of a polytope in order to guarantee D-stability of the entire set.

Most "facial" results pertain to continuous-time (CT) stability -- i.e., where \( D \) is the open left half complex plane. The seminal result \([3]\) for polynomial polytopes motivates the approach. In \([3]\) it is shown that a polynomial polytope of a particular simple structure (an "interval polynomial") is CT stable whenever four specially constructed vertices are CT stable. A more recent result \([1]\) demonstrates that, for an arbitrary polynomial polytope, checking all edges is sufficient to guarantee CT stability. With respect to polytopes in \( \mathbb{R}^{n \times n} \), it has been shown \([4]\) that 1) an arbitrary polytope is CT stable if all \((2n-4)\)-dimensional faces are CT stable and 2) there exist CT unstable polytopes such that all \((2n-5)\)-dimensional faces are CT stable; hence, the value \( 2n-4 \) is minimal. In this paper we extend the results of \([4]\) to D-stability where \( D \) may be any nonempty, open, convex, conjugate-symmetric subset \( D \) of \( \mathbb{C} \).

We note that for the cases \( n=0 \) and \( n=1 \) our problem has a trivial solution: D-stability of vertices guarantees D-stability of the polytope. To handle \( n \geq 2 \) we need to partition the family of stability sets \( D \) according to the following two assumptions.

Assumption A. \( D \) is of the form \( D=\{s|\text{Re} s>a, \text{Im} s<b\} \), where \(-a\leq \text{Re} s\leq a, \text{Im} s\leq b\).

Assumption B. \( D \) is a nonempty, open, convex, conjugate-symmetric set not satisfying Assumption A.

In addition, we define \( m_1(n)=\begin{cases} 2n-4, & n \geq 2 \\
2n-2, & n=2 \end{cases} \) and \( m_2(n)=2n-2 \). We intend to show that \( m_1 \) and \( m_2 \) are the values of \( m \) that we seek for cases A and B.

2. Sufficiency of \( m_1 \) and \( m_2 \)

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map \( f: \mathbb{R}^k \rightarrow \mathbb{R}^2 \) determines an affine isomorphism between \( \mathbb{R}^k \) and \( f(\mathbb{R}^k) \). Among other things, this implies that, for any polytope \( \mathcal{P} \subset \mathbb{R}^k \), \( f(\mathcal{P}) \) is also a polytope of the same dimension as \( \mathcal{P} \); furthermore, \( f \) sets up a one-to-one correspondence between the \( q \)-dimensional faces of \( \mathcal{P} \) and the \( q \)-dimensional faces of \( f(\mathcal{P}) \). In addition, \( f \) maps each \( k \)-dimensional half-plane in \( \mathbb{R}^k \) into another \( k \)-dimensional half-plane in the form \( \{x|\forall x \in \mathbb{R}^k, \exists y \in \mathbb{R} \} \).

Finally, we note that every polytope is compact and that any set of the form \( \{(x,y)|x \in \mathbb{R}^k, y \in \mathbb{R}^l\} \) is a polytope whose \( q \)-dimensional faces are generated by fixing \( k-q \) entries of \( x \) at either \( 0 \) or \( 1 \) and letting the remaining \( q \) entries vary independently over \( [-r,r] \).

With these observations in mind, we prove a result characterizing the affine structure of the set of D-unstable points in \( \mathbb{R}^{n \times n} \).

**Lemma 2.1** 1) If \( D \) satisfies Assumption A then for each D-unstable \( \mathcal{M} \subset \mathbb{R}^{n \times n} \) there exists an \((n^2-m_A)\)-dimensional half-plane \( \mathcal{M} \subset \mathbb{R}^{n \times n} \) such that \( a \in \mathcal{M} \) and \( b \in \mathcal{M} \) implies \( x \in \mathcal{M} \) is D-unstable.

2) If \( D \) satisfies Assumption B, then for each D-unstable \( \mathcal{M} \subset \mathbb{R}^{n \times n} \) there exists an \((n^2-m_B)\)-dimensional half-plane \( \mathcal{M} \subset \mathbb{R}^{n \times n} \) such that \( a \in \mathcal{M} \) and \( b \in \mathcal{M} \) implies \( x \in \mathcal{M} \) is D-unstable.

---

This work was supported in part by NSF Grant No. ECS-8612948 and by AFOSR Grant No. AFOSR-86-0087.
For every 

there exists a polytope \( \mathcal{P} \) containing an unstable point and such that all \((2n-5)\)-dimensional subpolytopes of \( \mathcal{P} \) are stable. Hence, we conclude that checking stability of \( k \)-dimensional subpolytopes of \( \mathcal{P} \) for any \( k < 2n-4 \) is, in general, not sufficient to guarantee stability of \( \mathcal{P} \).

Consider the polytope

\[
\mathcal{P} = \begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -1 \end{bmatrix} A(s) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leq 1
\]

A routine calculation shows that \( \mathcal{P} \) has characteristic polynomial \( p(s) = (s+1)^{n-4} \Delta(s) \), where

\[
\Delta(s) = s^4 - 2s^2 + (2x^T + y^T) s - (2x^T - y^T) s^3 - (x^T y - y^T x)^2
\]

From the Schwartz inequality, it is clear that all coefficients are strictly positive. The corresponding 4x4 Hurwitz matrix has its leading principal 3x3 minor equal to

\[
M_3(x,y) = 4x^T + 4y^T - 4(x^T y)^2 - 4(x^T y)^2 - 4(x^T y)^2
\]

Clearly, \( M_3 \leq 0 \) with equality iff \( x^T y = 0 \). Thus, \( \mathcal{P} \) consists entirely of stable points, except for the relative interior point corresponding to \( x^T y = 0 \). We conclude that checking \((2n-5)\)-dimensional faces (in this case the entire boundary of \( \mathcal{P} \)) is insufficient to guarantee stability.

Comments

1) The preceding example can be strengthened by adding \( sI \) to \( \mathcal{P} \), where \( s \) is sufficiently small. This yields a polytope with stable \((2n-5)\)-dimensional boundary, but containing a ball of strictly unstable points.

2) Since the union of all \((2n-5)\)-dimensional subpolytopes is nowhere dense, shifting the parameter set \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) by an arbitrarily small vector yields an unstable polytope with all \((2n-5)\)-dimensional subpolytopes stable.

3) The polytope \( \mathcal{P} \) described above can be transformed into a similar example with any given dimension either by removing parameters or by using \( \mathcal{P} \) as a face of a higher dimensional polytope.

Note that the constructions described in 1), 2), and 3) can be carried out simultaneously to given a stronger but algebraically messier version of the minimality proof offered above.
THE MINIMAL DIMENSION OF STABLE FACES REQUIRED TO GUARANTEE
STABILITY OF A MATRIX POLYTOPE: D-STABILITY1

J. Daniel Cobb

Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

We consider the problem of determining whether a polytope $\mathcal{P}$ of $n \times n$ matrices is D-stable -- i.e. whether each point in $\mathcal{P}$ has all its eigenvalues in a given nonempty, open, convex, conjugate-symmetric subset $D$ of the complex plane. Our approach is to check D-stability of certain faces of $\mathcal{P}$. In particular, for each $D$ and $n$ we determine the smallest integer $m$ such that D-stability of every $m$-dimensional face guarantees D-stability of $\mathcal{P}$.

1This work was supported in part by NSF Grant No. ECS-8612948 and by AFOSR Grant No. AFOSR-88-0087.
Let $\Omega \subseteq \mathbb{C}$ be nonempty, open, convex, and conjugate-symmetric (symmetric about the real axis), and define an $n \times n$ real matrix $M$ to be $D$-stable if each eigenvalue $\lambda$ of $M$ satisfies $\lambda \in \mathbb{D}$; otherwise, $M$ is $D$-unstable. We consider the problem of determining whether certain subsets of $\mathbb{R}^{n\times n}$ consist entirely of $D$-stable matrices. To facilitate discussion we begin with some definitions.

A (convex) polytope $\mathcal{P}$ in a vector space $V$ is the convex hull $\text{conv}(\mathcal{Q})$ of any nonempty finite subset $\mathcal{Q} \subseteq V$. The dimension of $\mathcal{P}$ is the dimension of the affine hull $\text{aff}(\mathcal{P})$ of $\mathcal{P}$. The relative boundary of $\mathcal{P}$ is the boundary of $\mathcal{P}$ as a subset of the topological space $\text{aff}(\mathcal{P})$. A face of $\mathcal{P}$ is any set of the form $\mathcal{M} \cap \mathcal{P}$, where $\mathcal{M}$ is a supporting hyperplane of $\mathcal{P}$. Finally, a $k$-dimensional half-plane in $V$ is any nonempty set of the form $\mathcal{M} = \mathbb{R} \cap S$, where $\mathbb{R}$ is a closed half-space, $S$ is a $k$-dimensional affine subspace, and $S \subseteq \mathcal{M}$. (Note that this implies that $\text{aff}(\mathcal{M})$ is simply $S$.)

In the robust control literature, considerable interest has been generated by the problem of determining whether a family of linear systems can be shown to consist entirely of $D$-stable systems by checking $D$-stability of certain representative members of that family. In many cases, such problems can be reduced to that of determining whether a polytope or other subset of $\mathbb{R}^n$ or $\mathbb{R}^{n \times n}$ consists entirely of $D$-stable points [1],[2]. (D-stability of a vector $x \in \mathbb{R}^n$ means simply that the polynomial $s^n + x_n s^{n-1} + \ldots + x_1$ has all its roots in $\mathbb{D}$, where $x_i$ is the $i$th entry of $x$.) We are primarily interested in the technique of checking $D$-stability of lower dimensional faces of a polytope in order to guarantee $D$-stability of the entire set.

Most "facial" results pertain to continuous-time (CT) stability -- i.e. where $\mathbb{D}$ is the open left half complex plane. The seminal result [3] for polynomial polytopes motivates the approach. In [3] it is shown that a
polynomial polytope, of a particular simple structure (an "interval polynomial") is CT stable whenever four specially constructed vertices are CT stable. A more recent result [1] demonstrates that, for an arbitrary polynomial polytope, checking all edges is sufficient to guarantee CT stability. With respect to polytopes in \( \mathbb{R}^{n \times n} \), it has been shown [4] that 1) an arbitrary polytope is CT stable if all \((2n-4)\)-dimensional faces are CT stable and 2) there exist CT unstable polytopes such that all \((2n-5)\)-dimensional faces are CT stable; hence, the value \(2n-4\) is minimal. In this paper we extend the results of [4] to D-stability where D may be any nonempty, open, convex, conjugate-symmetric subset of \( \mathbb{C} \).

We note that for the cases \( n=0 \) and \( n=1 \) our problem has a trivial solution: D-stability of vertices guarantees D-stability of the polytope. To handle \( n \geq 2 \) we need to partition the family of stability sets D according to the following two assumptions.

**Assumption A:** D is of the form \( D=(s \in \mathbb{C} | a < Re \, s < b) \), where \( -\infty < a < b < \infty \).

**Assumption B:** D is a nonempty, open, convex, conjugate-symmetric set not satisfying Assumption A.

In addition, we define \( m_A(n)=\begin{cases} 1, & n=2 \\ 2n-4, & n>2 \end{cases} \) and \( m_B(n)=2n-2 \). We intend to show that \( m_A \) and \( m_B \) are the values of m that we seek for cases A and B.

### 2. Sufficiency of \( m_A \) and \( m_B \)

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map \( f: \mathbb{R}^k \rightarrow \mathbb{R}^n \) determines an affine isomorphism between \( \mathbb{R}^k \) and \( f(\mathbb{R}^k) \). Among other things, this implies that, for any polytope \( \mathcal{P} \subset \mathbb{R}^k \), \( f(\mathcal{P}) \) is also a polytope of the same dimension as \( \mathcal{P} \); furthermore, \( f \) sets up a
one-to-one correspondence between the $q$-dimensional faces of $\mathcal{P}$ and the $q$-dimensional faces of $f(\mathcal{P})$. In addition, $f$ maps each $k$-dimensional half-plane in $\mathbb{R}^k$ into another $k$-dimensional half-plane (e.g., see [5]). Finally, we note that every polytope is compact and that any set of the form \( \{x \in \mathbb{R}^k \mid \|x\| \leq \gamma \} \), where $\gamma > 0$, is a polytope whose $q$-dimensional faces are generated by fixing $k-q$ entries of $x$ at either $\pm \gamma$ and letting the remaining $q$ entries vary independently over $[-\gamma, \gamma]$.

With these observations in mind, we prove a result characterizing the affine structure of the set of $D$-unstable points in $\mathbb{R}^{n \times n}$.

**Lemma 2.1** If $D$ satisfies Assumption A (respectively, Assumption B), then for each $D$-unstable $M \in \mathbb{R}^{n \times n}$ there exists an $(n^2 - m_A)$-dimensional (respectively, $(n^2 - m_B)$-dimensional) half-plane $\mathcal{H} \subset \mathbb{R}^{n \times n}$ such that a) $M \in \mathcal{H}$ and b) $N \in \mathcal{H}$ implies $N$ is $D$-unstable.

**Proof** Suppose Assumption A holds. If $a = -\infty$, $b = +\infty$, the statement is vacuously true; otherwise, we need to consider two cases.

**Case I** -- $M$ has a real eigenvalue $\lambda_0 \notin \mathbb{D}$. Let $T = [v \ W]$, where $v$ is an eigenvector corresponding to $\lambda_0$ and $W$ is chosen to make $T$ nonsingular. Clearly, the map $f: \mathbb{R}^{n^2-n+1} \to \mathbb{R}^{n \times n}$ determined by $f(\lambda, y, Z) = T \begin{bmatrix} \lambda & \gamma \\ 0 & 0 \end{bmatrix} T^{-1}$ is affine and one-to-one. If $\lambda_0 < a$, let $I = [-\infty, \lambda_0]$ and let $\mathcal{H}$ be the $(n^2-n+1)$-dimensional half-plane $\mathcal{H} = \{ f(\lambda, y, Z) \mid \lambda \in I, y \in \mathbb{R}^{1 \times n-1}, Z \in \mathbb{R}^{n-1 \times n-1} \}$. If $a = -\infty$, then $\lambda_0 > b$ so we set $I = (\lambda_0, +\infty)$ and construct $\mathcal{H}$ in the same way. In either case, $M \in \mathcal{H}$ and every matrix in $\mathcal{H}$ is $D$-unstable. Since $n^2-n+1 > n^2-m_A$, it remains to select any $(n^2 - m_A)$-dimensional half-plane $\mathcal{H}$ satisfying $M \in \mathcal{H}$.

**Case II** -- $M$ has a complex eigenvalue pair $\alpha_0 \pm i\beta_0$ with $\alpha_0 > 0$. Let $T = [u \ v \ W]$, where $u+iv$ is an eigenvector corresponding to $\alpha_0 + i\beta_0$ and $W$ is chosen to make $T$ nonsingular. If $\alpha_0 < a$, let $\mathcal{H}$ be the $(n^2 - 2n + 4)$-dimensional half-plane $\mathcal{H} = \{ f(\lambda, y, Z) \mid \lambda \in I, y \in \mathbb{R}^{1 \times n-1}, Z \in \mathbb{R}^{n-1 \times n-1} \}$.
half plane \( \bar{H} = \{ X \in \mathbb{R}^n \mid \text{tr } X \leq 2\alpha_0 \} \). (\text{tr } X \leq 2\alpha_0 \) describes a 4-dimensional half-plane, since \( \text{tr } X = \langle X, I \rangle \). \( \bar{H} \) contains only D-unstable points, since \( \text{tr } X \leq 2\alpha_0 \) implies \( X \) has at least one eigenvalue \( \lambda \) with \( \Re \lambda \leq \alpha_0 \). If \( \alpha_0 > b \), \( \bar{H} \) is defined by \( \text{tr } X \geq 2\alpha_0 \), and the same reasoning holds. In either case, \( M \in \bar{H} \).

Now suppose Assumption B holds. We again consider two cases. Case I -- \( X \) has a real eigenvalue \( \lambda_0 \notin D \): Again let \( T = [v \ w] \), where \( v \) is an eigenvector corresponding to \( \lambda_0 \). Since \( D \) is convex, either \((-\infty, \lambda_0) \cap D = \emptyset\) or \((\lambda_0, \infty) \cap D = \emptyset\). In the former case, let \( \bar{H} \) be the \((n^2-n+1)\)-dimensional half-plane \( \bar{H} = \{ X \in \mathbb{R}^n \mid \text{tr } X \leq 2\lambda_0 \} \). For the latter case, alter the definition of \( \bar{H} \) by substituting \( \langle \lambda, \lambda_0 \rangle \) for \( \langle \lambda, \lambda_0 \rangle \). In either case, \( M \in \bar{H} \) and every matrix in \( \bar{H} \) is D-unstable. Since \( n^2-n+1 > n^2 - m_B \), it remains to select any \((n^2-m_B)\)-dimensional half-plane \( \bar{H} \) satisfying \( M \in \bar{H} \).

Case II -- \( M \) has a complex eigenvalue pair \( \alpha_0 \pm i\beta_0 \notin D \): Let \( T = [u \ v \ w] \), where \( u + iv \) is an eigenvector corresponding to \( \alpha_0 \pm i\beta_0 \). Since \( D \) is convex, there exists a half-space \( M \in \mathbb{C} \) such that \( \alpha_0 + i\beta_0 \notin M \) and \( M \cap D = \emptyset \). Let \( \bar{H} \) be the \((n^2-2n+2)\)-dimensional half-plane

\[
\bar{H} = \{ X \in \mathbb{R}^n \mid \text{tr } X \leq 2\alpha_0 \} \subset \mathbb{R}^{n^2}.
\]

Clearly, \( \bar{H} \) contains only D-unstable points, and \( M \in \bar{H} \).

Next we prove an easy result concerning the intersection of affine sets.

**Lemma 2.2** Let \( V \) be a \( p \)-dimensional Euclidean space, \( \mathcal{H} \subset V \) a \( k \)-dimensional half-plane, and \( \Gamma \) a \( q \)-dimensional affine subspace with \( k + q > p \). Consider any vector \( x_0 \in \mathcal{H} \cap \Gamma \). There exists a \((k + q - p)\)-dimensional half-plane \( \bar{H} \) such that \( x_0 \in \bar{H} \cap \Gamma \).

-4-
Proof. By definition, \( H \cap S \), where \( R \) is a closed half-space and \( S \) is a \( k \)-dimensional affine subspace satisfying \( S \cap R \). There exists an affine subspace \( \tilde{S} \subseteq S \cap R \) with \( \dim \tilde{S} = k+q-p \) and \( x_0 \in \tilde{S} \). If \( S \cap R \), let \( \tilde{H} \cap \tilde{S} \) be any \((k+q-p)\)-dimensional half-space containing \( x_0 \). Then \( \tilde{H} \cap \tilde{S} \subseteq \tilde{S} \cap R \). If \( S \cap R \), let \( \tilde{H} \cap \tilde{S} \). Then \( x_0 \in \tilde{H} \) since \( x_0 \in \tilde{H} \cap \tilde{S} \cap R \). Also, \( \dim \tilde{H} = k+q-p \), since \( \tilde{H} \) is nonempty.

We are now in a position to prove our first main result.

**Theorem 2.3** Under Assumption A (respectively, Assumption B), D-stability of every matrix in every \( m_A \)-dimensional (respectively, \( m_B \)-dimensional) face of \( \mathcal{P} \) guarantees D-stability of every matrix in \( \mathcal{P} \).

**Proof.** Suppose Assumption A holds. Our arguments here are similar to those used in [2,Lemma 1]. If \( P_k \) is a D-unstable polytope of dimension \( k > m_A \), there exists a D-unstable matrix \( M \in P_k \). From Lemma 2.1, there is an \((n^2-m_A)\)-dimensional half plane \( H_1 \), consisting entirely of D-unstable points and containing \( M \). Since \( H_1 \) is unbounded, there exists an \( M_2 \in H_1 \) lying on the boundary of \( P_k \) and, hence, in one of its \((k-1)\)-dimensional faces \( P_{k-1} \). From Lemma 2.2, the intersection \( H_1 \cap \text{aff}(P_{k-1}) \) contains a \((k-m_A-1)\)-dimensional half-plane \( H_2 \) such that \( M_2 \in H_2 \). Proceeding inductively, we find that there exists an \( m_A \)-dimensional face \( P_m \) and a point \( M_{k-m} \in P_m \) such that \( M_{k-m} \) is D-unstable.

Under Assumption B, the same proof holds if we replace \( m_A \) by \( m_B \).

3. Minimality of \( m_A \) and \( m_B \)

Our next task is to show that \( m_A \) and \( m_B \) are the smallest integers such that D-stability of all \( m_A \)-dimensional or \( m_B \)-dimensional faces of \( \mathcal{P} \) guarantees D-stability of \( \mathcal{P} \) under Assumptions A or B, respectively. In order to prove
this we need a lemma which may be interpreted as a multivariable extension of L'Hospital's rule. For any \( k \times k \) matrices \( Q \) and \( R \), we use the notation \( Q > 0 \) and \( R < 0 \) to signify that \( Q \) is positive definite and \( R \) is negative definite, respectively.

**Lemma 3.1** Let \( 0 \in U \subset \mathbb{R}^k \) with \( U \) open, and let \( e_1, e_2 : U \to \mathbb{R}^2 \) be \( C^2 \) functions. In addition, suppose \( e_1(0) = e_2(0) = 0 \),

\[
\begin{align*}
\frac{\partial^2 e_1}{\partial x^2} |_{x=0} &= 0, \\
\frac{\partial^2 e_2}{\partial x^2} |_{x=0} &= 0
\end{align*}
\]

For every \( \delta > 0 \) there exists an \( \varepsilon > 0 \) such that \( \|x\| < \varepsilon \) implies \( e_2(x) < -\frac{1}{\delta} |e_1(x)| \).

**Proof** From [6, p.340], for every \( Q > 0 \) there exists an \( \varepsilon > 0 \) such that \( \|x\| < \varepsilon \) implies

\[
\frac{|e_1(x) - e_1(0) - \frac{\partial e_1}{\partial x} x - \frac{1}{2} x^T \partial^2 e_1}{x^T Q x} < \frac{1}{2(1+\delta)}
\]

Setting \( Q = -\frac{\partial^2 e_2}{\partial x^2} |_{x=0} \) yields

\[
\left| e_1(x) \right| < \delta, \quad \left| e_2(x) + \frac{1}{2} x^T Q x \right| < \delta
\]

and, from (1), \( e_2(x) < (\delta - \frac{1}{2}) x^T Q x < 0 \) for \( x \neq 0 \). Hence, for \( x \neq 0 \),

\[
\frac{|e_1(x)|}{|e_2(x)|} = \frac{|e_1(x)|}{|e_2(x) + \frac{1}{2} x^T Q x|} = \frac{|e_1(x)|}{|e_2(x) + \frac{1}{2} x^T Q x - \frac{1}{2}|} < \frac{1}{2 - \frac{1}{2} \left( \frac{\delta}{1+\delta} \right)} = \delta
\]

Thus, \( e_2(x) < -\frac{1}{\delta} |e_1(x)| \).

Now we can prove our second main result.
Theorem 3.2 Suppose $D$ satisfies Assumption A (respectively, Assumption B). For each $n$ there exists an $m_A$-dimensional (respectively, $m_B$-dimensional) polytope $P \subset \mathbb{R}^{n \times n}$ containing a $D$-unstable point and such that all $(m_A-1)$-dimensional (respectively, $(m_B-1)$-dimensional) faces of $P$ are $D$-stable.

Proof Under Assumption A we need to consider nine cases.

Case I -- $a>\infty, b<\infty, n=2$: Consider the affine, one-to-one map

$$f(x) = \begin{bmatrix} b & x \\ -x & \frac{a+b}{2} \end{bmatrix}$$

and the corresponding one-dimensional polytope $P = \{ f(x) | x \leq 1 \}$. The point in $P$ corresponding to $x=0$ is clearly $D$-unstable. It suffices to prove that the characteristic polynomials $\Delta^+$ and $\Delta^-$ of $f(x)-bl$ and $aI-f(x)$, respectively, are Hurwitz for all $x \neq 0$. This is in fact true, since $\Delta^+(s) = s^2 + \frac{b-a}{2}s + x^2$ and $\Delta^-(s) = s^2 + \frac{b-a}{2}s + (b-a)^2 + x^2$ have positive coefficients for $x \neq 0$.

Case II -- $a>\infty, b<\infty, n=3$: Let

$$f(x,y) = \begin{bmatrix} b & 1 & -x \\ -1 & b & -y \\ x & y & \frac{a+b}{2} \end{bmatrix}$$

It is sufficient to show that the characteristic polynomials $\Delta^+$ and $\Delta^-$ of $f(x,y)-bI$ and $aI-f(x,y)$, respectively, are Hurwitz for $[x, y] \neq 0$. A straightforward calculation yields $\Delta^+(s) = s^2 + \frac{b-a}{2}s + x^2 + y^2 + (1+2x^2+y^2)s + \frac{b-a}{2}$ and $\Delta^-(s) = s^2 + \frac{b-a}{2}s + (1+2x^2+y^2 + (b-a)^2)s + \frac{b-a}{2}(1+2x^2+y^2 + (b-a)^2)$. Each polynomial has positive coefficients for $[x, y] \neq 0$. The fact that they are Hurwitz follows from positivity of the second-order leading principal minors $M_2^+ = \frac{b-a}{2}(x^2+y^2)$ and $M_2^- = \frac{b-a}{2}(4+3x^2+3y^2+9(b-a)^2)$ of the corresponding $3 \times 3$ Hurwitz matrices.

Case III -- $b>\infty, a<\infty, n>3$: Let $f(x,y) = \begin{bmatrix} b & 1 & -x^T \\ -1 & b & -y^T \\ x & y & (a+b)^T \end{bmatrix}$ where $x,y \in \mathbb{R}^{n-2}$. A tedious calculation shows that $\Delta^+(s) = (s^2 + \frac{b-a}{2})^{n-4}\Delta^-(s)$ and
\[ \Delta(s) = (s + \frac{b-a}{2})^n \hat{\Delta}(s), \]

where

\[ \hat{\Delta}^+(s) = s^4 + (b-a)s^3 + (1+\frac{x^T x + y^T y}{2})s^2 + \frac{b-a}{2}(2+\frac{x^T x + y^T y}{2})s + x^T x y - (x^T y)^2 - \frac{1}{3}(b-a)^2 \]

\[ \hat{\Delta}^-(s) = s^4 + 3(b-a)s^3 + (1+\frac{x^T x + y^T y}{2})s^2 + (b-a)(1+\frac{3}{2}x^T x + \frac{3}{2}y^T y + \frac{3}{2}(b-a)^2)s 
+ x^T x y - (x^T y)^2 - (b-a)^2 \]

From the Schwartz inequality, \( \hat{\Delta}^+ \) and \( \hat{\Delta}^- \) have positive coefficients when \( \begin{vmatrix} x \\ y \end{vmatrix} = 0 \). Furthermore, the third-order leading principal minors of the corresponding 4x4 Hurwitz matrices of \( \hat{\Delta}^+ \) and \( \hat{\Delta}^- \) are

\[ M_3^+(s) = \left( \frac{b-a}{2} \right)^2 \left( 2 + \left( \frac{b-a}{2} \right)^2 \right) (x^T x + y^T y) + (x^T x - y^T y)^2 + 4(x^T y)^2 \]

\[ M_3^-(s) = \left( 2 + \frac{3}{2}x^T x + \frac{3}{2}y^T y - \frac{3}{4}(x^T x - y^T y)^2 + (x^T y)^2 \right) (b-a)^2 
+ \left( \frac{27}{4}x^T x + \frac{63}{8}y^T y \right) (b-a)^4 + \frac{81}{8} (b-a)^6 \]

Since \( M_3^+ \) and \( M_3^- \) are positive, \( \hat{\Delta}^+ \) and \( \hat{\Delta}^- \) and, hence, \( \Delta^+ \) and \( \Delta^- \) are Hurwitz.

The remaining six cases are handled similarly by choosing all eigenvalues in the interior of \( D \), except for one or two on the boundary of \( D \). For example, for \( a > -\infty \), \( b = \infty \), \( n > 3 \), set

\[ f(x,y) = \begin{bmatrix} a & 1 & -x^T \\ -1 & a & -y^T \\ x & y & (a-1)I \end{bmatrix} \]

Adopting Assumption B, suppose \( D \) is not of the form \{ \begin{vmatrix} s \end{vmatrix} \begin{vmatrix} a \Re s < b \end{vmatrix} \}. Since \( D \) is convex, there exists a real \( \alpha_0 \in \mathbb{R} \) such that the line \( L = \{ \alpha_0 + i\beta \} | \beta \in \mathbb{R} \) satisfies \( L \notin D \). Since \( D \) is conjugate symmetric and open, there exists a \( \beta_0 > 0 \) such that \( \alpha_0 + i\beta_0 \) are boundary points of \( D \), but \( \alpha_0 + i\beta \notin D \) when \( \beta > \beta_0 \). Furthermore, there
exists a $\delta > 0$ such that $a_0 \pm \delta d_0$. Again invoking convexity, the open diamond $d_\delta = \text{int conv}(a_0 \pm \delta a_0, a_0 \pm \delta b_0)$ is contained in $D$. To simplify the problem, consider the open diamond $d_\delta = \frac{1}{\beta_0}(d_\delta - a_0) = \text{int conv}(\pm 1, \pm \delta)$. We need only construct a single polytope $P$ containing a matrix with a pair of eigenvalues at $\pm 1$ and with all $m_B$-dimensional faces consisting of matrices with all eigenvalues in $d_\delta$; then $\beta_0 P - a_0 I$ would then satisfy the desired properties with respect to $d_\delta$.

Consider the $(n^2 - 2n + 2)$-dimensional polytope

$$
P_\varepsilon = \left\{ \begin{bmatrix} w & 1+x & y^T \\ -1+x & -w & z^T \\ y & z & 0 \end{bmatrix} \left| \begin{bmatrix} w \\ x \\ y \end{bmatrix} \leq \varepsilon \right. \right\}
$$

where $y, z \in \mathbb{R}^{n-2}$. Clearly, $P_\varepsilon$ has a D-unstable point $M$ at $w = x = 0, y = z = 0$. We will show that for sufficiently small $\varepsilon$, every point in $P_\varepsilon$ except $M$ is D-stable. Hence, $P = P_\varepsilon$ satisfies the desired properties.

Case I --- $n=2$: Each point in $P_\varepsilon$ has characteristic polynomial

$$
\Delta(w, x, s) = s^2 + (1-w^2 - x^2) s - (w(y^2 - z^2) + 2xyz)
$$

Let $g(w, x, y, z, a, \beta) = [\text{Re } \Delta(w, x, y, z, a + i\beta)]$. It is easy to see that $g$ is a polynomial function and, hence, analytic. A straightforward calculation shows

$$
\frac{\partial \Delta}{\partial (a, \beta)} \left| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right. = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}
$$
Thus, from the implicit function theorem, there exists a unique analytic function \( h: U \to \mathbb{R}^2 \) such that \( h(0) = [0] \) and \( g(w, x, y, z, h(w, x, y, z)) = 0 \) for every \( [w x y z]^T \in U \).

Next, let \( \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = h(0) \). A tedious computation shows

\[
\begin{bmatrix}
\frac{\partial e_1}{\partial (w, x, y, z)} \\
\frac{\partial e_2}{\partial (w, x, y, z)} \\
\frac{\partial^2 e_1}{\partial (w, x, y, z)^2} \\
\frac{\partial^2 e_2}{\partial (w, x, y, z)^2}
\end{bmatrix} = 0,
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

From Lemma 3.1, there exists an \( \varepsilon > 0 \) such that \( e_2(w, x, y, z) < \frac{1}{2} e_1(w, x, y, z) \) whenever \( 0 \neq \| [w x y z]^T \| < \varepsilon \). Since \( e_1 \) and \( e_2 \) are continuous, we may also assume \( |e_1| < 1; i = 1, 2 \). Returning to \( h = [h_1] \), it follows that \( h_2(w, x, y, z) < \frac{1}{\varepsilon} |h_1(w, x, y, z)|, |h_2(w, x, y, z)| < 1, \) and \( |h_1(w, x, y, z) - 1| < 1 \) for all \( [w x y z]^T \neq 0 \). Hence, \( \text{hed}_\varepsilon \).

Case III -- \( n > 3 \): We have

\[
\Delta(w, x, y, z, s) = s^4 + (1-w^2-x^2-y^2)z^2-(wy^2-2xy^2z-wz^2)z+y^2z^2-(y^2z)^2
\]

Let \( g(w, x, y, z, \alpha, \beta) = \begin{bmatrix} \text{Re} \Delta(w, x, y, z, \alpha + i\beta) \\ \text{Im} \Delta(w, x, y, z, \alpha + i\beta) \end{bmatrix} \). Again, \( g \) is a polynomial function; in this case

\[
\begin{bmatrix}
\frac{\partial g}{\partial (\alpha, \beta)}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Thus there exists an open \( U \subset \mathbb{R}^{2n-2} \) with \( 0 \in U \) and \( h: U \to \mathbb{R}^2 \) such that \( h(0) = [0] \) and \( g(w, x, y, z, h(w, x, y, z)) = 0 \) for every \( [w x y z]^T \in U \). Let \( \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = h(0) \). Then

\[
\begin{bmatrix}
\frac{\partial e_1}{\partial (w, x, y, z)} \\
\frac{\partial e_2}{\partial (w, x, y, z)} \\
\frac{\partial^2 e_1}{\partial (w, x, y, z)^2} \\
\frac{\partial^2 e_2}{\partial (w, x, y, z)^2}
\end{bmatrix} = 0,
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

-10-
Applying Lemma 3.1 as in Case II, it follows that \( h(w, x, y, z) \in U \) for every 
\[ [w x y z]^T \neq 0. \]

Note that Theorem 3.2 also implies that the half-planes constructed in Lemma 2.1 are maximal in the sense that there exists a D-unstable matrix \( M \) in \( \mathbb{R}^{n \times n} \) such that every half-plane containing \( M \) of dimension greater than \( n^2 - m_A \) or \( n^2 - m_B \) must also contain a D-unstable matrix. Indeed, if this were not the case, the arguments in Theorem 3.2 could be used to prove that \( m_A \) and \( m_B \) are not minimal.

4. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [1] and [3] can be extended to the case of \( n \times n \) matrices. We have shown that, without further information describing the particular structure of a polytope, either \((2n-4)\)-dimensional or \((2n-2)\)-dimensional faces need to be checked for D-stability, depending on the structure of \( D \). Since testing even one such face can be a formidable task when \( n \) is large, and since the number of \((2n-4)\)-dimensional and \((2n-2)\)-dimensional faces grow exponentially with \( n \), more work needs to be done before a computationally tractable algorithm can be devised for checking D-stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.

ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor C. L. DeMarco for his valuable suggestions during the course of this research.
REFERENCES


A REALIZATION THEORY FOR PERTURBED LINEAR SYSTEMS¹

Mingde Tan
J. Daniel Cobb

Department of Electrical and Computer Engineering
University of Wisconsin
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

In this paper we present a theory which characterizes LTI state-space realizations of perturbed rational transfer function matrices. Our approach is to model system perturbations as sequences in the space of rational matrices. First, we give a definition of convergence in the space of rational matrices which is motivated by the kinds of parameter uncertainties occurring in many robust control problems. A realization theory is then established under the constraint that the realization of any convergent sequence of rational matrices should also be convergent. Next, we consider the issue of minimality of realizations and propose a method for calculating the dimension of a minimal realization of a given transfer matrix sequence. Finally, necessary and sufficient conditions are discussed under which a sequence of state-space systems is a minimal realization and under which minimal realizations of the same transfer function sequence are state-space equivalent. Relationships with standard algebraic system theoretic results are discussed.

¹This work was supported by AFOSR Grant No. AFOSR-88-0087.
1. Introduction

The theory of state-space realizations for strictly proper rational matrices has been thoroughly studied (e.g. see [16]). More recently, techniques for handling improper transfer matrices have been devised (see [14]). In this paper we extend those ideas to the perturbational case -- i.e. where a system is described by a convergent sequence of rational matrices (possibly improper). A realization is then a sequence of (generalized) state-space systems. The problem is made nontrivial by imposing the constraint that the matrix entries of the realization sequence should also converge.

Part of our motivation for this problem comes from the study of robust control problems -- specifically from those dealing with order uncertainty and singular perturbations. For example, the robustness problems addressed in [1]-[3] are based on singularly perturbed system models. Physical systems are invariably subject to some variations in parameters, often resulting in changes in model order. It is desirable, therefore, to design compensators which meet performance criteria independent of system perturbations. Many robust control theories (e.g. [4]) emphasize input-output performance characteristics. Our intention is to develop some fundamental tools for examining robustness problems associated with a system's internal structural properties.

One way to approach this problem might be through the application of algebraic system theory (see, e.g., [17]). In this setting, the transfer function sequence is viewed as a rational function over the ring $\mathbb{c}$ of convergent real sequences using pointwise operations. Unfortunately, we will see that existing results in algebraic realization theory apply to our case only marginally. This is due to three key facts: 1) The ring $\mathbb{c}$ is not an
integral domain. 2) Most results in algebraic realization theory deal only with the case of proper transfer functions. 3) An abstract version of the Weierstrass decomposition for matrix pencils over a ring does not yet exist. Nevertheless, our feeling is that the properties of sequences of transfer functions are sufficiently important from the point of view of robust control theory that they deserve separate treatment, not only for the sake of mimicking standard results from the algebraic theory, but also in order to obtain deeper insight into the specific structure of realizations over this particular ring.

From an analytic perspective, considerable work dealing with perturbations of rational matrices has appeared (e.g., [4], [7]-[12]). In these papers various rational matrix topologies have been proposed, motivated by a variety of control problems. The closest of these to our work are [10]-[12], where a singular perturbation theory for transfer functions is developed and a specific form of realization is given. However, [10]-[12] do not explicitly address those problems dealing with the existence of realizations in general and, in particular, the minimal realization of perturbed systems. In [4] rational matrix convergence is characterized in terms of the "graph metric" which is used to address certain problems in local simultaneous stabilization. It is easy to show that the graph metric induces a topology which is very different from that corresponding to simple system parameter convergence. The work of [7] and [9] also treats the problem of topologizing the set of rational matrices and is closely related to ours, but again does not examine the realization problem. Our work is motivated solely by realization and robustness issues; our constructions are designed to yield the simplest definition of convergence corresponding to convergence of system parameters.
We are mainly concerned with the interplay between two types of LTI system representations. First, let $\mathcal{R}(s)$ be the set of all rational functions over $\mathcal{R}$, and let $\mathcal{R}(s)^{rm}$ be the set of $r \times m$ matrices over $\mathcal{R}(s)$. Next, consider (generalized) state-space systems

$$\dot{E}x = Ax + Bu$$

$$y = Cx.$$ (1)

where $E$ and $A$ are $n \times m$ real matrices satisfying the standard regularity assumption $\det(sE-A) \neq 0$. $B$ is $n \times m$, $C$ is $r \times n$, and $E$ may be singular. For the sake of brevity, we identify the system (1) with the matrix 4-tuple $\sigma=(E,A,B,C) \in \mathbb{R}^{2(n-m+r)}$. The transfer matrix of (1) is

$$H(s) = C(sE-A)^{-1}B = \frac{C \cdot \text{adj}(sE-A) \cdot B}{\det(sE-A)} \in \mathcal{R}(s)^{rm}. $$ (2)

Throughout the paper we assume that the values of $m$ and $r$ are fixed; we consider $n$ to be a variable.

**Definition 1.1**

1) A state-space system $\sigma \in \mathbb{R}^{n(n-m-r)}$ is said to have dimension $n$. In this case, we write $\dim \sigma = n$.

2) If a rational matrix $H$ is of the form (2), we say that $(E,A,B,C)$ is a realization of $H$.

With regard to (1) and (2), a (nonperturbational) realization theory already appears in [14]. We now summarize the main results of this theory.

**Theorem 1.2** [14]

1) Every rational matrix has a realization.

2) The minimal dimension over all realizations of $H$, denoted $\mu(\cdot)$, is

$$\mu(H(s)) = \nu(H_s(s)) + \nu(H_f(s)),$$

where $\nu(\cdot)$ is MacMillan degree, and $H_s$ and $H_f$ are the unique strictly proper rational matrix and polynomial matrix, respectively, satisfying $H = H_s + H_f$. 

3
3) A 4-tuple $\sigma$ is a minimal realization of some rational matrix $H$ if and only if $\sigma$ is controllable and observable (as defined in [8]).

4) If $\sigma_1 = (E_1, A_1, B_1, C_1)$ and $\sigma_2 = (E_2, A_2, B_2, C_2)$ are minimal realizations of the same rational matrix, there exist nonsingular matrices $M$ and $N$ such that $E_2 = ME_1N$, $A_2 = MA_1N$, $B_2 = MB_1$, and $C_2 = C_1N$.

The results of our present work may be considered to be a generalization of Theorem 1.2 to the case of rational matrix sequences $(H_k)$.

In Section 2 we will choose a natural definition for convergence of rational matrices. Working from this definition, we will consider sequences $H_k \rightarrow H$ in $\mathbb{R}(s)^{r \times m}$ and attempt to characterize those sequences $\{\sigma_k\}$ in $\mathbb{R}^{n(2n-m-r)}$ such that 1) $\sigma_k$ converges to some $\sigma$ in the matrix sense, 2) $\sigma_k$ is a realization of $H_k$ for sufficiently large $k$, and 3) $\sigma$ is a realization of $H$.

We view this approach as a way of modeling the possible perturbations in the internal structure of a system corresponding to a given perturbation in the input-output description $(H_k)$.

In our realization theory, we will see that Theorem 1.2, part 1) remains true (Section 5). Corresponding to the expression for $\mu$ in part 2), in Section 6 we will define and give an explicit expression for a degree function which equals the dimension of all "minimal realizations" of a sequence of transfer matrices. It will be shown that properties 3) and 4) do not hold as stated for sequences; however, we will discuss important special cases where similar statements do hold. In Sections 5 and 6 we will also discuss the connections between our work and the standard algebraic realization theory (see [17]).
2. Convergence in the Space of Rational Matrices

We first consider the problem of defining a topology on \( \mathbb{R}(s) \) and later on \( \mathbb{R}(s)^{r \times m} \). Convergence in \( \mathbb{R}(s) \) will be defined in the most natural way such that small perturbations in \( \mathbb{R}(s) \) correspond to small perturbations in the coefficients of numerator and denominator of the rational function. To begin, we must define convergence in the set \( \mathbb{R}[s] \) of all polynomials over \( \mathbb{R} \). Suppose \( P_k : k = 1, 2, \ldots \) and \( P \) are polynomials in \( \mathbb{R}[s] \).

**Definition 2.1** We say \( P_k \) converges to \( P \) if there exists an integer \( q \) such that \( \deg P_k \leq q; k = 1, 2, \ldots, \deg P \leq q \), and \( a_{1k} \rightarrow a_1; i = 0, \ldots, q \), where

\[
P_k(s) = a_{qs} s^q + \cdots + a_{1k} s + a_{0k}; k = 1, 2, \ldots
\]

\[
P(s) = a_q s^q + \cdots + a_1 s + a_0.
\]

**Remarks**

1) If we regard \( P_k \in \mathbb{R}[s]; k = 1, 2, \ldots \) as functions over \( \mathbb{C} \), we might be tempted to define \( P_k \rightarrow P \) when \( \lim_{k \rightarrow \infty} P_k(s) = P(s) \) for any \( s \in \mathbb{C} \). But we notice that in this definition, \( \deg P_k \) may not be bounded. For example, let \( P_k(s) = \frac{1}{k} s^k - 1 \). This observation brings us to a crossroads in the theory:

If we were to allow convergent polynomial sequences to have unbounded degree, the same would be true for sequences of rational functions. This would result in an undesirable situation where state-space realizations could have unbounded dimension. Hence, we insist on bounded degree based both on physical intuition and on a desire for mathematical elegance.

2) Definition 2.1 is equivalent to the following two conditions:

a) \( \{\deg P_k : k = 1, 2, \ldots\} \) is bounded.

b) \( \lim_{k \rightarrow \infty} P_k(s) = P(s) \) for every \( s \in \mathbb{C} \).

Indeed, necessity of a) and b) is obvious. On the other hand, if \( \{P_k\} \) satisfies a), \( P_k(s) \) and \( P(s) \) can be written as in Definition 2.1. Choose
q-1 distinct complex numbers \( \{s_1, \ldots, s_{q-1}\} \). Then, from b),

\[
V_{q-1} \rightarrow V_{q-1}
\]

where

\[
V = \begin{bmatrix}
  s_1^q & \cdots & s_1 \\
  s_2^q & \cdots & s_2 \\
  \vdots & \ddots & \vdots \\
  s_{q-1}^q & \cdots & s_{q-1}
\end{bmatrix},
\]

\[
\xi_k = \begin{bmatrix}
  a_{qk} \\
  \vdots \\
  a_{1k} \\
  a_{0k}
\end{bmatrix},
\]

\[
\xi = \begin{bmatrix}
  a_q \\
  \vdots \\
  a_1 \\
  a_0
\end{bmatrix}.
\]

We know that the Vandermonde matrix \( V \) satisfies \( \det V \neq 0 \) as long as \( s_i \neq s_j \), \( i \neq j \); therefore, \( V^{-1} \) exists, and \( \xi_k \rightarrow \xi \) as \( k \rightarrow \infty \).

3) We can define a topology on \( \mathbb{R}[s] \) which is consistent with our notion of convergence in Definition 2.1. To do so, identify every element in \( \mathbb{R}[s] \) with an element in \( \mathbb{R}^\infty \) according to

\[
p_m s^m \cdots p_1 s - p_0 \mapsto (p_0, p_1, \ldots, p_m, 0, 0, 0, \ldots).
\]

and let

\[
\mathcal{R}_{m+1} = \{(p_0, p_1, \ldots, p_m, 0, 0, \ldots) \in \mathbb{R}^\infty | p_i \in \mathbb{R}; \ i = 0, 1, 2, \ldots, m\}.
\]

Then

\[
\mathcal{R} = \bigcup_{k=1}^{\infty} \mathcal{R}_k
\]

is the set of all polynomials. On \( \mathcal{R}_k \), we take identification topology (e.g., see [19, p.120]) with respect to the bijections \( f_k: \mathbb{R}^k \rightarrow \mathcal{R}_k \) defined by

\[
f_k(a_1, a_2, \ldots, a_k) = (a_1, a_2, \ldots, a_k, 0, \ldots).
\]

That is, a set

\[
U = \{(a_1, \ldots, a_k, 0, \ldots) | (a_1, \ldots, a_k) \in V\}
\]

is open in \( \mathcal{R}_k \) if and only if \( V \) is open in \( \mathbb{R}^k \). On \( \mathcal{R} \) we impose the inductive limit topology [19, p.420] with respect to the \( \mathcal{R}_k \) — i.e., we impose on \( \mathcal{R} \) the finest topology which makes the natural imbeddings \( \mathcal{R}_k \subset \mathcal{R} \).
continuous. It is routine to prove that \( P_k \rightarrow P \) in the sense of Definition 2.1 if and only if \( P_k \) converges to \( P \) in \( \mathbb{R} \).

4) It is shown in [6, Lemma 4.3] that, if \( \{P_k\} \) is convergent in \( \mathbb{R}[s] \), then there exists convergent real sequences \( \{a_{ik}\}, \{\beta_{ik}\}, \) and \( \{\gamma_k\} \), with \( \alpha_{ik} \rightarrow 0 \) and \( \lim \gamma_k = 0 \), such that

\[
P_k(s) = \gamma_k \prod_{i}(s-\alpha_{ik})(s-\beta_{ik}).
\]

In particular, if the roots of \( P_k \) are bounded, then \( \deg P_k = \deg \lim P_k \).

In order to define convergence in \( \mathbb{R}(s) \), we adopt a standard quotient space construction over \( \mathbb{R}[s] \times (\mathbb{R}[s] - \{0\}) \) (e.g., see [18], p.136) and identify each rational function with a unique equivalence class under the relation

\[(a,b) \sim (c,d) \iff ad = bc.\]

We use the expression \( a/b \) to denote both a rational function as well as its corresponding equivalence class in \( \mathbb{R}[s] \times (\mathbb{R}[s] - \{0\}) \). Note that a similar construction may be employed in identifying rational matrices \( N/d \in \mathbb{R}^{r \times m}(s) \) with equivalence classes of pairs \( (N,d) \in \mathbb{R}[s]^{r \times m}(\mathbb{R}[s] - \{0\}) \).

Adopting ordinary quotient set topology on \( \mathbb{R}(s) \), we arrive at the following definition.

**Definition 2.2** Suppose \( h_k : k = 1, 2, \ldots \), and \( h \) are in \( \mathbb{R}(s) \). We say \( h_k \) converges to \( h \) in \( \mathbb{R}(s) \), if there exist \( n_k \rightarrow n \) and \( d_k \rightarrow d \) in \( \mathbb{R}[s] \), with \( d_k \neq 0 \), such that

\[
n_k/d_k = h_k \quad k = 1, 2, \ldots \quad \text{and} \quad n/d = h.
\]

Along similar lines, we now give three alternative definitions for convergence in \( \mathbb{R}(s)^{r \times m} \).

**Definition 2.3** Suppose \( H_k : k = 1, 2, \ldots \), and \( H \) are \( r \times m \) rational matrices with components \( h_{ijk} \) and \( h_{ij} \), respectively. We say \( H_k \) converges to \( H \) in \( \mathbb{R}(s)^{r \times m} \) if \( h_{ijk} \rightarrow h_{ij} \) in \( \mathbb{R}(s) \) as \( k \rightarrow \infty \).

---

7
Definition 2.3' Suppose $H_k = 1.2, \ldots$ and $H$ are $r \times m$ rational matrices. We say $H_k$ converges to $H$ in $\mathbb{R}(s)^{rm}$ if there exist $N_k \rightarrow N$ in $\mathbb{R}[s]^{rm}$ and $d_k \rightarrow d$ in $\mathbb{R}[s]$ such that $N_k/d_k = H_k = 1.2, \ldots$ and $N/d = H$. (Here we assume product topology on $\mathbb{R}[s]^{rm}$ and that the quotient space constructions above are applied componentwise on $\mathbb{R}[s]^{rm} \times (\mathbb{R}[s]-\{0\})$.)

Definition 2.3'' We say $H_k$ converges to $H$ in $\mathbb{R}(s)^{rm}$ if there exist $N_k \rightarrow N$ in $\mathbb{R}[s]^{rm}$ and $D_k \rightarrow D$ in $\mathbb{R}[s]^{m^2}$ with $D_k$ and $D$ nonsingular such that $N_k D_k^{-1} H_k = 1.2, \ldots$ and $N D^{-1} = H$.

Remarks

1) It is easy to show that Definitions 2.3, 2.3' and 2.3'' are equivalent. A fourth alternative definition is the same as 2.3'' except using left instead of right factorizations.

2) Note that a sequence which converges in the sense of Definition 2.3 also converges in identification topology with respect to the map $\mathcal{M}: \mathbb{R}^{n(2n+m+r)} \rightarrow \mathbb{R}(s)^{rm}$ defined by

$\mathcal{M}(E.A.B.C) = \frac{C \cdot \text{adj}(sE - A)B}{\det(sE - A)}$,

where $(E,A,B,C) \in \mathbb{R}^{n(2n+m+r)}$. The construction of the topology on $\mathbb{R}(s)^{rm}$ shows that $\mathcal{M}$ is continuous.

3) If $H_k \rightarrow H$ and $G_k \rightarrow G$, then $H_k + G_k \rightarrow H + G$ and $H_k G_k \rightarrow H G$: more generally, $\mathbb{R}(s)^{rm}$ is a topological ring with respect to identification topology on $\mathbb{R}(s)$ and the corresponding product topology on $\mathbb{R}(s)^{rm}$. In particular, relative topology on the subgroup of polynomial matrices $\mathbb{R}[s]^{rm}$ is the same as product topology with respect to Definition 2.1. Note that $\mathbb{R}(s)^{rm}$ is closed in $\mathbb{R}(s)^{rm}$.

We will show in Section 5 that our definition of convergence is the "right" definition for the realization problem, since a sequence in $\mathbb{R}(s)^{rm}$ converges in our sense if and only if it admits a convergent sequence of
state-space realizations. One view of the results of this paper is that they characterize local properties of the map \( H \).

3. Time-Scale Decomposition of Transfer Matrix Sequences

Clearly, any rational matrix \( H \) can be uniquely expressed as \( H = H_s - H_f \), where \( H_s \) is strictly proper and \( H_f \) is a polynomial matrix. We now generalize the decomposition to the sequential case: this must be carried out in a way that preserves convergence.

**Definition 3.1** 1) We say a convergent sequence \( \{H_k\} \) in \( \mathbb{R}(s)^m \) is a slow sequence if \( H_k \) is strictly proper for every \( k \) and there exists a bounded region \( \Lambda \subseteq \mathbb{C} \) such that all poles of each \( H_k \) lie in \( \Lambda \).

2) A convergent sequence \( \{H_k\} \) is called a fast sequence if for every \( \epsilon > 0 \) there exists a \( K > 0 \) such that \( k > K \) implies that each pole \( p \) of \( H_k \) satisfies \( p > M \) (all poles tend to infinity).

**Remarks**

1) The set of all slow sequences in \( \mathbb{R}(s)^m \) forms a proper subspace of the real vector space of all convergent sequences in \( \mathbb{R}(s)^m \). The same statement holds for fast sequences.

2) Any slow sequence can be expressed as \( H_{sk} = \frac{N_k}{d_k} \) where \( d_k \) is convergent and monic for every \( k \) and \( \deg N_k < \deg d_k \), where \( \deg N = \max(\deg n_{ij}) \) for any polynomial matrix \( N \). Thus \( \deg \lim N_k < \deg \lim d_k \). This shows that the limit of every slow sequence is strictly proper.

3) Since the limit of any fast sequence can have no finite poles, such a limit must be a polynomial matrix.

4) Every convergent sequence of polynomial matrices is a fast sequence.
3) If a sequence is both slow and fast, it must be strictly and have no poles whatsoever for large $k$: hence, the sequence must be identically zero for large $k$.

6) A sequence of matrices $\{H_k\}$ is slow (fast) if and only if each component sequence $\{h_{ijk}\}$ is slow (fast).

**Definition 3.2**

1) We say $H_k = H_{sk} + H_{fk}$ is a time-scale decomposition of $\{H_k\}$ when $\{H_{sk}\}$ and $\{H_{fk}\}$ are slow and fast sequences, respectively.

2) In a time-scale decomposition, $\{H_{sk}\}$ and $\{H_{fk}\}$ are called the slow part and the fast part of $\{H_k\}$.

Note that, from remarks 2) and 3) above, if $H_k = H_{sk} + H_{fk}$ is a time-scale decomposition of $\{H_k\}$, then $H_{sk} = H_s$ and $H_{fk} = H_f$, where $H_s$ and $H_f$ are the strictly proper part and the polynomial part of $H \equiv \lim_{k \to \infty} H_k$. Theorem 3.3 tells us that every convergent sequence $\{H_k\}$ has an essentially unique time-scale decomposition.

**Theorem 3.3**

1) For every convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{r \times m}$, there exist a slow sequence $\{H_{sk}\}$ and a fast sequence $\{H_{fk}\}$ such that $H_k = H_{sk} - H_{fk}$ for every $k$.

2) If $\{\hat{H}_{sk}\}$ and $\{\hat{H}_{fk}\}$ are slow and fast sequences, respectively, and $H_k = \hat{H}_{sk} - \hat{H}_{fk}$ for every $k$, then $H_{sk} = \hat{H}_{sk}$ and $H_{fk} = \hat{H}_{fk}$ for sufficiently large $k$.

**Proof**

1) We need only treat the case $r = m = 1$: the multivariable case can then be handled componentwise. If $h_k = h \in \mathbb{R}(s)$, we can find $n_k - n$ and $d_k - d$ with $n_k/d_k = h_k$ and $n/d = h$. Since $d_k - d$, from (3) we can write $d_k = \gamma_k d_{sk} + d_{fk}$, where

$$
\begin{align*}
d_{sk}(s) &= s^{\mu} + b_{\mu-1,k}s^{\mu-1} + \ldots + b_{0k} \\
d_{fk}(s) &= a_{\nu,k}s^{\nu} - a_{\nu-1,k}s^{\nu-1} - \ldots - a_{1k}s^{1} - 1,
\end{align*}
$$

with each $(b_{ik})$ convergent, $\gamma_k \to \gamma = 0$, and $a_{ik} \to 0$ as $k \to \infty$. Let

$$
n_k(s) = z_{pk}s^{p} - \ldots - z_{1k}s + z_{0k},
$$

...
and let $q = \max\{\nu, p, \mu\}$. We will show that there exist convergent polynomial sequences

\[ n_{sk} = x_\mu 1.k^{\mu-1} + \ldots + x_1 k + x_0 k \]
\[ n_{sk} = y_{q-1}.k^{q-1} + \ldots + y_1 k + y_0 k \]

such that $n_k d_k = n_{sk} - n_{sk} d_k$. Equivalently, we need to show that

\[ r_k (n_{sk} d_k - n_{sk} d_k) = n_k. \]  \hspace{1cm} (4)

Note that equation (4) may be written in matrix form

\[ \begin{bmatrix} A_{1k} & B_{1k} \\ A_{2k} & B_{2k} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} z_k \end{bmatrix}. \]  \hspace{1cm} (5)

where

\[ A_{1k} = \begin{bmatrix} 1 & \ldots & 0 \\ a_{1k} & \ldots & 1 \\ \vdots & \ddots & \vdots \\ a_{nk} & \ldots & a_{nk} \end{bmatrix} \]
\[ B_{1k} = \begin{bmatrix} b_{0k} & \ldots & b_{0k} \\ \vdots & \ddots & \vdots \\ b_{\mu-1.k} & \ldots & b_{\mu-1.k} \end{bmatrix} \]
\[ A_{2k} = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \]
\[ B_{2k} = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix} \]

with $A_{1k}$, $A_{2k}$, $B_{1k}$, $B_{2k}$, and $z_k$ having dimensions $\mu \times \mu$, $q \times \mu$, $\mu \times q$, $q \times q$, and $(\mu \times q) \times 1$, respectively. Also note that $A_{1k} \to I_{\mu}$, $B_{2k} \to I_{\nu} - M$, where $M$ is nilpotent and upper triangular, and $A_{2k} \to 0$ as $k \to \infty$. Hence, there exists a $K < \infty$ such that (5) has a unique solution when $k > K$. For $k \leq K$, let $h_{sk}$ be any strictly proper rational function and let $h_{sk} = h_{sk} - h_{sk}$.

2) We have $h_{sk} = \hat{h}_{sk} - \hat{h}_{sk}$ for sufficiently large $k$. Hence,

\[ \hat{h}_{sk} = \hat{h}_{sk} - \hat{h}_{sk}. \]  \hspace{1cm} (6)

But the left side of (6) is slow, and the right side is fast. Hence, both sides are identically zero for large $k$. 

To conclude this section, we note that a time scale decomposition of any transfer matrix sequence of the form $H_k^\infty = C_k^\infty (sE_k - A_k)^{-1}B_k^\infty$, where $\{E_k, A_k, B_k, C_k\}$ is convergent in $\mathbb{R}^{n(2n+m-r)}$ and det$(s \cdot \lim E_k \cdot \lim A_k) \neq 0$, can be achieved by invoking the perturbational form of the classical Weierstrass decomposition for matrix pencils as developed in [7]. Indeed, from [7, p.147] there exist convergent nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$, with $\lim M_k$ and $\lim N_k$ nonsingular, such that

$$M_k E_k N_k = \begin{bmatrix} I & 0 \\ 0 & A_k^\infty \end{bmatrix}, \quad M_k A_k N_k = \begin{bmatrix} A_k^\infty & 0 \\ 0 & 1 \end{bmatrix} \quad (7)$$

where $\lim A_k^\infty$ is nilpotent. Let

$$\begin{bmatrix} B_k^\infty \\ B_k^\infty \end{bmatrix} = M_k B_k^\infty, \quad \begin{bmatrix} C_k^\infty \\ C_k^\infty \end{bmatrix} = C_k N_k^\infty \quad (8)$$

Then

$$H_k(s) = C_k^\infty (sI - A_k^\infty)^{-1}B_k^\infty - C_k^\infty (sA_k^\infty I)^{-1}B_k^\infty \quad (9)$$

From Definition 3.1 it is clear that

$$H_k^\infty (s) = C_k^\infty (sI - A_k^\infty)^{-1}B_k^\infty \quad (10)$$

and

$$H_k^\infty (s) = C_k^\infty (sA_k^\infty I)^{-1}B_k^\infty \quad (11)$$

are slow and fast sequences, respectively. Hence, (9) is a time-scale decomposition of $\{H_k\}$.

4. The Characteristic Polynomial Sequence

In this section we investigate several useful properties of the sequence of characteristic polynomials corresponding to a convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$. We first extend the conventional definition of the characteristic polynomial to improper transfer matrices. Recall that the characteristic polynomial $\Lambda$ of a strictly proper rational matrix $H_s$ is defined as the least common monic denominator of all minors of $H_s$. 

12
Definition 4.1: If \( H \) is a rational matrix with \( H = H_s - H_r \) for some strictly proper \( H_s \) and polynomial matrix \( H_r \), the characteristic polynomial \( \Delta \) of \( H \) is defined as the characteristic polynomial of \( H_s \).

Consider the sequence of characteristic polynomials \( \{ \Delta_k \} \) corresponding to \( \{ H_k \} \). Since \( H_k = H_s \), it follows that \( \Delta_k \) divides \( \det(H_k)^{\min(r,m)} \) for each \( k \); thus, boundedness of \( \{ \deg d_k \} \) ensures boundedness of \( \{ \deg \Delta_k \} \). Let \( \eta = \lim \{ \deg \Delta_k \} \), and note that \( \deg \Delta_k \leq \eta \) for sufficiently large \( k \). For all such \( k \), \( \Delta_k \) can thus be uniquely identified with a point \( <\Delta_k> \) in the real projective space \( \mathbb{P}^\eta \) (see, e.g., [20]) according to

\[
\alpha_i s^i - \alpha_{i-1} s^{i-1} - \ldots - \alpha_0 \mapsto (0, \ldots, 0, \alpha_{i-1}, \ldots, \alpha_0) \in \mathbb{P}^\eta.
\]

In fact, there is a one-to-one correspondence between \( \mathbb{P}^\eta \) and the set of monic polynomials \( \Delta \) with \( \deg \Delta \leq \eta \). These observations lead to the following definition.

**Definition 4.2** Let \( \{ H_k \} \) be any convergent sequence in \( \mathbb{R}(s)^{r,m} \), and let \( \Delta_k \) be the characteristic polynomial of \( H_k \). Set

\[
\rho_k = \begin{cases} <\Delta_k>, & \deg \Delta_k \leq \eta \\ <1>, & \deg \Delta_k > \eta \end{cases}
\]

The sequence \( \{ \rho_k \} \) is called the characteristic polynomial (CP) of \( \{ H_k \} \).

It is easy to show that \( \{ \rho_k \} \) converges if and only if there exists a real sequence \( \{ \gamma_k \} \) such that \( (\gamma_k \Delta_k) \) converges to a non-zero limit \( \Delta \in \mathbb{R}(s) \). In this case, \( \lim \rho_k = <\Delta> \). We now present several pathological situations that can arise in dealing with the CP.

**Example 4.3**

1) The following example illustrates that when \( \{ H_k \} \) is convergent, the corresponding CP may not converge. Consider the sequence
Let \( H(s) = \frac{1}{s-1} \). We may write \( H_k = \sum_{i=0}^{k} d_i \), where \( d_k = \begin{cases} (s-1)(s-2)(s-3) & \text{k even} \\ (s-1)(s+2)(s+3) & \text{k odd} \end{cases} \). Thus \( H_k \to H \), but

\[
\Delta_k(s) = \begin{cases} (s-1)(s-2) & \text{k even} \\ (s-1)(s+2) & \text{k odd} \end{cases}
\]

Clearly, \( \{\rho_k\} \) is not convergent. Note, however, that \( \{H_k\} \) can be divided into two subsequences with convergent CP's according to \( H_k^{(1)} = H_{2k-1} \) and \( H_k^{(2)} = H_{2k} \).

2) In some cases, \( \{\rho_k\} \) may converge even though \( \{\Delta_k\} \) does not. Consider the rational sequence

\[
H_k(s) = \frac{1}{(s^2-1)(s-2)}
\]

In this case, \( \{H_k\} \) has CP determined by \( \Delta_k(s) = (s+k)(s+2) \), but \( \{\rho_k\} \) converges, since

\[
\lim_{k \to \infty} \Delta_k(s) = \frac{1}{k} (s-k)(s-2) \to s-2.
\]

3) Finally, we note that convergence of \( \{\rho_k\} \) (or even \( \{\Delta_k\} \)) does not guarantee that \( \lim \Delta_k \) is the characteristic polynomial of \( \lim H_k \). For example, let

\[
H_k(s) = \frac{s^2}{(s-1)(s-2)(s-1/k)}
\]

Then

\[
\Delta_k(s) = (s+1)(s+2) - (s-1)(s-2).
\]
But

\[ H_k(s) \rightarrow \frac{1}{s-1} \]

Next we examine some basic properties of the CP with respect to the time-scale decomposition. First we need a simple result for individual systems.

**Lemma 4.4** Suppose \( H = H_1 - H_2 \), where \( H_1 \) and \( H_2 \) have no common poles, and let \( \Delta_1 \) and \( \Delta_2 \) be the characteristic polynomials of \( H, H_1, \) and \( H_2 \), respectively. Then \( \Delta = \Delta_1 \Delta_2 \).

**Proof** From the definition of the CP we can assume without loss of generality that \( H_1 \) and \( H_2 \) are strictly proper. Suppose \( (A_1, B_1, C_1) \) and \( (A_2, B_2, C_2) \) are minimal realizations of \( H_1 \) and \( H_2 \); then \( \Delta_i(s) = \det(sI-A_i) \). If we let

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},
\]

then \( (A, B, C) \) is a minimal realization of \( H \) with CP

\[ \Delta(s) = \det(sI-A_1) \det(sI-A_2). \]

In particular, for any time-scale decomposition. Lemma 4.4 implies that, when \( k \) is sufficiently large, we have

\[ \Delta_k = \Delta_{sk} \Delta_{fk}, \]

where \( \Delta_k, \Delta_{sk}, \) and \( \Delta_{fk} \) are the characteristic polynomials of \( H_k, H_{sk}, \) and \( H_{fk} \), respectively.

**Lemma 4.5** Let \( H_k = H_{sk} + H_{fk} \) be a time-scale decomposition.

1) If \( \Delta_{sk} \) is the characteristic polynomial of \( H_{sk} \) and \( \Delta \in \mathbb{R}[s] \) is monic, then \( \langle \Delta_{sk} \rangle \rightarrow \langle \Delta \rangle \) if and only if \( \Delta_{sk} \rightarrow \Delta \).

2) If \( (\rho_{fk}) \) is the CP of \( H_{fk} \), then \( \rho_{fk} \rightarrow \langle 1 \rangle \).

3) The CP of \( \{H_k\} \) is convergent if and only if the CP of \( \{H_{sk}\} \) is convergent. When the two CP's converge, their limits coincide.
Proof 1) Sufficiency is obvious. To show necessity, observe that there must exist a real sequence \( (\gamma_k) \) such that \( \gamma_k \Delta_{sk} \to \Lambda \) and that \( \{\Delta_{sk}\} \) has bounded roots. From (3), \( (\gamma_k) \) converges. Since \( \Delta_{sk} \) and \( \Delta \) are monic, \( \gamma_k \approx 1 \).

2) We have \( \gamma_k = \Delta_{f_k} \), where
\[
\Delta_{f_k} = \begin{cases} \prod (s-\lambda_{ik}), & H_{f_k} \text{ is not a polynomial matrix.} \\ 1, & H_{f_k} \text{ is a polynomial matrix.} \end{cases}
\]

Here the \( \lambda_{ik} \) satisfy the property that, for every \( M<\infty \), there exists a \( K<\infty \) such that \( \lambda_{ik} > M \) for each \( i \) and each \( k>K \). Let
\[
\gamma_k = \begin{cases} \prod \frac{1}{s-\lambda_{ik}}, & H_{f_k} \text{ is not a polynomial matrix.} \\ \frac{1}{s-\lambda_{ik}}, & H_{f_k} \text{ is a polynomial matrix.} \end{cases}
\]

Then \( \gamma_k \Delta_{f_k} \to 1 \) in \( \mathbb{R}[s] \).

3) The result follows immediately from (12) and part 2).

The final result of this section focuses on the observation made in Example 4.3, part 1), that the CP of a sequence \( \{H_k\} \) in \( \mathbb{R}^{1,m}(s) \) which is not convergent can sometimes be decomposed into convergent subsequences. We can in fact demonstrate that a finite decomposition of this sort can always be achieved.

Theorem 4.6 If \( H_k \to H \), then \( \{\rho_k\} \) has finitely many limit points.

Proof From Lemma 4.5, part 3), we need only consider the case where \( \{H_k\} \) is a slow sequence. Let \( H_k = N_k/d_k \) and \( H = N/d \). From the definition of the characteristic polynomial, \( \Delta_k \) divides \( d_k^{\min(r,m)} \) for each \( k \). But \( d_k^{\min(r,m)} = d^{\min(r,m)} \), so the unique monic representative of each limit point of \( \{\rho_k\} \) must divide \( d^{\min(r,m)} \). The result then follows from the fact that any polynomial over \( \mathbb{R} \) has finitely many monic divisors.
Corollary 4.7 If $H_k \to H$, then there exist finitely many strictly increasing sequences $(k^{(i)}_j)_{i=1,\ldots,\pi}$ of positive integers such that

1) $(k^{(i)}_j | i=1,\ldots,\pi; j=1,2,\ldots} = \{1,2,3,\ldots\}.

2) $(k^{(p)}_j | j=1,2,\ldots} \cap (k^{(q)}_j | j=1,2,\ldots} = \emptyset$ when $p \neq q$.

3) each $(H_{k^{(i)}_j})$ has convergent CP.

Proof From Theorem 4.6, there are only finitely many limit points $\rho^1,\ldots,\rho^\pi \in \mathbb{P}^\pi$ of $(\rho_k)$. Since $\mathbb{P}^\pi$ is a compact Hausdorff space, each open subset $U$ of $\mathbb{P}^\pi$ satisfying $(\rho^1,\ldots,\rho^\pi) \subseteq U$ contains a tail of $(\rho_k)$. Indeed, otherwise there would exist a subsequence of $(\rho_k)$ with no limit point, contradicting compactness of $\mathbb{P}^\pi$. Let $U_1,\ldots,U_\pi$ be nonintersecting neighborhoods of $\rho^1,\ldots,\rho^\pi$, respectively; then there exists a $K<\infty$ such that $(\rho_k) \subseteq U_i$ for $k>K$. Let $k^{(i)}_j = j$: $j=1,\ldots,K$. The remaining $k^{(i)}_j$ may then be defined iteratively according to $k^{(i)}_j = \min\{k | \rho_k \in U_i\} - (k^{(i)}_q | q<j)$. If $V_i \subseteq U_i$ is another neighborhood of $\rho^1_i$, then by compactness of $\mathbb{P}^\pi$ there must be a tail of the subsequence $(\rho^{(i)}_{k^{(i)}_j})$ contained in $V_i$; hence, $\rho^{(i)}_{k^{(i)}_j} \to \rho^1_i$.

5. Existence of Realizations

We are now ready to formally define realizations of a given transfer matrix sequence $(H_k)$ and discuss their existence. We base our definition of a realization of $(H_k)$ on the standard definition of a realization of a single rational matrix $H$ as in Theorem 1.2.
Definition 5.1 1) Suppose \( \{H_k\} \) converges in \( \mathbb{R}(s)^{\mathbb{R}m} \). We say a sequence \( \{a_k\} \) in \( \mathbb{R}^{n(2m-m-r)} \) is a realization of \( \{H_k\} \), if there exists an integer \( K \) and a \( \sigma \in \mathbb{R}^{n(m+m-r)} \) such that \( a_k \) is a realization of \( H_k \) when \( k > K \) and \( a_k \to \sigma \) in \( \mathbb{R}^{n(2m-m-r)} \).

2) A realization \( \{a_k\} \) in \( \mathbb{R}^{n(m+m-r)} \) is said to have dimension \( n \).

Note that the dimension of a realization \( \{a_k\} \) is given simply by \( \operatorname{dim} a_k \) for any \( k \). If \( H_k \to H \), then continuity of \( H \) implies that \( \sigma \) is a realization of \( H \) in the conventional sense. We will show that there exists a realization for any convergent sequence \( \{H_k\} \); this generalizes part 1) of Theorem 1.2 to sequences and demonstrates that the definition of convergence in \( \mathbb{R}(s)^{\mathbb{R}m} \) outlined in Section 2 is the correct one for our purposes.

To simplify subsequent discussion, we will make use of the mapping \( \Phi: \mathbb{R}(s)^{\mathbb{R}m} \to \mathbb{R}(s)^{\mathbb{R}m} \) defined by

\[
\Phi(H)(s) = \frac{1}{s} H(\frac{1}{s}).
\]

It is easy to see that \( \Phi \) is an isomorphism on \( \mathbb{R}(s)^{\mathbb{R}m} \) and that \( \Phi^{-1} = \Phi \). Some elementary properties of \( \Phi \) follow.

Lemma 5.2 Let \( H \in \mathbb{R}(s)^{\mathbb{R}m} \), and let \( \{H_k\} \) be convergent in \( \mathbb{R}(s)^{\mathbb{R}m} \).

1) \((E,A,B,C)\) is a realization of \( H \) if and only if \((A,E,B,C)\) is a realization of \( \Phi(H) \).

2) \( \mu(H) = \mu(\Phi(H)) \)

3) If the characteristic polynomial of \( H \) is \( \Delta(s) = s^n - \eta_{n-1}s^{n-1} - \ldots - \eta_0 \), then the characteristic polynomial of \( \Phi(H) \) is \( \tau(s) = \gamma(s^n - \eta_{n-1}s^{n-1} - \ldots - \eta_0) \) for some \( \gamma \neq 0 \).

Proof 1) Suppose \( H(s) = (sE-A)^{-1}B \). Then \( \Phi(H)(s) = \frac{1}{s}(sE-A)^{-1}B = C(sA-E)^{-1}B \).

18
2) From 1), if \( H \) has a realization of degree \( n \), then so does \( \% (H) \). The converse follows from \( \% (\% (H)) = H \).

3) Let \((E, A, B, C)\) be a minimal realization of \( H \); then \((A, E, B, C)\) is a minimal realization of \( \% (H) \). Hence, \( \Delta(s) = r_1 \det(sE - A) \) for some \( r_1 > 0 \) (for details, see [14]), and

\[
\% (s) = \frac{\gamma_2 \det(sE - A)}{\gamma_1},
\]

where \( \gamma_1, \gamma_2 \) are constants.

Lemma 5.3 If \( \{H_k\} \) is a fast sequence, then \( \{\% (H_k)\} \) is a slow sequence.

Proof. Since all poles of \( H_k \) tend to infinity, we can write

\[
H_k(s) = \frac{\prod_{i=1}^{q} \left( s - a_{ik} \right)}{\prod_{i=1}^{q} \left( s - a_{ik} \right)},
\]

where \( \prod_{i=1}^{q} \left( s - a_{ik} \right) \) is convergent and each \( \alpha_{ik} \to 0 \) as \( k \to \infty \). Let \( q = \max (\deg \ n_{ij}, p-1) \).

Then

\[
\% (H_k)(s) = (-1)^{p-1} s^{q-1} \frac{s^{\deg N_k} - \prod_{i=1}^{q} \left( s - a_{ik} \right)}{s^{q-1} \prod_{i=1}^{q} \left( s - a_{ik} \right)}.
\]

Note that \( s^{\deg N_k} \left( \frac{1}{s} \right) \) is a polynomial matrix, each of whose elements has degree at most \( q \). Clearly, \( \% (H_k) \) has bounded poles, and the denominator of \( \% (H_k) \) has degree \( q+1 \), so \( \% (H_k) \) is strictly proper.

Now consider a time-scale decomposition \( H_k = H_{sk} H_{fk} \) of an arbitrary \( \{H_k\} \) in \( \mathbb{R}(s)^m \). Suppose \( \{H_{sk}\} \) and \( \{\% (H_{fk})\} \) have realizations of the form

\[
((I, A_{sk}^t S_{sk}^t, C_{sk}^t)) \quad \text{and} \quad ((I, A_{fk}^t, B_{fk}^t, C_{fk}^t)).
\]

Then each \( H_{fk} = \% (\% (H_{fk})) \) has \( (A_{fk}^t, I, B_{fk}^t, C_{fk}^t) \) as one of its realizations.
\[
\begin{bmatrix}
1 & 0 \\
0 & A_k
\end{bmatrix}, \begin{bmatrix}
A_{sk} & 0 \\
0 & I
\end{bmatrix}, \begin{bmatrix}
B_{sk} \\
B_{fk}
\end{bmatrix}, \begin{bmatrix}
C_{sk} & C_{fk}
\end{bmatrix}.
\]

It is easy to check that \(\{(E_k, A_k, B_k, C_k)\}\) is a realization of \(\{H_k\}\). Therefore, we only need to prove existence of realizations for slow sequences.

**Theorem 5.41** Every slow sequence has a realization of the form \(\{(I, A_k, B_k, C_k)\}\).

**Proof** First we treat the case \(r = m = 1\). Let \(H_k \in \mathbb{R} \times \mathbb{R}^2\), where \(\{S_k\}\) and \(\{d_k\}\) are convergent in \(\mathbb{R}\). Then

\[
d_k = s^k, \quad S_k = \beta_{q-1,k}s^q-1 - \beta_{q-2,k}s^{q-2} - \ldots - \beta_{0,k}.
\]

where \(\alpha_{1,k}\) and \(\beta_{1,k}\) converge as \(k \to \infty\). To obtain a realization of \(\{H_k\}\) of the desired form, set

\[
A_k = \begin{bmatrix}
0 & 1 \\
\ldots & \ldots & \ldots \\
-a_{0,k} & -a_{1,k} & \ldots & -a_{q-1,k}
\end{bmatrix}, \quad B_k = \begin{bmatrix}
0 \\
\ldots \\
0
\end{bmatrix}, \quad C_k = [\beta_{0,k} \beta_{1,k} \ldots \beta_{q-1,k}].
\]

Now we consider the general case. By the definition of convergence in \(\mathbb{R}(s)^{m \times m}\), every component sequence \(h_{i,j,k}\) is convergent. Suppose \(\{(I, A_{i,j}, B_{i,j}, C_{i,j})\}\) is a realization of \(\{h_{i,j,k}\}\). Let

\[
A_{i,j} = \text{diag}(A_{i,j} \mid j = 1,2,\ldots,m), \quad B_{i,k} = \text{diag}(B_{i,j,k} \mid j = 1,2,\ldots,m), \quad C_{i,k} = [C_{i,1,k} \ldots C_{i,m,k}]
\]

and

\[
A_k = \text{diag}(A_{i,j} \mid j = 1,2,\ldots,m), \quad B_k = \begin{bmatrix}
B_{1,k} \\
\vdots \\
B_{r,k}
\end{bmatrix}, \quad C_k = \text{diag}(C_{i,j} \mid i = 1,2,\ldots,r).
\]

A simple calculation verifies that \(\{(I, A_k, B_k, C_k)\}\) is a realization of \(\{H_k\}\).

Combining the time-scale decomposition with Lemma 5.3 and Theorem 5.4, we arrive at the following result.

**Corollary 5.5** Every convergent sequence in \(\mathbb{R}(s)^{m \times m}\) has a realization.
Theorem 5.4 (but not Corollary 5.5) may also be proven in an abstract algebraic framework as outlined in [17.Chapter 4]. Briefly, consider the commutative ring $c$ of convergent sequences in $\mathbb{R}$ using pointwise operations, and let the set of $r \times m$ proper rational real matrices be denoted by $\mathbb{R}_p(s)^{r \times m}$. A convergent sequence $\{H_k\}$ in $\mathbb{R}_p(s)^{r \times m}$ may then be viewed as a formal power series over the ring of $r \times m$ matrices with elements in $c$. Indeed, we may expand each element of each $H_k$ about $s=\infty$, yielding the series

$$H_k = \sum_{i=1}^{\infty} \left(\frac{1}{s}\right)^i H_{ik}.$$  

(13)

where the sequences $\{H_{ik}\}$ are convergent. From this point there are two ways to proceed. First, one can prove realizability by constructing a certain infinite-dimensional Hankel matrix from the $H_{ik}$. It must then be shown that the span of the columns of the Hankel matrix is a finitely generated module over $c$. A second approach is to show that the formal power series (13) is "rational" in a certain algebraic sense. This immediately guarantees realizability. Both conditions can be demonstrated in our framework fairly easily; however, our proof of Theorem 5.4 is more direct and is sufficient for our purposes.

6. Minimality

In Section 5 we showed that every convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{r \times m}$ has a convergent realization $\{\sigma_k\}$. In this section, we explore the issue of minimality of a realization.

**Definition 6.1** 1) If $n$ is the smallest integer such that $\{H_k\}$ has a realization of dimension $n$, and $\{\sigma_k\}$ is a realization of $\{H_k\}$ with $\dim \sigma_k = n$, then we say $\{\sigma_k\}$ is a minimal realization of $\{H_k\}$.
2) If a sequence of state-space systems \( \{ \sigma_k \} \) is a minimal realization of its transfer matrix sequence, we say \( \{ \sigma_k \} \) is minimal.

Obviously, all minimal realizations of \( \{ H_k \} \) have the same dimension. This fact enables us to define a degree function \( \delta \) on the set of convergent rational matrix sequences by setting \( \delta(\{ H_k \}) \) equal to the dimension of any minimal realization of \( \{ H_k \} \). In this section we develop a simple expression for \( \delta(\{ H_k \}) \) for slow sequences and then extend it to the general case. Next, we examine a natural conjecture for determining whether a sequence \( \{ \sigma_k \} \) is minimal and relate minimal realizations of the same \( \{ H_k \} \) in a manner analogous to Theorem 1.2, part 4). Finally, we relate our results to the realization theory outlined in [17] for algebraic systems over the ring \( C \).

In our development it will be helpful to exploit various properties of the mapping which takes each state-space system into a particular choice of numerator and denominator of its transfer function. Specifically, define

\[
\Gamma_n : \mathbb{R}^{n(n+m+r)} \rightarrow \mathbb{R}^{n(rm+1)}
\]

according to

\[
\Gamma_n(A,B,C) = (C \cdot \text{adj}(sI-A)B, \text{det}(sI-A)).
\]

Here we have identified \( \mathbb{R}_k \) as defined in Remark 3) after Definition 2.1. with \( \mathbb{R}^k \). Note that \( \Gamma_n \) is continuous: if \( \Gamma_n(A,B,C)=(N,d) \), then \((I,A,B,C)\) is a realization of \( N/d \). Also notice the distinction between \( \Gamma_n \) and \( \mathbb{N}_n \) as defined in Section 2. We denote \( \text{Im } \Gamma_n = \Gamma_n(\mathbb{R}^{n(n+m+r)}) \).

The following series of lemmas leads us to the first main theorem of this section.

**Lemma 6.2** Consider any pair \( (N,d) \) where \( d \) is monic, \( \deg d = n \), and \( N/d \) is strictly proper with characteristic polynomial \( \Delta \). Then \( (N,d) \in \text{Im } \Gamma_n \) if and only if \( \Delta \) divides \( d \).
Proof (Sufficient) Suppose

\[ d(s) = \Delta(s) \prod_{i=1}^{p} (s - \alpha_i). \]

and let (I.A.B.C) be a minimal realization of N/d. Define

\[ A^* = \begin{bmatrix} A & 0 \\ 0 & \Sigma \end{bmatrix}, \quad B^* = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}, \quad C^* = \begin{bmatrix} C & 0 \end{bmatrix}, \]

where

\[ \Sigma = \begin{bmatrix} -\beta_1 \\ -\beta_2 \\ \vdots \\ -\beta_p \end{bmatrix}. \]

Then

\[ \det(sI - A^*) = \det(sI - A) \prod_{i=1}^{p} (s - \alpha_i) \cdot d(s). \]

Since \( C(sI - A)^{-1}B = C(sI - A)^{-1}B = N(s)/d(s) \),

\[ C^* \text{adj}(sI - A^*)B^* = \det(sI - A^*) \frac{N(s)}{d(s)} = N(s). \]

Hence, \((N.d) = \Gamma_n(A^*, B^*, C^*)\).

(Necessary) Suppose \((N.d) = \Gamma_n(A, B, C)\); then \((I.A.B.C)\) is a realization of \(N.d\). From [16, Theorem 5-18], we can find a similarity transformation \( T \) such that

\[ T^{-1}A^* T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad CT = \begin{bmatrix} 0 & C_2 & C_3 \end{bmatrix}. \]

where \((A_{22}, B_2, C_2)\) is a minimal realization of \(N/d\). Note that \(\det(sI - A_{22}) = \Delta(s)\). Thus

\[ d(s) = \det(sI - A) = \det(sI - A_{11}) \det(sI - A_{33}) \Delta(s). \]

From Corollary 4.7, \((H_k)\) can be decomposed into \(\pi\) sequences \((H_k^{(i)}); i=1,2,\ldots,\pi\), where we define \(H_j^{(i)} = H_k^{(i)}\). Each sequence has convergent CP \(K_j\) satisfying

\[ \Delta_k^{(i)} \rightarrow \Delta_k^{(i)}; \quad i=1,2,\ldots,\pi. \]

23
where \( \Delta^{(1)} \) is monic. If \( \{H_k\} \) is slow, then, from Lemma 4.3, part 1), \( \Delta_k^{(1)} \rightarrow \Delta^{(1)} \). Let

\[
\Delta = \text{LCM} \{\Delta^{(1)}, \ldots, \Delta^{(\pi)}\},
\]

where LCM denotes the least common multiple. Also define

\[
\Delta_k^{(i)} = \frac{\Delta_k^{(1)}}{\Delta^{(i)}}, \quad i=1,2,\ldots,\pi.
\]

Note that each \( \Delta_k^{(i)} \) is a polynomial and that, if \( \{H_k\} \) is slow, \( \Delta_k^{(i)} \rightarrow \Delta \).

Lemma 6.3 Let \( \{H_k\} \) be a slow sequence with \( H_k \rightarrow H \), and suppose \( H \) has characteristic polynomial \( \Delta \). Then \( \Delta \) divides \( \Delta \).

Proof According to Corollary 4.7, \( \{H_k\} \) can be decomposed into \( \pi \) subsequences with convergent CP's \( \Delta_k^{(1)} \rightarrow \Delta^{(1)} \). If \( \Delta \) divides \( \Delta_k^{(i)} \) for each \( i \), then \( \Delta \) divides \( \Delta \). Hence, it suffices to treat the case where \( \{H_k\} \) has convergent CP \( \Delta_k \rightarrow \Delta \).

Let \( p = \min\{r,m\} \), and consider the sequence \( \{H_k\} \) of \( 1 \times \sum_{1=1}^{p} \binom{r}{i} \binom{m}{i} \) rational matrices, each \( H_k \) consisting of the minors of \( H_k \) of all orders. Obviously, \( H_k \rightarrow H \), where \( H \) is defined similarly. It follows from elementary arguments that \( H_k \) has characteristic polynomial \( \Delta_k \) (same as \( H_k \)) and that, for any polynomial \( q \), \( qH_k \) is a polynomial matrix if and only if \( \Delta_k \) divides \( q \). In particular, \( \Delta_k H_k \) is a polynomial matrix. Since \( R[s]^j \) is closed in \( R(s)^j \) for any \( j \), \( \Delta H \) is a polynomial matrix. Thus, the characteristic polynomial \( \Delta \) of \( H \) (and \( H \)) divides \( \Delta \).

Lemma 6.4 \( \Gamma_n \) is an open mapping.

Proof Note that \( \Gamma_n \) is multilinear; thus, it is a composition of functions on Euclidean spaces \( R^p \) of the form \( f(x_1, \ldots, x_p) = x_i x_j \) and \( g(x_1, \ldots, x_p) = x_1 \cdot \ldots \cdot x_p \). Since \( f \) and \( g \) are open, compositions of open maps are open, and products of opens sets are open. It follows that \( \Gamma_n \) is open.
Lemma 6.5 Let $X$ and $Y$ be topological spaces with $X$ first countable, and let $Q:X \rightarrow Y$ be an onto, open, and continuous map. For any convergent sequence $\{y_k\}$ in $Y$ with $y_k \rightarrow y \in Y$ and any $x \in Q^{-1}(y)$, there exist $x_k \in Q^{-1}(y_k)$: $k=1,2,...$ such that $x_k \rightarrow x$.

Proof: Let $\{U_i:i=1,2,...\}$ be a countable basis of neighborhoods of $x$ with $U_i \supseteq U_{i+1}$. Since $Q$ is open, each $V_i=Q(U_i)$ is a neighborhood of $y$. Hence, for any $V_i$, we can find an integer $K_i$ such that $y_k \in V_i$ when $k > K_i$. Furthermore, there must exist points $x_k^{(i)} \in U_i:k=K_i-1, K_i-2,...$ with $Q(x_k^{(i)})=y_k$. For $k \leq K_i$, select any $x_k^{(i)} \in Q^{-1}(y_k)$. This process defines sequences $\{x_k^{(i)}\}:i=1,2,...$ without loss of generality, we may assume $K_{i+1} > K_i$. If we let $x_k = x_k^{(i)}:k=K_i-1,...,K_i$, where $K_0=0$, the construction shows that each $U_i$ contains a tail of the sequence $\{x_k\}$. Hence, $x_k \rightarrow x$.

Lemma 6.6 Suppose $\{H_k\}$ is a slow sequence with $H_k \rightarrow H$. If there are pairs $(N_k,d_k),(N,d) \in \Gamma_n$ such that $N_k/d_k=H$, $N/d=H$, and $(N_k,d_k) \rightarrow (N,d)$, then $\{H_k\}$ has an n-dimensional realization.

Proof: Note that $R^{n(n+m+r)}$ is first countable. Thus, if we restrict the range of $\Gamma_n$ to $\text{Im} \Gamma_n$, we may use Lemmas 6.4 and 6.5 and the fact that $\Gamma_n$ is continuous to conclude that there exists a convergent sequence $(A_k,B_k,C_k) \rightarrow (A,B,C)$ in $R^{n(n+m+r)}$ such that $\Gamma_n(A_k,B_k,C_k) = (N_k,d_k)$ and $\Gamma_n(A,B,C) = (N,d)$. Notice that $(I,A_kB_kC_k)$ is a realization of $N_k/d_k=H_k:k=1,2,...$, and $(I,A,B,C)$ is a realization of $N/d=H$.

Lemma 6.7 If a slow sequence $\{H_k\}$ has an n-dimensional realization, then it has an n-dimensional realization of the form $\{(I,A_k,B_k,C_k)\}$.

Proof: Suppose $\{H_k\}$ has a realization having dimension n. Since $\{H_k\}$ is slow, the decomposition (7)-(11) shows that $\{H_k\}$ is of the form

$$H_k(s) = C_{sk}(sI-A_{sk})^{-1}B_{sk}.$$
where \( A_{sk} \) is \( q \cdot q \) with \( q \leq n \). Define

\[
A_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_k = \begin{bmatrix} H_{sk} \\ 0 \end{bmatrix}, \quad C_k = [C_{sk}, 0].
\]

**Theorem 6.8** For any slow sequence \( \{H_k\} \),

\[
\delta(H_k) = \deg \Delta.
\]

**Proof.** Let \( n_0 = \deg \Delta \). We first show that there exists an \( n_0 \)-dimensional realization of \( \{H_k\} \). Let \( \tilde{N}(i) = \Delta(i)H_k(i) \) and \( \tilde{N} = \Delta H_k \); then \( \tilde{N}(i) \) and \( \tilde{N} \) are polynomial matrices with \( (\tilde{N}(i), \tilde{\Delta}(i)) = (\tilde{N}, \tilde{\Delta}) \). Since all poles of \( H_k \) are bounded. Remark 4) after Definition 2.1 shows that \( \deg \tilde{\Delta}(i) = \deg \tilde{\Delta} = n_0 \). Thus, from (14) and Lemmas 6.2 and 6.3, \( (N_k, \Delta_k) \in \text{Im } \Gamma_{n_0} \). Suppose \( \{(N_k, \Delta_k)\} \) is constructed by setting \( \tilde{N}_j = \tilde{N}(i) \) and \( \tilde{\Delta}_j = \tilde{\Delta}(i) \) whenever \( H_j = H_k(i) \), where the \( k_j \) are defined by the decomposition of Corollary 4.7. Then \( (N_k, \Delta_k) \in \text{Im } \Gamma_{n_0} \). The desired result follows from Lemma 6.6.

It remains to show that \( n_0 \) is the minimal dimension over all realizations of \( \{H_k\} \). Suppose \( \{H_k\} \) has an \( n \)-dimensional realization. Then, from Lemma 6.7, it has an \( n \)-dimensional realization of the form \( ((I, A_k, B_k, C_k)) \).

Let \( (N_k, d_k) = \Gamma_n (A_k, B_k, C_k) \) and \( (N, d) = \Gamma_n (\lim(A_k, B_k, C_k)) \); from Lemma 6.2, \( \Delta_k \) divides \( d_k \) for every \( k \). Letting \( d_j^{(i)} = d_k^{(i)} \), \( \Delta_k^{(i)} \) divides \( d_k^{(i)} \). Since \( \Gamma_n \) is continuous, \( d_k^{(i)} \rightarrow d \); thus, closure of \( \text{Re}[s] < \text{Re}(s) \) guarantees that each \( \Delta_k^{(i)} \) divides \( d_k \). Thus \( \Delta \) divides \( d \), and

\[
n = \deg d \geq \deg \Delta = n_0.
\]

The following result offers one method of calculating \( \delta(H_k) \) for an arbitrary convergent sequence \( \{H_k\} \) in \( \mathbb{R}(s)^{\Gamma_M} \).

**Theorem 6.9** If \( H_k = H_{sk} + H_{fk} \) is a time-scale decomposition, then

\[
\delta(H_k) = \delta(H_{sk}) + \delta(H_{fk}).
\]
Proof. Suppose \( \{ (E_k, A_{k}, B_k, C_k) \} \) is a minimal realization of \( \{ H_k \} \). Appealing to (7)-(11), it suffices to show that \( \{ \sigma_{sk} \} = \{ (I, A_{sk}, B_{sk}, C_{sk}) \} \) and \( \{ \sigma_{fk} \} = \{ (A_{fk}, 1, B_{fk}, C_{fk}) \} \) are minimal realizations of \( \{ H_{sk} \} \) and \( \{ H_{fk} \} \). Suppose there exists a realization \( \{ \overset{\text{sn}}{\sigma} \} = \{ (E_{sk}, A_{sk}, B_{sk}, C_{sk}) \} \) of \( \{ H_{sk} \} \) with \( \dim \overset{\text{sn}}{\sigma}_{sk} < \dim \sigma_{sk} \). By Lemma 6.7, we may assume that \( E_{sk} = I \). Let

\[
E_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad \dot{A}_k = \begin{bmatrix} \dot{A}_{sk} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} \dot{B}_{sk} \\ B_{fk} \end{bmatrix}, \quad C_k = \begin{bmatrix} C_{sk} & C_{fk} \end{bmatrix}.
\]

Then \( \{ \overset{\text{sn}}{\sigma} \} = \{ (E_k, A_k, B_k, C_k) \} \) is a realization of \( \{ H_k \} \) with \( \dim \overset{\text{sn}}{\sigma}_{k} < \dim \sigma_{k} \). This is a contradiction. A similar argument shows minimality of \( \{ \sigma_{fk} \} \).

Thus, one way to find \( \delta(H_k) \) is to first perform a time-scale decomposition \( H_k = H_{sk} + H_{fk} \) and then to use Theorem 6.8 to find \( \delta(H_{sk}) \) and \( \delta(H_{fk}) = \delta(\mathbb{H}(H_{fk})) \). Fortunately, the next theorem simplifies this task and shows how to calculate \( \delta(H_k) \) without resorting to time-scale decomposition.

Recall that, for any \( H \in \mathbb{R}(s)^{m \times n} \), \( H_f \) denotes the polynomial part of \( H \).

**Theorem 6.10** Suppose \( H_k \rightarrow H \). Then

\[
\delta(H_k) = \max \lim_{i, k} \frac{\delta^i}{\delta_k} = \mu((H(i)_{pk})) + \Delta(i).
\]

Proof. Suppose \( H_k = H_{sk} + H_{fk} \) is a time-scale decomposition of \( \{ H_k \} \), and let

\[
H(i)_{skj} = H(i)_{skij}, \quad H(i)_{fjk} = H(i)_{fijk},
\]

It is clear that \( H(i)_{sk} \) and \( H(i)_{fk} \) have characteristic polynomials \( \Delta(i)_{sk} \) and \( \Delta(i)_{fk} \), respectively. From (12),

\[
\Delta(i) = \Delta(i)_{sk} \Delta(i)_{fk}.
\]

From (15) and Lemma 4.5, part 2), \( \Delta(i)_{sk} \rightarrow \Delta(i)_{fk} \). Hence, from Theorem 6.8.

\[
\delta(H_{sk}) = \deg \Delta(i)_{sk} = \deg \Delta(i)_{sk},
\]

Also, since each \( \Delta(i)_{sk} \) is convergent and monic.

\[
\deg \Delta(i)_{sk} = \deg \Delta(i)_{sk} \quad \text{for large} \ k.
\]
From (3), \(\Delta_{f_k}\) is of the form
\[
<\Delta_{f_k}> = \prod_{i=1}^{p} (\alpha_i - s_{1k}^{-1}) = <\alpha_i s_{1k}^{-p} \cdots s_{1k}^{-1} >,
\]
where \(\varepsilon_{1k} \to 0\) as \(k \to \infty\). Note that some of the \(\varepsilon_{1k}\) may vanish; so, from Lemma 5.2, part 3), \(\tau_k\) has characteristic polynomial of the form
\[
\tau_k(s) = s^q \cdot \varepsilon_{1k} s_{1k}^{-q-1} \cdots \varepsilon_{qk},
\]
where \(q\) may depend on \(k\). Therefore, the limit of any convergent subsequence of \(\{\tau_k\}\) is of the form \(s^q\). Suppose \(\tau\) is the least common multiple of these limits; then \(\tau = s^q\), where
\[
q = \lim_{k \to \infty} \deg(\tau_k) = \lim_{k \to \infty} \mu(\tau_{f_k}) = \lim_{k \to \infty} \mu(H_{f_k}).
\]
The last equality is obtained from Lemma 5.2, part 2). Arguing as in Lemma 5.2, part 2), \(\delta(\tau_{f_k}) = \delta(H_k)\) for any \(\{H_k\}\). Hence, from Theorem 6.8, we have
\[
\delta(H_{f_k}) = \delta(\tau_{f_k}) = \deg \tau = q.
\]
Theorem 1.2, part 2) and Theorem 6.8 show that \(\mu(H_{f_k}) = \deg \Delta_{f_k} + \mu((H_k)^{f})\). From Theorem 6.9.
\[
\delta(H_k) = \deg \Delta + \lim_{k \to \infty} \left( \deg \Delta_{f_k} + \mu((H_k)^{f}) \right)
\]
\[
= \lim_{k \to \infty} \left( \deg \Delta + \deg \Delta_{f_k} + \mu((H_k)^{f}) \right)
\]
\[
= \max_{i=1} \lim_{k \to \infty} \left( \deg \Delta + \deg \Delta_{i}^{(1)} - \mu((H_k)^{f}) \right).
\]
It remains to prove
\[
\deg \Delta(i) = \deg \Delta + \deg \Delta_{f_k}^{(1)}. \tag{16}
\]
By the definition of \(\Delta(i)\),
\[
\deg \Delta(i) = \deg \Delta + \deg \Delta_{i}^{(1)} - \deg \Delta(i). \tag{17}
\]
Since \(\deg \Delta_{i}^{(1)} = \deg \Delta(i)\), it follows from (15) that
\[
\deg \Delta(i) = \deg \Delta + \deg \Delta^{(1)} \tag{18}
\]
Combining (17) and (18), we obtain (16).
Corollary 6.11 Suppose $H_k \rightarrow H$. If the CP of $\{H_k\}$ is convergent, then

$$\delta(H_k) = \lim_{k} \mu(H_k).$$

Proof In this case, $\pi=1$ and $\delta^{(1)}_k = \Delta_k$, so

$$\delta(H_k) = \lim_{k} \left( \deg \Delta_k + \mu((H_k)^f) \right)$$

$$= \lim_{k} \left( \mu(H_k^s) - \mu((H_k)^f) \right)$$

$$= \lim_{k} \mu(H_k).$$

Our next goal is to generalize part 3) of Theorem 1.2. An obvious conjecture is that a realization $\{\sigma_k\}$ of $\{H_k\}$ is minimal if and only if each $\sigma_k$ is controllable and observable (as defined in [8]). While controllability and observability for every $k$ (or, indeed, for infinitely many $k$) are clearly sufficient for minimality, the next examples demonstrate how necessity can fail.

Example 6.12 1) Even for a slow $(H_k)$, minimality of $\{\sigma_k\}$ does not imply controllability and observability even at a single point. Consider the sequence $(H_k)$ in Example 4.3, part 1). We can decompose $(H_k)$ into two subsequences

$$H_k^{(1)} = \frac{s-2}{(s+1)(s+2+\frac{1}{2k-1})}$$

$$H_k^{(2)} = \frac{s+3}{(s+1)(s+3+\frac{1}{2k})}.$$ 

For $H_k^{(1)}$ and $H_k^{(2)}$, the CP's are

$$\Delta_k^{(1)} = (s-1)(s+2+\frac{1}{2k-1}) \rightarrow (s+1)(s-2) = \Delta^{(1)}$$

$$\Delta_k^{(2)} = (s-1)(s+3+\frac{1}{2k}) \rightarrow (s+1)(s-3) = \Delta^{(2)}.$$ 

Thus

$$\Delta(s) = \text{LCM } \{\Delta^{(1)}, \Delta^{(2)}\} = (s+1)(s+2)(s-3).$$

Since $(H_k)$ is a slow sequence, Theorem 6.9 indicates that $\delta(H_k) = \deg \Delta = 3$. 

29
Every minimal realization \( \{ \sigma_k \} \) of \( \{ H_k \} \) must have dimension 3, but \( \mu(\sigma_k) = 2 \) for each \( k \); hence, no \( \sigma_k \) is controllable and observable.

2) In this example, the CP converges, but controllability and observability on a subsequence is the most that can be achieved. Let

\[
H_k(s) = \begin{cases} 
1 & \text{k even} \\
\frac{1}{(s-1)^2} & \text{k odd}
\end{cases}
\]

A simple calculation yields \( \delta(H_k) = 2 \), but \( \mu(H_k) = 1 \) when \( k \) is even, so any realization must contain infinitely many terms which are not controllable and observable.

The next result offers a weak extension of Theorem 1.2, part 3) to the sequential case.

**Theorem 6.13** Consider a convergent sequence \( \{ \sigma_k \} = \{ (E_k, A_k, B_k, C_k) \} \), and let

\[ H_k = C_k (sE_k - A_k)^{-1} B_k. \]

If the CP of \( \{ H_k \} \) converges and \( \{ \sigma_k \} \) is minimal, then there exists a subsequence \( \{ \sigma_{k_j} \} \) such that \( \sigma_{k_j} \) is controllable and observable for every \( j \). If, in addition, \( \{ H_k \} \) is a slow sequence, then \( \lim \sigma_k \) is controllable and observable, so \( \sigma_k \) is controllable and observable for every sufficiently large \( k \).

**Proof** From Corollary 6.11, there exists a subsequence \( \{ H_{k_j} \} \) of \( \{ H_k \} \) such that

\[ \delta(H_{k_j}) = \mu(H_{k_j}) \]

for all \( j \). Therefore, each \( \sigma_{k_j} \) has dimension \( \mu(H_{k_j}) \) and must be controllable and observable. If \( \{ H_k \} \) is a slow sequence and the CP converges, Lemma 4.5, part 1) shows that \( \Delta_k \rightarrow \Delta \), where \( \Delta \) is the characteristic polynomial of \( H = \lim H_k \). Since each \( \Delta_k \) is monic, for large \( k \) we have \( \mu(H_k) = \deg \Delta_k = \deg \Delta = \mu(H) \). From Corollary 6.11, \( \dim \sigma = \delta(H_k) = \mu(H_k) \); hence, \( \sigma \) is controllable and observable.

Restricting attention to slow sequences, Example 6.12, part 1) has special significance from an abstract algebraic perspective. It is easy to
show that algebraic controllability and observability over the ring \( \mathbb{C} \) of convergent real sequences, as defined in [17, Chapter 2], is equivalent to controllability and observability of \( \sigma_k \) for sufficiently large \( k \). Thus, linear systems over \( \mathbb{C} \) do not satisfy the property that minimality implies algebraic controllability and observability.

We conclude this section by examining the problem of extending Theorem 1.2, part 4) to the sequential case. The following examples show that, in our case, two minimal realizations of the same sequence \( \{H_k\} \) may not be related by nonsingular transformations (cf. [17, Theorem 4.19]).

**Example 6.14** 1) In fact, two minimal realizations may not be related by nonsingular transformation for any value of \( k \). To see this, let \( \{H_k\} \) again be the slow sequence given in Example 4.3, part 1). In Example 6.12, part 1), we showed that \( \delta(H_k)=3 \). Consider the two minimal realizations \( \{(I,A_{1k},B_k,C_{1k})\} \) and \( \{(I,A_{2k},B_k,C_{2k})\} \), where

\[
\begin{align*}
A_{1k} & = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{3}{k}) & -(11+\frac{4}{k}) & -(6+\frac{1}{k}) \end{bmatrix}, & k \text{ even} \\
& = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{2}{k}) & -(11+\frac{3}{k}) & -(6+\frac{1}{k}) \end{bmatrix}, & k \text{ odd} \\
A_{2k} & = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{5}{k}) & -(11+\frac{1}{k}) \end{bmatrix} \quad \text{for } k \text{ even} \\
& = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(6+\frac{5}{k}) & -(11+\frac{1}{k}) & -(6+\frac{2}{k}) \end{bmatrix} \quad \text{for } k \text{ odd} 
\end{align*}
\]

\[
B_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_{1k} = \begin{bmatrix} 6 & 5 \\ 1 \end{bmatrix}, \quad C_{2k} = \begin{bmatrix} 6+\frac{2}{k} & 5+\frac{1}{k} & 1 \end{bmatrix}, \quad \text{for } k \text{ even}
\]

\[
= \begin{bmatrix} 6+\frac{2}{k} & 5+\frac{1}{k} \\ 1 \end{bmatrix}, \quad \text{for } k \text{ odd}
\]

Suppose there exist convergent nonsingular matrix sequences \( \{M_k\} \) and \( \{N_k\} \) such that
for each \( k \). Then \( N_k = M_k^{-1} \) and \( A_{2k} = M_k A_{1k} M_k^{-1} \). But a simple calculation shows that \( A_{1k} \) and \( A_{2k} \) have different spectra, yielding a contradiction for each value of \( k \).

2) When the CP converges. Theorem 6.13 implies that there exists a subsequence on which every minimal realization is controllable and observable. Hence, Theorem 1.2. part 4) guarantees that for any two minimal realizations there exist nonsingular sequences \( \{M_k\} \) and \( \{N_k\} \) which relate the various matrices. However, it may not be possible to find \( \{M_k\} \) and \( \{N_k\} \) which converge. Consider the sequence given by

\[
H_k(s) = \frac{1}{k s + 1}
\]

with realizations \((\frac{1}{k}, -1, \frac{1}{k}, 1)\) and \((\frac{1}{k}, -1, \frac{1}{k})\). We have immediately \( M_k \cdot \frac{1}{k} = 1 \) so \( M_k = k \).

However, there is still one interesting case where a result is possible.

**Theorem 6.15** Suppose \( \{H_k\} \) is a convergent sequence in \( \mathbb{R}(s)^{FM} \) and let \((E_{1k}, A_{1k}, B_{1k}, C_{1k})\): \( i = 1, 2 \) be two minimal realizations of \( \{H_k\} \) with \((E_{1k}, A_{1k}, B_{1k}, C_{1k})\) controllable and observable for every \( i, k \). Further assume that each \((E_i, A_i, B_i, C_i) = \lim(E_{1k}, A_{1k}, B_{1k}, C_{1k})\) is controllable and observable. Then there exist nonsingular matrix sequences \( \{M_k\} \) and \( \{N_k\} \) with \( M_k = M \) and \( N_k = N \). and \( M \) and \( N \) nonsingular, such that \( M_k A_{1k} N_k = A_{2k}, M_k B_{1k} = B_{2k}, \) and \( C_{1k} N_k = C_{2k} \) for every \( k \).

**Proof** Applying the decomposition (7), (8) to \((E_{1k}, A_{1k}, B_{1k}, C_{1k})\) yields decomposing matrices \( \tilde{M}_{1k} = \tilde{M}_i \) and \( \tilde{N}_{1k} = \tilde{N}_i \) and decomposed system matrices \( A_{isk}, A_{ifik}, B_{isk}, B_{ifik}, C_{isk}, \) and \( C_{ifik} \), with \( \lim A_{ifik} \) nilpotent. This determines in two ways the same time-scale decomposition \( H_k = H_{sk} + H_{fk} \) given by

32
\[ H_{sk}(s) = C_{isk}(sI-A_{isk})^{-1}B_{isk} \]
\[ H_{fk}(s) = C_{ifk}(sA_{ifk}-I)^{-1}B_{ifk} \]

Note that, for sufficiently large \( k \), each of the subsystems \((I,A_{isk},B_{isk},C_{isk})\) and \((A_{ifk},B_{ifk},C_{ifk})\) must be controllable and observable. From (16, p. 208), the similarity transformation \( T_{sk} = (V_{2sk}^TV_{2sk})^{-1}V_{2sk}^TV_{sk} \) where \( V_{isk} \) is the observability matrix of the pair \((A_{isk},C_{isk})\), takes \((I,A_{isk},B_{isk},C_{isk})\) into \((I,A_{2sk},B_{2sk},C_{2sk})\). Furthermore, \( (T_{sk}) \) converges to the nonsingular matrix \( T_{s} = (V_{2s}^TV_{2s})^{-1}V_{2s}^TV_{1s} \). A similar construction yields \( T_{fk} \rightarrow T_{f} \). A straightforward calculation shows that the sequences

\[
M_k = M_{2k}^{-1}\begin{bmatrix} T_{sk} & 0 \\ 0 & T_{fk} \end{bmatrix} M^{-1}_{1k} \quad N_k = N_{1k}^{-1}\begin{bmatrix} T_{sk} & 0 \\ 0 & T_{fk} \end{bmatrix}^{-1}\]

yield the desired result.

Our final result follows with the aid of Theorem 6.13.

**Corollary 6.16** If \( \{H_k\} \) is a slow sequence with convergent CP and \( \{(E_{ik},A_{ik},B_{ik},C_{ik})\}; i=1,2 \) are any two minimal realizations, then, for sufficiently large \( k \), there exist nonsingular matrix sequences \( \{M_k\} \) and \( \{N_k\} \) and nonsingular matrices \( M \) and \( N \) such that \( M_k \rightarrow M \), \( N_k \rightarrow N \), \( M_k E_k = M_k^{-1} E_k \), \( M_k A_{1k} = M_k^{-1} A_{1k} \), \( M_k B_k = M_k^{-1} B_k \), and \( C_{1k} = C_{2k} \) for every \( k \).

7. Concluding Remarks

The problem discussed in this paper is the realization of convergent transfer matrix sequences with convergent generalized state-space sequences. Just as state-space sequences may be decomposed according to time-scale behavior, a time-scale decomposition for any rational matrix sequence may also be achieved. We have shown that convergence of the CP of a sequence of rational matrices is a crucial issue in the minimal realization problem. It was proved that, when the characteristic polynomial of a rational matrix
sequence is not convergent, the rational sequence can be decomposed into finitely many subsequences in such a way that each subsequence has convergent CP. Our results demonstrate that the general problem can be reduced to finitely many subproblems, each of which can be handled using a simpler theory. It is hoped that our results will complement the robustness literature at large.
REFERENCES


Linear Compensator Designs Based Exclusively on Input–Output Information are Never Robust with Respect to Unmodeled Dynamics

J. DANIEL COBB

Abstract—We investigate the effects of unmodeled, higher order dynamics or parasitics on the stability of linear control systems. We first describe a class of perturbations of a given state equation which cannot be distinguished from the original on the basis of input–output measurements alone. Then it is shown that, given any plant–compensator pair, such perturbations of such a system can always be found which destabilize the closed-loop configuration. Finally, the effect of destabilizing perturbations on output behavior is explored.

I. INTRODUCTION

The effects of high-frequency or parasitic phenomena on closed-loop system performance have long been studied. A popular framework for addressing this issue has been that of singular perturbation theory (see, e.g., [1], [2]). The point of view that parasitics are ultimately connected with unmodeled plant dynamics has become quite popular in recent years, sometimes with surprising consequences. For example, it was shown by Rohrs et al. [3] and Ioannou and Kokotovic [4] that high-frequency phenomena can lead to instability in adaptive control schemes. Adaptive controllers being highly nonlinear, a natural question to ask is whether parasitics could have a similar destabilizing effect on control systems which are based on linear compensators. This was answered in the affirmative by Khalil in [5] and [6]. A notable effort to circumvent these difficulties in the case of linear, time-invariant systems was made by Vidyasagar, culminating in the results of [6] and [7].

Our work is most similar to [7], but differs primarily in that we investigate the stability of a closed-loop system when both the plant and compensator are perturbed. The idea of perturbing both systems has been largely neglected in the literature (with the notable exception of [6]), even though one can easily make a strong case for considering such perturbations. Indeed, one need only recognize that a compensator, like the plant, is a physical system governed by a mathematical model which is inherently subject to uncertainty.

In light of examples such as those contained in [4] and [5], even arbitrarily small model errors are to be feared since such effects have the capability of destabilizing a system just as certainly as larger errors do. In fact, those examples illustrate that in some cases, small errors can cause greater instability than do larger ones.

In this paper, we intend to show that, when uncertainties in both plant and compensator are taken into account, even strictly proper compensators are subject to parasitic destabilization. Hence, properness of the compensator is really not the pivotal issue here as it is in [7]. We will show that, if only input–output information concerning the plant and compensator is available, robust compensation can never be achieved.

The results of this paper are by nature primarily negative. We do not claim to have a clear understanding yet of exactly what constitutes sufficient information for robust compensation, although we do mention a possible approach to finding an answer in Section V. It is hoped that our results will stimulate further discussion in an area which has been neglected by all but a handful of researchers.

II. PRELIMINARIES

We study systems characterized by the linear, time-invariant state equations

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \] (1)

and perturbations of (1) given by

\[
\begin{bmatrix} I & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = C_1 x + C_2 \xi
\] (2)

where the submatrices in (2) satisfy

\[
A_{11} - A_{12} A_{22}^{-1} A_{21} = A, \quad B_1 - A_{12} A_{22}^{-1} B_2 = B \] (3)

\[
C_1 - C_2 A_{22}^{-1} A_{21} = C_1 - C_2 A_{22}^{-1} B_2 = D \] (4)

and $A_{22}$ is nonsingular. If we set $\epsilon = 0$ in (2) and eliminate $\xi$, (1) is obtained; hence, (2) with $\epsilon = 0$ may be thought of as a state augmentation of (1). Setting $\epsilon > 0$ in (2) constitutes a perturbation of that augmentation. For the moment, we allow $A_{22}$ to be either stable or unstable.

To aid our analysis, we will use the decomposition for singularly perturbed systems developed in [10] where it is shown that there exist real matrix-valued analytic maps $\epsilon \mapsto M_\epsilon$ and $\epsilon \mapsto N_\epsilon$, defined on some interval $[0, \beta)$, such that $M_\epsilon$ and $N_\epsilon$ are square and nonsingular for each $\epsilon$ and

\[
M_\epsilon = \begin{bmatrix} I & 0 \\ 0 & e^t \end{bmatrix}, \quad N_\epsilon = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \] (5)

with $A_{12}$ and $A_{11}$ analytic and $A_{22}$ nilpotent. According to [10], the matrices $M_\epsilon$ and $N_\epsilon$ are unique up to change of bases; hence, we may take $M_0$ and $N_0$ to be any matrices which achieve the decomposition (5) at $\epsilon = 0$. For example, let

\[
M_0 = \begin{bmatrix} I & -A_{12} A_{22}^{-1} \\ 0 & I \end{bmatrix}, \quad N_0 = \begin{bmatrix} I & 0 \\ -A_{12} A_{22}^{-1} & A_{22}^{-1} \end{bmatrix}.
\]

Next, define

\[
\begin{bmatrix} B_{10} \\ B_{20} \end{bmatrix} = M_0 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \begin{bmatrix} C_{10} \\ C_{20} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} N_0.
\] (6)

Equations (5) and (6) yield the decoupled state equations

\[
\begin{bmatrix} I & 0 \\ 0 & A_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_{20} \end{bmatrix} u
\]

\[y = C_1 x_1 + C_2 x_2 \] (7)

where

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N_0^{-1} \begin{bmatrix} x \\ \xi \end{bmatrix}.
\]

Manuscript received August 1, 1987; revised August 27, 1987. Paper recommended by Past Associate Editor, S. P. Bhattacharyya. This work was supported by NSF Grant ECS-8612948.

The author is with the Department of Electrical and Computer Engineering, University of Wisconsin, Madison, WI 53706.

IEEE Log Number 8718553.
We now present a series of technical results which will be useful in Sections III and IV.

**Lemma 1**: \(A_0 = A, B_0 = B, C_0 = C, B_0 = B, C_0 = C, A_{12} \), and \(A_1 = \epsilon F \), for every \( \epsilon \in (0, \beta) \) where \( F_0 = A_{11}^{-1} \).

**Proof**: From (5) and (6), we have

\[
\begin{bmatrix}
A_0 & 0 \\
0 & 1
\end{bmatrix} = M_0 \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \quad N_0 = \begin{bmatrix}
A & 0 \\
0 & I
\end{bmatrix},
\]

\[
\begin{bmatrix}
B_0 \\
B_0
\end{bmatrix} = M_0 \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} = \begin{bmatrix}
B \\
B
\end{bmatrix},
\]

\[
\begin{bmatrix}
C_0 & C_0 & 1
\end{bmatrix} = \begin{bmatrix} C_1 & C_1 & N_0 = 1 & C & C & A_{12}^{-1} \end{bmatrix}.
\]

Let

\[
M_0 = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}, \quad N_0 = \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}
\]

and note that

\[
N_0^{-1} = \begin{bmatrix}
I & 0 \\
A_1 & A_2
\end{bmatrix} M_0 \begin{bmatrix}
I & 0 \\
0 & A_1
\end{bmatrix} N_0^{-1}.
\]

We thus have \(e^{M_2t} = A_{11}^{-1} A_{12} \), so \(A_1 = e^{M_2} \), where \( F_0 = M_2^{-1} L_{12} = A_{12}^{-1} \).

From \(A_1 = e^{M_2} \), we immediately obtain the well-known result that the eigenvalues of \(2 \) which tend to infinity as \( \epsilon \to 0^+ \) are "close" to those of \((1/e)A_{22} \) (see, e.g., [2, Corollary 2.1]). One useful way of stating this result is the following.

**Lemma 2**: If \( \mu \) is an eigenvalue of \( A_{11} \), \( \gamma > 0 \), and \( R < \infty \), then there exists \( \epsilon_0 > 0 \) such that \((2)\) has an eigenvalue \( \lambda \) satisfying \( |\lambda| > R \) and \( |\lambda - \arg(1/e)\mu| \leq \gamma \) whenever \( 0 < \epsilon < \epsilon_0 \).

**Proof**: From (7), the eigenvalues of \((1/e)F_1^{-1} \) are also eigenvalues of \(2 \). Since \( F_1^{-1} = A_2 \) and \( F_1^{-1} \) is continuous in \( \epsilon \) and each \( F_1^{-1} \) has an eigenvalue \( \mu \), with \( \mu \to \mu \) as \( \epsilon \to 0^+ \). Choose \( \epsilon_0 \) so that \((1/e)\mu \to R \) and \( |\lambda - \arg(1/e)\mu| \to \gamma \) whenever \( 0 < \epsilon < \epsilon_0 \) and let \( \lambda = (1/e)\mu \). Then \( \lambda \) is an eigenvalue of \(2 \), \( |\lambda| > R \) and \( |\lambda - \lambda(e)| \leq \gamma \) whenever \( 0 < \epsilon < \epsilon_0 \).

Suppose the transfer matrices of \((1)\) and \(2 \) are \( P \) and \( P \), respectively.

We will need conditions under which an eigenvalue of \((2) \) is also a pole of \( P \).

**Lemma 3**: If \( (A_{12}, B_2, C_2) \) is controllable and observable, there exists \( \epsilon_0 > 0 \) and \( R < \infty \) such that every eigenvalue \( \lambda \) of \(2 \) satisfying \( |\lambda| > R \) is also a pole of \( P \), whenever \( 0 < \epsilon < \epsilon_0 \).

**Proof**: An eigenvalue \( \lambda \) of \(2 \) is a pole of \( P \) if

\[
\lambda I - A_{11} - A_{12} \\
- A_{21} e^{\lambda I - A_{22}} - A_{22}
\end{bmatrix} = M_1^{-1} \begin{bmatrix}
\lambda I - A_{11} & 0 \\
0 & \lambda I - F_1^{-1} - F_1
\end{bmatrix} N_1 = \begin{bmatrix}
I & 0 \\
0 & F_1
\end{bmatrix} N_1^{-1} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

and

\[
M_2^{-1} \begin{bmatrix}
\lambda I - A_{11} & 0 \\
0 & \lambda I - F_1^{-1} - F_1
\end{bmatrix} N_2 = \begin{bmatrix}
I & 0 \\
0 & F_1
\end{bmatrix} N_2^{-1} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

have full rank. Choose \( R > 0 \) and \( \epsilon > \epsilon_0 \) so that \( \epsilon I > \epsilon_0 \) and \( \lambda I > \lambda_0 \) has full rank.

III. INPUT-OUTPUT EQUIVALENCE

In this section, we explore the relationship between the nominal and perturbed systems \((1)\) and \((2)\) and discuss the conditions under which they are indistinguishable if only input-output information is available. Consider the process of obtaining or verifying an input-output model of a physical system. We are allowed to take measurements by applying an input signal starting at \( t = 0 \) and by observing the output; it is assumed that no direct access to internal states is possible. Once a nominal model is obtained, a controllable and observable realization can be chosen, yielding the state equation \((1)\). Since we have no direct control over internal states, the input is specified through the input ports, and since \( u(0) \) necessarily occurs long after the system was built, the system may be assumed initially at rest. Hence, we choose \( x(0) = 0 \) and \( \xi(0) = 0 \) in \((1)\) and \((2)\).

We define the class of admissible input signals \( \mathcal{U} \) to be all \( \epsilon \) functions \( u(t) = \mathcal{U} \) satisfying \( \max \{\|u(t)\| < K_u \} \) and \( \max \{\|\dot{u}(t)\| < K_N \} \), and \( u(0) = 0 \) where the constants \( s \leq 0, K_N \leq \infty \), and \( K_N \leq \infty \) are independent of \( u \).

From an engineering standpoint, it is not unreasonable to place such restrictions on \( u \). Indeed, in any real-world scenario, there is a maximum length of time one would be willing to invest in collecting data, as well as a maximum amplitude of voltage, force, or other input quantity that could possibly be generated using available technology. Furthermore, there is always an upper bound on the rate at which \( u \) can be made to vary (e.g., every amplifier has a maximum slew rate). Thus, the constants \( s, K_N \), and \( K_N \) are often very large, but are finite. Since no input is applied prior to \( t = 0 \) and since \( K_N \leq \infty \), we must have \( u(0) = 0 \).

We would be sure in serious trouble if, in order to design a robust compensator, we needed the capability of generating inputs over arbitrarily large intervals of time \( o \) with arbitrarily large amplitudes or rates of change.

Associated with any real-world measuring device is a minimum error which can be detected. For example, if a function \( y \) represents an output voltage, velocity, or other physical quantity of interest, there must exist a number \( \delta > 0 \), characteristic of the measuring device alone, such that another output \( y' \) cannot be distinguished from \( y \) if

\[ \sup \{\|y(t) - y'(t)\| \leq \epsilon, 0 \leq t \leq \tau\}. \]

(10)

For the remainder of the paper, we assume a fixed source of input signals and measurements and, consequently, a fixed set \( \mathcal{U} \) and number \( \delta > 0 \).

The quantities \( \mathcal{U} \) and \( \mathcal{U} \) together determine an equivalence between systems: two systems are indistinguishable under input-output measurements if for every \( u \in \mathcal{U} \), the output functions \( y \) and \( y' \) of the two systems satisfy \((10)\). The next result applies this idea to the nominal and perturbed models \((1)\) and \((2)\).
valid model from an input-output perspective is given by (2) with $\varepsilon$ sufficiently small and $A_{22}$ strictly stable.

**IV. CLOSED-LOOP DESTABILIZATION**

We are now ready to investigate the effects that the system perturbations in Section II have on a closed-loop configuration. Consider the feedback compensator governed by

$$z = Fz + Gy, \quad u = Hz + v.$$  \hfill (11)

We consider only compensators with strictly proper transfer matrices since the results of [7] indicate that nonsymmetrically proper compensators are never robust with respect to unmodeled dynamics. Perturbations of (11) are of the form

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} y = H_1 z + H_2 \xi + v$$  \hfill (12)

where

$$F_{11} - F_{12}F_{22}^{-1}F_{21} = F, \quad G_1 - F_{12}F_{22}^{-1}G_2 = G$$  \hfill (13)

and $F_2$ is nonsingular. The discussion of Section III applies equally well to both plant and compensator.

Combining (1) and (13) in a standard feedback configuration yields

$$\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & BH \\ GC & F + GH \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ GD \end{bmatrix} u$$  \hfill (15)

where

$$F_{11} - F_{12}F_{22}^{-1}F_{21} = F, \quad G_1 - F_{12}F_{22}^{-1}G_2 = G$$  \hfill (13)

$$H_1 - H_2F_{22}^{-1}F_{21} = H, \quad -H_2F_{22}^{-1}G_2 = 0$$  \hfill (14)

Combining the perturbed systems (2) and (12) gives

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A_{11} & B_1H_1 \\ A_{12} & B_1H_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} y = C_1 x + C_2 \xi.$$  \hfill (16)

Let (15) and (16) have transfer matrices $H$ and $H_2$, respectively.

**Proof:** Since $X$ is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of $X$. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\mu)| < \gamma$. The result then follows from Lemmas 2 and 3.

Now consider behavior of the output $y(t)$ in the closed-loop system (16). Theorem 3 shows that under certain conditions, the instability described in Theorem 2 also has a pronounced effect on $y(t)$. Let $\Delta$ denote Lebesgue measure.

**Theorem 3:** Suppose $R < \infty$, $\beta_1, \beta_2 > 0$, $(X, Y, Z)$ is controllable and observable, and $X$ is nonsingular with an eigenvalue in the sector $S$. Then there exists $\varepsilon_0 > 0$ such that $H_2$ has a pole $\rho \in S$ satisfying $|\rho| > R$ whenever $0 < \varepsilon < \varepsilon_0$.

**Proof:** Since $X$ is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of $X$. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\mu)| < \gamma$. The result then follows from Lemmas 2 and 3.

Now consider behavior of the output $y(t)$ in the closed-loop system (16). Theorem 3 shows that under certain conditions, the instability described in Theorem 2 also has a pronounced effect on $y(t)$. Let $\Delta$ denote Lebesgue measure.

**Theorem 3:** Suppose $R < \infty$, $\beta_1, \beta_2 > 0$, $(X, Y, Z)$ is controllable and observable, and $X$ is nonsingular with an eigenvalue in the sector $S$. Then there exists $\varepsilon_0 > 0$ such that $H_2$ has a pole $\rho \in S$ satisfying $|\rho| > R$ whenever $0 < \varepsilon < \varepsilon_0$.

**Proof:** Since $X$ is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of $X$. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\mu)| < \gamma$. The result then follows from Lemmas 2 and 3.

Now consider behavior of the output $y(t)$ in the closed-loop system (16). Theorem 3 shows that under certain conditions, the instability described in Theorem 2 also has a pronounced effect on $y(t)$. Let $\Delta$ denote Lebesgue measure.

**Theorem 3:** Suppose $R < \infty$, $\beta_1, \beta_2 > 0$, $(X, Y, Z)$ is controllable and observable, and $X$ is nonsingular with an eigenvalue in the sector $S$. Then there exists $\varepsilon_0 > 0$ such that $H_2$ has a pole $\rho \in S$ satisfying $|\rho| > R$ whenever $0 < \varepsilon < \varepsilon_0$.

**Proof:** Since $X$ is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of $X$. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\mu)| < \gamma$. The result then follows from Lemmas 2 and 3.

Now consider behavior of the output $y(t)$ in the closed-loop system (16). Theorem 3 shows that under certain conditions, the instability described in Theorem 2 also has a pronounced effect on $y(t)$. Let $\Delta$ denote Lebesgue measure.

**Theorem 3:** Suppose $R < \infty$, $\beta_1, \beta_2 > 0$, $(X, Y, Z)$ is controllable and observable, and $X$ is nonsingular with an eigenvalue in the sector $S$. Then there exists $\varepsilon_0 > 0$ such that $H_2$ has a pole $\rho \in S$ satisfying $|\rho| > R$ whenever $0 < \varepsilon < \varepsilon_0$.

**Proof:** Since $X$ is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of $X$. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\mu)| < \gamma$. The result then follows from Lemmas 2 and 3.
The divergence of the output of the closed-loop system described in Theorem 3 is referred to in analysis texts as "almost uniform convergence to infinity." In view of the arbitrarily tight bounds that may be placed on an input or initial condition which generate this divergent behavior, we conclude that, if the assumptions of the theorem are met, unbounded instability at the output of a closed-loop configuration can result from arbitrarily small noise impinging on it.

So far we have demonstrated that the existence of destabilizing perturbations of the plant and compensator is guaranteed if a certain linear algebra problem admits a solution. Indeed, if any $A_{12}$, $B_1$, and $C_2$ are chosen, (3) and (4) may be satisfied by simply selecting $A_{12}$ and $A_{21}$ arbitrarily and solving for $A_{11}$, $B_1$, and $C_1$. A similar remark applies to (13) and (14). It is sufficient, therefore, to find $A_{12}$, $B_1$, $C_2$, $F_2$, $G_2$, and $H_2$ such that (1) $A_{12}$ and $F_2$ are strictly stable, (2) $X = (Y, Z)$ is controllable and observable, and (3) $X$ is nonsingular with an eigenvalue in $S$ and (4) and (14) are satisfied. Theorems 2 and 3 further indicate that, if (1–4) are met, the resulting instability in (16) becomes progressively worse as $\epsilon \to 0$ since $R$ may be chosen arbitrarily large. Thus, arbitrarily small uncertainty can lead to arbitrarily large instability.

We now address the linear algebra problem 1–4). We really need to find only one solution in order to demonstrate the existence of destabilizing perturbations; however, it is possible to do better. To obtain an understanding of just how many destabilizing perturbations actually exist, let (1), (2), (11), and (12) have orders $n$, $n + \bar{n}$, $\bar{n}$, and $k + \bar{k}$ respectively; define $\theta = (n + \bar{n}) (n + \bar{n} + m + p + (k + \bar{k}) (k + \bar{k} + p + m)$.

Consider the variety in $H^2$ consisting of all $(A_{11}, A_{12}, B_1, C_1, F_2, F_3, G_2, H_2)$ such that (3)–(6) and (13) and (14) are satisfied, and let $V \subset \mathbb{H}^2$ denote the intersection of that variety with the subset in which $A_{12}$ and $F_2$ are strictly stable. $V$ may be interpreted as the set of all possible state augmentations of (1) and (11) of order $\bar{n}$ and $\bar{k}$ respectively. Finally, let $\Gamma \subset \mathbb{H}^2$ be the set of all points for which $(X, Y, Z)$ is controllable and observable and $X$ is nonsingular with an eigenvalue in $S$. We are interested in properties of the set $V \cap \Gamma$.

**Theorem 4:**

1) $V \cap \Gamma$ is relatively open in $V$.

2) $V \subset \Gamma$ is nonempty if $k \geq 2$ and either a) $D = 0$ and $\bar{n} \geq 2$ or b) $D \neq 0$ and $\bar{n} \geq 2$.

**Proof:**

1) This is obvious since $\Gamma$ is open in $\mathbb{H}^2$.

2) Suppose $D = 0$ and consider

\[
T(s) = \begin{bmatrix} 2s/(s+1)^4 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix}, \quad U(s) = \begin{bmatrix} 2s/(s+1)^4 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix}
\]

Then $(A_{12}, B_1, C_2)$ and $(F_2, G_2, H_2)$ be controllable and observable realizations of $T$ and $U$, respectively. Then $A_{12}$ and $F_2$ are strictly stable, $C_2 A_{12} B_1 = T(0) = D$, and $-H_2 F_2 G_2 = U(0) = 0$. Note that $T$ and $U$ have degrees $\bar{n}$ and $\bar{k}$. Since $(X, Y, Z)$ has transfer function

\[
V(s) = (I - T(s) U(s))^{-1} T(s)
\]

and $V$ has characteristic polynomial $\Delta(s) = (s + 1)^4 e^{-s} - 2s^2 e^s$, it follows that $(X, Y, Z)$ is controllable and observable and $X$ is nonsingular with a unit eigenvalue.

Now suppose $D \neq 0$. There exist nonsingular matrices $M$ and $N$ such that

\[
MN = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}
\]
where \( r = \text{rank } D \). Let

\[
T(s) = M^{-1} \begin{bmatrix}
1/(s+1) & \cdots & 1/(s+1) \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1/(s+1)
\end{bmatrix} N^{-1},
\]

\[
U(s) = N \begin{bmatrix}
2^{x+1} & 1/(s+1)^{x+1} & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 1/(s+1) & 0 & \cdots & 0 \\
\end{bmatrix} M.
\]

Then

\[
V(s) = (I - T(s) U(s))^{-1} T(s) = M^{-1}
\]

has characteristic polynomial \( \Delta(s) = (s + 1)^{x} (s + 1)^{n+1} - 2^{x+1} s^{n+1} \). Reasoning similarly as for part \( a \), we conclude that \( A_{22} \) and \( F_{22} \) are strictly stable, \( 4 \) and \( 14 \) hold, \( (X, Y, Z) \) is controllable and observable, and \( X \) is nonsingular with a unit eigenvalue.

To complete the proof, we need only choose \( A_{22}, A_{31}, F_{22}, \) and \( F_{31} \) arbitrarily and solve for the remaining matrices from (3), (4) and (13), (14).

Part 1) of Theorem 4 demonstrates that, in a certain sense, the high-frequency effects which bring about closed-loop instability do not correspond to the complement of a generic set, and hence cannot be dismissed as merely a pathological case.

V. CONCLUSIONS

We have shown that input-output information alone is insufficient for designing robust linear compensators. This conclusion leads one immediately to ask what further information is actually required to allow a robust design. Although we cannot give a clear answer yet, we can offer some insight. The development of our results indicates the high-frequency behavior in (2) and (12) plays a role in destabilization. Such behavior is winding numbers of simple, closed, piecewise-smooth curves. This insight. The development of our results indicates the high-frequency transfer matrices, but the stability conditions themselves involve only the

\[
\text{REFERENCES}
\]


WEDNESDAY, 4:30 PM - 6:30 PM

**MINISYMPOSIUM 10**
Room: Cathedral Hill A
GEOMETRIC METHODS IN LINEAR CONTROL THEORY 2
Chair: Joyce O'Halloran

4:30 PM
On the Topological Structure of the Orbit Space of Controllable Generalized Linear Systems

**Abstract:** We study the topology of the orbit space of controllable descriptor systems modulo restricted system equivalence. Using a scaling action, we prove that this space is an analytic manifold. Using the Weierstrass decomposition, we obtain an analytic stratification of this manifold. By decomposing the strata into generalized Hermite cells, and using tools from Borel-Moore homology, we compute the singular homology groups for this space in the complex cases. Consequently, the orbit space of controllable descriptor systems is a smooth compactification of the orbit space of controllable state space systems modulo change of basis in the state space.

Uwe Helmke, Department of Mathematics, University of Regensburg, 8400 Regensburg, West Germany; Mark A. Shayman, Electrical Engineering Department and Systems Research Center, University of Maryland, College Park, Maryland 20742

5:00 PM
A Geometric Decomposition Theorem for Singularly Perturbed Systems with Applications to Convergence of Solutions

Let $E$ and $A$ be $n \times n$ matrix-valued analytic functions of a nonnegative real parameter $s$ with $sE(s)-A(s)$ a regular matrix pencil for all $s$. The limiting behavior of the solutions of the family of differential equations

$$E(s)x-A(s)x$$  

(1)

as $s \to 0^+$ is at best only partially understood. Some sufficient conditions are known for certain types of convergence, but no necessary and sufficient conditions are known for nontrivial topologies. We first present a new geometric decomposition theorem for (1). We then show that this result enables us to prove a necessary and sufficient condition under which, for any initial vector, the corresponding solutions of (1) converge uniformly on compact subsets of $[0, \infty)$ with bounded peaking in $[0,1]$.

Daniel Cobb
Department of Electrical and Computer Engineering
University of Wisconsin
Madison, WI 53706-1691

---

**5:30 PM**
Approximation techniques for parabolic control systems: a variational approach

In this paper we consider the linear quadratic regulator problem for a class of boundary control problems for parabolic systems. The problem is formulated using a variational approach and an approximation theory is developed for solutions of the associated operator Riccati equation. Our study includes strongly-damped elastic systems.

H.T. Banks
K. Ito
Center for Control Sciences
Brown University
Providence, RI 02912

6:00 PM
Uniform stabilization/exact controllability from the boundary of thermoelastic plates

We consider the small vibrations of a thin homogeneous, thermally and elastically isotropic plate of uniform thickness. It is known that thermal dissipation alone is sufficient to strongly, but not uniformly, stabilize the elastic components of the motion. It will be shown that the introduction of additional dissipation through the action of bending and twisting moments, and shear force, at the boundary leads to uniform asymptotic stability and to explicit asymptotic energy estimates. The more difficult question of exact controllability of the elastic dynamics by means of such boundary forces and moments will also be considered. It may be shown that if the thermal diffusivity is sufficiently small, the reachable set of the elastic components of the motion is essentially the same as for purely elastic plate motion.

John E. Lagnese
Department of Mathematics
Georgetown University
Washington, DC 20057