THESIS

APPROXIMATE INTERVAL ESTIMATION METHODS FOR THE RELIABILITY OF SYSTEMS USING COMPONENT DATA WITH EXPONENTIAL AND WEIBULL DISTRIBUTIONS

by

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Two approximate parametric interval estimation methods for system reliability using component test data are developed and evaluated. One method can be applied to any coherent system with components which have exponential failure times with possibly different failure rates and different mission operating times. This method estimates the ratios of component failure rates which are then used to develop the approximate lower confidence limit. These ratio estimates are developed with and without jackknife methods and the two results are compared. This procedure is very accurate and simple to compute, requiring the use of standard chi-square tables. This ratio method is subsequently extended to coherent systems with components whose failure times have a Weibull distribution. A nearly exact parametric lower confidence limit for $P(X > x)$ is developed and evaluated where $x$ is given and $X$ has a normal distribution with unknown mean and variance. This procedure is also simple to evaluate and requires the use of Student $t$ tables.
Approximate Interval Estimation Methods for the Reliability of Systems Using Component Data with Exponential and Weibull Distributions

by

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ABSTRACT

Two approximate parametric interval estimation methods for system reliability using component test data are developed and evaluated. One method can be applied to any coherent system with components which have exponential failure times with possibly different failure rates and different mission operating times. This method estimates the ratios of component failure rates which are then used to develop the approximate lower confidence limit. These ratio estimates are developed with and without jackknife methods and the two results are compared. This procedure is very accurate and simple to compute, requiring the use of standard chi-square tables. This ratio method is subsequently extended to coherent systems with components whose failure times have a Weibull distribution. A nearly exact parametric lower confidence limit for $P(X > x)$ is developed and evaluated where $x$ is given and $X$ has a normal distribution with unknown mean and variance. This procedure is also simple to evaluate and requires the use of Student t tables.
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I. BACKGROUND

Parametric confidence interval procedures for the reliability of mechanical systems are much less developed than procedures for electrical systems. This is due to the more complicated failure distributions used to model mechanical hardware. The failure rate of a series system of independent components, with exponentially distributed failure times and equal mission times, is the sum of the failure component failure rates. This property has permitted development of numerous methods for system reliability of series and other coherent systems using component failure data. The *Weibull* and *extreme value* distributions have been used in life testing methods for both electrical and mechanical devices. Several methods have been used for obtaining point estimates of parameters for these distributions and for the reliability function itself, Harter and Moore [Ref. 1: pp. 889-901], Mann [Ref. 2: pp. 231-256].

The derivation of simple confidence limits for the reliability function for the *extreme value* distribution with parameters \( \xi \) and \( \delta \) has posed problems, because methods based on a pivotal quantity such as \( (\hat{\xi} - \xi) \delta \delta \) are inadequate. Lawless [Ref. 3: pp. 355-364], Johns and Lieberman [Ref. 4: pp. 135-175] and Thoman, Bain and Antle [Ref. 5: pp. 363-372] have developed nearly exact procedures for confidence limits for the reliability function of the *Weibull* and *extreme value* distributions. Schneider and Weissfeld [Ref. 6: pp. 179-186] have developed interval estimation methods for percentiles of the *Weibull* and *extreme value* distribution based on censored data. Although extensive methods have been developed for interval estimates of the reliability of a single component with the *Weibull* or *extreme value* failure distribution, very few parametric interval methods have been developed for system reliability using component test data with *Weibull* failure distributions.

Two approximate parametric interval estimation methods for reliability of coherent systems using component test data are developed and evaluated in this thesis. Evaluations of these procedures are performed using computer simulation for series systems only. One of these methods is developed for the reliability of a series system whose components have exponential failure distributions and different mission times. This procedure was found to be quite accurate and can be applied to coherent systems in general. This first procedure is then extended to the case where components of the sys-
tem have *Weibull* failure distributions. The method used is an extension of a non-parametric method developed by Myhri, J., Rosenfeld, A. and Saunders, S. [Ref. 7: pp.213-227].

The *normal* distribution is used extensively in some mechanical reliability models. Maximum likelihood and minimum variance unbiased estimators for $P(X \geq x)$ when both $\mu$ and $\sigma^2$ are unknown were developed over thirty years ago, Lieberman and Resnikoff [Ref. 8: pp. 457-516], Folks and others [Ref. 9: pp. 43-50], and Barton [Ref. 10: pp. 227-229]. Exact interval estimation procedures for $P(X \geq x)$ were developed by Owen and Hua [Ref. 11: pp.285-31], using the non-central $\chi$ distribution. Letting $X$ denote strength and $Y$ denote stress, mechanical reliability is sometimes modeled as the value for $P(X > Y)$. Approximate interval estimation procedures for this probability when $X$ and $Y$ are assumed to be normal have been developed by Church and Harris [Ref. 12: pp. 49-54] when the mean and variance of $Y$ are known. Downton [Ref. 13: pp. 551-558] modifies their procedure slightly to get more accurate bounds and suggests an approximate procedure when the means and variances of both $X$ and $Y$ are unknown. Lower confidence intervals for $P(X \geq x)$ obtained under the assumption of normality for $X$ and $Y$ can lead to serious error when either $X$ or $Y$ or both are truncated well into the tails. Consequently, $P(X \geq x)$ may be a more reasonable model of mechanical reliability where $x$ is chosen conservatively. A very accurate approximate lower confidence limit procedure for $P(Y \geq x)$ is developed and evaluated in this thesis. It can be computed easily.
II. INTERVAL ESTIMATION PROCEDURE - EXPONENTIAL CASE

A system of independent components is coherent if an increase in reliability of any one of its components does not cause a degradation in system reliability. Suppose a coherent system has \( k \) components. We assume that the failure distribution of component \( i \) is exponential with failure rate \( \lambda_i \). Then system reliability \( R_s \) can be written as a function of \( \lambda_i, t_i \; i = 1, 2, \cdots, k \); i.e.,

\[
R_s(t) = g(\lambda_1, \lambda_2, \cdots, \lambda_k, t_1, t_2, \cdots, t_k)
\]

where \( t_i = t_i(t) \) is the operating time for component \( i, \; i = 1, 2, \cdots, k \).

Let \( \lambda_m = \max(\lambda_1, \lambda_2, \cdots, \lambda_k) \) and \( r_i = \lambda_i/\lambda_m \; i = 1, 2, \cdots, k \). Then one can write

\[
R_s(t) = g(\lambda_m, r_1, \cdots, r_k, t_1, t_2, \cdots, t_k)
\]

If the \( r_i \) were known and \( \hat{\lambda}_m(U_{(\alpha)}) \) were an upper \( 100(1 - \alpha)\% \) confidence limit for \( \lambda_m \), the corresponding lower confidence limit for \( R_s(t) \) would be

\[
\hat{R}_s(t)_{L(\alpha)} = g(\hat{\lambda}_m(U_{(\alpha)}), r_1, \cdots, r_k, t_1, \cdots, t_k)
\]

Specifically, if the system is a series system of independent components, so that

\[
R_s(t) = \exp\left\{-\sum_{i=1}^{k} \lambda_i t_i\right\} = \exp\left\{-\lambda_m \sum_{i=1}^{n} r_it_i\right\}
\]

then,

\[
\hat{R}_s(t)_{L(\alpha)} = \exp\left\{-\hat{\lambda}_m(U_{(\alpha)}) \sum_{i=1}^{n} r_it_i\right\}
\]

If \( n_i \) items of component \( i \) are tested until failure, \( T_i \) denotes the total test time accumulated by all \( n_i \) items and \( n = \sum_{i=1}^{k} n_i \), then \( 2\lambda_n \sum_{i=1}^{k} r_i T_i \) is \( \chi^2_n \). See Bain and Engelhardt [Ref. 14].

An upper confidence limit for \( \lambda_m \) is
where $\chi^2_{n}$ is the 100(1 - $\alpha$)th percentile of the $\chi^2_n$ distribution. Corresponding equations for truncated testing are similar.

If the $r_i$ are unknown, the following methods estimate the values for $r_i$ from the data. One method uses the likelihood ratio estimate for $r_i$. The second method uses a jacknifed version of the first method. The two resulting confidence limits $\hat{R}_{s,\lambda(a)}$ and $R_{s,\lambda(a)}$, with and without jacknifing $\hat{r}_i$ respectively, are compared for relative accuracy. Quenouille [Ref. 15: pp. 353-360] first reported a method for estimating ratios that reduced the bias without increasing the variance. Miller [Ref. 16: pp. 1-15] gives an excellent review of the jackknife method which includes a discussion on the application of jacknifing to estimating ratios.

A. LOWER CONFIDENCE LIMIT $R_{s,\lambda}$ WITHOUT JACKNIFING

In this case, the maximum likelihood estimate of the ratio $r_i = \frac{\hat{\lambda}_i}{\hat{\lambda}_m}$ is

$$\hat{r}_i = \frac{\hat{\lambda}_i}{\hat{\lambda}_m}$$

(2.3)

where $\hat{\lambda}_i = n_i/T_i$, and $\hat{\lambda}_m = \max (\hat{\lambda}_1, \cdots, \hat{\lambda}_k)$. The resulting approximate upper confidence limit for $\lambda_m$ is

$$\hat{\lambda}_{m,U(a)} = \frac{\chi^2_{n,2n}}{k} \frac{1}{2 \sum_{i=1}^{k} r_i T_i}$$

(2.4)

where $n = \sum_{i=1}^{k} n_i$, and

$$T_i = \sum_{j=1}^{T_y} T_{ij} \quad i = 1, 2, \cdots, k$$

(2.5)

and $T_{ij}$ denotes the failure time of the $j$th test for component $i$.

The resulting approximate confidence bound $R_{s,\lambda(a)}$ is given by
The definitions for $\hat{\lambda}_i$, $\hat{r}_i$, $n_i$, and $T_i$ in Section A are also used in this section. Let $\hat{\lambda}_{i(o)}$ denote the estimate for $\lambda_i$ by removing $T_i$ from the data; i.e.,

$$\hat{\lambda}_{i(o)} = \frac{n_i - 1}{\sum_{l=1}^{n_i} T_{il}} \quad j = 1, 2, \ldots, n_i, \quad l \neq j$$  \hspace{1cm} (2.7)

and

$$\hat{\lambda}_{m(o)} = \frac{n_m - 1}{\sum_{l=1}^{n_m} T_{ml}} \quad j = 1, 2, \ldots, n_m, \quad l \neq j$$  \hspace{1cm} (2.8)

Then the jacknifed ratio estimate $\hat{r}_i^*$ is given by

$$\hat{r}_i^* = n^* \hat{r}_i - \frac{(n^* - 1) \sum_{j=1}^{n^*} \hat{r}_{ij(o)}}{n^*}$$  \hspace{1cm} (2.9)

where $n^* = \min(n_1, n_2, \ldots, n_k)$ and

$$\hat{r}_{ij(o)} = \frac{\hat{\lambda}_{i(o)}}{\hat{\lambda}_{m(o)}} \quad j = 1, 2, \ldots, n^*$$  \hspace{1cm} (2.10)

Now define $\hat{\lambda}_{i(o)}^*$ by

$$\hat{\lambda}_{i(o)}^* = \frac{2 \hat{r}_i^* - T_i}{2 \sum_{i=1}^{k} \hat{r}_i^* T_i}$$  \hspace{1cm} (2.11)

The corresponding confidence bound $\hat{R}_{s,t,\alpha}$ is given by

$$R_{S,L(\alpha)} = \exp\left\{ - \hat{\lambda}_{m,U(\alpha)} \sum_{i=1}^{k} \hat{r}_i^* T_i \right\}$$  \hspace{1cm} (2.6)
\[
\hat{R}_{S,L(a)} = \exp\left\{- \sum_{i=1}^{k} \hat{r}_i^* t_i \right\} \tag{2.12}
\]

When the \( n_i, \ i = 1, 2, \ldots, k \) differ considerably, this jackknife estimation procedure can be unbalanced. That is, the number of data points used to compute the jackknife estimate will differ from one component to another. It was decided to use this jacknifing procedure and determine the effect of differing sample sizes by examining the results of the simulations.

In equation (2.9), the jackknife estimate is constructed by using only the first \( n' \) observations from each component to obtain \( \hat{r}_{(n)} \) where \( n' = \min(n_1, n_2, \ldots, n_k) \). Of course it is rather arbitrary to take the first \( n' \) observations from \( n_i \), since any \( n' \) of the collection of \( n_i \) values could be used.

C. SIMULATIONS AND RESULTS FOR EXPONENTIAL CASE

1. Simulation
   
   a. Simulation language and package

   The programming language used to simulate this problem was VS FORTRAN on an IBM 3033. Also LEXPN in LLRANII was used to generate observed exponential random variates. SHSORT was used to perform the sort routines.

   b. Cases and Input parameters

   The six input parameters below determine the conditions for each simulation run. System reliability is determined by five parameters. Values of these parameters are given in the tables that show the results. The test plan simulated was to test \( n_i \) items until all fail.

   - number of component types, \( k = 5 \) and 15
   - system reliability, \( R_s \) : 0.9 and 0.975
   - significance level, \( \alpha \) : 0.05 and 0.2
   - component time, \( t_i \) : small to large (see tables)
   - sample size, \( n_i \) : small to large (see tables)

Reliability of a series system is expressed as \( R_s = \exp\left\{ - \sum_{i=1}^{k} \lambda_i t_i \right\} \). We chose arbitrarily to determine the failure rates, \( \lambda_t \), from this equation by assuming all \( \lambda_t \) to be equal, consequently,
c. Replications

The procedure was replicated 1000 times for each case to get 1000 values of \( R_{S,L(\cdot)} \) and 1000 values of \( \hat{R}_{S,L(\cdot)} \). We order each set of the 1000 values of \( R_{S,L(\cdot)} \) and \( \hat{R}_{S,L(\cdot)} \) in ascending order. Then the two approximate confidence bounds, \( R_{S,L(\cdot)(1000(1-\alpha))} \) and \( \hat{R}_{S,L(\cdot)(1000(1-\alpha))} \), for \( R_S \) are the 1000(1 - \( \alpha \))th order values of these two sets of data. Finally, for each of the two ordered sets of data, we find the order indices \( j_i \) and \( j_2 \) for which \( R_{S,L(\cdot)(j_i)} \) and \( \hat{R}_{S,L(\cdot)(j_2)} \) are closest to \( R_S \) for their respective sets. Then \( j_i / 1000 \) and \( j_2 / 1000 \) are called the two corresponding simulated true confidence levels.

2. Results

Tables 1 through 3 show the results of the 3 cases simulated. The results indicate that this interval estimation method using estimates of failure rate ratios will yield quite accurate lower confidence limits for system reliability when components have unknown constant failure rates. The jackknife method also yields very accurate confidence limits which are slightly conservative. That is the 100(1 - \( \alpha \)) percentile points of \( \hat{R}_{S,L(\cdot)} \), given in the tables, are slightly less than the true value of \( \hat{R}_S \). Consequently, \( P(\hat{R}_{S,L(\cdot)} \leq R_S) > 1 - \alpha \). Alternatively, one can say \( \hat{R}_{S,L(\cdot)} \) is a conservative 100(1 - \( \alpha \)) percent lower confidence limit procedure for \( R_S \).
Table 1. RELIABILITY OF A SERIES SYSTEM WITH SMALL NUMBER (LESS THAN 10) OF SAMPLE SIZES - EXPONENTIAL CASE

<table>
<thead>
<tr>
<th>Number of Components</th>
<th>Reliability of System</th>
<th>$\alpha$</th>
<th>Lower Confidence Limit</th>
<th>True Confidence Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>W/O</td>
<td>WITH Jacknifing</td>
<td>W/O Jacknifing</td>
</tr>
<tr>
<td>5</td>
<td>.90</td>
<td>.2</td>
<td>0.9017</td>
<td>0.8904</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.8985</td>
<td>0.8889</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9750</td>
<td>0.9720</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9746</td>
<td>0.9729</td>
</tr>
<tr>
<td>15</td>
<td>.90</td>
<td>.2</td>
<td>0.9012</td>
<td>0.8901</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9012</td>
<td>0.8933</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9745</td>
<td>0.9724</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9746</td>
<td>0.9725</td>
</tr>
</tbody>
</table>

Sample sizes for 5 components are 9, 7, 10, 8, 6
Sample sizes for 15 components are 6, 7, 5, 6, 7, 8, 9, 5, 8, 7, 9, 10, 7, 9, 6
Component times $t_i$ are 2, 3, 7, 8, 10, 3, 7, 10, 1, 7, 8, 3, 10, 1, 8

Table 2. RELIABILITY OF A SERIES SYSTEM WITH MEDIUM NUMBER (LESS THAN 30) OF SAMPLE SIZES - EXPONENTIAL CASE

<table>
<thead>
<tr>
<th>Number of Components</th>
<th>Reliability of System</th>
<th>$\alpha$</th>
<th>Lower Confidence Limit</th>
<th>True Confidence Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>W/O</td>
<td>WITH Jacknifing</td>
<td>W/O Jacknifing</td>
</tr>
<tr>
<td>5</td>
<td>.90</td>
<td>.2</td>
<td>0.8999</td>
<td>0.8978</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.8990</td>
<td>0.8983</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9750</td>
<td>0.9745</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9748</td>
<td>0.9745</td>
</tr>
<tr>
<td>15</td>
<td>.90</td>
<td>.2</td>
<td>0.8992</td>
<td>0.8960</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.8980</td>
<td>0.8981</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9751</td>
<td>0.9738</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9750</td>
<td>0.9741</td>
</tr>
</tbody>
</table>

Sample sizes for 5 components are 30, 21, 10, 15, 26
Sample sizes for 15 components are 3, 7, 10, 15, 20, 15, 7, 5, 30, 20, 10, 7, 13, 21, 30
Table 3. RELIABILITY OF A SERIES SYSTEM WITH LARGE NUMBER (LESS THAN 100) OF SAMPLE SIZES - EXPONENTIAL CASE

<table>
<thead>
<tr>
<th>Number of Components</th>
<th>Reliability of System</th>
<th>( \alpha )</th>
<th>Lower Confidence Limit</th>
<th>True Confidence Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>W:\O Jacknifing</td>
<td>WITH Jacknifing</td>
</tr>
<tr>
<td>5</td>
<td>.90</td>
<td>.2</td>
<td>0.8999</td>
<td>0.8991</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.8990</td>
<td>0.8992</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9750</td>
<td>0.9750</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9751</td>
<td>0.9750</td>
</tr>
<tr>
<td>15</td>
<td>.90</td>
<td>.2</td>
<td>0.9000</td>
<td>0.8988</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9000</td>
<td>0.8996</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9750</td>
<td>0.9747</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9750</td>
<td>0.9750</td>
</tr>
</tbody>
</table>

Sample sizes for 5 components are 30, 63, 75, 98, 26
Sample sizes for 15 components are 15, 40, 35, 17, 26, 67, 50, 65, 80, 32, 95, 100, 15, 45, 30
III. INTERVAL ESTIMATION PROCEDURE - WEIBULL CASE

A. LOWER CONFIDENCE LIMIT $R_{s,L}$

Consider a series system with $k$ components. Let the time to failure $X_i$ of component $i$ have a Weibull distribution with density

$$f_i(t_i) = \lambda_i^{\beta_i} \beta_i t_i^{\beta_i-1} \exp\{- (\lambda_i t_i)^{\beta_i}\}, \quad t_i > 0. \quad (3.1)$$

Then

$$R_i(t) = \exp\{- (\lambda_i t)^{\beta_i}\}, \quad t_i > 0 \quad (3.2)$$

and

$$R_{s}(t) = \exp\left\{- \sum_{i=1}^{k} \lambda_i^{\beta_i} t_i^{\beta_i}\right\} = \exp\left\{- \lambda_*^{\beta_*} \sum_{i=1}^{k} r_i t_i^{\beta_i}\right\}, \quad t > 0. \quad (3.3)$$

where $\lambda_* = \max \lambda_i$ and $r_i = \lambda_i / \lambda_*$. If the $\beta_i$ are known, $X_i$ has constant failure rate $\lambda_i^{\beta_i}$ and the procedures in Chapter II can be used to obtain $\hat{R}_{s,L(0)}$ and $R_{s,L(0)}$ with $T_v$ replaced by $T_v^{\beta_i}$ in equation (2.5). It is well known that $Y = \ln X_i$ has an extreme value distribution with CDF

$$F_Y(y) = 1 - \exp\left\{- \frac{y-\gamma}{\delta}\right\}$$

where $\gamma = \ln(1/\lambda_i)$ and $\delta_i = 1/\beta_i$.

Engelhardt and Bain [Ref. 17: p. 323] have developed the following simple unbiased estimators for $\gamma_i$ and $\delta_i$, using ordered values $Y_j = \ln X_{j0}$

$$\hat{\gamma}_i = \frac{1}{\hat{\beta}_i} = \frac{- \sum_{j=1}^{s} Y_j + \frac{s}{n - s} \sum_{j=s+1}^{n} Y_j}{n_i k_{n_i}} \quad (3.4)$$

where $s = [0.84 n_i] = \text{largest integer} \leq 0.84 n_i$ and $X_{j0}$ is the $j$th order statistics from the sample of size $n_i$ of $X_i$. Also
\[ \hat{\xi}_i = \ln\left( \frac{1}{\lambda_i} \right) = \bar{y}_i + \gamma \delta \]  

(3.5)

where \( \gamma = 0.5772 \) and \( \bar{y}_i = \sum_{j=1}^{n_i} Y_j / n_i \). Let

\[ \hat{\beta}_i = \frac{1}{\delta_i} \]  

(3.6)

and

\[ T_y = X_i \hat{\beta}_i \quad i = 1, 2, \ldots, n_i \quad j = 1, 2, \ldots, k. \]  

(3.7)

We approximate the distribution of \( T_y \) by the exponential distribution with failure rate \( \hat{\lambda}_i = \lambda_i \) and proceed as in Chapter II. Define

\[ \hat{\lambda}_i = \frac{n_i}{T_i} \]  

(3.8)

where \( T_i = \sum_{j=1}^{n_i} T_j \) \( i = 1, 2, \ldots, k \). Let \( \hat{\lambda}_m = \max_i \hat{\lambda}_i \) and

\[ \hat{r}_i = \frac{\hat{\lambda}_i}{\hat{\lambda}_m} \]  

(3.9)

Then an approximate upper confidence limit for \( \hat{\lambda}_m \) is given by

\[ \hat{\lambda}_{m,U(n)} = \frac{\chi^2_{2,2n}}{2 \sum_{i=1}^{k} \hat{r}_i T_i} \]  

(3.10)

and the corresponding approximate lower confidence limit \( R_{S,L(n)} \) for \( R_S(t) \) is given by

\[ R_{S,L(n)} = \exp\left\{ -\hat{\lambda}_{m,U(n)} \sum_{i=1}^{k} \hat{r}_i \hat{\beta}_i \right\} \]  

(3.11)

where \( n = \sum_{i=1}^{k} n_i \).

This procedure is labeled the Formula procedure in the tables that follow in this section. Its distinguishing feature is the equation for \( \hat{\beta}_i \) given by equation (3.6). An alternative procedure, labeled the Newton - Raphson procedure, estimates \( \beta \) using the
maximum likelihood procedure which is obtained using a Newton - Raphson approximation method. Equations for \( \hat{\beta} \) under the Newton - Raphson procedure are provided in Appendix A.

**B. SIMULATIONS AND RESULTS**

1. Simulation
   
   a. Simulation language and package

   The programming language used to simulate this problem was VS FORTRAN on an IBM 3033. Also LEXPN in LLRANII was used to generate observed exponential random variates. SHSORT was used to perform the sort routines.

   b. Cases and Input parameters

   The six input parameters below determine the conditions for exercising the simulation runs. System reliability is determined by five parameters. Values of these parameters are given in the tables that show the results. The test plan simulated was to test \( n_i \) items until all fail.

   * number of component types, \( k \): 5 and 15
   * system reliability, \( R_s \): 0.9 and 0.975
   * significance level, \( \alpha \): 0.05 and 0.2
   * component time, \( t_i \): small to large (see tables)
   * sample size, \( n_i \): small to large (see tables)

   Reliability of a series system is expressed as
   
   \[
   R_s = \exp\left\{-\sum_{i=1}^{k} \lambda_i t_i \right\}.
   \]

   Failure rates \( \lambda_i \) can be determined from that equation by assuming all \( \lambda_i t_i \) to be equal, consequently,

   \[
   \lambda_i = \frac{-\ln R_s}{k t_i}.
   \]

   c. Replications

   The procedure was replicated 1000 times for each case to get 1000 values of \( R_{S,L(0)} \) and 1000 values of \( \hat{R}_{S,L(0)} \). We order each set of the 1000 values of \( R_{S,L(0)} \) and \( \hat{R}_{S,L(0)} \) in ascending order. Then the two approximate confidence bounds, \( R_{S,L(0)(1000(1-\alpha))} \) and \( \hat{R}_{S,L(0)(1000(1-\alpha))} \), for \( R_s \) are the 1000(1 - \( \alpha \))th order values of these two sets of data. Finally, for each of the two ordered sets of data, we find the order indices \( j_1 \) and \( j_2 \) for which \( R_{S,L(0)(j_1)} \) and \( \hat{R}_{S,L(0)(j_2)} \) are the closest to \( R_s \) for their respective sets. Then \( j_1 / 1000 \) and \( j_2 / 1000 \) are called the two corresponding true confidence levels.
2. Results

Tables 4 and 5 display the results of the simulations and determine the accuracy of \( \hat{R}_{L,0.01} \) given in equation (3.11) as a lower confidence limit procedure for system reliability, \( R_s \), for parameter values \( \lambda, \beta, t, \) and \( R_s \) given in the tables. The terms in the tables have the same meaning as the corresponding terms in Tables 1 through 3 which were explained in Section 2.C.2. The procedure would be exact for the Formula method if the values in the Formula column equal the corresponding numbers in the same row in the Reliability of System column. In Table 4 for example, the 80th percentile point of .9205 for the Newton-Raphson procedure is more accurate than the Formula procedure which has an 80th percentile point of .9324. The last column shows, however, that what we have called an 80% lower confidence limit procedure is in fact closer to a 61% procedure. The accuracy improves if the sample size is increased from 15 to 30 as indicated in Table 5. The accuracy improves even more if the number of components in the system increases from 5 to 15 as indicated in Tables 4 and 5.

The results in Tables 4 and 5 show that the \( \hat{R}_{L,0.01} \) method given by equation (3.11) is too optimistic - especially for small sample sizes and for systems with a small number of components. Modifications to this procedure are needed. These tables also indicate that the Newton-Raphson method is superior to the Formula method.
Table 4. ACCURACY OF $R_{5_L}$ AS A LOWER CONFIDENCE LIMIT FOR $R_s$ WITH SAMPLE SIZES OF 15 - WEIBULL CASE

<table>
<thead>
<tr>
<th>Number of Components</th>
<th>Reliability of System</th>
<th>$\alpha$</th>
<th>Lower Confidence Limit</th>
<th>True Confidence Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Formula</td>
<td>Newton Raphson</td>
</tr>
<tr>
<td>5</td>
<td>.90</td>
<td>.2</td>
<td>0.9249</td>
<td>0.9125</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9424</td>
<td>0.9321</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9837</td>
<td>0.9824</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9879</td>
<td>0.9845</td>
</tr>
<tr>
<td>15</td>
<td>.90</td>
<td>.2</td>
<td>0.9207</td>
<td>0.9036</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9333</td>
<td>0.9198</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9808</td>
<td>0.9758</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9850</td>
<td>0.9810</td>
</tr>
</tbody>
</table>

Initial beta for 5 components are 2.2, 2.4, 2.6, 2.8, 2.5
Initial beta for 15 components are 2.2, 2.4, 2.6, 2.8, 2.5, 2.3, 2.7, 2.5, 2.2, 2.6, 2.9, 2.8, 2.4, 2.6, 2.1

Component times $t_i$ are 2, 3, 7, 8, 10, 3, 7, 10, 1, 7, 8, 3, 10, 1, 8

Table 5. ACCURACY OF $R_{3_L}$ AS A LOWER CONFIDENCE LIMIT FOR $R_s$ WITH SAMPLE SIZES OF 30 - WEIBULL CASE

<table>
<thead>
<tr>
<th>Number of Components</th>
<th>Reliability of System</th>
<th>$\alpha$</th>
<th>Lower Confidence Limit</th>
<th>True Confidence Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Formula</td>
<td>Newton Raphson</td>
</tr>
<tr>
<td>5</td>
<td>.90</td>
<td>.2</td>
<td>0.9249</td>
<td>0.9125</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9424</td>
<td>0.9321</td>
</tr>
<tr>
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<td>.2</td>
<td>0.9837</td>
<td>0.9824</td>
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<tr>
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<td></td>
<td>.05</td>
<td>0.9879</td>
<td>0.9845</td>
</tr>
<tr>
<td>15</td>
<td>.90</td>
<td>.2</td>
<td>0.9207</td>
<td>0.9036</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9333</td>
<td>0.9198</td>
</tr>
<tr>
<td></td>
<td>.975</td>
<td>.2</td>
<td>0.9808</td>
<td>0.9758</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.05</td>
<td>0.9850</td>
<td>0.9810</td>
</tr>
</tbody>
</table>

Initial beta for 5 components are 2.2, 2.4, 2.6, 2.8, 2.5
Initial beta for 15 components are 2.2, 2.4, 2.6, 2.8, 2.5, 2.3, 2.7, 2.5, 2.2, 2.6, 2.9, 2.8, 2.4, 2.6, 2.1
IV. INTERVAL ESTIMATION PROCEDURE - NORMAL CASE

A. BACKGROUND

In recent years it has become popular to model mechanical reliability as $P(X > Y)$ where $X$ denotes strength and $Y$ denotes stress. This formulation of reliability is an average of $P(X > y | Y = y)$, since

$$P(X > Y) = E_Y P(X > y) = \int P(X > y) f_Y(y) dy$$

An alternative model is the worst case approach. In the worst case model, one selects a worst case value of $y$, say $y_o$, and then designs the strength, $X$, of the component so that $P(X > y_o) = R_o$ where $R_o$ is a reliability requirement. If $X$ has a normal distribution then this requirement imposes constraints on the mean and variance of $X$. This is usually done in a manner to comply with standard "safety factor" procedures.

The average model, $P(X > Y)$, uses two random variables and is subject to more random error than the worst case model. The accuracy of the expression $P(X > Y)$ for values of this expression close to one is highly suspect due to deviations in the tail probabilities of both $X$ and $Y$ from those assumed in the model. It is common to assume that both $X$ and $Y$ have normal distributions. Truncated normal distributions would be more appropriate for many types of mechanical equipment. Harris and Soms [Ref. 18: pp. 650-663] discuss implications of this problem. Very significant errors in point and interval estimation for reliability are readily demonstrated when $X$ is truncated normal but assumed to be normal in the more simple model which specifies that $R = P(X > Y)$. Table 10 shows this effect when $X$ is truncated above at $A + 1.645 \sigma$, where $Z$ is the 100 $(1 - \alpha)$th percentile point of the standard normal distribution. Church and Harris [Ref. 12: pp. 49-54] and Downton [Ref. 13: pp. 551-558] have developed approximate confidence intervals for $P(X > Y)$ when $X$ is normal with unknown mean and variance and $Y$ has the standard normal distribution.
Minimum variance unbiased estimators (MVUE) for \( R = P(X > y) \) are well known when \( X \) is normal with unknown mean and known variance and also when the variance is unknown. In the former case, Lieberman and Resnikoff [Ref. 8] developed the result
\[
\hat{R} = \Phi\left( \frac{\bar{X} - y}{\sigma \sqrt{\frac{n-1}{n}}} \right)
\]
which is MVUE for \( R \) where \( \Phi \) is the standard normal cumulative distribution function. When the variance is unknown, several versions of an integral expression for \( P(X > y) \) have been developed by Lieberman and Resnikoff [Ref. 8], Basu [Ref. 19] and Folks and others [Ref. 9].

Exact lower confidence interval estimates for \( P(X > y) \) when \( X \) is normal with unknown mean and variance involve the non-central \( t \)-distribution. Owen and Hua [Ref. 11] developed tables of the lower 90% and 95% confidence limit values \( R_L \) for \( P(X > y) \) based on the non-central \( t \)-distribution. These values are tabulated for values of \( k \) in the range -3.0 (.2) 6.0 and sample sizes \( n = 2 \) (1) 18, \( 21 \) (3) 30, \( 40 \) (20) 100, where \( k = (\bar{X} - y)/s \) and \( \bar{X} \) and \( s \) are the sample mean and standard deviation. Their tables are essentially exact. An approximation to their exact tabulated values is given by \( R'_L \) where \( R'_L = \Phi(y') \),
\[
y' = k - \left\{ \frac{1}{n} + \frac{k^2}{2(n - \sqrt{k})} \right\}^{1/2} t_{s,n-1}, \tag{4.1}
\]
\( k = (\bar{X} - y)/s \) and \( t_{s,n-1} \) is the 100(1 - \( \alpha \))th percentile of the \( t \) distribution with \( n - 1 \) degrees of freedom. Equation (4.1) was developed in this thesis. It is an extensive ad hoc modification of an equation developed by Church and Harris [Ref. 12]. Tables 6 and 7 display values of \( R_L = \Phi(y), y', R'_L \) and \( R'_L - R_L \) for \( k = 1, 2, 3, 4 \), sample sizes \( n = 10, 18, 30 \) and confidence levels 90% and 95%. \( R_L \) and \( y \) denote the "exact" lower confidence limits and corresponding \( \Phi^{-1}(R_L) \) values from Owen and Hua [Ref. 11]. Both \( y' \) and \( R'_L \) are given by equation (4.1). The accuracy of the approximate confidence interval is quite good relative to the values for \( R_L \) given by Owen and Hua [Ref. 11].
Table 6. APPROXIMATE ($R_x^*$) AND EXACT ($R_L$) 90% CONFIDENCE LIMITS FOR $P(X > Y_o)$

<table>
<thead>
<tr>
<th>n</th>
<th>$t_{0.05}$</th>
<th>k</th>
<th>$R_x;\Phi(y)$</th>
<th>$y$</th>
<th>$y'$</th>
<th>$R_x;\Phi(y')$</th>
<th>$R'_L - R_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.3830</td>
<td>1</td>
<td>0.68156</td>
<td>0.47194</td>
<td>0.45454</td>
<td>0.67528</td>
<td>-0.00628</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0.89130</td>
<td>1.23397</td>
<td>1.20199</td>
<td>0.86517</td>
<td>-0.00613</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0.97453</td>
<td>1.95262</td>
<td>1.88991</td>
<td>0.97025</td>
<td>-0.00428</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0.99602</td>
<td>2.65320</td>
<td>2.54950</td>
<td>0.99437</td>
<td>-0.00165</td>
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<tr>
<td>18</td>
<td>1.3334</td>
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<td>0.73037</td>
<td>0.61325</td>
<td>0.61133</td>
<td>0.72950</td>
<td>-0.00087</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0.92569</td>
<td>1.44512</td>
<td>1.44083</td>
<td>0.92488</td>
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<tr>
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<td>0.98684</td>
<td>-0.00071</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0.99877</td>
<td>3.02679</td>
<td>3.00614</td>
<td>0.99859</td>
<td>-0.00018</td>
</tr>
<tr>
<td>30</td>
<td>1.3114</td>
<td>1</td>
<td>0.75937</td>
<td>0.70424</td>
<td>0.70508</td>
<td>0.75960</td>
<td>0.00023</td>
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<td>0.94256</td>
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</tr>
<tr>
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<td></td>
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<td>0.99205</td>
<td>-0.00028</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0.99945</td>
<td>3.26207</td>
<td>3.25926</td>
<td>0.99940</td>
<td>-0.00005</td>
</tr>
</tbody>
</table>

Table 7. APPROXIMATE ($R_x^*$) AND EXACT ($R_L$) 95% CONFIDENCE LIMITS FOR $P(X > Y_o)$

<table>
<thead>
<tr>
<th>n</th>
<th>$t_{0.01}$</th>
<th>k</th>
<th>$R_x;\Phi(y)$</th>
<th>$y$</th>
<th>$y'$</th>
<th>$R_x;\Phi(y')$</th>
<th>$R'_L - R_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.8331</td>
<td>1</td>
<td>0.63052</td>
<td>0.33311</td>
<td>0.27702</td>
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<tr>
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<td>2</td>
<td>0.85187</td>
<td>1.04477</td>
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<tr>
<td></td>
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<td>3</td>
<td>0.95565</td>
<td>1.70308</td>
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<td>0.93655</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>0.99031</td>
<td>2.33813</td>
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<td>-0.00956</td>
</tr>
<tr>
<td>18</td>
<td>1.7396</td>
<td>1</td>
<td>0.69504</td>
<td>0.51007</td>
<td>0.49292</td>
<td>0.68896</td>
<td>-0.00608</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>0.90348</td>
<td>1.30221</td>
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</tr>
<tr>
<td></td>
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<td>-0.00322</td>
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<td></td>
<td></td>
<td>4</td>
<td>0.99735</td>
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</tr>
<tr>
<td>30</td>
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<td>0.61789</td>
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<tr>
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<tr>
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<td>0.98855</td>
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<tr>
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<td>3.04028</td>
<td>0.99873</td>
<td>-0.00021</td>
</tr>
</tbody>
</table>
Tables 8 and 9 display the results of computer simulations with 1,000 replications to check the accuracy of the $R_i$ method for 80% and 90% lower confidence limits for $P(X > y) \equiv R$ for $y = 3$, with various values of $\sigma$ and $\mu$ determined so that $R$ equals the values shown. The procedure would be exact at the 80% level if the values in the column labelled $\alpha = .2$ equal the corresponding values of $R$ in the same row. The "true" confidence level corresponds to the index $i(R)$ of 1,000 ordered values of $R_{[:i]}$ for which $R_{[:i]}(R) = R$. For example, in the seventh row of Table 8, $R = .950$, $\sigma = 20$, $N = 10$, $R_{[:20],300}$ was .9575, and $R_{[:10],900}$ was .9533. Also $R_{[:20],886} = .950$ and $R_{[:10],886} = .950$. Tables 8 and 9 indicate that the $R_i$ procedure given by equation (4.1) is quite accurate at the 80% level of confidence for the cases simulated.
Table 8. ACCURACY ANALYSIS OF R₉ PROCEDURE FOR 80% AND 90% LOWER CONFIDENCE LIMITS FOR P(X > 3) WHEN X IS NORMALLY DISTRIBUTED

<table>
<thead>
<tr>
<th></th>
<th>σₓ</th>
<th>n</th>
<th>$R_{L,100α(3)}$</th>
<th>True Confidence Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>10</td>
<td>0.9624</td>
<td>0.7310</td>
<td>0.8970</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>0.9546</td>
<td>0.7560</td>
<td>0.8820</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>0.9517</td>
<td>0.7800</td>
<td>0.8980</td>
</tr>
<tr>
<td>0.95</td>
<td>10</td>
<td>0.9606</td>
<td>0.7470</td>
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</tr>
<tr>
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<td>0.8770</td>
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<td>75</td>
<td>0.9509</td>
<td>0.7900</td>
<td>0.8770</td>
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<td>0.7380</td>
<td>0.8860</td>
</tr>
<tr>
<td></td>
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<td>0.7410</td>
<td>0.8730</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>0.9511</td>
<td>0.7830</td>
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Table 9. ACCURACY ANALYSIS OF $R_L^*$ PROCEDURE FOR 80% AND 90% LOWER CONFIDENCE LIMITS FOR $p(X > 30)$ WHEN X IS NORMALLY DISTRIBUTED

<table>
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<tr>
<th>R</th>
<th>$\sigma_x$</th>
<th>n</th>
<th>$R_{L,(\alpha)=1}$</th>
<th>True Confidence Level</th>
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<tr>
<td></td>
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<td>$\alpha = .2$</td>
<td>$\alpha = .1$</td>
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<tr>
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<td></td>
<td></td>
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<td>$R_{L,(\alpha)=1}$</td>
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Table 10 displays the inaccuracies of the $R_L$ lower confidence limit procedure for $P(X > 3)$ using equation (4.1), which assumes $X$ is $N(\mu, \sigma^2)$, when in fact $X$ has the distribution of a normal random variable with mean $\mu$ and variance $\sigma^2$ that has been truncated at $\mu + 1.645\sigma$. Note that $\mu + 1.645\sigma > > 3$, because $P(X > 3) = .95, .99$ and .995.

Examination of Table 10 reveals gross inaccuracies; consequently, even when the distribution of $X$ is truncated far above the value $y$, the exact lower confidence limit for $P(X > y)$ can be greatly in error when computed under the assumption that $X$ is normal. This problem will be compounded when one is computing "exact" confidence intervals for $P(X > Y)$ assuming both $X$ and $Y$ are normal when in fact one or both may be truncated normal. This suggests that modeling mechanical reliability as $P(X > Y)$ may be more risky than more standard engineering approaches for modeling mechanical reliability which include the notion of safety factors.
<table>
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<th>(\sigma_x)</th>
<th>n</th>
<th>(R_{L, 1000(1-\alpha)})</th>
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<th>(\alpha = .1)</th>
<th>True Confidence Level</th>
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V. CONCLUSIONS AND RECOMMENDATIONS

The lower interval estimation procedure for reliability of coherent systems which was developed in Chapter II appears to be accurate, easy to use, and applicable to coherent systems. Although this procedure assumes that failure times of all components of the system are independent and have exponential probability distributions, it can be easily extended to systems with component failure distributions that can be transformed into exponential failure distributions; e.g., Weibull distribution with known shape parameter. This procedure has potential for being combined with a similar procedure for systems with cyclical components. The combined procedure would provide for use of binomial component test data and exponential component test data to compute lower confidence limits on the reliability of coherent systems with both cyclic and continuous components. Such an extension could use a failure rate ratio estimation procedure similar to that developed in this thesis. Such a method should be explored.

The interval estimation method for the reliability of a system with components that have Weibull failure times is not sufficiently accurate to be applied to systems that have 10 or fewer components each with ten or fewer tests. This procedure needs further study and refinement.

The approximate lower confidence limit for component reliability $P(X > y)$ when $X$ is normally distributed with unknown mean and variance is very accurate. It does not require an extensive set of tables such as those developed by Owen and Hua[Ref. 11], but only requires the use of the standard $t$ tables.
APPENDIX A. MLE OF WEIBULL PARAMETERS BY THE NEWTON - RAPHSON METHOD

Let $X \sim WEL(\alpha, \beta)$. Then the likelihood function for the first $r$ ordered observations from a random sample of size $n$ is given by

$$f(x_{1:n}, \ldots, x_{n:n}) = \frac{n!}{(n-r)!} \left[ \prod_{i=1}^{r} f_X(x_{i:n}) \right] [1 - F_X(x_{r:n})]^{n-r}$$

$$= \frac{n!}{(n-r)!} \left( \frac{\beta}{\alpha} \right)^r \prod_{i=1}^{r} \left( \frac{x_{i:n}}{\alpha} \right)^{\beta-1} \exp \left\{ - \left[ \sum_{i=1}^{r} \left( \frac{x_{i:n}}{\alpha} \right)^{\beta} + (n-r) \left( \frac{x_{r:n}}{\alpha} \right)^{\beta} \right] \right\}$$

Setting the partial derivatives of this likelihood with respect to $\alpha$ and $\beta$ equal to zero gives the MLE's $\hat{\alpha}$ and $\hat{\beta}$ as solutions to the equations

$$\sum_{i=1}^{n} x_i^{\hat{\beta}} \ln x_i \quad \frac{1}{\hat{\beta}} = \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

and

$$\hat{\alpha} = \left[ \frac{\sum_{i=1}^{n} x_i^{\hat{\beta}}}{n} \right] \frac{1}{\hat{\beta}}$$

where $n = r$. It can be shown that these equations have unique solutions which are the maximum likelihood estimates. The NEWTON - RAPHSON method for solving an equation $g(\hat{\beta}) = 0$ is to determine successive approximations $\hat{\beta}_n$, where $\hat{\beta}_{n+1} = \hat{\beta}_n - g(\hat{\beta}_n)/g'(\hat{\beta}_n)$. Therefore, the estimates of $\alpha$ and $\beta$ can be solved by letting

24
The derivative of $g(\hat{\beta})$ is

$$g'(\hat{\beta}) = \frac{\sum_{l=1}^{n} x_i^{\hat{\beta}} \ln x_i}{\sum_{l=1}^{n} x_i^{\hat{\beta}}} - \frac{1}{\hat{\beta}} - \frac{1}{n} \sum_{l=1}^{n} \ln x_i + \left(\frac{1}{\hat{\beta}}\right)^2.$$
APPENDIX B. FORTRAN CODE FOR INTERVAL ESTIMATION
PROCEDURE - EXPONENTIAL CASE

PROGRAM EXPONE
C
******************************************************************************

C * THIS IS A PROGRAM TO COMPUTE THE TRUE CONFIDENCE LIMIT OF A SERIES
C SYSTEM WITH / WITHOUT JACKNIFE METHOD AND COMPAR THE DIFFERENCE OF THOSE
C RESULTS.
C *

C BELOWS ARE GIVEN OR ASSUMED.
C *
C * N ; NUMBER OF COMPONENTS TYPE
C * RSYS ; RELIABILITY OF A SERIES SYSTEM
C * ALPHA ; SIGNIFICANCE LEVEL
C * TIME ; TEST TIME
C * SAMPLE ; SAMPLE SIZE FOR EACH TYPE OF COMPONENT
C *

C THESE ARE VARIABLES USED.
C *
C * B ; TEMPORARY ARRAY FOR EXPONENTIAL RANDOM VARIATE
C * BIGLAM ; LARGEST VALUE OF LAMBD
C * CASE ; COUNTER
C * CHISQR ; CHI-SQUARE VALUE FOR GIVEN ALPHA AND DF
C * CLR1 ; TRUE CONFIDENCE LIMIT FROM NON-JACKNIFING
C * CLR2 ; TRUE CONFIDENCE LIMIT FROM JACKNIFING
C * CTIME ; TEST TIME OF INDIVIDUAL COMPONENT
C * DF ; DEGREE OF FREEDOM FOR CHI-SQUARE.
C * DIFF ; ABSOLUTE VALUE OF DIFFERENCE BETWEEN Ri & R2
C * LAMBDA ; FAILURE RATE OF EACH COMPONENT TYPE
C * LAMHAT ; LAMDA HAT FOR JACKNIFING
C * LAMHST ; FINAL LAMDA HAT FROM NON-JACKNIFING
C * LAMHST ; FINAL LAMDA HAT FROM JACKNIFING
C * LAMMAX ; LARGEST VALUE OF LAMHAT
C * LOMIT ; LAMBA WITH 1 COMPONENT OMITTED
C * KEY1 ; TEMPORARY ARRAY
C * KEY2 ; TEMPORARY ARRAY
C * R ; INITIALLY COMPUTED RELIABILITY FOR EACH
C * COMPONENT TYPE
C * R1 ; COMPUTED RELIABILITY BY NON-JACKNIFING
C * R2 ; COMPUTED RELIABILITY BY JACKNIFING
C * RATI01 ; RATIO OF LAMDBA FOR NON-JACKNIFING
C * RATI02 ; RATIO OF LAMDBA FOR JACKNIFING
C * ROMIT ; RATIO WITH 1 COMPONENT OMITTED
C * ROMSUM ; USED FOR JACKNIF, SUM OF R WITH OMIT 1
C * RSTAR ; FINALLY COMPUTED R VALUE ( R STAR
C * RVAL1 ; R( 500 * (1-ALPHA) ) FOR NON-JACKNIFING
C * RVAL2 ; R( 500 * (1-ALPHA) ) FOR JACKNIFING
C * T ; TOTAL TEST TIME OF EACH COMPONENT TYPE
C * ZALPHA ; RIGHT PERCENTILE POINT(NORMAL DISTRIBUTION)
C *
PARAMETER (NN = 500)

REAL TIME(15), ALPHA(2), ZALPHA(2), RSYS(2), LAMBDA(15), P
REAL LAMMAX, LAMHAT(15), B(100), RSTAR(15), LAMHM
REAL RATIO1(15), RATIO2(15), BIGLAM, DIFF(15)
REAL LOMIT(15,100), CTIME(15,100), ROMIT(15,100), T(15)
REAL ROMSUM, SUM1, SUM2, LAMHST, R1(NN), R2(NN), KEY1(NN)
REAL RVAL1(15), RVAL2(15), CLR1(15), CLR2(15), KEY2(NN)
REAL DIFFR1(15), DIFFR2(15), R(15)

INTEGER SAMPLE(15), N(2), DF, CASE

DATA N / 5,0 /
DATA RSYS / .9, .975 /
DATA TIME / 2,3,7,8,10,3,7,10,1,7,8,3,10,1,8 /
DATA SAMPLE / 30,63,75,98,26,15,7,5,30,20,10,7,13,21,30 /
DATA ALPHA / .2, .05 /
DATA ISEED / 1736 /

C /* SUBROUTINE ZTABLE COMPUTES THE RIGHT PERCENT POINT ZALPHA */
C /* FROM RIGHT CUMULATIVE PROBABILITY ALPHA */

DO 10 I = 1, 2
   CALL ZTABLE(ALPHA(I), ZALPHA(I))
10 CONTINUE

CASE = 0

/* II IS INDEX FOR N */

DO 150 II = 1, 2

C /* COMPUTE THE DEGREE OF FREEDOM FOR CHI-SQUARE */

DF = 0.
DO 20 I = 1, N(II)
   DF = DF + SAMPLE(I)
20 CONTINUE

DF = 2 * DF

C /* FINDING COMPONENT TYPE THAT HAS THE MINIMUM NUMBER */
C /* OF SAMPLE SIZE */

NSTAR = 999
DO 25 I = 1, N(II)
   IF (SAMPLE(I) .LE. NSTAR) THEN
      NSTAR = SAMPLE(I)
   ENDIF
25 CONTINUE
DO 140 JJ = 1, 2

C /* COMPUTE LAMBDA FROM THE GIVEN EQUATION AND */
C /* FIND THE BIGGEST LAMBDA */

BIGLAM = 0
DO 30 K = 1, N(II)
    LAMBDA(K) = ( -ALOG(RSYS(JJ)) / N(II)) / TIME(K)
    IF( LAMBDA(K) .GE. BIGLAM) BIGLAM = LAMBDA(K)
    R(K) = EXP( - LAMBDA(K) * TIME(K) )
30 CONTINUE

C /* COMPUTE CHI-SQUARE VALUE FOR GIVEN ALPHA AND DF */

DO 130 LL = 1, 2

   IF ( DF .EQ. 1 ) THEN
       P = ALPHA(LL) / 2
       CALL ZTABLE(P, ZALPHA(LL))
       CHISQR = ZALPHA(LL) ** 2
   ELSE IF ( DF .EQ. 2 ) THEN
       CHISQR = -2 * ALOG(ALPHA(LL))
   ELSE IF ( DF .GE. 3 ) THEN
       CHISQR = DF * ( 1 - 2./(9 * DF) +
                       ZALPHA(LL) * SQRT( 2./(9 * DF)) ) ** 3
   ENDIF

CASE = CASE + 1
DIFF(CASE) = 0
DO 120 L = 1, NN
    LAMMAX = -99.
    DO 50 I = 1, N(II)

C /* GENERATE EXPONENTIAL RANDOM */
C /* NUMBERS WITH MU = 1 */

CALL LEXPN(ISEED, B, SAMPLE(I), 1, 0)
    T(I) = 0
    DO 40 J = 1, SAMPLE(I)

C /* CONVERT TO EXPONENTIAL RANDOM */
C /* NUMBERS WITH MU = LAMBDA AND */
C /* ADD THOSE FOR EACH COMPONENT */
C /* TYPE */
B(J) = B(J) / LAMBDA(I)
T(I) = T(I) + B(J)
CTIME(I,J) = B(J)
40 CONTINUE

LAMHAT(I) = SAMPLE(I) / T(I)
C /* FINDING MAXIMUM LAMBDA HAT AND ITS */
C /* INDEX */
IF ( LAMHAT(I) .GE. LAMMAX ) THEN
  M = I
  LAMMAX = LAMHAT(I)
ENDIF
50 CONTINUE

C /* RATIO1 IS FOR WITHOUT JACKNIFE */
C /* RATIO2 IS FOR WITH JACKNIFE */
DO 60 I = 1, N(I)
  RATIO1(I) = LAMBDA(I) / BIGLAM
  RATIO2(I) = LAMHAT(I) / LAMMAX
60 CONTINUE

C /* PART OF JACKNIFE METHOD FOR LAMBDA */
C /* WITH OMIT 1 VARIABLE EACH TIME */
DO 90 I = 1, N(I)
  DO 80 J = 1, SAMPLE(I)
    LOMIT(I,J) = 0.
    DO 70 K = 1, SAMPLE(I)
      IF (J .NE. K) THEN
        LOMIT(I,J) = LOMIT(I,J) + CTIME(I,K)
      ENDIF
    70 CONTINUE
    LOMIT(I,J) = (SAMPLE(I)-1) / LOMIT(I,J)
  80 CONTINUE
90 CONTINUE

29
```c
/* ADAPT ABOVE RESULT TO OUR EQUATION TO */
/* GET THE RELIABILITY AND TRUE CONFIDENCE */

SUM1 = 0.
SUM2 = 0.
SUM3 = 0.
SUM4 = 0.

DO 110 I = 1, N(II)

/* NON JACKNIFING ( ORIGINAL ) */
SUM1 = SUM1 + RATIO1(I) * T(I)
SUM2 = SUM2 + RATIO1(I) * TIME(I)

/* WITH JACKNIFING */
ROMSUM = 0.
DO 100 J = 1, NSTAR
   ROMIT(I,J) = LOMIT(I,J) / LOMIT(M,J)
   ROMSUM = ROMSUM + ROMIT(I,J)
100 CONTINUE

RSTAR(I) = NSTAR * RATIO2(I) -
         (NSTAR - 1) * ROMSUM / NSTAR
SUM3 = SUM3 + RSTAR(I) * T(I)
SUM4 = SUM4 + RSTAR(I) * TIME(I)
110 CONTINUE

/* R1 ; RELIABILITY OF A SYSTEM WITHOUT */
/* JACKNIFE */
/* R2 ; RELIABILITY OF A SYSTEM WITH */
/* JACKNIFE */

LAMHM = CHISQR / (2 * SUM1)
R1(L) = EXP(-LAMHM * SUM2)

LAMHST = CHISQR / (2 * SUM3)
R2(L) = EXP(-LAMHST * SUM4)

IF ( ABS(R1(L) - R2(L)) .GE. DIFF(CASE) ) THEN
   DIFF(CASE) = ABS(R1(L) - R2(L))
   DIFFR1(CASE) = R1(L)
   DIFFR2(CASE) = R2(L)
ENDIF

120 CONTINUE

/* NONIMSL LIBRARY 'SHSORT' WILL SORT R1, R2 */
/* BY SHELL SORT ALGORITHM */
```
CALL SHSORT(R1, KEY1, NN)
CALL SHSORT(R2, KEY2, NN)

MM = NN * (1 - ALPHA(LL))

RVAL1(CASE) = R1(MM)
RVAL2(CASE) = R2(MM)

C /* SUBROUTINE FINDJ FINDS THE INDEX OF R1, R2 */
C /* WHICH THE VALUE OF IT IS CLOSEST TO RSYS. */

CALL FINDJ(R1, NN, RSYS(JJ), J1)
CALL FINDJ(R2, NN, RSYS(JJ), J2)

CLR1(CASE) = J1 / FLOAT(NN)
CLR2(CASE) = J2 / FLOAT(NN)

130 CONTINUE
140 CONTINUE
150 CONTINUE

WRITE(6,600)
WRITE(6,650)
WRITE(6,670)

CASE = 1
DO 210 II = 1, 2
   DO 200 JJ = 1, 2
      DO 190 LL = 1, 2
         WRITE(6,700) N(II), RSYS(JJ), ALPHA(LL), RVAL1(CASE), RVAL2(CASE), DIFF(CASE), CLR1(CASE), CLR2(CASE)
   190 CONTINUE
200 CONTINUE
WRITE(6,777) ( SAMPLE(J), J=1,N(II) )
WRITE(6,999) ( R(J), J=1,II )

210 CONTINUE

600 FORMAT('1',5(/),7X,'**** RELIABILITY OF SERIES SYSTEM *****')
650 FORMAT(/,T50,'R1, CL1 ; WITHOUT JACKNIFING',
   *,/,'T50',R2, CL2 ; WITH JACKNIFING')
670 FORMAT(/,T6,'NUMBER OF',T19,'RELIABILITY',T33,'ALPHA',T46,'R1',
   *,T56,'R2',T63,'MAX (R1 - R2)',T79,'TRUE',T87,'TRUE',/,
   *,T6,'COMPONENTS',T19,'OF SYSTEM',T79,'C. L. 1',T87,'C. L. 2',
   *,/,'T5,89('-'))

31
SUBROUTINE ZTABLE(ALPHA, ZALPHA)
C
C << SUBROUTINE ZTABLE COMPUTES RIGHT PERCENT POINT ZALPHA FROM >>
C
C << RIGHT CUMULATIVE PROBABILITY ALPHA >>
C
REAL ALPHA, ZALPHA

IF ((ALPHA .GT. 0.0) .OR. (ALPHA .LT. 1)) THEN
    W = - ALOG(4 * ALPHA * (1 - ALPHA))
    ZALPHA = SQRT(W * (2.06118 - (5.72622 / (W + 11.6406))))
    IF (ALPHA .GT. 0.5) THEN
        ZALPHA = - ZALPHA
    ENDIF
ENDIF
RETURN
END

********************************************************************

SUBROUTINE SHSORT(A, KEY, N)
DIMENSION A(N), KEY(N)
M1=1
6 M1=M1*2
   IF (M1 .LE. N) GO TO 6
   M1=M1/2-1
   MM=MAX0(M1/2,1)
   GO TO 21
20 MM=MM/2
   IF (MM .LE. 0) GO TO 100
21 K=N-MM
22 DO 1 J=1,K
II=J
11 IM=II+MM
   IF (A(IM) .GE. A(II)) GO TO 1
   TEMP=A(II)
   IT=KEY(II)
   A(II)=A(IM)
   KEY(II)=KEY(IM)
   A(IM)=TEMP
   KEY(IM)=IT
   II=II-MM
   IF (II .GT. 0) GO TO 11
   CONTINUE
GO TO 20
100 RETURN
END

*******************************************************************************
*******************************************************************************

SUBROUTINE FINDJ(A, NN, R, J)
C
C   < SUBROUTINE FINDJ FINDS THE INDEX OF ARRAY A WHICH THE VALUE OF IT IS
C   << CLOSEST TO R. >>
C
REAL A(NN), R, VALUE
INTEGER J

VALUE = ABS(A(NN) - R)

DO 100 I = NN-1, 1, -1
   IF (ABS(A(I) - R) .LT. VALUE) THEN
      VALUE = ABS(A(I) - R)
      J = I + 1
   ELSE
      ENDIF
100 CONTINUE

RETURN
END

*******************************************************************************
*******************************************************************************
APPENDIX C. FORTRAN CODE FOR INTERVAL ESTIMATION

PROCEDURE - WEIBULL CASE

PROGRAM WEIBUL

C
C

********************************************************************

*  THIS PROGRAM COMPUTES THE RELIABILITY OF A SERIES SYSTEM  *
*  USING TWO DIFFERENT METHODS FOR THE CASE WHEN FAILURE    *
*  OF EACH COMPONENTS IS WEIBULLY DISTRIBUTED.               *
*  AND ALSO COMPARES THE ESTIMATE OF SHAPE PARAMETERS ( BETA ) *
*  WHICH IS COMPUTED FROM TWO DIFFERENT METHODS.             *
*  
*  THE SERIES SYSTEM CONSIDERED IN THIS PROGRAM HAS       *
*  N COMPONENT TYPES AND EACH COMPONENT HAS SAME NUMBER OF *
*  SAMPLE SIZES.                                           *
*  
*  BELOWS ARE GIVEN OR ASSUMED ;                           *
*  
*  BETA ; SHAPE PARAMETER OF WEIBULL DISTRIBUTION          *
*  ALPHA ; SIGNIFICANCE LEVEL                              *
*  X ; TEST TIME                                           *
*  RSYS ; RELIABILITY OF A SERIES SYSTEM                   *
*  
*  THESE ARE VARIABLES USED ;                              *
*  
*  BETHAT ; BETA HAT FROM THE FORMULA                      *
*  BETNEW ; BETA HAT FROM NEWTON - RAPHSON METHOD          *
*  CHISQR ; CHI-SQUARE VALUE FOR GIVEN ALPHA AND DF        *
*  CLR ; TRUE CONFIDENCE LIMIT FROM THE FORMULA            *
*  CLRNEW ; TRUE CONFIDENCE LIMIT FROM THE N-R METHOD      *
*  EXPO ; EXPONENTIAL RANDOM NUMBERS WITH MU=1             *
*  KSUBM ; K SUB M VALUE                                   *
*  LAMBDA ; SCALE PARAMETER OF WEIBULL DISTRIBUTION        *
*  LAMBIG ; MAXIMUM VALUE OF LAMBDA FROM THE N-R METHOD    *
*  LANHAT ; ESTIMATES OF LAMBDA FROM THE FORMULA           *
*  LANNAX ; MAXIMUM VALUE OF LAMBDA FROM THE FORMULA       *
*  LAMMU ; FINALLY COMPUTED LAMBDA FROM THE FORMULA        *
*  LAMSTR ; LAMBDA TO THE BETA                             *
*  LANNW ; ESTIMATES OF LAMBDA FROM THE N-R METHOD         *
*  LAMHNR ; FINALLY COMPUTED LAMBDA FROM THE N-R METHOD    *
*  RATIO ; RATIO OF LAMBDA FOR THE FORMULA                 *
*  RATNEW ; RATIO OF LAMBDA FOR THE N-R METHOD             *
*  RHT ; R( 500 * (1-ALPHA) ) FOR THE FORMULA             *
*  RHNTNEW ; R( 500 * (1-ALPHA) ) FOR THE N-R METHOD       *
*  XTOBET ; X TO THE BETHAT                                *
*  XHAT ; SUM OF XIJHAT                                    *
*  XIJHAT ; W TO THE BETHAT                                *
*  XIJNEW ; W TO THE BETNEW                                *
*  XNEW ; SUM OF XIJNEW                                   *
*  XTNBET ; X TO THE BETHAT                                *
*  W,WEIB ; WEIBULL RANDOM NUMBERS                         *

********************************************************************
C
* ZALPHA ; RIGHT PERCENTILE POINT (NORMAL DISTRIBUTION) *
**

PARAMETER ( N=5, M=30 , NN= 500 )
PARAMETER ( N=15, M=30 , NN= 500)

REAL   BETA(N), LAMBDA(N), X(N), XHAT(N), XTOBET(N), XX(N,M)
REAL   W(N,M), Y(N,M), BETHAT(N), WBHAT(N,M), LAMHAT(N)
REAL   RATIO(N), LAMSTR(N), LAMMAX, RSYS(2), KSUBM(N), LAMMU
REAL   EXPO(M), ALPHA(2), ZALPHA(2), RHAT(10), RSL(NN), CHISQR
REAL   B(M%), KEY(M), KEY1(NN), KEY2(NN), XIJHAT(N,M)
REAL   WEIB(M), BETNEW(N), XBENEW(N), XNEW(N), XIJNEW(N,M)
REAL   LAMNEW(N), LAMBIG, RATNEW(N), LAMUNR, RSLNEW(NN)
REAL   RHTNEW(10), CLR(10), CLRNEW(10)
REAL   BHTBAR, BNRBAR, BHTMSE, BNRMSE

INTEGER  S(N), DF, ICOUNT, SAMPLE(N), CASE

DATA ISEED / 1736 /
DATA ALPHA / .2, .05 /
DATA RSYS / .90, .975 /

C DATA BETA / 1.2,1.4,1.6,1.8,1.5 /
C DATA BETA / 1.2,1.4,1.6,1.8,1.5,1.3,1.7,1.5,1.2,1.6,1.9,1.8,1.4, 1.6,1.1/
C DATA BETA / 2.2,2.4,2.6,2.8,2.5 /
DATA BETA / 2.2,2.4,2.6,2.8,2.5,2.3,2.7,2.5,2.2,2.6,2.9,2.8,2.4, 2.6,2.1/

C DATA X / 2,3,7,8,10/
DATA X / 2,3,7,8,10,5,7,10,1,7,8,3,10,1,8/

C DATA SAMPLE / 9,7,10,8,6/
C DATA SAMPLE /30,21,10,15,26/
C DATA SAMPLE / 30,63,75,98,26 /

C DATA SAMPLE / 6,7,5,6,7,8,9,5,8,7,9,10,7,9,6/
C DATA SAMPLE / 3,7,10,15,20,15,7,5,30,20,10,7,13,21,30/
C DATA SAMPLE / 15,40,35,17,26,67,50,65,80,32,95,100,15,45,30/

DATA SAMPLE / 15 * 15 /

CASE = 1

DO 10 I = 1, N
   IF ( SAMPLE(I) .LE. 15) KSUBM(I) = 1.40
   IF ( SAMPLE(I) .LE. 30) KSUBM(I) = 1.50
10 CONTINUE

DO 220 JJ = 1, 2

C /* FINDING THE LAMBDA AND LAMBDA STAR FROM THE GIVEN DATA */
C DO 20 I = 1, N
\[
\text{LAMBDA(I)} = \left( \frac{\text{ALOG(RSYS(J))}}{N} \right) \\
\text{LAMSTR(I)} = \frac{\text{LAMBDA(I)} \times \text{BETA(I)}}{X(I)} \\
\text{S(I)} = 0.84 \times \text{SAMPLE(I)} - 0.5
\]

20 CONTINUE

C /* SUBROUTINE ZTABLE COMPUTES THE RIGHT PERCENT POINT ZALPHA */
C /* FROM RIGHT CUMULATIVE PROBABILITY ALPHA */

DO 40 I = 1, 2
CALL ZTABLE(ALPHA(I), ZALPHA(I))
40 CONTINUE

C /* COMPUTE THE DEGREE OF FREEDOM FOR CHI-SQUARE */

DF = 0
DO 60 I = 1, N
DF = DF + SAMPLE(I)
60 CONTINUE
DF = 2 * DF

C /* COMPUTE CHI-SQUARE VALUE FOR GIVEN ALPHA AND DF */

DO 200 LL = 1, 2

IF ( DF .EQ. 1 ) THEN
P = ALPHA(LL) / 2
CALL ZTABLE(P, ZALPHA(LL))
CHISQR = ZALPHA(LL) ** 2
ELSE IF ( DF .EQ. 2 ) THEN
CHISQR = -2 * ALOG(ALPHA(LL))
ELSE IF ( DF .GE. 3 ) THEN
CHISQR = (1 - 2./(9 * DF) + 
ZALPHA(LL) * SQRT( 2./(9 * DF))) ** 3
ENDIF

LAMMAX = 0
LAMBIG = 0
DO 180 ITER = 1, NN

DO 140 I = 1, N

/* GENERATE EXPONENTIAL RANDOM NUMBER WITH MU=1 */
CALL LEXPN(ISEED, EXPO, SAMPLE(I), 1, 0)

/* CONVERT EXPONENTIAL RANDOM NUMBER WITH MU=1 */
/* TO MU=LAMBDAS AND GET WEIBULL RANDOM NUMBERS */
DO 80 J = 1, SAMPLE(I)
XX(I,J) = EXPO(J) / LAMSTR(I)
W(I,J) = XX(I,J) ** (1/BETA(I))
Y(I,J) = ALOG(W(I,J))
WEIB(J) = W(I,J)
80 CONTINUE

36
\[ B(J) = Y(I,J) \]

CONTINUE

/* SUBROUTINE SHSORT WILL SORT ARRAY B IN */
/* ASCENDING ORDER BY SHELL SORT ALGORITHM */
CALL SHSORT(B, KEY, SAMPLE(I))

/* FINDING BETA HAT (BETHAT) BY THE FORMULA */

SUM1 = 0
SUM2 = 0

DO 100 J = 1, S(I)
    SUM1 = SUM1 + B(J)
    CONTINUE

DO 120 J = S(I)+1, SAMPLE(I)
    SUM2 = SUM2 + B(J)
    CONTINUE

BETHAT(I) = (SAMPLE(I)) * KSUBM(I) /
            (S(I) * SUM2 / (SAMPLE(I) - S(I)) - SUM1)

TEMP = BETA(I)

/* SUBROUTINE NEWTON WILL COMPUTE THE ESTIMATE */
/* OF BETA (BETNEW) BY NEWTON-RAPHSON METHOD */
CALL NEWTON(WEIB, SAMPLE(I), TEMP, BETNEW(I))

XTOBET(I) = X(I) ** BETHAT(I)
XBENEW(I) = X(I) ** BETNEW(I)

XHAT(I) = 0.
XNEW(I) = 0.

DO 130 J = 1, SAMPLE(I)
    XIJHAT(I,J) = W(I,J) ** BETHAT(I)
    XHAT(I,J) = XHAT(I) + XIJHAT(I,J)
    XIJNEW(I,J) = W(I,J) ** BETNEW(I)
    XNEW(I,J) = XNEW(I) + XIJNEW(I,J)
    CONTINUE

LAMHAT(I) = SAMPLE(I) / XHAT(I)
LAMNEW(I) = SAMPLE(I) / XNEW(I)

IF (LAMHAT(I) .GT. LAMMAX) THEN
    LAMMAX = LAMHAT(I)
ENDIF

IF (LAMNEW(I) .GT. LAMBIG) THEN
    LAMBIG = LAMNEW(I)
ENDIF

CONTINUE

SUM3 = 0
SUM3N = 0
SUM4 = 0
SUM4N = 0

DO 160 I = 1, N
   RATIO(I) = LAMHAT(I) / LAMMAX
   RATNEW(I) = LAMNEW(I) / LAMBIG
   SUM3 = SUM3 + RATIO(I) * XHAT(I)
   SUM4 = SUM4 + RATIO(I) * XTOBET(I)
   SUM3N = SUM3N + RATNEW(I) * XNEW(I)
   SUM4N = SUM4N + RATNEW(I) * XBENEW(I)

160 CONTINUE

LAMMU = CHISQR / (2 * SUM3)
RSL(ITER) = EXP(-LAMMU * SUM4)

LAMUNR = CHISQR / (2 * SUM3N)
RSLNEW(ITER) = EXP(-LAMUNR * SUM4N)

180 CONTINUE

CALL SHSORT(RSL, KEY1, NN)
CALL SHSORT(RSLNEW, KEY2, NN)

KK = (1 - ALPHA(LL)) * NN

/* RHAT IS THE RELIABILITY OF THE SERIES SYSTEM COMPUTED */
/* BY THE FORMULA */
/* RHTNEW IS THE RELIABILITY OF THE SERIES SYSTEM COMPUTED */
/* BY THE NEWTON - RAPHSON METHOD */

RHAT(CASE) = RSL(KK)
RHTNEW(CASE) = RSLNEW(KK)

/* SUBROUTINE FINDJ WILL FIND THE TRUE CONFIDENCE LIMIT */
CALL FINDJ(RSL, NN, RSYS(JJ), J1)
CALL FINDJ(RSLNEW, NN, RSYS(JJ), J2)

CLR(CASE) = J1 / FLOAT(NN)
CLRNEW(CASE) = J2 / FLOAT(NN)

CASE = CASE + 1

200 CONTINUE

220 CONTINUE

DO 240 I = 1, N
   WRITE(6,500) BETA(I), SAMPLE(I)

240 CONTINUE

WRITE(6,550) N, NN
WRITE(6,600)
WRITE(6,650)
WRITE(6,670)
CASE = 1
DO 280 JJ = 1, 2
   DO 260 LL = 1, 2
      WRITE(6,700) N,RSYS(JJ),ALPHA(LL),RHT(CASE),RHTNEW(CASE),
                  CLR(CASE), CLRNEW(CASE)
   CASE = CASE + 1
   CONTINUE
260 CONTINUE
280 CONTINUE

500 FORMAT(T10,'INITIAL BETA :',3X,F4.1,
      */T45,'NUMBER OF SAMPLE SIZES :',2X,I3)
550 FORMAT(T10,'NUMBER OF COMPONENTS :',2X,I3,
      */T10,'NUMBER OF REPLICATIONS :',2X,I3)
600 FORMAT(1',5(,'; '*50** RELIABILITY OF SERIES SYSTEM ***')
650 FORMAT(///,T35,'Ri',CL1 ; USING THE FORMULAR',
      */T35,'R2',CL2 ; USING THE NEWTON RAHPSON METHOD')
670 FORMAT(///,T6,'NUMBER OF',T19,'RELIABILITY',T33,'ALPHA',T46,'R1',
      T56,'R2',T66,'TRUE',T74,'TRUE',/
      T6,'COMPONENTS',T19,'OF SYSTEM',T66,'C.L. 1',T74,'C.L. 2',
      */T5,76('-'))
700 FORMAT(/,T8,15,T22,F5.3,T33,F5.2,T43,F7.4,T53,F7.4,T65,
      */F7.4,T73,F7.4)

STOP
END

********************************************************************
********************************************************************
********************************************************************
SUBROUTINE ZTABLE(ALPHA,ZALPHA)
C    C
C    << SUBROUTINE ZTABLE COMPUTES RIGHT PERCENT POINT ZALPHA FROM >>
C    C
C    RIGHT CUMULATIVE PROBABILITY ALPHA
C    C
REAL ALPHA, ZALPHA

IF ((ALPHA .GT. 0.0) .OR. (ALPHA .LT. 1)) THEN
   W = - ALOG(4 * ALPHA + ( 1 - ALPHA ))
   ZALPHA = SQRT(W * (2.06118 - ( 5.72622 / (W + 11.6406))))
   IF (ALPHA .GT. 0.5) THEN
      ZALPHA = - ZALPHA
   ENDIF
ENDIF

39
```fortran
ENDIF
RETURN
END

******************************************************************************
*ornmenterture SHARET(A,KEY,N)
DIMENSION A(N),KEY(N)
M1=1
6 M1=M1*2
   IF (M1 .LE. N) GO TO 6
   M1=M1/2-1
20 MM=MAX0(M1/2,1)
   GO TO 21
21 K=N-MM
22 DO 1 J=1,K
   II=J
11 IM=II+MM
   IF (A(IM) .GE. A(II)) GO TO 1
       TEMP=A(II)
   IT=KEY(II)
   A(II)=A(IM)
   KEY(II)=KEY(IM)
   A(IM)=TEMP
   KEY(IM)=IT
   II=II-MM
   IF (II .GT. 0) GO TO 11
1 CONTINUE
   GO TO 20
100 RETURN
END

******************************************************************************
******************************************************************************

SUBROUTINE NEWTON(TIME, M, BETA, BETNEW )

*C
*C << SUBROUTINE NEWTON WILL COMPUTE THE ESTIMATE OF BETA (BETHAT) >>
*C << BY NEUON - RAPHSON METHOD >>
*C
REAL      TIME(M), LNTIME(100), A(3), LNTS
INTEGER   R

DATA TS / 6 /
C     = 0
ITER = 0

40```
R = M
LNTS = ALOG(TS)

DO 20 I = 1, R
    LNTIME(I) = ALOG(TIME(I))
    C = C + LNTIME(I)
20 CONTINUE

C = C / R

30 DO 60 J = 1, 3
    SUM = 0
    DO 40 K = 1, R
        IF ( J .EQ. 1 ) THEN
            SUM = SUM + TIME(K) ** BETA
        ELSE
            SUM = SUM + ((TIME(K) ** BETA)*(LNTIME(K) ** (J-1)))
        ENDIF
    40 CONTINUE
    A(J) = SUM

60 CONTINUE

/* FUNCTION FPRIME IS THE DERIVATIVES OF FUNCTION F */
QUOT = A(2) / A(1)
FPRIME = A(3) / A(1) - (QUOT**2) + ((1/BETA)**2) - C

/* BETA IS UPDATED EACH TIME AND CHECK IF IT CONVERGES */
BETA = BETA - F / FPRIME

ITER = ITER + 1
IF (BETA .GT. 25.) GOTO 100
IF (ABS(F) .GT. 0.0001) GOTO 30
ALPHA = (A(1) / R) ** (1/BETA)
BETNEW = BETA
RETURN

100 WRITE(6,*) 'DID NOT CONVERGE'
WRITE(6,*) 'TRY AGAIN WITH BETTER ESTIMATE OF BETA'
RETURN
END
SUBROUTINE FINDJ(A, NN, R, J)

<< SUBROUTINE FINDJ FINDS THE INDEX OF ARRAY A WHICH THE
<< VALUE OF IT IS CLOSEST TO R.  >>

REAL A(NN), R, VALUE
INTEGER J

VALUE = ABS(A(NN) - R)

DO 100 I = NN-1, 1, -1

    IF (ABS(A(I) - R) .LT. VALUE) THEN
        VALUE = ABS(A(I) - R)
        ELSE
            J = I + 1
            ENDIF

    RETURN

100 CONTINUE

RETURN

END
APPENDIX D. FORTRAN CODE FOR INTERVAL ESTIMATION
PROCEDURE - NORMAL CASE

PROGRAM NORMAL

******************************************************************************
* This program determine the accuracy of an approximate
* lower confidence bound for P( X > Y ).
* Program NORMAL is the case when "Y" is given a value "Y0"
* & "X" is normally distributed with unknown parameters.
******************************************************************************

REAL Y0, P, ZP, MUX, SUMX, SUMX2, XBAR
REAL TEMP1, TEMP2, TEMP3, SIGHAT
REAL R(3), SIGMAX(3), ALPHA(2), X(100), Y(100)
REAL RL1(54,1000), AR1(1000), BRL1(54), CIR1(54)
REAL TRL1(54), TC11(54), T2RL1(54), T2C11(54)
REAL TRL1(2), TC11(2)
REAL ZHAT(1000), ZKNIFE(54), ZVAR(54), ZBAR(54)
REAL NUJUNK(54)

INTEGER NUMX(3), NUMY(3), CASE, LINE

DATA R / .95, .99, .995 /, SIGMAX / .5, 1, 20 /, NUMX / 10, 25, 75 /
DATA NUMY / 10, 25, 75 /, ALPHA / .2, .1 /
DATA ISEED, JSEED / 4875, 7981 /
DATA Y0 / 400 /
CASE = 0

C /* II IS THE INDEX FOR R. R(1) = .95, R(2) = .99, R(3) = .995 */
DO 500 II = 1, 3

C /* P IS THE RIGTH(UPPER) CUMULATIVE PROBABILITY */
P = 1 - R(II)

C /* SUBROUTINE ZTABL2 COMPUTES RIGHT PERCENT POINT ZP */
C /* FROM RIGHT CUMULATIVE PROBABILITY P */
CALL ZTABL2( P, ZP )

C /* JJ IS THE INDEX FOR SIGMA OF X. */
C /* SIGMAX(1) = .5, SIGMAX(2) = 1, SIGMAX(3) = 3 */
DO 400 JJ = 1, 3
MUX = ZP * SIGMAX(JJ) + YO

C /* KK IS THE INDEX OF NUMBER OF X. */
C /* NUMX(1) = 10, NUMX(2) = 25, NUMX(3) = 75 */
DO 300 KK = 1, 3

C /* LL IS THE INDEX FOR ALPHA. */
C /* ALPHA(1) = .2, ALPHA(2) = .1 */
DO 200 LL = 1, 2
CALL ZTABL2(ALPHA(LL), ZALPHA)

CASE = CASE + 1
TEMP = (NUMX(KK) - 1.) / NUMX(KK)
TEMPO = SQRT(TEMP)
MUJUNK(CASE) = MUX

C /* REPLICATE 1000 TIMES FOR EACH CASES */
DO 100 I = 1, 1000

C /* USE NORMAL RANDOM NUMBER GENERATOR TO GET */
C /* NUMX(KK) NUMBER OF X. */
CALL LNORM(ISEED,X,NUMX(KK),2,0)

C /* THIS PART IS TO GET SAMPLE MEAN(XBAR) AND */
C /* SAMPLE VARIANCE(XVAR) */
DO 50 MM = 1, NUMX(KK)
X(MM) = SIGMAX(JJ) * X(MM) + MUX
50 CONTINUE

C /* SUBROUTINE VAR WILL COMPUTE SAMPLE MEAN AND */
C /* SAMPLE VARIANCE */
CALL VAR(X, NUMX(KK), XBAR, XVAR)

C /* NOW WE COMPUTE THE LOWER CONFIDENCE BOUND */
TEMP1 = (XBAR - YO) / SQRT(XVAR * TEMPO)
TEMP2 = 1. / NUMX(KK) + 
((XBAR - YO)**2) / (2*(NUMX(KK)+1)*XVAR)
TEMP1 = TEMP1 - ZALPHA * SQRT(TEMP2) * TEMPO

C /* SUBROUTINE ZTABLE1 COMPUTE RIGHT CUMULATIVE PROBA- */
C /* BILITY FROM RIGHT PERCENT POINT. */
CALL ZTABLE1(TEMP1, ARL1(I))
ARL1(I) = 1. - ARL1(I)

CALL JKNIFE(X, NUMX(KK), YO, ZHAT(I))

CONTINUE

CALL VAR(ZHAT, 1000, ZBAR(CASE), ZVAR(CASE))

SIGHAT = 1./NUMX(KK) + ZBAR(CASE)**2 / (2*(NUMX(KK)+1))

SIGHAT = SQRT(SIGHAT) * TEMP

SIGHAT = ZBAR(CASE) / TEMP0 - SIGHAT * ZALPHA

CALL ZTABL1(SIGHAT, ZKNIFE(CASE))

ZKNIFE(CASE) = 1 - ZKNIFE(CASE)

C /* NON-IMSL LIBRARY 'SHSORT' WILL SORT ARL1 BY SHELL */
C /* SORT ALGORITHM. */

CALL SHSORT(ARL1, KEY, 1000)

DO 150 I = 1, 1000

RL1(CASE,I) = ARL1(I)

CONTINUE

C /* SUBROUTINE FINDJ FINDS THE INDEX OF ARL20 WHICH THE */
C /* VALUE OF IT IS CLOSEST TO R. */

CALL FINDJ(ARL1, R(II), J)

C /* DIVIDING THE INDEX BY 1000 WILL GIVE US TRUE */
C /* CONFIDENCE LEVEL. */

CIRL1(CASE) = J / 1000.

MM = 1000 * (1 - ALPHA(LL))

BRL1(CASE) = RL1(CASE,MM)

CALL TNYVAL(R(II), SIGMAX(JJ), NUMX(KK),

* ALPHA(LL), ZALPHA, MUX, YO, TRL1, TCI1)

T1RL1(CASE) = TRL1(1)

T1CI1(CASE) = TCI1(1)

T2RL1(CASE) = TRL1(2)

T2CI1(CASE) = TCI1(2)

CONTINUE

CONTINUE

CONTINUE

CONTINUE

CONTINUE

CONTINUE

CONTINUE
WRITE(6,600)
WRITE(6,630)
I = 1
LINE = 0
DO 550 II = 1, 3
  DO 550 JJ = 1, 3
    DO 550 KK = 1, 3
      IF (LINE .GE. 3) THEN
        WRITE(6,800)
        LINE = 0
      ENDIF
      WRITE(6,710) R(II), SIGMAX(JJ), NUMX(KK),
      & BRL1(I), BRL1(I+1), CIRL1(I), CIRL1(I+1),
      & ZKNIFE(I), ZKNIFE(I+1),
      & MUJUNK(I), MUJUNK(I+1)
      I = I + 2
      LINE = LINE + 1
    550 CONTINUE
WRITE(6,610)
WRITE(6,650)
I = 1
LINE = 0
DO 560 II = 1, 3
  DO 560 JJ = 1, 3
    DO 560 KK = 1, 3
      IF (LINE .GE. 3) THEN
        WRITE(6,800)
        LINE = 0
      ENDIF
      WRITE(6,700) R(II), SIGMAX(JJ), NUMX(KK),
      & T1RL1(I), T1RL1(I+1), T1C11(I), T1C11(I+1)
      I = I + 2
      LINE = LINE + 1
    560 CONTINUE
WRITE(6,620)
WRITE(6,650)
I = 1
LINE = 0
DO 570 II = 1, 3
  DO 570 JJ = 1, 3
    DO 570 KK = 1, 3
      IF (LINE .GE. 3) THEN
        WRITE(6,800)
        LINE = 0
      ENDIF

WRITE(6,700) R(II), SIGMA(JJ), NUMX(KK),
    T2RL1(I), T2RL1(I+1), T2CI1(I), T2CI1(I+1)
*
I = I + 2
LINE = LINE + 1
570 CONTINUE

600 FORMAT('1',5(/),7X,'**** Y IS GIVEN A VALUE OF YO ****')
610 FORMAT('1',5(/),7X,'**** Y IS GIVEN A VALUE OF YO ****',
    //,20X,'-- X IS TRUNCATED NORMAL WITH PROBABILITY .95 --')
620 FORMAT('1',5(/),7X,'**** Y IS GIVEN A VALUE OF YO ****',
    //,20X,'-- X IS TRUNCATED NORMAL WITH PROBABILITY .90 --')
630 FORMAT(3(/),5X,'CASE ',6X,'RL1(1000*(1-ALPHA))',
    //4X,'TRUE CONFIDENCE LEVEL',5X,'JACKNIFE',
    //,5X,'R SX N',3X,'ALPHA = .2',12X,'1',11X,'.2',
    //12X,'1',12X,'1',11X,'.2',12X,'1',/3X,13('-'),6X,25('1'),4X,21('1'),2X,16('1'),/)
650 FORMAT(3(/),5X,'CASE ',6X,'RL1(1000*(1-ALPHA))',
    //4X,'TRUE CONFIDENCE LEVEL',//,5X,'R SX N',3X,
    // 'ALPHA = .2',12X,'1',11X,'.2',12X,'1',//,3X,13('-'),6X,
    //25('1'),4X,21('1'),/)
700 FORMAT(3X,F4.3,2X,F3.1,2X,12,F7.4)
710 FORMAT(3X,F4.3,1X,F4.1,2X,I2,4(7X,F7.4),1X,2(2X,F6.4),2(2X,F9.5))
800 FORMAT(/)

RETURN
END

**************************************************************************
**************************************************************************
SUBROUTINE ZTABL1(ZALPHA, ALPHA)
C
C << SUBROUTINE ZTABL1 COMPUTE RIGHT CUMULATIVE PROBABILITY FROM RIGHT PERCENT POINT. >>
C
REAL PI, ALPHA, ZALPHA
PARAMETER ( PI = 3.141592 )

IF ( ZALPHA .EQ. 0.0 ) THEN
   ALPHA = 0.5
ELSE
   ALPHA = EXP( -2 * ZALPHA ** 2 / PI )
   ALPHA = SQRT(1 - ALPHA * (1 + 2*(PI-3)*ZALPHA**4/(3*PI**2))
   ALPHA = 0.5 * ( 1 - ALPHA )
   IF ( ZALPHA .LT. 0.0 ) THEN
      ALPHA = 1 - ALPHA

47
SUBROUTINE ZTABL2(ALPHA,ZALPHA)

REAL ALPHA, ZALPHA

IF ((ALPHA .GT. 0.0) .OR. (ALPHA .LT. 1)) THEN

W = - ALOG(4 * ALPHA * ( 1 - ALPHA ))
ZALPHA = SQRT(W * (2.06118 - ( 5.72622 / (W + 11.6406))))

IF (ALPHA .GT. 0.5) THEN
ZALPHA = - ZALPHA
ENDIF

RETURN
END

SUBROUTINE TTABLE(ALPHA, NU, TALPHA)

REAL ALPHA, ZALPHA, TALPHA, A1, A2, A3
INTEGER NU

IF ((ALPHA .GT. 0.0) .OR. (ALPHA .LT. 1)) THEN

CALL ZTABL2(ALPHA, ZALPHA)

A1 = (ZALPHA**2 + 1) / 4
A2 = (5*ZALPHA**4 + 16*ZALPHA**2 + 3) / 96
A3 = (3*ZALPHA**6 + 19*ZALPHA**4 + 17*ZALPHA**2 - 15) / 384

TALPHA = ZALPHA * (1 + A1/NU + A2/NU**2 + A3/NU**3)

IF ( ALPHA .GT. 0.5 ) THEN
TALPHA = - TALPHA
ENDIF

RETURN
END
SUBROUTINE SHSORT(A,KEY,N)
DIMENSION A(N),KEY(N)
M1=1
6 M1=M1+2
IF (M1 .LE. N) GO TO 6
M1=M1/2-1
MM=MAX0(M1/2,1)
GO TO 21
20 MM=MM/2
IF (MM .LE. 0) GO TO 100
21 K=N-MM
22 DO 1 J=1,K
II=J
11 IM=II+MM
IF (A(IM) .GE. A(II)) GO TO 1
TEMP=A(II)
IT=KEY(II)
A(II)=A(IM)
KEY(II)=KEY(IM)
A(IM)=TEMP
KEY(IM)=IT
II=II-MM
IF (II .GT. 0) GO TO 11
1 CONTINUE
GO TO 20
100 RETURN
END

SUBROUTINE VAR(Z, NUMZ, ZBAR, ZVAR)
REAL Z(NUMZ), SUMZ, SUMZ2, ZBAR, ZVAR
SUMZ = 0
SUMZ2 = 0
DO 100 I = 1, NUMZ
      SUMZ = SUMZ + Z(I)
100 CONTINUE
\[ \text{SUMZ2} = \text{SUMZ2} + Z(I)^2 \]

100 CONTINUE

\[ \text{ZBAR} = \frac{\text{SUMZ}}{\text{NUMZ}} \]
\[ \text{ZVAR} = \text{ABS} \left( \frac{\text{SUMZ2}}{\text{NUMZ}} - \text{ZBAR}^2 \right) \]

RETURN
END

******************************************************************************
*
******************************************************************************

SUBROUTINE FINDJ(A, R, J)
C
C << SUBROUTINE FINDJ FINDS THE INDEX OF ARRAY A WHICH THE VALUE OF IT IS CLOSEST TO R. >>
C
REAL A(1000), R, VALUE
INTEGER J

VALUE = ABS(A(1000) - R)

DO 100 I = 999, 1, -1
    IF (ABS(A(I) - R) .LT. VALUE) THEN
        VALUE = ABS(A(I) - R)
        ELSE
            J = I + 1
            RETURN
        ENDIF

100 CONTINUE

RETURN
END

******************************************************************************
*
******************************************************************************

SUBROUTINE TNYVAL(R, SIGMAX, NUMX, ALPHA, ZALPHA, MUX, YO,
                   TBRL1, TCIR1)
C
C << SUBROUTINE TNYVAL IS THE CASE WHEN X IS TRUNCATED NORMAL >>
C << AND Y IS GIVEN A VALUE >>
C
REAL R, SIGMAX, ALPHA, TALPHA, MUX, P(2), ZP(2), A(2)
REAL X(100), X1, TEMP1, TEMP2, Y0
REAL TARL1(1000), KEY(1000), TRL1(1000), TBRL1(2), TCIR1(2)

INTEGER NUMX, CASE, COUNT

50
DATA P / .95, .90 /
DATA ISEED, JSEED / 4875, 7981 /

DO 400 I = 1, 2
    P(I) = 1 - P(I)
    COUNT = 0
    CALL ZTABL2(P(I), ZP(I))
    A(I) = MUX + ZP(I) * SIGMAX

DO 200 J = 1, 1000
    CALL LNORM(ISEED, X1, 1, 2, 0)
    XTEMP = MUX + SIGMAX * X1
    IF (XTEMP .LE. A(I)) THEN
        COUNT = COUNT + 1
        X(COUNT) = XTEMP
    ELSE
        GOTO 100
    ENDIF
    IF (COUNT .LT. NUMX) GOTO 100
    CALL VAR(X, NUMX, XBAR, XVAR)

C /* NOW WE COMPUTE THE LOWER CONFIDENCE BOUND */
    TEMP1 = (XBAR - Y0) / SQRT(XVAR**((NUMX-1) / NUMX))
    TEMPr2 = 1. / NUMX + ((XBAR - Y0) ** 2) / (2**(NUMX+1) ** XVAR)
    TEMP1 = TEMP1 - ZALPHA * SQRT(TEMPr2)

C /* SUBROUTINE ZTABL1 COMPUTE RIGHT CUMULATIVE PROBABILITY FROM RIGHT PERCENT POINT. */
    CALL ZTABL1(TEMP1, TARL1(J))
    TARL1(J) = 1. - TARL1(J)

200 CONTINUE

C /* NON-IMSL LIBRARY 'SHSORT' WILL SORT TARL1 BY SHELL SORT ALGORITHM. */
    CALL SHSORT(TARL1, KEY, 1000)

DO 300 J = 1, 1000
    TRL1(J) = TARL1(J)

51
CONTINUE

C /* SUBROUTINE FINDJ FINDS THE INDEX OF TARL1 WHICH THE */
C /* VALUE OF IT IS CLOSEST TO R. */
CALL FINDJ(TARL1, R, J)

C /* DIVIDING THE INDEX BY 1000 WILL GIVE US TRUE */
C /* CONFIDENCE LEVEL. */
TCIR1(I) = J / 1000.
MM = 1000 * (1 - ALPHA)
TBRL1(I) = TRL1(MM)

400 CONTINUE
RETURN
END

***************************************************************************
SUBROUTINE JKNIFE(X, N, YO, ZHAT)
***************************************************************************
C << THIS ROUTINE IS FOR THE 'JACKNIFE' METHOD. >>
C << DUMMY PARAMETER IS ZHAT. >>
REAL X(N),XOMIT(100),XOMBAR(100),XOMVAR(100),ZOMHAT(100),ZHAT
INTEGER M, N
M = N - 1
SUM = 0.
DO 200 I = 1, N
   DO 100 J = 1, M
      IF (J .GE. I) THEN
         XOMIT(J) = X(J+I)
      ELSE
         XOMIT(J) = X(J)
      ENDIF
100 CONTINUE
   CALL VAR(XOMIT, M, XOMBAR(I), XOMVAR(I))
   ZOMHAT(I) = ( XOMBAR(I) - YO ) / SQRT(XOMVAR(I))
   SUM = SUM +ZOMHAT(I)
200 CONTINUE
   ZHAT = SUM / N
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LIST OF REFERENCES


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