SATELLITE MOTION IN AN
AXI-SYMMETRIC GRAVITATIONAL FIELD
PART 2: PERTURBATIONS DUE TO
AN ARBITRARY $J_2$

by

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SATELLITE MOTION IN AN AXI-SYMMETRIC GRAVITATIONAL FIELD
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SUMMARY

This Report continues the presentation of the untruncated orbital theory begun in Technical Report 88067. The effects of the general zonal harmonic, $J_l$, are now covered, the main results being a trio of formulae for perturbations in the spherical-polar coordinates introduced in the previous paper. The formulae are only first-order in $J_l$, but, in conjunction with the second-order results for $J_2$ published in Part 1, the complete set of formulae may be regarded as constituting a second-order theory, the Earth's $J_2$ being much larger than $J_l$ for $l > 2$.

The mean elements of the theory are defined in such a way that, for each $J_l$, the coordinate-perturbation formulae have their simplest possible form, with no occurrence of zero denominators. The general formulae are used in a derivation of the results for $J_3$, given in Part 1, and in a derivation of results for $J_4$.

Numerical comparisons with reference orbits are held over to a later report (Part 3).

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INTRODUCTION

This Report is the second of an intended trilogy devoted to satellite motion about an axi-symmetric primary, i.e. about a gravitating solid of revolution. Thus it continues the exposition of Ref 1, which will henceforth be referred to as 'Part 1'. Part 1 brought together the principles of an approach to orbit modelling in which lengthy expressions for short-period perturbations in the usual osculating elements are compressed into concise expressions for perturbations in a particular set of spherical-polar coordinates; it then proceeded into the presentation of a complete second-order theory for perturbations due to the zonal harmonic $J_2$, and a complete first-order theory for $J_3$. When the primary body is the Earth, $J_3$ (and every subsequent $J_k$) is of order $J_2^2$, so Part 1 may be regarded as describing (for $J_2$ and $J_3$ only) a complete second-order theory for Earth satellites, where 'first order' refers to effects of relative magnitude $10^{-3}$. Though Part 1 has only recently been published, a résumé of the theory had been given much earlier.

Two other papers are relevant to the maturation of the trilogy: a recent one on mean elements (as used in Part 1), with particular reference to the relation between mean semi-major axis and mean mean motion; and a much earlier (and more important) paper, of similar title to the trilogy's, that established formulae for secular and long-period perturbations due to the general $J_k$ (so general, in fact, that $k$ could be negative, the formulae then being applicable to lunisolar perturbations). The present Report effectively combines the new approach of Part 1 with the general principles and notation of Ref 4, the result being a complete theory for the zonal harmonics; secular and long-period perturbations are applied to mean orbital elements, and short-period perturbations to coordinates.

Part 3 of the trilogy will be largely devoted to the way in which the mean elements evolve over periods of time longer than just a small number of orbital revolutions. This topic, which was given limited attention in Part 1, is entirely neglected in Part 2. It is intended that Part 3 will also give details of the Fortran program(s) written to evaluate the accuracy of the overall approach, using harmonics up to $J_4$. (The variations in the mean elements are computed by a technique that involves a numerical component of an otherwise analytical model, aspects of this technique were described in the paper that originally outlined the author's philosophy of coupling a hybrid computational procedure to the coordinate-perturbation approach.)

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Other authors have published first-order formulae for satellite perturbations due to the geopotential; they usually address the subject more generally than here, by covering the tesseral harmonics as well as the zonal harmonics. The first entirely general results were derived by Groves\(^6\), in an analysis of formidable complexity, whilst the classic reference is the text-book of Kaula\(^7\). The very generality of the formulae in Refs 6 and 7 makes it difficult to write down expressions for individual effects, however, and it is not even easy to show that the two sets of formulae are formally equivalent (the full first-order expression for the perturbation in mean anomaly is omitted in Ref 6, and the supplementary terms are only added as an afterthought in Ref 7).

Much of the difficulty in the general analysis arises from the need, when covering the tesseral harmonics as well as the zonal harmonics, to allow for the rotation of the primary. The uniformity of this rotation with time makes it natural to work with \( M \) (mean anomaly), rather than \( v \) (true anomaly), as integration variable, but this inevitably leads to infinite summations. When the analysis is restricted to the zonal harmonics, however, use of \( v \) (rather than \( M \)) leads to expressions that are free of infinite summation, and Zafiropoulos\(^8\) has recently published untruncated formulae for the first-order perturbations in the orbital elements due to the general \( J_2 \). The formulae of Ref 8 are much more explicit than those in Refs 6 and 7, but this is unfortunately at the expense of some very long expressions - it takes more than five pages to express the basic formulae, and even then the supplementary terms of the perturbation in \( M \) are again absent. Now it will emerge from the present Report that the formulae of Zafiropoulos can be expressed much more concisely. The real breakthrough comes, however, when the short-period perturbations in elements are replaced by perturbations in coordinates. If it were not for the rotation of the primary, this procedure could be immediately extended to the tesseral harmonics; for orbits of sufficiently low eccentricity there is no difficulty, and very simple general formulae were given in Refs 5 and 9, having originally been derived during a study\(^10\) of Navstar/GPS.

As with Part 1, a List of Symbols is appended to the Report; it is almost entirely consistent with the List of Part 1, the few exceptions being noted. The meaning of every new symbol is fully specified in the text, but only minimal

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* Appendix A, which is in the nature of a postscript, outlines what is involved in the extension for a non-rotating primary, and a separate paper is planned for later publication.
explanation is given for those carried over from Part 1. This is true, in particular, for standard symbolism: thus we note, straight away, that the orbital elements used are $a$, $e$, $i$, $\Omega$, $\omega$ and $M$, an arbitrary one of which is denoted (generically) by $\zeta$; also $M = \zeta + \zeta$, where $f$ is shorthand for $\int_0^t \mathrm{d}t$, the integral being taken from epoch to current time. We continue to make use of the quasi-elements $\psi$, $\varphi$ and $L$, (real) only defined at the differential level; thus, $\psi = d\omega + c\, d\Omega$ (where $c = \cos i$), $d\varphi = d\phi + q\, d\psi$ (where $q^2 = 1 - e^2$) and $dL = dM + q\, d\psi$.

As explained in Part 1, each osculating element, $\zeta$, may be regarded as the sum of a mean element, $\bar{\zeta}$, and a short-period perturbation, $\delta \zeta$, so that

$$\zeta = \bar{\zeta} + \delta \zeta.$$ (1)

A 'semi-mean' element, $\bar{\zeta}$, is also needed (see section 3.2 of Part 1), but in Part 2 we will usually ignore the distinction between $\bar{\zeta}$ and $\bar{\zeta}$. The effect of this neglect is that we do not distinguish between the quantities denoted by $\delta \zeta$, $\delta \varphi$, and $\delta L$ in Part 1, normally using $\delta \zeta$ here in the sense of $\delta \varphi$ of Part 1; towards the end of the Report (in deriving the perturbations due to $J_2$, in section 8.5), we remind the reader of the additional terms (split between $\zeta$ and $\delta \zeta$, as explained in Part 1) that are needed to express (first-order) perturbations in full. Not even the distinction between osculating elements and mean elements is of significance in evaluating the right-hand sides of equations in general, since second-order perturbations are not taken into account in Part 2, but the following important distinction (on left-hand sides) between $\zeta$ and $\dot{\zeta}$ is worth noting: Lagrange's planetary equation for $\zeta$ constitutes the starting point of analysis for the element $\zeta$, whereas a formula for $\dot{\zeta}$ is part of the goal of that analysis.

The analysis is greatly facilitated by using, instead of $\omega$ and $u$ (argument of latitude), the modified quantities $\omega'$ and $u'$, where

$$\omega' = \omega - \frac{1}{2} \zeta$$ and $$u' = u - \frac{1}{2} \zeta.$$ (2)

To avoid any confusion, it is remarked that the use of the accent (prime sign) here has a connotation entirely different from that which distinguishes a (osculating semi-major axis) from $a'$; the latter quantity is an absolute constant of the motion (under zonal harmonics only), which (as shown in Part 1)

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constitutes the best choice for mean semi-major axis \((a)\), to whatever order the analysis is conducted. With \(u'\) now introduced, it is convenient to define here the much-used quantities \(c^k_j\) and \(s^k_j\); thus

\[
\begin{align*}
  c^k_j &= \cos (jv + ku') \\
  s^k_j &= \sin (jv + ku')
\end{align*}
\] (3)

When there is no ambiguity in regard to \(k\), the superfix (but never the suffix) will often be omitted. (Warning: \(C_j\) and \(S_j\), as used in Part 1, identify with \(-C_{j-2}^0\) and \(-S_{j-2}^0\) here.)

As the primary is assumed axi-symmetric, we start from the potential function \(\nu/r + \frac{1}{2} U_4\), where the individual terms of the disturbing function are given (in the usual notation) by

\[
U_4 = -\frac{1}{2} \sum (R/r)^2 P_4(\sin \theta).
\] (4)

The value of \(l\) in the summation is normally taken to run from 2 upwards, since the cases \(l = 0\) and \(l = 1\) are essentially trivial, but the general formulae to be developed cover the case \(l = 1\) without difficulty; both 'trivial' cases are instructive and are interpreted in section 8, following Ref 4. If the concept of an axi-symmetric primary is generalized to allow for mass outside the orbital region, as well as inside, then (4) can be extended to cover negative \(l\), as in Ref 4; the only change needed in the expression for \(U_4\) is that \(P_l\) is replaced by \(P_{-l-1}\). Our overall requirement is to integrate the planetary equations for the general \(U_4\), thereby obtaining the first-order contributions to each \(\delta\) and \(\epsilon\), and then to combine the \(\delta\) (for the six elements) into \(\delta r\), \(\delta b\) and \(\delta w\), the corresponding perturbations in the spherical-polar coordinate system attached to the mean orbital plane; the latter is specified by \(\Gamma\) and \(\Pi\), and the transformation from the \((r, b, w)\)-coordinate system to the usual rectangular equatorial system is described in detail in Part 1.

In section 2 we decompose \(U_4\) as

\[
U_4 = \sum_k U_k^k.
\] (5)

* This notation leads to more concise expressions than if the trigonometric argument was \(jv + ku'\), as was originally planned; the disadvantage is that the Kepler-constant quantities are now \(C_k\) and \(S_k\), rather than \(C_0\) and \(S_0\).
where \( 0 \leq k \leq l \) and \( U_k^l \) is only non-zero when \( k \) has the same parity as \( l \) (or as \(-l-1\), in the extension to \( l < 0 \)). The decomposition arises as \( S \), in \( U_k^l \), is effectively replaced by \( l \), and this involves the introduction of certain families of inclination functions. The functions \( A_k^l(i) \) were originally introduced in Ref 4 and are used again, but quantities \( A_{k,k} \), proportional to the \( A_k^l(i) \) values, are actually more convenient. Related functions, and associated quantities, are also introduced, and recurrence relations are given. These relations (and corresponding relations for the eccentricity functions, referred to in the next paragraph) are required here in the development of the theory, but they are also important as computational aids in the implementation of the theory. The \( U_k^l \) can be treated separately in all the analysis up to the derivation of the \( \delta r \) and the \( \delta w \), but there is a complication in the derivation of \( \delta b \); this will be handled by the introduction of another index, \( k' \), which is always of the opposite parity to \( l \) (and hence \( k \)).

Following the elimination of \( S \), we must also eliminate \( r \), using the basic formula of the ellipse

\[
\frac{p}{r} = 1 + \omega \cos \nu ,
\]

before the planetary equations can be integrated. This involves families of eccentricity functions, which are introduced in section 3. The functions \( B_j^l(e) \) were originally introduced in Ref 4, but the quantities \( B_{k,j} \), proportional to them, are actually more useful. Recurrence relations are given, and these are even more important than the relations for the inclination functions. It is implicit in the use of the \( B_{k,j} \) that every \( U_k^l \) could be further decomposed, into \( U_k^l \), say, but we prefer not to do this, postponing the introduction of the \( B_{k,j} \) until \( U_k^l \), in each planetary equation, has been eliminated in favour of an explicit expression, thus the notation \( U_k^l \) is not needed.

The development of each planetary equation, via first the \( A_{k,k} \) and then the \( B_{k,j} \), is the topic of section 4. As already remarked, we avoid infinite summations by retaining \( \nu \) as an argument of the equation (as opposed to eliminating it in favour of \( M \)), and indeed we make it the integration variable by applying the relation

\[
\frac{dv}{dt} = n q^{-3} \left( \frac{p}{r} \right)^2
\]
Each equation now expresses \( d/dv \), rather than \( \dot{\xi} \), as a (finite) sum of terms in \( v \). Each such term is just a multiple of \( C_k^j \) or \( S_k^j \), as it turns out, so the integration of the equation is immediate. Terms with \( k + j \neq 0 \) lead to the \( \delta \xi \) by definition. Terms with \( k + j = 0 \), on the other hand, are effectively constant over the short term, and contribute directly to \( \dot{\xi} \); when \( k = 0 \) (and so also \( j = 0 \)), the integrated contribution is a secular perturbation, whilst the terms with \( k \neq 0 \) contribute to the long-period perturbation. The distinction is important for Earth satellites, because the secular variation of \( \omega \) (due to \( J_2 \)) must be allowed for in integrating the perturbation, but the subject was dealt with in Part 1 and will be picked up again in Part 3; there will be no further reference in Part 2 to this coupling between \( J_2 \) and the other \( J_k \).

Formulae for the \( \xi_{\ell k} \) (\( \ddot{x} \) due to \( U_k^j \)) are collected in section 5, whilst the reduction of the appropriate \( \delta \xi \) to formulae for \( \delta r \), \( \delta b \) and \( \delta \omega \) is the subject of the next two (and much longer) sections. As described in previous pages there is an important distinction between the integrations required for the two types of term: for the \( \xi_{\ell k} \), the process leads to definite integrals (see Parts 1 and 3), necessarily zero if taken over zero time from epoch; in \( \ell k \)-components of the \( \delta \xi \), on the other hand, the process leads to epoch-independent indefinite integrals that (apart from the complication of semi-mean elements) satisfy (1). But indefinite integrals contain arbitrary constants, where a 'constant' in the present context is any quantity that is independent of the fast-varying \( v \), i.e. would be a true constant for motion in a fixed ellipse. It is only when these constants are all assigned that (at the first-order level) the mean elements, \( \bar{z} \), are fully defined.

It has been noted that enormous advantage accrues from taking \( \bar{z} \) to be the exact quantity \( a' \) (defined by the energy integral, as explained in Part 1), but there are no immediately compelling reasons for associating particular constants with any of the other five elements. We therefore base our choice on the philosophy of making the expressions for \( \delta r \), \( \delta b \) and \( \delta \omega \) as simple as possible. These expressions, which constitute the most important results of the Report, are presented in their general form in section 6; each of the three expressions involves a summation over the index \( j \), with the integration constants for the elements (other than \( a' \)) not yet taken into account.
In the general formulae just referred to, certain values of \( j \) in the summations would involve terms of zero denominator, and it is by the elimination of all these terms that the integration constants (other than for \( a \) ) are chosen. This is the subject matter for section 7, which completes the entire analysis. An outline of the material in this section is as follows. First, the formula for the mandatory constant in \( \delta a \) (for each \( U_k^\delta \) ) is recorded, essentially as a matter of completeness. Second, the constants are derived for \( \delta e \) and \( \delta M \) that validate the omission of the terms with particular \( j \) that would otherwise arise in \( \delta r \). Third, the constants are derived for \( \delta l \) and \( \delta Q \) that do the same thing for \( \delta b \). Fourth, special terms (with particular \( j \) ) in \( \delta w \), that could not be included in section 6 because they are induced by the constants in \( \delta e \) and \( \delta M \), are obtained. Finally, the constant in \( \delta w \) (for each \( U_k^\delta \) ) is derived.

Examples of the general formulae of section 6, together with the special terms in \( \delta w \) derived in section 7, are given in section 8: first, for the trivial cases \( \ell = 0 \) and \( \ell = 1 \), the interest in which has been remarked; then for \( \ell = 2 \) and \( \ell = 3 \), leading (as a useful overall check) to results already known from Part 1; finally, for \( \ell = 4 \), leading to formulae not hitherto published.

2 FUNCTIONS OF INCLINATION REQUIRED IN EXPANDING THE POTENTIAL

Following Ref 4, we expand \( P_\ell(\sin B) \), required in (4), via the addition theorem for zonal harmonics (or Legendre polynomials); thus

\[
P_\ell(\sin B) = \sum_{k=0}^{\ell} u_k \frac{(\ell - k)!}{(\ell + k)!} P_k^\ell(\cos W) P_k(c) \cos ku'.
\]  

(8)

Here \( u_0 = 1 \), \( u_k = 2 \) if \( k > 0 \), and the Legendre function \( P_k^\ell \) is defined by

\[
P_k^\ell(c) = s^k \frac{d^k P_k(c)}{dc^k}.
\]  

(9)
The second factor (the k'th derivative) in (9) is a polynomial in c, which (with k ≥ l) does not vanish when c = 1, its value then being

\( (l + k)!/(2^k k! (l - k)!) \). Hence this factor may be normalized, in a certain useful sense, and we write

\[
\frac{d^k P_l(c)}{dc^k} = \frac{(l + k)!}{2^k k! (l - k)!} A_k^l(1),
\]

where \( A_k^l(1) \) is a pure polynomial in \( a = \sin \theta \) if \( k \) has the same parity as \( l \), but has an additional factor \( c \) if \( k \) and \( l \) are of opposite parity; in each case the constant term in the polynomial is unity, by the normalization.

Explicit expressions for the \( A_k^l(1) \) are given in Table 1, for values of \( l \) and \( k \) up to 6.

We can now rewrite (8) as

\[
P_l(\sin \theta) = \sum_{k=0}^l a_{lk} s^k A_k^l(1) \cos ku,
\]

where the constant, \( a_{lk} \), is given by

\[
a_{lk} = \frac{P_k'(0)}{(2^k k!)}.
\]

A different constant, \( C_k^l \), was used in Ref 4, incorporating a factor associated with the eccentricity functions of section 3; it is given by \(-2^{-k} \binom{l-1}{k-1} a_{lk}\), where \( \binom{m}{n} \) is the usual binomial coefficient, \( m \) here (and throughout the paper) being used to denote a general integer, with negative values allowed; when \( k < 0 \), \( P_k^l \) in (12) must be replaced by \( P_{k-1}^l \), so that \( a_{lk} = C_{k-1,l-k} \), but the relation of \( a_{lk} \) to the \( C_k^l \) of Ref 4 is unchanged. It is clear, from the last paragraph, that \( P_k^l(0) \) (or \( P_{k-1}^l(0) \)) vanishes when \( k \) and \( l \) (or \( -l - 1 \)) are of opposite parity, and it may be shown that when the parity is the same,\n
* This "normalization, which has nothing to do with the standard normalization of the spherical harmonics and their J-coefficients, leads directly to \( A_k^l(1) = 1 \), one of the pair of starting values for the recurrence relation (21). For some purposes a different normalization is preferable such that \( A_k^l(1) \) is defined for all \( l \geq 0 \), and \( A_k^l(1) = 1 \); the family of normalized functions can then be extended in a unified manner when the orbital theory is to cover the tesseral harmonics (Appendix 1).
In substituting (11) into (4), it is of great benefit to introduce a new quantity, $A_{tk}$, defined by

$$A_{tk} = p_k (R/p)^k \sigma_{tk} \delta^k A_k(i), \quad (14)$$

where $p (a(1 - a^2))$ is the semi-latus rectum (or parameter) of the orbit; the equation applies when $\ell < 0$, so long as the suffix of $A$ is replaced by $-\ell - 1$. The use of $A_{tk}$ permits us to write the general term of (5), using also the notation of (3), as

$$U_k^c = -\frac{2^{c}}{p} (R/p)^{k} \sigma_{tk} \delta^k A_{tk} c^k. \quad (15)$$

It will be noted that, whereas $A_k(i)$ is defined and useful regardless of parity, $A_{tk}$ (and hence $U_k^c$) is only non-zero when $k$ and $t$ are of the same parity. The zeroes come from $\sigma_{tk}$, for which the non-zero values, up to $k = \ell = 6$, are given by the like-parity entries of Table 2. (The Table has been extended back to $\ell = -7$, to illustrate the identity of $\sigma_{tk}$ with $a_{-\ell-1,tk}$ when $\ell < 0$.) However, a use will be found for quantities that behave in the opposite way from $\sigma_{tk}$ and $A_{tk}$, anticipating which we define (with bold letters to make the distinction)

$$a_{tk} = u_k (t - k + 1) P_{k+1}(0) / (2^k c^t \ell) \quad (16)$$

and

$$A_{tk} = J_k (R/p)^k a_{tk} \delta^k A_k(i). \quad (17)$$

where $k$ has been replaced by $\delta$ to signify that we now have quantities that are non-zero only when $k$ and $\ell$ are of opposite parity. Half of Table 2 (for all $\ell$) is devoted to $a_{tk}$, since these quantities can be included with $a_{tk}$ on a chequer-board basis.
We will require derivatives of the inclination functions. It is evident from (10) that
\[ \frac{d}{dI} \{ A_k^l \} = -\frac{(l - k)(l + k + 1)}{2(k + 1)} s A_k^{l-1} \]  
(18)
from this and (14) it follows that the (partial) derivative of \( A_{k1} \) with respect to \( I \) is given by
\[ A_{k1} = J_k (R/p) s_k s^{k-1} \left\{ k_0 A_k^{l-1} - \frac{(l - k)(l + k + 1)}{2(k + 1)} r A_k^{l+1} \right\} \]  
(19)
where \( r = s^2 \). The quantity in (curly) brackets is the \( D_k^{l}(r) \) of Ref 4. We will also require, finally, the particular combinations of \( A_{k1} \) and \( A_{i1} \), denoted by \( A_{k1}^* \) and \( A_{i1}^* \), and given by
\[ A_{k1}^* = k s^{-1} A_{k1} + c^{-1} A_{i1}^* \]  
(20)
the \( s^{-1} \) and \( c^{-1} \) factors do not imply singularities, as they must always cancel via \( k A_{k1} \) and \( A_{i1}^* \) respectively.

The \( A_{k1}^*(r) \) and \( A_{i1}^*(r) \) (and hence the \( A_{k1}^* \)) may be computed with the aid of the aid of recurrence relations. A fixed \( k \) was stipulated in Ref 4 for the formula
\[ (l + k) A_{k1}^*(r) = (2l - 1) c A_{k1}^{l-1} - (l - k - 1) A_{k1}^{l+1} \]  
(21)
valid for \( l \geq k + 2 \) with the starting values \( A_{k1}^*(r) = 1 \) and \( A_{k1}^{k+1}(r) = c \); (21) is even valid for \( l = k + 1 \), if an arbitrary (but finite) \( A_{k1}^{k-1}(r) \) is assumed. However, it is usually more useful to stipulate a fixed \( k \); the required formula was given by Herson\(^1\), being
\[ A_{k1}^*(r) = c A_{k1}^{k+1} - \frac{(l - k - 1)(l + k)}{4(k + 1)(k + 2)} r A_{k1}^{k+2} \]  
(22)
valid for \( l - 2 \geq k \geq 0 \) with the starters \( A_{k1}^*(r) = 1 \) and \( A_{k1}^{k+1}(r) = c \); (22) is also valid for \( k = l - 1 \), with an arbitrary (finite) \( A_{k1}^{k+1}(r) \). Either of the two preceding 'pure' three-term recurrence relations, (21) or (22), can be used with just one 'mixed' such relation to generate all the relations connecting the \( A_{k1}^*(r) \); perhaps the simplest mixed relation (with neither \( l \) nor \( k \) fixed) is
(l + k) a^k(1) = (l - k) A^k_{l-1}(1) + 2k A^{k-1}_{l}(1). \tag{23}

For the \( a_{l,k} \) we have the relation, for proceeding along a 'fixed diagonal' of Table 2 (with \( l > 0 \) and a constant value of \( l - k \)),

\[ a_{l,k} = \frac{\xi + k - 1}{\xi - k} \alpha_{l-1,k-1} \tag{24} \]

whilst to proceed to a lower diagonal we have

\[ a_{l,k} = -\frac{\xi (k + 1)}{\xi - k} a_{l-1,k+1} \tag{25} \]

These relations suffice to generate all the \( a_{l,k} \) from \( a_{0,0} = 1 \). Similar relations permit the generation of all the \( a_{l,k} \) from \( a_{1,0} = -1 \), they can be dispensed with, however, since it follows from (12), (13) and (16) that

\[ a_{l,k} - \frac{\xi - k + 1}{\xi} a_{l+1,k} = -\frac{\xi + k}{\xi} a_{l-1,k} \tag{26} \]

Though it is the \( A_{l,k} \) (and \( A_{l,k} \)) that we actually require to carry through the paper, recurrence relations are not offered for these. To preserve parity if one suffix is fixed, it would be necessary to use alternate values of the other; there seems little point in doing this, though a valid relation could easily be obtained, for example by applying (21) three times. There are simple relations between the \( A_{l,k} \) and \( A_{l,k} \), however. We will need two of these in section 7.3, namely,

\[ \xi \left( \frac{A_{l,k+1}}{u_{k+1}} - \frac{A_{l,k-1}}{u_{k-1}} \right) = 2kcs^{-1} \frac{A_{l,k}}{u_k} \tag{27} \]

and

\[ \xi \left( \frac{A_{l+1,k+1}}{u_{k+1}} + \frac{A_{l-1,k-1}}{u_{k-1}} \right) = -\frac{2A_{l,k}}{u_k} \tag{28} \]

Postscript. In regard to equations (27) - (31), it should have been noted that \( A_{l,k+1}/u_{k+1} \) and \( A_{l,k}/u_{k} \) actually reduce to \( -ic^{-1} A_{l,k+1}/u_{k+1} \) and \( 2c^{-1} A_{l,k+1}/u_{k+1} \) respectively, results that are implicit in the analysis of section 6.2; the \( u_k \) factors could be avoided by allowing negative \( k \) and \( \xi \) (see the footnote of page 40).
these being true for \( 1 \leq k \leq \ell \) (with \( \ell \) and \( k \) of the same parity). For \( k = 0 \) we only have one relation, given by direct addition of (27) and (28) (and really only one, involving direct subtraction, when \( k = \ell \)); thus,

\[
\ell a_{\ell,0} = -2a_{\ell,0}.
\]

We could use (27) and (28) to obtain expressions for \( A_{k \ell} \), defined by (20), but instead derive them directly. On substituting for \( A_{k \ell} \) from (14) and for \( A_{k \ell} \) from (19), then in forming \( A_{k \ell} \) we find that the term in \( A_{k \ell}^{(1)} \) cancels out and we get (for \( 0 \leq k \leq \ell \), and \( \ell \) and \( k \) of like parity)

\[
A_{k \ell} = J_2 (R/p) \ell a_{k \ell} \frac{(\ell - k)(k + 1)}{2(\ell + 1)} c^{-1} s^{k+1} A_{k}^{(1)}(1). \tag{30}
\]

For \( A_{k \ell} \), on the other hand, the combination of \( A_{k}^{(1)} \) and \( A_{k}^{(1+1)} \) is such that (22), with \( k \) replaced by \( k - 1 \), is immediately applicable, leading to (but only for \( 1 \leq k \leq \ell \) now)

\[
A_{k \ell} = 2k J_2 (R/p) \ell a_{k \ell} c^{-1} s^{k-1} A_{k}^{(1)}; \tag{31}
\]

for \( k = 0 \), this would give a false value of zero, the correct result being the same as is given by (30), since \( A_{k \ell}^{(1)} = -A_{k \ell}^{(0)} \). The formulae (30) and (31) are used in the analysis of \( \delta \beta \) in section 6.2. (See also the footnote to page 15.)

3 FUNCTIONS OF ECCENTRICITY USED IN THE SUBSEQUENT ANALYSIS

The term \( U_2 \) of the potential, specified by (4), has now been decomposed into the \( U_2^{(1)} \) defined by (15), the latitude \( (8) \) having been eliminated. The longitude was absent from \( U_2 \) from the beginning, because of axial symmetry, so it remains to eliminate the radius vector \( (r) \). This can be done by appeal to (6), but (as noted in section 1) we will in practice postpone the use of (6) until the setting up of each planetary equation, so the present section is preparatory in nature. Further, it is not \( (p/r)^{2+1} \) in (15), that must be eliminated, but \( (p/r)^{2+1} \) as a factor \( (p/r)^2 \) is retained to effect the change of integration variable defined by (7).

It is evident from (6) that an expansion of the form

\[
(p/r)^{x-1} = \sum_{j=0}^{x-1} u_j B_{j,j} \cos j\psi
\]

(32)
It is possible, for $k > 1$, and we regard $B_{ij}$ as defined by this expansion.

We shall find it useful, and entirely natural, to extend the definition of $B_{ij}$ to negative $j$, by defining

$$ S_{j-i} = B_{tj} I a $$

and to take $B_{ij} = 0$ when $|j| = |l|$. On this basis, and using the notation of (3), we can replace (30) by

$$ (pr)^{l+1} \cos \nu \sin \nu = 2n \pi (n+1) X_{l+1,1} $$

To demonstrate (34), we first replace the index $j$ by $-(n+1)$ in (35), where $r^2 = -1$ and the summation runs from $-\infty$ to $\infty$. Only when $m = 0$, which is the case with which we are concerned, is $X_{n,m} = 1$, a simple (finite) expression in $e$, and it is precisely because of this that we change the variable from $t$ to $v$ in the planetary equations, thus avoiding infinite expansions in $h$.

Hansen's functions $X_{n,m}$ of eccentricity are defined uniquely by the existence of the expansion, for all integral $n$ and $j$, regardless of sign,

$$ B_{ij} = \sum_{k=0}^{\infty} X_{k,i-j} $$

where the summation effectively runs from $j = -\infty$ to $j = +\infty$, so that there is no need for explicit summation limits.

To make use of some results from Ref. 4, we first demonstrate that the $B_{ij}$ are directly related to the Hansen $X$ functions of classical celestial mechanics 12, such that

$$ (pr)^{l+1} X_{l+1,1} $$

and then integrate over a revolution of $M$. We get (from the real part of the result)

$$ (pr)^{l+1} \cos \nu \sin \nu = 2n \pi (n+1) X_{l+1,1} $$

To demonstrate (34), we first replace the index $j$ by $-(n+1)$ in (35), where $r^2 = -1$ and the summation runs from $-\infty$ to $\infty$. Only when $m = 0$, which is the case with which we are concerned, is $X_{n,m} = 1$, a simple (finite) expression in $e$, and it is precisely because of this that we change the variable from $t$ to $v$ in the planetary equations, thus avoiding infinite expansions in $h$.

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$$ (pr)^{l+1} X_{l+1,1} $$

and then integrate over a revolution of $M$. We get (from the real part of the result)
But from (33),
\[
2\pi \int_0^{(p/r)^{k+1}} \cos jv \, dM - 2\pi \sum_m B_{tm} (p/r)^2 \cos mv \cos jv \, dM .
\]
\[ \tag{37} \]

We apply (7) to change the integration variable to \( v \) on the right-hand side of (37); only if \( m = \pm j \) do we retain a non-zero term, and in fact
\[
2\pi \int_0^{(p/r)^{k+1}} \cos jv \, dM = 2\pi q^3 B_{kj} .
\]
\[ \tag{38} \]

Then (34) is immediate from (36) and (38).

Some comments related to the notation are worth making before we proceed further. In principle we are reserving the suffix \( k \) for the \( A \) functions and \( j \) for the \( B \) functions, but in section 5, where only the value \(-k\) arises for \( j \), we will naturally encounter \( B_{tk} \). We would also rather naturally change the notation from \( j \) to \( k \) in (35) if we were following the traditional path in which the integration variable is \( M \) and the expansion of (15) is by (35) directly. After replacement of \( \ell \) by \(-l+1\), the Hansen function would then appear as \( X_{m}^{\ell-1,k} \), which is nowadays (following Kaula) usually expressed (when \( \ell \geq 0 \)) as \( G_{ipq} \); here \( p = \ell(l-k) \) which must be integral (assuming \( t \) and \( k \) to be of the same parity), and \( q = m-k \). Introducing also the notation \( G_{iq} \), which the present author has recommended as preferable to \( G_{ipq} \), we may extend (38) by writing
\[
q^{-(2l-1)} B_{kj} = X_{0}^{l-1,j} \cdot G_{ik}\frac{K-l-j}{-j} - q^j G_{0}-j(e) .
\]
\[ \tag{39} \]

Gooding and King-Hele have recently reported on the \( G \) functions that are relevant to resonant satellite orbits. Ref 13 includes the listing of a Fortran program (by Alfred Odell) that computes the functions for arbitrary values of \( \ell \), \( k \), and Kaula's \( q \), by quadrature.

We can now use the identity (34) to tie into the analysis of Ref 4. Thus, we may express the \( q \)-polynomial \( B_{kj} \), when \( \ell \geq 1 \) and \( 0 \leq j < \ell \), in terms of a normalized such polynomial, the connecting relation being
\[
B_{kj} = \left( \frac{\ell-j}{\ell-1} \right) (e/2)^j B_{k}^j(e) ,
\]
\[ \tag{40} \]
$B_i^k(e)$ in (40) is a polynomial in $e^2$, with constant term unity by the normalization. Explicit expressions for the $B_i^k(e)$ are given in Table 3, for values of $i$ and $j$ up to 7 and 6 respectively. There is an evident resemblance between the $B_i^k(e)$ and the $A_k^j(l)$, a significant difference being that the new functions run from $l = 1$ and not $l = 0$; the resemblance is not fortuitous, since it can be shown that
\begin{equation}
B_i^k(e) = \frac{1}{(i-1+j)!} (ie/2)^{i-j} q^{j-1} P_{i-1}^{(j-1)} (q^{-1}) .
\end{equation}
from which it follows that
\begin{equation}
B_i^k(e) = q^{j-1} A_{i-1}^j (\tan^{-1} e) .
\end{equation}

In contradistinction with the $A_k^j$, however, it is usually much better to work with the $B_i^j(e)$ directly (in recurrence relations, for example), rather than through $B_i^k(e)$ and (40). One reason for this is that only alternate values of the $A_k^j$ are non-zero, whereas (for $|j| < i$ and $e = 0$) all the $B_i^j(e)$ are non-zero. Further, no difficulty arises with the $B_i^j(e)$ when $j < 0$ (as already noted, and see also Appendix B), whereas $B_i^j(e)$ would then be infinite (if $|j| < i$). We can even allow $i$ to be negative (or zero) as well as $j$. The validity of this follows from the universality of (34) - the universality is brought out by Table 4, which lists $B_i^j(e)$ for $i$ running from -3 to +4 and $j$ from -1 to +3.

The entries in Table 4 form triangular blocks of four types. First, for $i > 0$ and $|j| < i$, we have the quantities that can be expressed by (40) when $j \geq 0$. Secondly, for $i > 0$ and $|j| \geq i$, we have (two blocks of) zeroes. Thirdly, for $i \leq 0$ and $|j| \leq i$, we have quantities that, when $j = 0$, can be expressed by a formula complementary to (40), viz
\begin{equation}
B_i^j = \left(\frac{e}{2}\right)^j \frac{1}{(i-1+j)!} q^{j-1} P_{i-1}^{(j-1)} (q^{-1}) .
\end{equation}
a formula equivalent to this was given in Ref 4, the application being (as noted in section 1) to secular and long-period perturbations associated with exterior (rather than interior) mass. Finally, for $i \leq 0$ and $|j| > i$, we have (two blocks of) quantities that are most conveniently expressed in terms of $\beta$ (not now denoting latitude, as previously) and $q$, rather than $e$ and $q$, where
\[ \beta = \frac{e}{1 + q}, \quad (44) \]

no attention was paid to these quantities (or their equivalents) in Ref 4, there being no application for them, but the formula for \( B_{0,1} \) is derived here, for completeness, in Appendix B. (Other entries in this last pair of blocks are then derivable from recurrence relations.) Before leaving Table 4, we note that the formula for the \( X \) or \( G \) function corresponding to \( B_{0j} \) is immediate from the Table, in view of (34); thus it is only necessary to apply the factor \( q^{-\epsilon} \), which introduces a negative power of \( q \), when there is not one already present and cancels it out when there is.

In regard to derivatives of the eccentricity functions, it can be shown (by working from (41)), and easily verified from Table 3 that, for \( 1 \leq j < \ell \),

\[ \frac{d}{de}[B_{j}^{2}(e)] = 2j e^{-1} \left[ B_{j, j}^{2}(e) - B_{j}^{2}(e) \right]. \quad (45) \]

The universal formula for the derivative of \( B_{Kj} \) is

\[ B_{Kj}' = (\ell - 1) B_{K-1, j-1} - j e^{-1} B_{Kj}. \quad (46) \]

For \( 1 \leq j < \ell \), this follows from (45); for general entries in Table 3, it can be verified with the aid of \( q' \) and \( \beta' \), which may be expressed as \(-e/q\) and \( \beta/eq \) respectively. However, because we only introduce the \( B_{Kj} \) after each planetary equation has been set up, we effectively only use (46) in expressing the rates of change of the mean elements. Since this involves

\[ \frac{d}{de} (q A_{ik} B_{ik}) = q^{-1} \delta_{ik} \left[ q^{2} B_{ik}' + (2\ell - 1) e B_{ik} \right], \quad (47) \]

we define

\[ E_{ik} = q^{2} B_{ik}' + (2\ell - 1) e B_{ik}; \quad (48) \]

then (46), re-expressed via the recurrence relation (56), leads to

\[ E_{ik} = e^{-1} (te^{2} - k) B_{ik} + (t - k) B_{i,k-1} \quad (49) \]
(By symmetry, there is a parallel expression for $E_{4k}$ that involves $B_{4k}$ and $B_{4,k+1}$.) Table 5 gives explicit expressions for the $E_{4k}$, with $k$ running from $-3$ to $+3$ as in Table 4; only the entries in which $k$ has the same parity as $l$ (or $-l-1$ if $l < 0$) are useful in practice, and entries for $k > l$ (or $-l+1$ if $l > 0$) are omitted entirely (for $k > l > 0$ they would all be zero).

The $E_{4k}$ are related to the $E_{2k}^{(e)}$ of Ref 4 by

$$E_{4k} = e^{-1} (e/2)^k \binom{k-1}{k} E_{2k}^{(e)} , \quad (50)$$

when $k > 0$; for $k < 0$, the extra factor $q^{2l+1}$ is required (cf (43), where the additional factor, in relation to (40), is $q^{2l-1}$).

The $B_{4k}$ may be computed from recurrence relations for the $B_{2k}^{(e)}$, which will now be given, but in developing the theory it is more useful to have such relations for the $B_{4k}$ themselves, so these will also be given. For fixed $j$ ($\geq 0$), the recurrence formula (from Ref 4) is

$$(l + j - 1) B_{4k}^{(e)} = (2l - 3) B_{4k-1}^{(e)} - (l - j - 2) q^{2} B_{4k-2}^{(e)} , \quad (51)$$

valid for $l \geq j + 3$ with the starters $B_{4j+1}^{(e)}$ and $B_{4j+2}^{(e)}$ both unity. For fixed $k$ ($\geq 3$), on the other hand, the formula is

$$B_{4k}^{(e)} = B_{4k+1}^{(e)}(e) + \frac{(l - j - 2)(l - j + 1)}{4(j + 1)(j + 2)} e^{2} B_{4k+2}^{(e)} , \quad (52)$$

valid for $l - 3 \geq j \geq 0$ with the starters $B_{4k+1}^{(e)}$ and $B_{4k+2}^{(e)}$ again both unity. The resemblance of (51) and (52) to (21) and (22) respectively follows from the remark leading up to (42).

The recurrence relations for the $B_{4k}$, that correspond to (51) and (52), respectively, and are valid for all $k$ and $j$, are (when symmetrically expressed)

$$k(l - 1) q^{2} B_{4k-1,j} - (2l - 1) B_{4k,j} + (l^{2} - j^{2}) B_{4k+1,j} = 0 \quad (53)$$

and

$$(j - k) e B_{4k+1,j+1} + 2j B_{4k,j} + (j + k) e B_{4k+1,j} = 0 ; \quad (54)$$
the mandatory symmetry in (54), to satisfy the unchanging value of $B_{t,j}$ under the operation $j \rightarrow -j$, is evident.

As remarked for the inclination functions, either of the 'pure' relations, (53) and (54), can be used with a 'mixed' relation* to generate all the recurrence relations connecting the $B_{t,j}$. Here the mixed relations are, in particular, those that connect three out of four of the $B_{t,j}$ lying 'around a square' of index duplets; if the square consists of the duplets $(t, j)$, $(t - 1, j)$, $(t, j + 1)$ and $(t + 1, j + 1)$, then the four mixed relations connecting them (all of which we shall require in the sequel) are

\[ (t - j) e B_{t,j} - (k - j) e B_{t+1,j} + (k + j + 1) e B_{t+1,j+1} = 0, \]  
\[ k e B_{t,j} + (k + j) e B_{t+1,j+1} - (k + j + 1) e B_{t+1,j+1} = 0, \]  
\[ (t - j) e B_{t+1,j} + k e B_{t,j+1} - (k + j + 1) e B_{t+1,j+1} = 0, \]

and

\[ (t - j) e B_{t+1,j} + e B_{t,j+1} - (k + j + 1) e B_{t+1,j+1} = 0. \]  

If we re-order the terms in the last two relations and replace $j$ by $j + 1$, we get relations which are symmetric pairings of (55) and (56), viz

\[ (t + j) e B_{t+1,j} + k e B_{t,j-1} = 0 \]  
\[ k e B_{t+1,j} - (k + j + 1) e B_{t+1,j+1} - (k + j) e B_{t+1,j+1} = 0, \]  

and

\[ k e B_{t+1,j} - (k + j + 1) e B_{t+1,j+1} - (k + j + 1) e B_{t+1,j+1} = 0. \]

Of this set of relations, (55) and (59) can be obtained at once from (54) and the relation equivalent to one given (for the Hansen functions) by Zafiropoulos8, viz

\[ k e (B_{t,j-1} - B_{t,j+1}) = 2j B_{t+1,j}; \]  

this is of a different 'shape' from our triangles-around-the-square relations, but is perhaps the simplest recurrence relation of all.

* Note added in proof: Ref 19 indicates that, for inclination functions, pure relations are computationally preferable to mixed relations (see also Ref 20).
RATES OF CHANGE OF OSCULATING ELEMENTS

In this section we use Lagrange's planetary equations to develop the rate of change of each of the orbital elements \( (a, e, i, q, \omega, \text{ and } M) \) due to \( U_k \), the term of the disturbing function specified by (15). Each rate of change is to be with respect to \( v \), rather than \( t \), expressed as a finite trigonometric series (assuming \( k > 0 \), as we now always do, except in section 8.1), with \( v \) as the variable. The \( v \)-independent terms of each \( \frac{dc}{dv} \) are then isolated; they effectively contribute to the time rate of change, \( \frac{d\zeta}{dt} \), of the mean element \( \zeta \), expressions for the \( \zeta \) being held over to section 5. The remaining terms of \( \frac{dc}{dv} \) can at once be integrated to provide contributions to the short-period perturbation, \( \delta r \). The result of the integration is, in fact, so 'immediate' (apart from the question of the integration 'constants' already referred to in section 1) that we will not bother to write down formal expressions for the five \( \delta c \) other than \( \delta a \); this is to emphasize the fact that it is the combinations of the \( \delta c \) into \( \delta r \), \( \delta b \) and \( \delta w \) that are of interest (being the topic of section 6), not the \( \delta c \) themselves.

The perturbation \( \delta a \) is a special case because it can be obtained without integration. As in Part I, however, we also derive \( \frac{da}{dv} \) from the appropriate planetary equation, as a prototype for the derivation of the other \( \frac{dc}{dv} \). By bringing in the quasi-elements, \( \phi \) and \( \rho \), it is possible to develop each equation in terms of the partial derivative of \( U_k \) with respect to a single quantity.

4.1 Semi-major axis

As in Part I, there is an absolute constant of the motion, which we denote by \( a' \), such that

\[
a = a' (1 + 2aU/u) ;
\]

(62)

this is an exact relationship for any time-independent disturbing function, \( U \), and in particular for the axi-symmetric \( U_k \). It follows that there is no long-term variation in \( a \), to whatever order of magnitude the perturbation analysis is conducted. Further, the short-period perturbation, \( \delta a \), is given exactly, on substituting for \( U_k \) from (15), thus

\[
\delta a = -2a'^2 q^{-2} A_{kk} (p/r)^{k+1} c_k^k .
\]

(63)
This does not mean that an exact perturbation can be written down for semi-major axis, however, as the right-hand side of (63) is expressed in terms of osculating elements: as soon as mean elements are introduced, the result is no more than a first-order perturbation expression, as with any other $\xi$.

To present $\delta a$ in the form appropriate for use in section 6.1, we combine $C_0$ with one of the factors $p/r$. Thus,

$$\delta a = -a q^{-2} A_k (p/r)^k \left( e C_{k-1} + 2 C_0 + e C_1 \right).$$

(64)

We retain another $p/r$ factor explicitly, and expand the remaining $(p/r)^{k+1}$ by (33). By this means the term $2 C_0$, for example, in (64) is effectively transformed, for each $j$, into $C_j + C_{-j}$. But each pair of terms (such as this) for positive $j$, in the infinite summation of (33), is matched by the same pair (in reverse order) for negative $j$, so we can express the result of the expansion as

$$\delta a = -a q^{-2} A_k (p/r)^k \sum B_{kj} \left( e C_{j-1} + 2 C_j + e C_{j+1} \right).$$

(65)

We now develop an expression for $\frac{da}{dv}$ ab initio, using the general procedure that involves the planetary equation for $\dot{a}$. This equation is

$$\frac{da}{dt} = \frac{2}{na} \frac{3U}{\delta M},$$

(66)

and on substituting for $U_k^2$ we get

$$\frac{da}{dt} = -2 a q^{-2} A_k \frac{3}{\delta M} \left( (p/r)^{k+1} C_0 \right).$$

(67)

The $M$-differentiation is immediate, since $p/r$ is given by (6) and $3v/\delta M$ is $q^3 (p/r)^2$ (cf. (7)). We transform from $dt$ to $dv$ (again using (7)), and all this leads to

$$\frac{da}{dv} = a q^{-2} A_k (p/r)^k \left[ (k - \ell - 1) e S_{-1} + 2 \ell S_0 + (k + \ell + 1) e S_1 \right].$$

(68)
Finally, we split \((p/r)^l\) into \((p/r)^{l-1}\) and \(p/r\); then, applying (33) and taking advantage of the matching of terms for positive and negative \(j\), we get

\[
\frac{\text{d}a}{\text{d}v} = 4a q^{-2} A_{ik} \sum B_{kj} \left\{ (k - l - 1) e^2 S_{j-2} + 2(2k - l - 1) e S_{j-1} + 2k(2 + e^2) S_j + 2(2k + l + 1) S_{j+1} + (k + l + 1) e^2 S_{j+2} \right\}. \quad (69)
\]

This may be regarded as a prototype for all the \(d\xi/dv\); in addition, (69) is used in the derivation of \(d\alpha/dv\) in section 4.2. The equivalence of this result with the \(v\)-derivative of \(5a\) obtained by the special procedure may be verified, most easily by the \(v\)-differentiation of (64).

In dealing with the subsequent \(\zeta\), we will be isolating the component of \(d\xi/dv\) that leads to secular and long-period perturbations. We know that for \(d\alpha/dv\) this component must be zero, both from the special procedure and from the form of (66) (since a term of \(U\) that is free of short-period variation must tautologically have zero \(M\)-derivative), but it is instructive (as part of the prototype for the other \(\zeta\) ) to obtain this result from (69). The terms independent of \(v\) in the overall \(j\)-sum are the terms in \(S_{-k}\) \(\triangleq S_{-k}^k\). Since \(B_{kj} = B_{kj}\), the combination of all such terms involves the factor

\[
(k - l - 1) e^2 B_{k,k-2} + 2(2k - l - 1) e B_{k,k-1} + 2k(2 + e^2) B_{k,k} + 2(2k + l + 1) e B_{k,k+1} + (k + l + 1) e^2 B_{k,k+2},
\]

and it follows from three applications of (54) that this is zero.

4.2 Eccentricity

We develop the perturbation in \(e\) by first obtaining the perturbation in \(p\), since

\[
\dot{p} = q^2 \dot{a} - 2ae \dot{e} \quad (70)
\]

and the planetary equation for \(p\) is just

\[
\frac{dp}{dt} = 2a \frac{du}{dw}. \quad (71)
\]

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On substituting for \(U_i^j\) we get

\[
p = -2 \alpha a q^{-1} A_{nk} (p/r)^{l+1} \frac{\partial}{\partial \alpha} (C_0) .
\]  

(72)

Transformation of the integration variable to \(v\), by (7), yields

\[
\frac{dp}{dv} = 2 kp A_{nk} (p/r)^{l+1} S_0 ;
\]

hence we get, from (33) and the usual argument concerning positive and negative \(j\),

\[
\frac{dp}{dv} = 2 kp A_{nk} \sum B_{kj} S_j .
\]

(73)

(74)

We get the long-term variation by setting \(j = -k\); thus

\[
\frac{dp}{dv} = 2 kp A_{nk} B_{nk} S_{-k} .
\]

(75)

The expression for \(\xi\) is now immediate from (70), since \(\xi = 0\), but it is not given here, as the complete list of the \(\xi\) is given in section 5.

For the \(v\)-derivative of the short-period perturbation, \(\delta e\), we have all the terms with \(j + k = 0\) in the expression given by the combination of (69) and (73), according to (70). This combination leads to

\[
\frac{de}{dv} = \frac{1}{4} A_{nk} \sum B_{kj} \left\{ (k - l - 1) e S_{j+2} + 2(2k - l - 1) S_{j+1} \\
+ 6 k e S_j + 2(2k + l + 1) S_{j+1} + (k + l + 1) e S_{j+2} \right\} .
\]

(76)

As already indicated, we will not write down the expression for \(\delta e\), involving \(C_{j-2}\) etc., given immediately by integration of (76); immediate, that is, apart from the "constant term" in \(C_{-k}\), that we are not yet in a position to assign. This term effectively replaces the infinite term (in \(C_{-k}\)) that would arise if we had not removed the \(S_{-k}\) term from (76) in advance.
4.3 Inclination

We can develop the perturbation in $i$ from the perturbations in $p$ and $p_{O}^{2}$, since

$$\frac{d(p_{O}^{2})}{dt} = c^{2} \dot{p} - 2pc\dot{i} \tag{77}$$

and the planetary equation for $p_{O}^{2}$ is just

$$\frac{d(p_{O}^{2})}{dt} = \frac{2q_{c} 3\nu}{na \; 3\nu} \tag{78}$$

But $U_{k}^{i}$ is independent of longitude, and hence of $\Omega$, so $p_{O}^{2}$ is an invariant. Using (77), therefore, we have $\dot{t}$ at once from (75), whilst $\dot{i}$ will be based on the expression for $d\dot{i}/dv$ derived from (74), viz

$$\frac{d\dot{i}}{dv} = \cos^{-1} A_{ik} \sum B_{kj} S_{j} \tag{79}$$

(Since $A_{ik}$ contains $s^{k}$ as a factor, there will never be a non-zero multiple of an uncancelled $s^{-1}$.)

4.4 Right ascension of the node

The perturbation in $\Omega$ comes from the planetary equation

$$\frac{d\Omega}{dt} = \frac{1}{na^{2} \Phi} 3\nu \tag{80}$$

On substituting for $U_{k}^{i}$ we get

$$\dot{\Omega} = - nq^{-3} s^{-1} (p/r)^{l+1} A_{ik} C_{0} \tag{81}$$

We now apply (7) and (33) as usual, getting an isolated contribution to $\dot{\Omega}$, together with the expression for $d\dot{\Omega}/dv$, viz

$$\frac{d\dot{\Omega}}{dv} = - s^{-1} A_{ik} \sum B_{kj} C_{j} \tag{82}$$
4.5 Argument of perigee

Introduction of $\psi$ gives us a one-term planetary equation, since

$$\frac{d\psi}{dt} = \frac{q}{na^2 e^3} \frac{3U}{e}. \quad (83)$$

The $e$-derivative is much the most complicated of the partial derivatives of $U_k$, since $e$ is an argument of each of the four factors on the right-hand side of (15). Thus we get

$$\dot{\psi} = -e^{-1} q \frac{3}{\delta e} \left( A_{l_k} q^{-2} (p/r)^{l+1} C_0 \right). \quad (84)$$

But

$$\frac{3}{\delta e} (q^{-2} A_{l_k}) = 2(l + 1) e^{-1} A_{l_k} \quad (85)$$

and, using the expressions for $\delta r/\delta e$ and $\delta v/\delta e$ (equations (41) and (42) of Part 1),

$$\frac{3}{\delta e} \left( (p/r)^{l+1} C_0 \right) = q^{-2} (p/r)^{l+1} \left[ (l + 1) (\cos v - e = e \sin^2 v) C_0 \right]$$

$$- k \sin v (2 + e \cos v) S_0 \right), \quad (86)$$

so that (84) reduces to

$$\dot{\psi} = e^{-1} q^{-3} A_{l_k} (p/r)^{l+1} \left[ k \sin v (2 + e \cos v) S_0 \right]$$

$$- (l + 1) \cos v (1 + e \cos v) C_0 \right). \quad (87)$$

We make the standard expansions of the trigonometric products in (87), and then apply (7) and (33) as usual. This leads to

$$\frac{d\psi}{dv} = -e^{-1} A_{l_k} \sum B_{l_j} \left[ (l + 1 - k) e C_{l-2} + 2(l + 1 - 2k) C_{l-1} \right]$$

$$+ 2(l + 1) e C_j + 2(l + 1 + 2k) C_{j+1} \right. + (l + 1 + k) e C_{j+2} \right]. \quad (88)$$
To get \( \ddot{\psi} \), we pick out the coefficient of \( C_k \). Thus

\[
\ddot{\psi} = -\frac{1}{2} \left[ \frac{n}{e} \right]^2 A_{\ell k} \left\{ (\ell + 1 - k)e B_{\ell+1,k-2} + 2(\ell + 1 - 2k)B_{\ell,k-1} \\
+ 2(\ell + 1)\kappa B_{\ell,k} + 2(\ell + 1 + 2k)B_{\ell+1,k+1} + (\ell + 1 + k)e B_{\ell+1,k+2} \right\} C^{-k} - \frac{1}{2} \left( \frac{n}{e} \right)^2 A_{\ell k} \left\{ (\ell e^2 - k)e B_{\ell,k} + (\ell - k)e B_{\ell,k-1} \right\} C^{-k}.
\] (89)

But this can be simplified by three applications of (54), which lead to

\[
\ddot{\psi} = -\frac{1}{2} \left( \frac{n}{e} \right)^2 A_{\ell k} \left\{ (\ell e^2 - k)e B_{\ell,k} + (\ell - k)e B_{\ell,k-1} \right\} C^{-k}.
\] (90)

We can now introduce the quantity \( E_{\ell k} \), to get a concise expression, since by (49) we have

\[
\ddot{\psi} = -\frac{1}{2} \left( \frac{n}{e} \right)^2 A_{\ell k} E_{\ell k} C^{-k}.
\] (91)

To get \( \dot{\psi} \) and the appropriate terms of \( \frac{d\omega}{dv} \), we combine (91) and (the residual terms of) (88) with \( \dot{\psi} \) and (82), respectively, using

\[
\dot{\psi} = \dot{\psi} - \ddot{\psi} - \ddot{\psi}.
\] (92)

4.6 Mean anomaly

We start by studying \( \phi \), since our final one-term planetary equation is

\[
\frac{d\phi}{dt} = -2 \frac{3u}{na}. \tag{93}
\]

Remembering that \( A_{\ell k} \), in \( U_\ell^k \), is itself a function of semi-major axis, we obtain

\[
\dot{\phi} = -2(\ell + 1)q n^{-2} (p/r)^{\ell+1} A_{\ell k} C_0 \] (94)

and hence

\[
\frac{d\phi}{dv} = -2(\ell + 1)q A_{\ell k} \sum B_{kj} C_j. \tag{95}
\]

In particular,

\[
\dot{\phi} = -2(\ell + 1)q A_{\ell k} B_{\ell k} C^{-k} \tag{96}
\]
and from (91) we therefore also have

\[ \dot{\theta} = -nq A_{\ell k} \left[ 2(\ell + 1)B_{\ell k} - e^{-1} E_{\ell k} \right] C_{-k}, \]  

(97)

since

\[ \dot{\sigma} = \dot{\rho} - q\dot{\psi}. \]  

(98)

From (88), similarly, the \( v \)-derivative of \( \delta \sigma \) is given by the \( v \)-dependent terms of

\[ \frac{d\sigma}{dv} = \frac{d}{dv} \left[ 3q A_{\ell k} \sum B_{\ell j} \left( (\ell + 1 - k) e C_{j-2} + 2(\ell + 1 - 2k) C_{j-1} - 6(\ell + 1) e C_{j} + 2(\ell + 1 + 2k) C_{j+1} + (\ell + 1 + k) e C_{j+2} \right) \right] . \]  

(99)

But

\[ \dot{H} = \dot{\sigma} + \dot{\psi}, \]  

(100)

where (with \( \tau \) standing for time)

\[ f = \int_0^t n \, dt \]  

(101)

and (assuming only \( U_{\ell k} \) to be operating)

\[ n - n' = 3nq^{-2} A_{\ell k} (p/r)^{\ell+1} C_0 \]  

(102)

by (63) and Kepler's third law.

From (101) and (102) it follows that

\[ \frac{df}{dv} = \frac{n'}{v} + 3q A_{\ell k} \sum B_{\ell j} C_j, \]  

(103)

by the usual procedure. We may then write

\[ \dot{f} = n' + 3nq A_{\ell k} B_{\ell k} C_{-k}. \]  

(104)
from which \( \ddot{\dddot{N}} \) is available on combining with (97). Finally, on combining the residual terms (those with \( k + J = 0 \)) of (103) with (99), we find that the \( v \)-derivative of \( \dddot{M} \) is given by the \( v \)-dependent terms of

\[
\frac{d\dddot{M}}{dv} = \dddot{v} e^{-1} A_{lk} \sum B_{lj} \left\{ (l + 1 - k)e C_{j-2} + 2(l + 1 - 2k)C_{j-1} - 6(l - 1)e C_j + 2(l + 1 + 2k)C_{j+1} + (l + 1 + k)e C_{j+2} \right\}. \tag{105}
\]

To conclude, we note that a very much simpler result than (105) is available for the non-singular \( \dddot{L} \). Thus from (88) and (105) we get

\[
\frac{dL}{dv} = -(2l - 1)q A_{lk} \sum B_{lj} C_j. \tag{106}
\]

5 SECULAR AND LONG-PERIOD ELEMENT RATES

In this section we collect the expressions for the rates of change of the mean elements, i.e. the \( \dot{\xi} \) associated with \( U_k^L \). As we have seen in section 4, this simply amounts to listing the components of the \( dC/dv \) that are multiples of either \( S_k^L \) or \( C_k^L \). When \( k = 0 \), the rate of change is secular; for \( k > 0 \), it is long-period. We shall not be concerned with the build-up of actual perturbations from the \( \dot{\xi} \), since this is fully dealt with in Parts 1 and 3; suffice it to say that there is no difficulty in the secular perturbations, but that (even in a first-order analysis) difficulties arise with the long-period perturbations, in particular due to the singularities associated with zero \( e \) and zero \( s \).

Another point must be mentioned before we list the \( \dot{\xi} \). As the expressions arise from terms in \( dC/dv \), but were treated (in section 4) as if from terms in \( dC/dM \), each \( \dot{\xi} \) produces a short-period component of the perturbation in \( \xi \); i.e. a contribution to \( \dot{\xi} \) is induced. These contributions may be amalgamated into components of \( \dot{\delta}r, \dot{4}b \) and \( \dot{3}w \), as done for \( J_3 \) in Part 1 (section 7). The issue relates to the definition of semi-mean elements (section 3.2 ibid), which is outside the scope of Part 2; it should be clear, from equations (120) - (122) in section 6, however, that no difficulty arises in the amalgamating.
In the list of the $\xi$ that follows, we note that the maximum value of $k$ is $\ell - 2$, since $B_{kk} = E_{kk} = 0$. We attach an explicit subscript $(1k)$ to each $\xi$; then our first result is

$$\xi_{1k} = 0.$$  \hfill (107)

For $e$, it follows from (70) and (75) that

$$\xi_{ek} = -k\cos^2 A_{ek} B_{ek} s_{ek}^k.$$  \hfill (108)

There will always be a positive power of $e$ to cancel the factor $e^{-1}$, it will be noted, coming from $kB_{ek}$.

For $I$, similarly, it follows from (75) and (77) that

$$\xi_{ik} = \kappa \cos^2 A_{ik} B_{ik} s_{ik}^k.$$  \hfill (109)

Here there will always be a positive power of $s$, coming from $kA_{ik}$, to cancel the factor $s^{-1}$.

For $a$, our analysis of (81) gives

$$\xi_{ak} = -n A_{1k} B_{1k} c_{1k}^k.$$  \hfill (110)

The formula is expressed in this way, with a factor $s$ on the left-hand side, to avoid the possibility of an uncancellable $s^{-1}$ on the right-hand side. For long-period perturbations, there is a singularity difficulty here, which can be dealt with as indicated in Part I (section 3.5). For secular perturbations (and here is our first non-zero $\xi$ when $k = 0$) there is no problem, since in the expression for $A_{1k}$, given by (19), $A_{1k}^2(I)$ appears with the multiplying factor $k$, and $A_{2k}^2(I)$ with the factor $f$.

For $\omega$, we use the final result, (91), for $\xi$. Then from (92) and (110) it follows that

$$e \xi_{ek} = n(\cos A_{1k} B_{1k} - s A_{1k} E_{1k}) c_{1k}^k.$$  \hfill (111)
Again the formula is expressed like this to make the right-hand side non-singular; and again (because $E_{\ell k}$ contains a factor $e$ when $k = 0$, as seen from Table 5) there is no difficulty with secular perturbations.

For $\hat{M}$, we combine the results for $\mathring{\theta}$ and $\mathring{\iota} - n'$, given by (97) and (104) respectively; thus

$$e \mathring{\theta}_{\ell k} = nq A_{\ell k} \left[ E_{\ell k} - (2\ell - 1)e B_{\ell k} \right] C_{-k}^k. \tag{112}$$

From (48), this may also be written as

$$e \mathring{\theta}_{\ell k} = nq A_{\ell k} B_{\ell k} C_{-k}^k. \tag{113}$$

From the definition of $L$, we may also combine (112) with (91); this gives the non-singular result

$$\mathring{\hat{M}}_{\ell k} = - (2\ell - 1) nq A_{\ell k} B_{\ell k} C_{-k}^k. \tag{114}$$

As usual, a factor $e$ can be cancelled from both sides of (113) when $k = 0$. However, there is a simpler way of dealing with secular perturbations in $\hat{M}$, as indicated in Part 1; Ref 3 was largely devoted to this topic, and the rest of this section conforms with the account therein.

The basic idea is that we represent the secular perturbations in mean anomaly by modifying the value of the mean mean motion. In view of (113), in fact, we write

$$\hat{M} = n'(1 + e^{-1}q^3 \sum A_{\ell k} B_{\ell k} C_{-k}^k), \tag{115}$$

where the summation is now on $k$, and we have set $c_0^0$ to unity. This agrees with equation (11) of Ref 3, since $A_{\ell k,0}$ here may be identified with $-J_{\ell k} C_{\ell k}(R/p)A_{\ell k}(1)$ from that paper.

The logic for using $a'$ as mean semi-major axis ($\pi$) is, as we have seen, compelling, so if $\hat{M} = n'$, we do not retain Kepler's third law in its simplest form. This is of no consequence, however, and we simply write, from (115),

$$n'^2 a'^3 = u(1 + 2e^{-1}q^3 \sum A_{\ell k,0} B_{\ell k,0}). \tag{116}$$
Ref 3 gives, as equation (15), an explicit version of $\phi_{16}$ with even values of $\ell$ up to 8; the two versions can easily be verified as equivalent if we note that $Q_k$ (in Ref 3) is just a normalized form of $B_{k,0}^\ell$ such that

$$B_{k,0}^\ell = \frac{1}{2}(\ell - 1)(\ell - 2)\alpha Q_k.$$  \hfill (117)

The following recurrence relation was given in Ref 3:

$$(\ell - 1)Q_{\ell} = (2\ell - 5)Q_{\ell-1} - (\ell - 4) q^2 Q_{\ell-2}; \hfill (118)$$

this is valid for $\ell \geq 4$, with $Q_2 = 0$ and $Q_3 = 1$. The relation may be obtained from (117) and (46), together with

$$(\ell - 1)(\ell - 3) B_{k-1,1} = (\ell - 2)(2\ell - 5) B_{k-2,1} - (\ell - 2)(\ell - 3) q^2 B_{k-3,1} \hfill (119)$$

which derives from (53) on replacing $(l, j)$ by $(\ell - 2, 1)$.

It is very convenient that $Q_2 = 0$. It means that for first-order analysis associated with $J_2$ (the dominant harmonic of the geopotential) $M$ is the same as $n'$.

6 PERTURBATIONS (SHORT-PERIOD) IN COORDINATES - GENERAL CASE

In this section we develop general expressions for the $\delta r$, $\delta b$ and $\delta w$ that can be derived from the first-order $\delta \zeta$ via the formulae (taken from section 3.3 of Part 1)

$$\delta r = (r/a) \delta a - (a \cos v) \delta e + (ae^{-1} \sin v) \delta H , \hfill (120)$$

$$\delta b = \left( \cos u' \right) \delta i + (a \sin u') \delta \Omega \hfill (121)$$

and

$$\delta w = \delta \phi + q^{-2} \sin v (1 + p/r) \delta e + q^{-3}(p/r)^2 \delta H . \hfill (122)$$

Special cases (derived from the choice of integration constants in the $\delta \zeta$) are reserved to section 7, but in counting the number of terms associated with the general $U_k$ we have regard to the basis on which these constants are chosen.
The $\delta \xi$ are available at once as the $v$-integrals of the expressions for $d\xi/dv$ in section 4. Generation of the expressions for $\delta r$ and $\delta w$ is essentially straightforward in that the analysis starts with the $\delta \xi$ due to $u^k_2$ (which $\delta \xi$ we can denote by $\delta \xi_{2k}$) and finishes with $\delta r_2$ and $\delta w_2$. With $\delta b$, however, there is a complication, due to the appearance of $u'$ in (121), as opposed to $v$ in (120) and (122); as already noted in section 1, we deal with the difficulty by deriving $\delta b_{2k}$, rather than $\delta b_{2k}$, where $k$ has values of opposite parity to those of $k$.

We do not give expressions for $\delta r$, $\delta b$, and $\delta w$, but (as is clear from Part I) these are immediately available from the expressions for $\delta r$, $\delta b$, and $\delta w$, just by replacing $S_j$ and $C_j$ by (respectively) $(k + j) \overline{R} C_j$ and $-(k + j) \overline{R} S_j$. We can do better than this if we allow for the (overall) rate of change of $\delta \xi$, replacing $(k + j) \overline{R}$ by $(k + j) \overline{R} + k \delta$, assuming $C_j$ and $S_j$ still to be shorthand for $C^k_j$ and $S^k_j$.

6.1 The perturbation $\delta r$

We have to apply (120) with $\delta a$, $\delta e$, and $\delta M$ given by (65) and (the integrals of) (76) and (105). We find that the integrals combine in a very natural way, as a result of which we can write (with $\delta r$ short for $\delta r_{2k}$)

\[
\delta r = \frac{r/a}{2} \delta a = \frac{1}{2} a_{2k} \sum B_{2k} \left\{ e \left[ \frac{k - \frac{1}{2}}{k + \frac{1}{2}} \right] + 3 \left[ \frac{k - \frac{1}{2}}{k + \frac{1}{2}} \right] C_{j-1} \right\} + \frac{1}{2} \sum \left[ \frac{2k - \frac{1}{2}}{k + \frac{1}{2}} \right] C_j + e \left[ \frac{3k - \frac{1}{2}}{k + \frac{1}{2}} \right] C_{j+1} \right\} (123)
\]

The simplest way to incorporate (65) is to note that this can be decomposed into

\[
\sum a_{2k} \sum B_{2k} \left\{ \left[ \frac{k + \frac{1}{2}}{k + \frac{1}{2}} \right] C_{j+1} - 3 \left[ \frac{k + \frac{1}{2}}{k + \frac{1}{2}} \right] C_{j-1} \right\} + \frac{1}{2} \sum \left[ \frac{2k + \frac{1}{2}}{k + \frac{1}{2}} \right] C_j + e \left[ \frac{3k + \frac{1}{2}}{k + \frac{1}{2}} \right] C_{j+1} \right\} (124)
\]

By this trick, we can combine (123) and (124) at once, to get, say,

\[
\delta r = - \frac{1}{2} a_{2k} \sum B_{2k} R_1 .
\]
where \( R_j \) (or \( R_{2k,j} \)) to display all the index parameters is given by

\[
R_j = e(j + 1 - 1) \left( \frac{1}{k + j - 2} + \frac{3}{k + j} \right) C_{j-1} + 2 \left( \frac{24 + 8 - 1}{k + j - 1} + \frac{24 - 8 + 1}{k + j + 1} \right) C_j + e(j - 1 + 1) \left( \frac{3}{k + j} + \frac{1}{k + j + 2} \right) C_{j+1}.
\]  

(126)

It can be seen that (125) is a summation in which \( R_j \), as given by (126), has three components; each component is expressed as the sum of two multiples of the same 'C quantity'. Let us separate the first multiple from the second (in each component of \( R_j \)), feeding them back separately into the summation of (125), so that we have two distinct summations that we can denote by \( L_\) and \( L_\). Thus \( L_\) involves \( \sum B_{k,j} R_{j-} \), where

\[
R_{j-} = \frac{1 + 1}{k + j - 2} eC_{j-1} + 2 \frac{24 + 8 - 1}{k + j - 1} C_j + 3 \frac{3 + 1}{k + j} eC_{j+1}.
\]  

(127)

Now we have seen (in section 3) that all sums over \( B_{k,j} \) can be regarded as running from -- to ++. It follows that we can rearrange the three sets of terms in \( \sum B_{k,j} R_{j-} \) such that (with \( j \) now used in a different way)

\[
\sum B_{k,j} R_{j-} = \sum (k + j - 1)^{-1} [(j + 1)e B_{k,j+1} + 2(2j + 1 - 1) B_{k,j} + 3(j - 1)e B_{k,j-1}] C_j.
\]  

(128)

We now invoke the recurrence relation (54) to eliminate \( B_{k,j+1} \), so that the quantity in curly brackets in (128) becomes

\[
2(j + 1 - 1) B_{k,j} + 2(j - 1)e B_{k,j-1} x,
\]

and then simplify further, using (58) with both \( t \) and \( j \) reduced by 1, to reduce this to \( 2(t - 1)q^2 B_{k-1,j} \). (We get the same result by using (54) to eliminate \( B_{t,j-1} \) first, and then simplifying further via (56).) Thus

\[
\sum B_{k,j} R_{j-} = 2(t - 1)q^2 \sum (k + j - 1)^{-1} B_{k-1,j} C_j.
\]  

(129)
Similarly,
\[
\sum B_{k,j} R_j = -2(\ell + 1)q^2 \sum (k + j + 1)^{-1} B_{k-1,j} C_j.
\] (130)

The final result we require now follows from (125), (129) and (130).
Because of its importance, we write \( C_j \) in full. Thus
\[
\delta \varphi = - (\ell - 1)^2 \delta \varphi = - \frac{A_{k,j}}{k+\ell} \sum \frac{1}{(k+j+1)(k+j-1)} B_{k-1,j} C_j \cos (ku' + jv). \] (131)

Equation (131) provides a general formula for \( \delta \varphi \) due to \( U_\ell \), valid for \( \ell \geq 1 \). (This restriction on \( \ell \) has been operative from the beginning of section 4.) In view of the fact that \( k \) only takes non-negative values of the same parity as \( \ell \), it should be noted that \( j \) takes all values, but with
\( B_{k-1,j} \) only non-zero if \( |j| \leq \ell - 2 \). (This applies if \( \ell \geq 2 \); but the case \( k = 1 \) is trivial because there is an overall factor \( \ell - 1 \).

If \( j = -k \pm 1 \), there is a zero denominator in (131), and terms with these values of \( j \) must be excluded from the formula; they are associated with the terms in \( \delta \varphi \) that were hived off in the generation of the \( \delta \varphi \). In section 7 we shall determine constants for \( \delta \varphi_{k,j} \) and \( \delta M_{k,j} \) such that the terms with these two values of \( j \) are forced to zero. It will be noted that all the cosine terms occurring in (131), for a given \( J_k \) and all possible \( k \), are distinct, except that if \( k = 0 \) (\( \ell \) even) then equal and opposite values of \( j \) lead to identical terms in \( \cos jv \).

We use the remarks in the last paragraph to provide a pair of formulae for \( N_{k,j} \), the total number of terms required to express \( \delta \varphi \) for a given value of \( \ell \). One formula applies when \( \ell \) is odd, the other when \( \ell \) is even. In both cases the number of \( j \) values for each \( k \) (regardless of the excluded values, if any) is \( 2\ell - 3 \), if \( \ell \geq 2 \).

If \( \ell \) is odd, there are \( \frac{1}{2}(\ell + 1) \) possible values of \( k \), so a priori the value of \( N_{k,j} \) is \( \frac{1}{2}(\ell + 1)(2\ell - 3) \), if \( \ell \geq 3 \). But this must be reduced by the number of excluded values of the duplet \( (k,j) \). If \( k = \ell \), \( j \) cannot be \(-k \pm 1 \), so there is no value to exclude. If \( k = \ell - 2 \), \( j \) can (a priori) be \(-k + 1 \) and this value must be excluded. If \( k \leq \ell - 4 \), it will always be necessary to exclude both \(-k + 1 \) and \(-k - 1 \). Thus the total number of exclusions is \( \ell - 2 \). Subtracting this from the a-priori value, we get
\[ N_{tr} = \xi^2 - \frac{1}{6}(3\xi - 1) \tag{132} \]

Values for \( \xi \) up to 15 (including \( N_{1,r} = 0 \), which is the correct value, even though the above analysis only applies for \( \xi \geq 3 \)) are given in Table 6.

If \( \xi \) is even, there are \( \frac{1}{2}(\xi + 1) \) possible values of \( k \), so a priori the value of \( N_{tr} \) is \( \frac{1}{2}(\xi + 2)(2\xi - 3) \). The exclusions are as recorded before, amounting to \( \xi - 1 \) now if \( \xi \geq 4 \), but it is also natural for \( N_{tr} \) not to count the 'duplications' that arise when \( k = 0 \); there are \( \xi - 3 \) of these duplications if \( \xi \geq 4 \), viz for \( 2 \leq |j| \leq \xi - 2 \) (we cannot 'discount' for \( |j| = 1 \), since both values have already been 'excluded'). Thus the total number of exclusions is effectively increased to \( 2\xi - 4 \), and this is the right number even when \( \xi = 2 \) (not covered by the argument that applies for \( \xi \geq 4 \) only).

Subtracting this value from the a-priori value, we get

\[ N_{tr} = \xi^2 - \frac{1}{6}(3\xi - 2) \tag{133} \]

Table 6 gives values for \( \xi \) up to 16. It is remarkable that, as a result of the discounting of the duplications, we have a formula that is so close to what the improper use of (132) would give, the value by (133) being larger by just \( \frac{1}{6} \).

Further, if we did not discount, the formula for \( \xi \geq 4 \), viz \( \xi^2 - \frac{1}{6}(\xi + 4) \), would give 1, instead of the correct 2, when \( \xi = 2 \).

It is noted, in conclusion, that, due to the multiplier \((r/a)\) of \( 6a \) in (120), it would not be a simple matter to null the 'constant' terms of \( 6r \) with a choice of \( a = a' \), but that in any case we would prefer not to make such a choice. Further, the constants in \( 6a \) and \( 6r \) are not the same, partly due to the multiplier of \( 6a \) referred to, but mainly to the way in which the terms in \( 6e \) and \( 6M \) combine. For even \( \xi \), we will have, in particular, a coefficient of \( C_0^0 = 1 \) equal to \( (\xi - 1)p_{a_{k,0}^{1-1,0}} \). (See (159) for the constant in \( 6a \).)

6.2 The perturbation \( \delta b \)

We get \( \delta b \) from (121), where \( \delta l \) and \( \delta n \) are given by the integrals of (79) and (82). This is on the assumption that \( \delta b \) is associated with \( U_{2}^{2} \), following the decomposition of \( U_{2} \) by (5). We shall shortly find, however, that it is much more convenient to decompose the total

* In software in particular, we would rather double a computed quantity than have to compute it again; in the general analytical formula, (131), however, there is no easy way to indicate a special situation when \( k = 0 \) and \( j = 0 \).
\( \delta b \) (associated with \( U_k \)) as \( \sum \delta b_{kk}' \), where the summation is for values of \( k \) that are of opposite parity to \( t \) and we no longer associate the individual \( \delta b \) (\( = \delta b_{kk}' \)) with specific components of \( U_k \).

In relation to \( U_k^2 \), we get

\[
db_{kk} = - \sum B_{k} \left( \frac{k \cos^{-1} \frac{1}{k + j}}{k + j} A_{kk} C_{j}^{k} \cos u' + \frac{1}{k + j} A_{kk} C_{j}^{k} \sin u' \right). \tag{134}
\]

The trigonometrical products can be replaced by sums, in the usual way, and we can then invoke the notation of (20) to write

\[
db_{kk} = - \frac{1}{2} \sum B_{k} (k + j)^{-1} \left( A_{kk}^{1} C_{j}^{k+1} + A_{kk}^{2} C_{j}^{k} \right). \tag{135}
\]

This expression may be contrasted with (125) and (126) for \( \delta r \). In view of the difference in superfix, as well as suffix (which alone varied in the terms of \( \mathbb{R} \)), in the two \( C \) terms of (135), we would now like to combine a pair of terms with different \( k \) indices, before the summation over the \( j \) index operates. We note that \( A_{kk}^{1} \) and \( A_{kk}^{2} \), though under the summation sign in (135), are actually independent of \( j \).

With the philosophy just referred to, we make the new decomposition

\[
\delta b = \sum \delta b_{kk}' \tag{136}
\]

where each \( \delta b \) (\( = \delta b_{kk}' \)) is of the form

\[
\delta b = \sum T_{j} B_{k} C_{j} \tag{137}
\]

and we require an expression for \( T_{j} \) (or \( T_{kkj} \) to display all the index parameters). We note first that since (for non-trivial results) \( k \) runs from 0 or 1 to \( t \) (taking alternate values), it follows that, in principle, \( k \) runs from -1 or 0 to \( t + 1 \) (again alternate values, but of opposite parity to \( k \)); for the minimum value of \( k \), only the term in \( A_{kk}^{1} \), in (135), contributes to \( T_{j} \), whilst for the maximum value of \( k \), only the term in \( A_{kk}^{2} \) contributes; for intermediate values (if any), both terms contribute. But we can straight away dismiss the 'maximum value' \( (k = t + 1) \), because \( A_{kk}^{2} \) is just a multiple of \( s^{k} \); from this it follows that \( A_{kk}^{1} \), defined by (20), is zero. (Also \( B_{kk} = 0 \) anyway!) We shall find that we do not require the 'minimum value' \( (k = -1) \) either.
To evaluate $T_j$, in general, we use (30) and (31) for $A_{ik}$ and $A_{ik}^*$, respectively. It is fortunate that we require the first with $k = k_1$ and the second with $k = k + 1$, since this means that we pick up the same inclination function, $A_k^*(1)$, for both; moreover, it is aesthetically satisfying to have a direct application for the inclination functions for which subscript and superscript are of opposite parity, as opposed to merely an application in the propagation of like-parity functions*. We have

$$T_j = -J \left\{ \frac{\kappa + 1}{\kappa + j + 1} a_{\kappa,k+1} + \frac{(\kappa - \kappa + 1)(\kappa + \kappa)}{a_{\kappa,k+1}} a_{\kappa,k+1} \right\} (R/p)^K a^K A_k^*(1). \quad (138)$$

The quantity in curly brackets in (138) is a pure constant, in which the $a_{\kappa,k}$ are given by (12); thus the first $a$ involves $P_{k+1}^k(0)$ and the second involves $P_{k}^{k-1}(0)$, these being given by (13). By relating these to $P_{k+1}^k(0)$, we may express the aforementioned quantity (after some algebraic reduction) as

$$-\frac{(t - \kappa + 1)}{2^{\kappa+1}} P_{k+1}^k(0) \left\{ \frac{u_{\kappa+1}}{\kappa + j + 1} - \frac{u_{\kappa+1}}{\kappa + j - 1} \right\}. \quad (139)$$

But $P_{k+1}^k(0)$ is related to $a_{\kappa,k}$ by (16), and thus to $A_{\kappa,k}$ by (17). Hence (138) gives

$$T_j = \frac{t A_{\kappa,k}}{2u_\kappa} \left( \frac{u_{\kappa+1}}{\kappa + j + 1} - \frac{u_{\kappa+1}}{\kappa + j - 1} \right). \quad (139)$$

The preceding is 'general' in that it applies for $1 \leq k \leq t - 1$ (or, more precisely, with 2 as lower bound when $t$ is odd); further, if $k \geq 2$ we can obviously cancel the three appearances of $u$. We still have to cover the cases $k = 0$ ($t$ odd) and $k = -1$ ($t$ even), in which (in principle) only the first term in curly brackets is to be taken. To count only non-zero terms when $T_j$ is substituted in (137), the restriction on $j$ is that $|j| \leq k - 1$ (of an upper bound of $t - 2$ in the analysis for $r$); we shall be excluding the values $j = k \pm 1$, of course.

* This strengthens the view (noted in other papers, and in section 8 of Ref 13 in particular) that $\nu$ (of either parity) is a much better index parameter than Kaula's $p$ (where $2p = t - k$, referred to in section 3 here). When the analysis includes the tesseral harmonics (Ref 9, and see Appendix A here), $k$ takes negative values (with $|k| \leq t$) as well as positive, but the factor $u_k$ in (12) is not required.
When \( K = 0 \), we require just \( 2/(j + 1) \) from the curly brackets in (139); but for each \( J > 0 \), half this quantity may be combined with half the corresponding quantity for \( J < 0 \), to give \( 1/(j + 1) + 1/(-j + 1) \), which can be rewritten as \( 1/(j + 1) - 1/(j - 1) \). It follows that (139), with the three occurrences of \( u \) deleted, again gives the right results (counted separately for \( j > 0 \) and \( j < 0 \)). The modified formula may be seen to apply, finally, for \( j = 0 \).

When \( K = -1 \), the position is more complicated. First, the expression by (139) is not even legitimate now, since \( A_{\delta K} \) is not defined for \( \kappa < 0 \); the illegitimacy arose in the substitution for \( A_{\delta K} \kappa \) since (31) does not apply when \( K = 0 \). But (20) indicates that \( A_{\delta, 0} = -A_{\delta, 0} \), and this suggests that we can relate the required term, involving \( u_{\kappa+1} \) with \( \kappa = -1 \), to the term in \( u_{\kappa-1} \) when \( \kappa = 1 \). Since \( C_{\delta, j} = C_{\delta, j} \), the relating will involve the transposition of positive and negative values of \( j \), and this is also necessary to identify \( u_{\kappa+1}/(\kappa+j+1) \) for \( \kappa = -1 \) with \( u_{\kappa-1}/(\kappa+j-1) \) for \( \kappa = 1 \). In short, we can deal with \( \kappa = -1 \) just by doubling the second term in curly brackets in (139) that is associated with \( \kappa = 1 \). This means that, yet again\(^*\), we get the right result from (139) if we cancel the three appearances of \( u \).

We can now write down the final result we require, by substituting (139) into (137) and expressing \( C_{\delta, 0} \) in full. Thus

\[
\delta B_{\delta K} = -2 A_{\delta K} \sum_{j} \frac{1}{(\kappa + j + 1)(\kappa + j - 1)} B_{\delta j} \cos (\kappa u' + jv). \tag{140}
\]

As already indicated, this formula is unlike (131)\(^*\) the corresponding one for \( \delta \), in that it cannot be taken in isolation as relating to a sub-component of \( U_k \). It is like (131) in one respect, however, in that terms of \( \delta B_{\delta K} \) with \( j = -\kappa + 1 \) are excluded. In section 7 we shall determine constants for \( \delta B_{\delta K} \) and \( \delta B_{\delta, K} \) (\( k \), not \( \kappa \), now being the appropriate symbol) such that these terms are forced to zero.

\(^*\) The universality of this procedure (cancelling the \( u \)) stems from the original introduction of \( U_k \) into the definition of \( \delta B_{\delta K} \). If we dispensed with this factor, but used positive values of \( \kappa \) as well as negative ones (see also Appendix A), then we would find nothing special about the values \( \pm 1 \) and \( 0 \) in the first place.
We proceed to obtain a pair of formulae for $N_{ib}$, the total number of terms (without duplication of $C_j$) required to express $\delta b$ for a given value of $k$. Whether $k$ is odd or even, the number of $j$ values for each $\kappa$ (not discounting excluded values) is $2\kappa - 1$, correct for all $k \geq 2$ this time.

If $k$ is even (which we have seen to be the simpler case), there are $\frac{1}{2}k$ possible values of $\kappa$, so a priori the value of $N_{ib}$ is $\frac{k}{2}(2\kappa - 1)$. When $\kappa = k - 1$, $j$ can be $-\kappa + 1$ but not $-\kappa - 1$, so there is just one value to exclude. For all other $\kappa$, values of $-\kappa \pm 1$ are both possible, so the total number of exclusions is $\kappa - 1$. Subtracting this from the a-priori value, we get

$$N_{ib} = k^2 - \frac{1}{4}(3k - 2).$$ (141)

Interestingly, this is the same as $N_{Ir}$ given by (133). Values for $k$ up to 16 are given in Table 6.

If $k$ is odd, there are $\frac{1}{2}(k + 1)$ possible values of $\kappa$, so a priori the value of $N_{ib}$ is $\frac{1}{2}(k + 1)(2\kappa - 1)$. There is again a single exclusion if $\kappa = k - 1$, and two otherwise, so there are $k$ basic exclusions (assuming $k \geq 3$). In addition, however, there are $k - 2$ duplications when $\kappa = 0$, and these can be discounted for $N_{ib}$ (though not for $\delta b$ itself - see also the footnote in section 6.1, so the effective number of exclusions is $2k - 2$; this value applies even when $k = 1$. Subtracting this total number of exclusions from the a-priori values we get

$$N_{ib} = k^2 - \frac{1}{4}(k - 1),$$ (142)

which is one more than for the corresponding $N_{Ir}$. Values for $k$ up to 15 are given in Table 6.

In conclusion, it is worth remarking that if the planetary equations are used in Gauss's form, as opposed to Lagrange's, (and this is done in Ref 8), then the resulting form of the expressions for $\delta r$ and $\delta A$ is such as to provide an easier route to our $\delta b$ (with $\kappa$, rather than $k$, effectively involved from the outset). For both $\delta r$ and $\delta A$, however, the approach via Lagrange's form of the equations is much simpler.
6.3 The perturbation $\delta w$

The analysis for $\delta w$ is much more like the $\delta r$ analysis than the $\delta b$ analysis, because each $U_k^2$ can again be treated separately throughout. There are two complications, however. First, \((122)\) effectively involves $\cos 2v$ and $\sin 2v$, not just $\cos v$ and $\sin v$ (we see this at equation \((143)\), following), and this means that the values $j = -k \pm 2$ are special as well as $j = -k \pm 1$. Second, we cannot take $\delta w$ to be zero for any of these special cases, since the constants in $\delta e$ and $\delta M$ must now be assumed to have been already assigned, formula for the four special $\delta w$ will be obtained in section 7.4. Actually, a fifth special case emerges, corresponding to $J = -k$ and a zero denominator $k + J$; $\delta w$ for this case can be set to zero, since we still have (for each $k$) the constant in $\delta w$, as yet unassigned, available for the purpose - the constants for $\delta w$ are determined in section 7.5.

We start by rewriting \((122)\) as

$$\delta w = 2q^{-2} (\delta e \sin v + eq^{-1} \delta M \cos v) + \frac{4q^{-2}}{\epsilon} (\delta e \sin 2v + eq^{-1} \delta M \cos 2v) + 4e^2q^{-3} \delta M + q^{-1} \delta L, \quad (143)$$

where $\delta e$, $\delta M$ and $\delta L$ are available from the integrals of \((76)\), \((105)\) and \((106)\). The integrals for $\delta e$ and $\delta M$ combine in a very natural way and we get

$$2q^{-2} (\delta e \sin v + eq^{-1} \delta M \cos v) = 4q^{-2} A_{tk} \left[ B_{kj} \left\{ \frac{1+k-k}{k+j-\frac{1}{2}} \right\} + \frac{1}{k+j-1} S_j - 2 \left( \frac{1+k-2k}{k+j-1} + \frac{1+k+2k}{k+j+1} \right) S_j \right] + e \left\{ \frac{1+k-k}{k+j-1} + \frac{1+k+k}{k+j+2} \right\} + k_{j+1}, \quad (144)$$

$$\delta e^{-2} (\delta e \sin 2v + eq^{-1} \delta M \cos 2v) = \frac{4q^{-2}}{\epsilon} A_{tk} \left[ B_{kj} \right. \left. \left\{ 3e \left( \frac{1+k-k}{k+j} \right) S_{j-2} + 2 \frac{1+k+2k}{k+j+1} S_{j-1} + e \left( \frac{1+k+k}{k+j+2} + \frac{1+k-k}{k+j-1} \right) S_j \right] \right] + 2 \frac{1+k-2k}{k+j-1} S_{j+1} + 3e \left( \frac{1+k-k}{k+j} \right) S_{j+2}, \quad (145)$$
\[
\frac{1}{4} e^{2q^{-3}} 5H = \frac{1}{4} e^{q^{-2}} A_{kk} \sum B_{kj} \left\{ e^{\frac{1 + e - k}{k + j - 2}} S_{j-2} + 2 \frac{1 + e - 2k}{k + j - 1} S_{j-1} + 6e \frac{1 - k}{k + j} S_j \right. \\
+ 2 \frac{1 + e + 2k}{k + j + 1} S_{j+1} + 6e \frac{1 + k + e}{k + j + 2} S_{j+2} \} \tag{146}
\]

and
\[
q^{-1} 5L = (1 - 2e) A_{kk} \sum B_{kj} \frac{1}{k + j} S_j . \tag{147}
\]

We substitute the last four results in (143) and at the same time (as in the analysis for 6r) change the interpretation of \(j\) so that we can use the same \(S_j\) in each term. This leads to
\[
5w = \frac{1}{4} e^{2q^{-2}} A_{kk} \sum \left\{ 3e^2 \left( \frac{1 - e + k}{k + j + 2} + \frac{1 + e - k}{k + j} \right) B_{k,j+2} + 2e \left( \frac{1 + e - 2k}{k + j - 1} B_{k,j+1} + \left( e^2 \frac{1 + e + k}{k + j + 2} + 8 \frac{1 + e - 2k}{k + j - 1} + \frac{e^2}{k + j - 2} B_{k,j} + 2e \left( \frac{1 + e + k}{k + j} + 3 \frac{1 + e - 2k}{k + j + 1} + 6 \frac{1 - e - k}{k + j - 1} + \frac{1 + e - 2k}{k + j - 2} B_{k,j-1} + 3e^2 \left( \frac{1 + e - k}{k + j} + \frac{1 - e - k}{k + j - 2} B_{k,j-2} \right) S_j \right) \right) \right\} . \tag{148}
\]

Though the algebra is tedious, we can now eliminate \(B_{k,j+2}\) and \(B_{k,j-2}\) by the appropriate versions of (54). If we express the result as
\[
5w = \frac{1}{4} e^{q^{-2}} A_{kk} \sum \left( V_{j,1} B_{k,j+1} + V_{j,0} B_{k,j} + V_{j,-1} B_{k,j-1} \right) S_j , \tag{149}
\]
the formulae for \(V_{j,1}\), \(V_{j,0}\) and \(V_{j,-1}\) are initially very complicated. For \(V_{j,1}\) in particular, we start with
\[
2e \left( \frac{3(k - e + 1)}{(j + e + 1)(k + j + 2)} - \frac{k - 4e + 2}{k + j + 2} + \frac{6(k - e + 1)}{k + j} \right) - \frac{3(k - e - 1)}{(j + e + 1)(k + j)} - \frac{3k}{k + j} - \frac{2(k - e - 1)}{k + j - 1} .
\]

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\( V_{j,-1} \) is symmetrically related to this, but \( V_{j,0} \) is a great deal more complicated. All three formulae can be greatly simplified, however; for \( V_{j,0} \) this was done by a technique akin to partial fractions. The resulting expressions are

\[
V_{j,1} = 2e(k + J) \left( \frac{1}{k + J + 2} - \frac{6}{k + J + 1} + \frac{3}{k + J} + \frac{2}{k + J - 1} \right) \quad (150)
\]

\[
V_{j,0} = 2 \left( \frac{k + 2k + 1}{k + J + 1} - \frac{2k - 1}{k + J} + \frac{2 - 2k + 1}{k + J - 1} \right) - 2e^2 \left( \frac{k + k + 1}{k + J + 2} - \frac{2(k + 1)}{k + J} + \frac{2 - k + 1}{k + J - 2} \right) \quad (151)
\]

and

\[
V_{j,-1} = 2e(4 - J) \left( \frac{2}{k + J + 1} + \frac{3}{k + J} - \frac{6}{k + J - 1} + \frac{1}{k + J - 2} \right) \quad (152)
\]

As a result of this remarkable simplification, it will be observed that \( V_{j,1} B_{k,j+1} \) and \( V_{j,-1} B_{k,j-1} \), in (149), have been expressed in a very suitable form for the application of (56) and (60), with \( k \) replaced by \( 4 \) in both relations, to eliminate \( B_{k,j+1} \) and \( B_{k,j-1} \), respectively, in favour of \( B_{k,j} \) (already present in (149)) and \( B_{k+1,j} \). Thus, if we now write

\[
6w = \frac{1}{2} \sum_{k} (W_{k,0} B_{k,j} + W_{k,1} B_{k+1,j}) \quad (153)
\]

we get

\[
W_{k,0} = 2 \left( \frac{k + k + 1}{k + J + 2} - \frac{k + 1}{k + J} + \frac{k - k + 1}{k + J - 2} \right) \quad (154)
\]

and

\[
W_{k,-1} = -2(k - 1) \left( \frac{1}{k + J + 2} - \frac{2k + 1}{k + J + 1} + \frac{6}{k + J} - \frac{2k + 1}{k + J - 2} \right) \quad (155)
\]
The final result we require follows from the substitution of (154) and (155) into (153). Writing \( S_j \) in full, we get

\[
\delta u_{jk} = \pm_k \sum_j \frac{1}{(k+j)(k+j-1)} \left[ \frac{1}{2(k+1) - k(k+j)} \right] B_{kj} \\
- \frac{6(k-1)}{(k+j)(k+j-1)} B_{k-1,j} \sin (ku' + jv). \quad (156)
\]

Equation (156) is the general formula for \( \delta w \) due to \( U_1^k \). As with (131) and (140), for \( \delta r \) and \( \delta b \) respectively, it applies for all \( \ell \geq 1 \); like (140) but unlike (131), on the other hand, values of \( |j| \) up to \( \ell - 1 \) are required to cover all the non-zero terms. For each \( k \), zero denominators exist for five different values of \( j \): for four of these values \( (j = k \pm 1 \) and \( j = -k \pm 2 \) \), special formulae are required, in place of (156), as already noted; only for the fifth value \( (j = -k \) \) can a term (for each \( k \) \) be actually excluded.

Before proceeding to a pair of formulae for \( N_{\delta w} \), the total number of terms required to express \( \delta w \) for a given value of \( \ell \), we note (and make allowance for) one specific null term that arises for each even value of \( \ell \). For \( k = 2 \) and \( j = 1 \), we see from (156) that the coefficient of \( B_{kj} \) is identically zero (i.e. independently of \( k \)). But \( B_{\ell-1,j} \) is itself zero when \( j = \ell - 1 \), so this specific term of \( \delta w_{\ell,2} \) always vanishes. Proceeding to \( N_{\delta w} \), we first note that the number of \( j \) values for each \( k \) (regardless of any exclusion) is \( 2\ell + 1 \) (for all \( \ell \geq 1 \)).

If \( \ell \) is odd, then a priori the value of \( N_{\delta w} \) is \( \frac{1}{2}(\ell + 1)(2\ell - 1) \). There is one excluded value of \( j \) for each \( k = \ell \), so there are \( \frac{1}{2}(\ell - 1) \) exclusions altogether. It follows that

\[
N_{\delta w} = \ell^2, \quad (157)
\]

and values for \( \ell \) up to 15 are given in Table 6.

If \( \ell \) is even, the a-priori value of \( N_{\delta w} \) is \( \frac{1}{2}(\ell + 2)(2\ell - 1) \). There is again an excluded \( j \) for each \( k = \ell \), amounting to \( \ell \) basic exclusions, but there are now two other sources of discounted terms. We have just remarked on the particular zero term that arises for \( k = 2 \); we might prefer to allow zero actually to be computed in a general computer program, but here we regard this
term as an exclusion. The other source of discounted terms consists of the 
1 - 1 duplications that occur when \( k = 0 \) (see the footnote of section 6.1).
Thus the number of effective exclusions is \( \frac{1}{2} \), and from subtraction we obtain
\[
N_{ew} = z^2 - 1 .
\]
(158)

Values for \( z \) up to 16 are given in Table 6.

6.4 Universality of results (non-elliptic orbits)

Equations (131), (1'0) and (156) give, on summing over \( k \) or \( k' \) as
appropriate, general formulæ for the perturbations \( \delta r, \delta b \) and \( \delta w \),
respectively, due to \( U_k \). It is being tacitly assumed, in the rest of the
Report, that we are only considering elliptic orbits. It is worth remarking
here, therefore, that (as follows by a continuity argument) the formulæ are also
valid for parabolic and hyperbolic orbits. The formulæ are effectively
universal\(^a\), in other words, though they inevitably fail for rectilinear orbits
(with infinities arising from zero \( p \)).

7 THE SPECIAL CASES, AND INTEGRATION CONSTANTS

The main results in this section, obtained in section 7.4, are the formulæ
required to supplement (156), the general formulæ for \( \delta w \). These formulæ,
covering the cases \( j = -k \pm 1 \) and \( -k \pm 2 \), are forced by the 'constants' for
\( \delta e \) and \( \delta M \), which are determined so that certain terms (those for \( j = -k \pm 1 \))
can be excluded from \( \delta r \). Though we have omitted (in section 4) the full
expressions for the short-period perturbations, \( \delta \zeta \), in the elements, we give
here the adopted 'constants' for all the \( \zeta \). Five of the elements have
constants chosen to suit \( \delta r \), \( \delta b \) and \( \delta w \); for completeness, we start with
the semi-major axis, for which the constants are mandated by the use of \( a' \)
as \( A \).

7.1 Mandatory constants for \( \delta a \)

We go back to the original expression for \( \delta a \) due to \( U_k \), viz (63). We
can expand the complete factor \( (p/r)^{k+1} \) in terms of the \( B_{k+2} \) (of the
expansion via the \( B_{kj} \) in (65))). On taking just the term of the expansion with
\( j = -k \), we isolate the constant term that (for each \( k \), and a given \( J_k \)) is
mandated by taking \( \Pi = a' \).
The result can be written in the form (for the 'constant' component of \( \delta a_{lk} \))

\[
\delta a_{lk}(c) = -2aq^{-2} A_{lk} B_{l+k+2, k} \cos kw.
\]  

(159)

7.2 Constants for \( \delta e \) and \( \delta M \)

The task in this section is to derive the formulae for \( \delta e_{lk}(c) \) and \( \delta M_{lk}(c) \) that will legitimize our taking the terms in \( \delta a_{lk} \) for \( j = -k + 1 \) and \( -k - 1 \) to be zero. These 'constants' will complete the formulae, for \( \delta e \) and \( \delta M \), given by the integrals of (76) and (105) respectively.

We start by observing that (131), the general formula for \( \delta a_{lk} \), was obtained by combining the two different denominators from (129) and (130). If we do not combine the denominators, we can rewrite the formula as

\[
\delta a_{lk} = -\frac{4(k-1)p A_{lk}}{[k+j-1] - [k+j+1]} B_{l-1,j} C_j.
\]  

(160)

The first denominator here is associated with the \( L \) summation of section 6.1. If this summation still applied for \( j = -k + 1 \), then the result would be an infinite coefficient of \( B_{l-1,-k+1} C_{-k+1} \). We actually want this coefficient to be \(-\frac{4(k-1)p A_{lk}}{[k+j-1] - [k+j+1]} \), since it will then neutralize the coefficient, \( \frac{4(k-1)p A_{lk}}{[k+j+1]} \), that arises without difficulty from the second denominator in (160). The situation is similar when \( j = -k - 1 \) and we want the coefficient of \( B_{l-1,-k-1} C_{-k-1} \), from the second term of (160), to be \(-\frac{4(k-1)p A_{lk}}{[k+j-1] - [k+j-1]} \) (and not infinity) to neutralize the first term. (It is recalled that infinite coefficients are avoided, simply because we deal separately, in section 5, with the relevant terms of \( de/dv \) and \( dM/dv \).) What we do, therefore, is to obtain the coefficients of \( C_{-k+1} \) and \( C_{-k-1} \) that would apply in the absence of the constants \( \delta e_{lk}(c) \) and \( \delta M_{lk}(c) \); we can then derive the appropriate values of these constants to cancel these putative coefficients.

So what would the first-denominator coefficient of \( C_j \) be, with \( j = -k + 1 \), in the absence of the constants? There would then be no contribution from equation (123), but still a contribution from the complementary (124), given by \( \delta a \). Its value may be obtained from the first term of each pair in (124) - the second term does not apply because it feeds separately into the second-denominator coefficient of \( C_{-k+1} \) which behaves normally as we have seen.
But equation (124) was written down before the 'rearrangement', from (127) to (128), in which the use of \( j \) changed. This change affects the \( B \) subscripts, and it may be seen that the required first-denominator coefficient of \( C_{k+1} \) is

\[-\frac{1}{4} A_{jk} \left( eB_{k,-k+2} + 4B_{k,-k+1} + 3eB_{k,-k} \right) .\]

The (normally behaved) second-denominator coefficient, on the other hand, may be written

\[-\frac{1}{4} A_{jk} \left( (2-1) q^2 B_{k-1,-k+1} \right) .\]

To cancel the combined coefficient by use of \( \delta e_{jk}(c) \) and \( \delta m_{jk}(c) \), let us suppose that

\[\delta e_{jk}(c) = A_{jk} x C_k \quad (161)\]

and

\[\delta m_{jk}(c) = A_{jk} e^{-1} q y S_{k-1} \quad (162)\]

where \( x \) and \( y \) are quantities to be determined. On combining these for a contribution to \( \delta r \) (of (120)), we get a coefficient of \( C_{k+1} \) given by

\[-\frac{1}{4} A_{jk} (x + y) ,\]

so that one equation to be satisfied by \( x \) and \( y \) is

\[2(x + y) + eB_{k,-k+2} + 4B_{k,-k+1} + 3eB_{k,-k} - (2-1) q^2 B_{k-1,-k+1} = 0 . \quad (163)\]

The complementary contribution to \( \delta r \) from (161) and (162) leads to a term in \( C_{k+1} \), of coefficient

\[-\frac{1}{4} A_{jk} (x - y) ,\]

and this combines with two other coefficients of \( C_{k+1} \), obtained as in the last paragraph; the result is another equation in \( x \) and \( y \),

\[2(y - y) + eB_{k,-k+2} + 4B_{k,-k+1} + 3eB_{k,-k} - (2-1) q^2 B_{k-1,-k+1} = 0 . \quad (164)\]
Solution of (163) and (164) gives

\[ x = \frac{1}{4} (t - 1)q^2 \left( B_{k-1},-k+1 + B_{k-1},-k-1 \right) - 8eB_{k,-k} \]
\[ - \frac{1}{2} (B_{k},-k+1 + B_{k},-k-1) - e(B_{k},-k+2 + B_{k},-k-2) \]  
(165)

and

\[ y = \frac{1}{4} (t - 1)q^2 \left( B_{k-1},-k+1 - B_{k-1},-k-1 \right) \]
\[ - \frac{1}{2} (B_{k},-k+1 - B_{k},-k-1) - e(B_{k},-k+2 - B_{k},-k-2) \].  
(166)

To get the formulae for \( \delta \theta_{tk}(c) \) and \( \delta M_{tk}(c) \) that we require, it remains to substitute (165) and (166) into (161) and (162). In doing this, we make two simplifications: we eliminate \( B_{k-1},-k+1 \) and \( B_{k-1},-k-1 \) by use of (56) and (56), respectively (with \( t \) replaced by \( t - 1 \) in each case); and we write \( B_{k},k \) etc rather than \( B_{k},-k \).

Finally, then, we have

\[ \delta \theta_{tk}(c) = - \frac{1}{4} A_{tk} \left[ eB_{k},k+2 - (t + k - 4)B_{k},k+1 + 2(t + 2)eB_{k} \right] \]
\[ - (t - k - 4)B_{k},k-1 + eB_{k},k-2 \cos ku' \]  
(167)

and

\[ \delta M_{tk}(c) = \frac{1}{4} e^{-1} q A_{tk} \left[ eB_{k},k+2 - (t + k - 4)B_{k},k+1 \right. \]
\[ - 2keB_{k} + (t - k - 4)B_{k},k-1 - eB_{k},k-2 \left. \right] \sin ku' \].  
(168)

The formulae could, of course, be reduced to a smaller number of terms, by use of the fixed-\( t \) recurrence relation, (54), but the coefficients would then be much more awkward; no genuinely simpler versions of (167) and (168) have been found.

7.3 Constants for \( \delta 1 \) and \( \delta 0 \)

In this section we derive formulae for \( \delta 1_{tk}(c) \) and \( \delta 0_{tk}(c) \) to legitimate our taking the terms for \( j = -k + 1 \) and \( -k - 1 \) in (140), the general expression for \( \delta b_{tk} \), to be zero. The analysis is somewhat simpler than that in the preceding section, in spite of the complexity entailed by the need to work with both \( k \) and \( \kappa \).
As with $\delta_{l,k}$, we start by observing that (140) was obtained by combining two denominators, which appear separately in the preceding (139). When $j = -k + 1$, the second denominator becomes zero and no longer operates; from the first alone we get, as the effective term in (140), $\frac{1}{L} A_{l,k} B_{z,-k+1} C_{k+1}^k$.

When $j = -k - 1$, similarly, the first denominator in (139) does not operate, and (140) effectively reduces to $\frac{1}{L} A_{l,k} B_{z,-k-1} C_{k-1}^k$. These terms have to be cancelled by the use of $\delta_{l,k}(c)$ and $\delta_{0,k}(c)$, with appropriate $k$, so we suppose that

$$\delta_{l,k}(c) = x C_{-k}^k$$

and

$$\delta_{0,k}(c) = s^{-1} y S_{-k}^k.$$  

(In the last section we were able to include $A_{l,k}$ in the corresponding expressions, (161) and (162), but there is no common factor available now.) On combining (169) and (170) for a contribution to $\delta b$ (of (121)), we get

$$\frac{1}{L}(x - y) C_{-k}^{k+1} + \frac{1}{L}(x + y) C_{-k}^{k-1}.$$  

If we postpone consideration of any difficulties associated with the extreme values of $k$ that arise, then we get a pair of equations for $x$ and $y$, on identifying $x$ with $k + 1$ and $k - 1$ respectively, in the coefficients of $C_{-k+1}$ and $C_{-k-1}$ that have been recorded. Thus the equations are

$$2(x - y) + \frac{1}{L} A_{l,k+1} B_{z,-k+1} = 0$$

and

$$2(x + y) + \frac{1}{L} A_{l,k-1} B_{z,-k-1} = 0.$$  

Solution gives

$$x = -\frac{1}{L} (A_{l,k+1} + A_{l,k-1}) B_{z,-k}$$

and

$$y = \frac{1}{L} (A_{l,k+1} - A_{l,k-1}) B_{z,-k}.$$  

The result is

$$\delta b = 2(x - y) C_{-k}^{k+1} + 2(x + y) C_{-k}^{k-1}.$$
The expressions in brackets can be replaced by simpler expressions, following (27) and (28), and we also replace $B_{k,-k}$ by $B_{lk}$. Then substitution in (169) and (170) gives us the formulae we require. Thus

$$\delta t_{lk}(c) = \frac{1}{4} A_{ik} B_{lk} \cos k\omega'$$

(175)

and

$$\delta G_{lk}(c) = \frac{1}{4} \cos^2 A_{ik} B_{lk} \sin k\omega'$$

(176)

We now have to show that (175) and (176) are valid for the extreme values of $k$ as well as 'general values'. For $k = 1$, the maximum value of $k$, this involves a value of $\ell + 1$ for $\ell$ in $g_b$, but this is all right as $B_{k,k} = 0$ (of section 6.2); thus $x = y = 0$ for $k = 1$. For the minimum value of $k$ (0 or 1, according to whether $\ell$ is even or odd), it is a little more complicated: we consider the two cases separately.

If $\ell$ is even, with $k = 0$, $C_{12}^1$ and $C_{02}^1$ are the same and we cannot separate $x + y$ from $x - y$ in the combination of (169) and (170). Thus there is only one equation to be satisfied (instead of both (171) and (172)), and it involves only the identification of $\kappa$ with $k + 1$ (not also with $k - 1$).

The equation reduces to

$$x = -\frac{1}{4} A_{ik} B_{lk,0}$$

(177)

which is consistent with (175), in view of (29). The value of $y$ is indeterminate, but this is appropriate for the coefficient of $\sin k\omega'$, which is itself zero when $k = 0$. If $\ell$ is odd (with $k = 1$), on the other hand, the validity of (175) and (176) follows in essence from the argument just prior to the establishment of equation (140), and the details are omitted.

A last point in this section is noted as no more than a curiosity. Whereas the rest of $\delta t_{lk}$ and $\delta G_{lk}$ contain $A_{lk}$ and $A_{ik}$, respectively, as a factor of every term, as indicated by (79) and (82), these factors are reversed in the 'constant' terms, as indicated by (175) and (176). The point was noted, for $\ell = 3$, in Part 1.
7.4 Forced terms in \( \delta w \)

We now have, for each \( U_k \), only one 'constant' at our disposal; denoted by \( \delta \omega_k(c) \), it will be determined in section 7.5 so as to validate the nulling of the term for \( j = -k \) in the formula, (156), for \( \delta w_{kk} \). For \( j = -k \pm 1 \) and \( -k \pm 2 \), on the other hand, we are forced to accept non-null terms that arise, via (122), from (167) and (168), the formulae for \( \delta \omega_k(c) \) and \( \delta \omega_k(c) \). We derive the formulae for these terms in the present section, which must therefore be regarded (from the reference viewpoint) as a completion of section 6.

For each of the four special values of \( j \), in principle we embark on a procedure that is similar to that employed in section 7.2. The basis of this procedure is that we re-determine the expression for \( \delta w \) with the appropriate terms in \( \delta \epsilon \) and \( \delta \mu \) (which in their general form would lead to infinities) replaced by 'constants' proportional to \( x \) and \( y \) (as defined by (161) and (162)); the only difference (in principle) from section 7.2 is that we do not have unknowns to solve for, so that the procedure is 'direct'. In practice, however, because the basic formula for \( \delta w \), (122), is so much more complicated than the corresponding formula for \( \delta r \), (120), it is better to proceed a little differently from this; instead of developing our four special formulae more or less ab initio, we start four times from the (final) general formula for \( \delta w \), (156), and modify it each time in the appropriate manner.

We start with \( j = -k + 1 \); it will be useful to have a shorthand for the denominator that becomes zero, so we define

\[
d = k + j - 1
\]

for general \( j \) and will eventually set \( d = 0 \). In terms of \( d \), we can rewrite (154) and (155), which on substitution into (153) give the final (156), as

\[
W_{k,0} = 2\left(\frac{k + 1}{d + 3} - \frac{4 - 1}{d + 1} + \frac{1 - k + 1}{d - 1}\right) \quad (179)
\]

and

\[
W_{k,-1} = -2(k + 1) \left(\frac{1}{d + 3} - \frac{4}{d + 2} + \frac{6}{d + 1} - \frac{4}{d} + \frac{1}{d - 1}\right). \quad (180)
\]
There is no difficulty with (179), but in (180) a zero denominator appears when $d$ is set to zero. We will find that this denominator disappears when the 'replacement procedure' is complete.

The terms we have to replace come from the first term of (143), for which the general expression is given by (144). In the latter equation the 'offending terms' consist of the first of each pair; on changing the way $j$ is used, as usual, the combination of the three terms in question is

$$
q^{-2} A_{jk} \left\{ e \frac{1 + k - j}{k + j - 1} B_{k,j+1} + 2 \frac{1 + j - 2k}{k + j - 1} B_{k,j} + 3e \frac{1 + j - k}{k + j - 1} B_{k,j+1} \right\} S_j
$$

where the zero denominators are evident as soon as we set $j = -k + 1$. The replacement term, also based on the first term of (143), emerges when $\delta e$ and $\delta M$ are set, following (161) and (162), to $A_{jk} x C_{-k}$ and $A_{jk} e^{-1} q y S_{-k}$ respectively. The resulting term in $S_{-k+1}$ is $A_{jk} q^{-2} (x+y) S_{-k+1}$ (in the analysis for $j = -k + 1$ it will be the term in $S_{-k+1}$ that we need), so the required change in $\delta w$ is

$$
q^{-2} A_{jk} \left\{ x + y + \frac{q \delta}{q - 2} [e(k - l - 1) B_{l,-k+2} + 2(2k - l - 1) B_{l,-k+1} + 3e(k + l - 1) B_{l,-k+1}] S_{-k+1} \right\} + \frac{q^{-1} \delta}{q - 2} [e(j - l) B_{l,-k+1} + 3e(j - l) B_{l,-k+1}] S_{-k+1}
$$

But $x + y$ is given by (163), so this can be expressed as

$$
q^{-2} A_{jk} \left\{ (l + 1) q^{2} B_{l,-k+1} - \frac{q \delta}{q - 2} [e(j - l) B_{l,-k+1} + 3e(j - l) B_{l,-k+1}] S_{-k+1} \right\} + \frac{q^{-1} \delta}{q - 2} [e(j - l) B_{l,-k+1} + 3e(j - l) B_{l,-k+1}] S_{-k+1}
$$

We set $j = -k + 1$ in this expression and then invoke versions of (56) and (58); as a result it simplifies to

$$
q^{-2} A_{jk} [(l + 1) q^{2} B_{l,-k+1} - \frac{4 \delta}{q - 2} q B_{l,-k+1} S_{-k+1}]
$$

Since only $B_{l,-k+1}$, i.e. $B_{l,-j}$, is present in this expression, the change in $\delta w$ can be represented as a change in $W_{k,-1}$ (of (153) and (155)) of amount $4(l + 1)(1 - 2d^{-1})$; thus (180) is to be replaced by

$$
W_{k,-1} = -2(l + 1) \left( \frac{1}{d + 3} - \frac{4}{d + 2} + \frac{6}{d + 1} - 2 + \frac{1}{d - 1} \right)
$$
It is now legitimate to set $d = 0$ in (179) and (181), the result being

$$W_{k,0} = -\frac{1}{4}(2k - k + 2)$$

and

$$W_{k,-1} = -\frac{1}{4}(L - 1).$$

On substituting in (153), we get the first of our four special formulae; it can be written (with a change of sign in the second suffix of each B)

$$\delta W_{k,-k+1} = -\frac{1}{2} A_k k [2(k + 2) B_{k,k-1} + (k - 1) B_{k-1,k-1}] \sin (kw' + v). \quad (182)$$

The second special case is with $j = -k - 1$. This is the twin of the first case, so it involves

$$d = k + j + 1 \quad (183)$$

and the expression for $x - y$ given by (164). There is no point in a virtual repetition of the analysis in detail, so we proceed direct to our second special formula; it can be written

$$\delta W_{k,-k-1} = -\frac{1}{2} A_k k [(k + 2) B_{k,k+1} + (k - 1) B_{k-1,k+1}] \sin (kw' - v). \quad (184)$$

The symmetry between (182) and (184) is obvious. For $k = 0$, of course, the equations reduce to the same formula, and the footnote of section 6.1 is again relevant.

For our third special case, we require to set $j = -k + 2$, so we start by defining

$$d = k + j - 2. \quad (185)$$

Then (154) and (155) take the form

$$W_{k,0} = 2 \left[ \frac{L + k + 1}{d + 4} - 2 \frac{L + 1}{d + 2} + \frac{k - k + 1}{d} \right] \quad (186) \mid$$

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and

\[ W_{k, -1} = -2(l - 1) \left( \frac{1}{d + 4} - \frac{1}{d + 3} + \frac{1}{d + 2} - \frac{1}{d + 1} + \frac{1}{d} \right), \quad (187) \]

with the potentially zero denominator in both expressions.

The terms of the general (156) that we have to replace now come from the second term of (143), for which the general expression is given by (145). The 'offending terms' are the last three of the six that occur in the latter equation; with the usual re-interpretation of \( j \), these three terms in combination give

\[ \frac{1}{d} \cdot eq^{-2} \cdot A_{l, k} \left\{ \frac{1}{k + J - 2} \cdot B_{l, j} + \frac{2}{k + J - 2} \cdot B_{l, j-1} + 3e \cdot \frac{1}{k + J - 2} \cdot B_{l, j-2} \right\} \cdot S_{l} \]

The replacement term is now \( \frac{1}{d} \cdot A_{l, k} \cdot eq^{-2} \cdot (x + y) \cdot S_{-k+2} \), so the required change in \( \delta_{W} \) is

\[ \frac{1}{d} \cdot eq^{-2} \cdot A_{l, k} \left\{ (x + y) + \frac{1}{d} \cdot [e(k - l - 1) B_{l, -k+2} + 2(2k - l - 1) B_{l, -k+1} + 3e(k + l - 1) B_{l, -k}] \right\} \cdot S_{-k+2} \]

With \( x + y \) given by (163), this can be expressed as

\[ \frac{1}{d} \cdot eq^{-2} \cdot A_{l, k} \left\{ (l - 1) q^2 \cdot B_{l-1, -k+1} - d^{-1} [e(j + k - 1) B_{l, -k+2} + 2(2j + k - 1) B_{l, -k+1} + 3e(j - k - 1) B_{l, -k-1}] \right\} \cdot S_{-k+2} \]

We set \( j = -k + 2 \) in this expression and then invoke versions of (54), (57) and (58) to eliminate \( B_{l-1, -k+1} \), \( B_{l, -k+1} \) and \( B_{l, -k} \) in favour of \( B_{l-1, -k+2} \). This simplifies the result (for the change) to

\[ \frac{1}{d} \cdot A_{l, k} (2d^{-1} - 1) \left\{ (l - k + 1 + d) B_{l, -k+2} - [l - 1] B_{l-1, -k+2} \right\} \cdot S_{-k+2} \]

It now follows, from (186) and (187), that the altered values of \( W_{l, 0} \) and \( W_{l, -1} \), to be substituted in (153), are given by

\[ W_{l, 0} = 2 \left( \frac{d + k + 1}{d + 4} - 2 \cdot \frac{d + 1}{d + 2} \right) + (l - k - 1 + d) \quad (188) \]
and

$$W_{k,-1} = -(\lambda - 1) \left( \frac{2}{d+4} - \frac{8}{d+3} + \frac{12}{d+2} - \frac{8}{d+1} + 1 \right). \quad \text{(189)}$$

\(\nu_{9}\) can now set \(d = 0\), getting

$$W_{k,0} = -\frac{1}{2}(\lambda + k + 5) \quad \text{(190)}$$

and

$$W_{k,-1} = \Psi(\lambda - 1). \quad \text{(191)}$$

Thus the substitution gives, for our third formula,

$$\delta \omega_{k,-k+2} = -\chi A_k \left[ 3(\lambda + k + 5)B_{k,k-2} - 19(\lambda - 1)B_{k-1,k-2} \right] \sin (k \omega' + 2\nu). \quad \text{(192)}$$

Our final special case, with \(j = -k - 2\), is the twin of the preceding (third) case and involves

$$d = k + j + 2. \quad \text{(193)}$$

We will not go through the analysis in detail, in view of the symmetry, but proceed directly to the final formula; it is

$$\delta \omega_{k,-k+2} = -\chi A_k \left[ 3(\lambda - k + 5)B_{k,k+2} - 19(\lambda - 1)B_{k-1,k+2} \right] \sin (k \omega' - 2\nu). \quad \text{...(194)}$$

7.5 Constants for \(\delta \omega\)

To complete section 7, it remains to determine the constant, \(\delta \omega_{k,k}(0)\), that legitimizes our taking the term for \(j = -k\) in (156), the general expression for \(\delta \omega_{k,k}\), to be zero. We already have \(\delta \omega_{k,k}(0)\), given by (168), so we only need to determine \(\delta \omega_{k,k}(0)\), the constant in \(\delta \omega_{k,k}\), for \(\delta \omega_{k,k}(0)\) to be known at once.
We start by now defining
\[ d = k + j \]  
so that (154) and (155), in their general form with potentially zero denominators, are rewritten as
\[ W_{i,0} = 2 \left( \frac{d + k + 1}{d + 2} - 2 \frac{k + 1}{d} + \frac{k + 1}{d - 2} \right) \]  
and
\[ W_{i,-1} = -2(l - 1) \left( \frac{1}{d + 2} - \frac{k}{d + 1} + \frac{1}{d - 1} + \frac{1}{d - 2} \right). \]

The zero denominators will disappear when, for the coefficient of \( S_{-k} \) in (153), we replace a quantity occurring in the general analysis with a quantity based on \( \delta_{j}^{tk}(c) \) (\( \delta_{j}^{tk}(c) \) not being involved); then \( \delta_{i}^{tk}(c) \) is defined to null this resulting coefficient.

The quantity to be replaced derives from the combination of (146) and (147). With our usual re-interpretation of \( j \), we can write the resulting coefficient of \( S_{-k} \) as the sum of
\[ \frac{1}{2} e q^{2} A_{j}^{k} d^{-1} \left[ a(1 + 1 - k)B_{k,j} + 2(1 + l - 2k)B_{k,j+1} \right. \]
\[ + 6e(1 - l)B_{k,j} + 2(1 + l + 2k)B_{k,j-1} + a(1 + l + k)B_{k,j-2} \]
and
\[ (1 - 2l) A_{j}^{k} d^{-1} B_{k,j}, \]
associated with \( \frac{1}{2} e^{2} q^{-3} \delta M \) and \( q^{-1} \delta L \) respectively. The quantity that has to replace this coefficient is available immediately from (168), but it is more convenient to back-track a little and take it instead as \( \frac{1}{2} e q^{2} A_{j}^{k} y \), with \( y \) given by (166). Thus the replacement coefficient may be written, with \( j \) rather than \(-k\) in the \( B \) subscripts, as
\[ -\frac{1}{2} e q^{2} A_{j}^{k} \left\{ a B_{k,j+2} + 2 B_{k,j+1} - a B_{k,j-1} - q B_{k,j-2} \right. \]
\[ + (\xi - 1)q^{2} (B_{k-1,j+1} - B_{k-1,j-1}) \right\} . \]
On subtracting from this the coefficient being replaced, we get

\[- \frac{1}{2} \frac{A_{2k} q^{-2} d^{-1}}{A_{2k}} \left\{ 3e^2(l + j + 1)B_{j+2} + 6e(l + 2j + 1)B_{j+1} \right\} - 2 \left[ 4(2l - 1) + e^2(l - 5) \right] B_{l+j} + 6e(l - 2j + 1)B_{l,j-1} + 3e^2(l - j + 1)B_{l,j-2} - 3eq^2 d(l - 1)(B_{l-1,j+1} - B_{l-1,j-1}) \right\}.

By application of versions of the five relations (54) to (58), we can eliminate

$B_{l,j+2}$ and $B_{l,j-2}$, then $B_{l,j+1}$ and $B_{l,j-1}$, and finally $B_{l-1,j+1}$ and $B_{l-1,j-1}$. This reduces the foregoing expression to

\[- \frac{1}{2} \frac{A_{2k} q^{-2} d^{-1}}{A_{2k}} \left[ 2(l + 1) - 3jd \right] B_{l,j} + 6(l - 1)B_{l-1,j} \right\},

which represents (when $j = -k$) the adjustment required to the coefficient of $S_{-k}$ in $\delta w$. On adding the appropriate contributions to (196) and (197) we get

\[ W_{l,0} = 2 \left( \frac{l + k + 1}{d + 2} - \frac{4}{d - 2} \right) \frac{k - k + 1}{d - 2} - 3 \right]\] (198)

and

\[ W_{l,-1} = -2(l + 1) \left( \frac{1}{d + 2} - \frac{4}{d - 1} - \frac{4}{d - 1} - \frac{1}{d - 2} \right). \] (199)

We can now set $d = 0$ (i.e. $j = -k$), getting

\[ W_{l,0} = 8k \] (200)

and

\[ W_{l,-1} = 0. \] (201)

These results mean that, in the absence of $\delta L_{kk}(0)$, we would have

\[ k A_{kk} B_{kk} \] as the coefficient of $S_{-k}$ in $\delta w_{kk}$; so, to null this, we take

\[ \delta L_{kk}(0) = - kq A_{kk} B_{kk} \sin k\omega'. \] (202)
From (168) and the definition of \( \psi \), we now get

\[
\delta \psi_{\lambda k}(c) = -\frac{1}{4} \epsilon^{-1} A_{\lambda k} \left[ e B_{\lambda, k+2} - (\lambda + k - 4)B_{\lambda, k+1} + 2k e B_{\lambda k} + (\lambda - k - 4)B_{\lambda, k-1} - e B_{\lambda, k-2} \right] \sin \omega' .
\]  

(203)

Finally, \( \delta \omega_{\lambda k}(c) \) is given by (175), so our desired formula for \( \delta \omega_{\lambda k}(c) \) is

\[
\delta \omega_{\lambda k}(c) = -\frac{1}{4} \epsilon^{-1} A_{\lambda k} \left[ e B_{\lambda, k+2} - (\lambda + k - 4)B_{\lambda, k+1} + 2k e^{-2} B_{\lambda k} + (\lambda - k - 4)B_{\lambda, k-1} - e B_{\lambda, k-2} \right] \sin \omega' .
\]  

(204)

It is remarked that as \( \delta \omega_{\lambda k}(c) \) is free of singularity (as would be expected), since \( A_{\lambda k} \) contains \( \epsilon^k \) as a factor, so that \( \epsilon^{-1} A_{\lambda k} \) is non-singular. We also have the non-singularity of \( \delta \omega_{\lambda k}(c) \), given by (176), for the same reason.

8 RESULTS EXEMPLIFIED FOR \( \lambda \) FROM 0 THROUGH 4

To illustrate the main results of this Report, derived for general \( \lambda (>0) \), we use them to derive results for the particular cases \( \lambda = 1, 2, 3 \) and 4. We start with an analysis for \( \lambda = 0 \), a case not covered by the general formulae - their failure for \( \lambda \leq 0 \) stems from the fact that the expansion (33) is then inherently infinite, and not just 'effectively' so (cf Table 4). Both the cases \( \lambda = 0 \) and \( \lambda = 1 \) (analysed next) are actually trivial, since the 'perturbed motion' can (in each case) be looked at from a viewpoint which makes it pure Keplerian (unperturbed). The interest in these cases then lies in the interpretation of the perturbation formulae, which relate to the nominal mean elements \( \xi \), in terms of the 'true' (fixed) elements of the effective Keplerian orbit - the elements of the latter will be denoted by \( \xi_T \).

For \( \lambda = 2 \) and 3 we write down, from the general results, the specific formulae for \( \delta r, \delta b \) and \( \delta \omega \) that were given before in Refs 1 ('Part 1') and 2. Both these papers gave also the specific \( \xi \) that complement \( \delta r, \delta b \), and \( \delta \omega \), and Part 1 gave the \( \delta \xi \) that underlie them - the \( \delta \xi \) for \( \lambda = 2 \) are well known, having been given by many authors. For \( \lambda = 4 \) we summarize a complete (first-order) solution, giving the \( \xi \) as well as \( \delta r, \delta b \) and \( \delta \omega \), the coordinate perturbations \( (\delta r, \delta b, \delta \omega) \), like the general formulae from which they are derived, have not been published before.
8.1 The trivial (but exceptional) case $\lambda = 0$

From (4), we have

$$U_0 = - \frac{\mu J_0}{r}.$$  \hspace{1cm} (205)

This is confirmed by (15), in which $k$ is restricted to zero so that $U_0 = U_0^0$; $a_{0,0}$ and $A_0^0$ are both unity, so $A_{0,0} = J_0$ by (14). Thus the effect of $J_0$ is to reduce the power of the central force as indicated in Ref 4, the value of the overall 'true' power being given by

$$\mu_T = \mu (1 - J_0).$$  \hspace{1cm} (206)

The orbit can be fully represented by $\mu_T$ and the 'true elements' $\zeta_T$, but it is instructive to exhibit the behaviour of the osculating elements (as well as the perturbations $d\nu$, $db$ and $dw$) relative to the $\mu$ originally assumed. As the general results of the paper do not apply when $\lambda = 0$, it is simplest to derive formulae from the original planetary equations directly.

There are no out-of-plane effects, even as a 'trivial' phenomenon, since we at once get

$$\delta l = \delta \nu = \delta b = 0. \hspace{1cm} (207)$$

so that

$$\Gamma = \Lambda_T \text{ and } \bar{\nu} = \bar{\nu}_T. \hspace{1cm} (208)$$

From (62) and (205) it follows that

$$\delta a = a - a' = -2J_0a^2/r. \hspace{1cm} (209)$$

This is a first-order relation, as usual, with $a$ on the right-hand side interpreted as $a'$ ($= \bar{\nu}$); it becomes exact if $a'$ is interpreted as $aa'$. There should be no surprise that (osculating) a variate around the orbit (unless it is circular): this results from the use of the 'wrong' $\mu$; with the 'right' $\mu (u_T)$, the osculating $a$ would have the fixed value $a_T$. Since (with $r_T = r$)
where all three expressions identify with \( V^2 \), \( V \) being the orbital speed, it follows that

\[
a_T = a'(1 - \frac{1}{a})
\]

(211)

this is an exact relation.

The planetary equation for \( \mu \), (71), gives

\[
\frac{\mu}{T} \left( \frac{2 \mu - 1}{a_T} \right) = \mu \left( \frac{2 \mu - 1}{a} \right) = \mu \left( \frac{2 \mu - 1}{a_T - \frac{2J_0}{r}} \right),
\]

(210)

and it is found best to take the integral of this to be

\[
\delta p = p - \mathcal{P} - 2J_0 \alpha.
\]

(213)

Then (209) and (213) lead to the non-singular perturbation for \( e \) given by

\[
\delta e = e - \mathcal{P} - J_0 \cos \nu.
\]

(214)

it turns out that we cannot get a simpler expression for our eventual \( \delta r \) by altering the implicit constant in (214), based on the explicit constant in (213). We introduce \( \nu_T \), and hence \( e_T \), by noting that

\[
\delta \frac{\mu}{T} = \nu_T \mathcal{P}_T,
\]

(215)

since both quantities identify with \( h^2 \), where \( h \) is the angular momentum. It follows from this, using (206) and (213), that

\[
\mathcal{P}_T = \mathcal{P} - J_0 a(1 + e_2).
\]

(216)

Thence, using (211), we have

\[
e_T = \mathcal{P}(1 + J_0).
\]

(217)
The planetary equation for $\psi$, (83), leads to

$$\frac{d\psi}{dv} = -J_0 e^{-1} \cos v,$$  \hspace{1cm} (218)

and we take the integral of this to be

$$\delta \psi = \psi - \overline{\psi} = -J_0 e^{-1} \sin v,$$  \hspace{1cm} (219)

since (it turns out) we cannot get a simpler expression for $\delta \omega$ by changing the (implicit) constant. Hence also

$$\delta \omega = \omega - \overline{\omega} = -J_0 e^{-1} \sin v.$$  \hspace{1cm} (220)

We introduce $\omega_T$ via $v_T$, noting that (6), taken with and without $T$-suffixes (and with $r_T = r$), yields

$$(p - p_T)/r = (e - e_T) \cos v - e(v - v_T) \sin v.$$  \hspace{1cm} (221)

This leads to

$$v - v_T = J_0 e^{-1} \sin v$$  \hspace{1cm} (222)

and hence (taking $u = u_T$)

$$\omega - \omega_T = -J_0 e^{-1} \sin v.$$  \hspace{1cm} (223)

From (220), we now see that

$$\omega_T = \overline{\omega}. $$  \hspace{1cm} (224)

The planetary equation for $\rho$, (93), leads to

$$\frac{d\rho}{dv} = -2J_0 q r/p,$$  \hspace{1cm} (225)

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and this introduces a difficulty in the analysis for \( M \), since (for the first time) we effectively have a negative power of \( 1 + e \cos v \). The simplest way to deal with the situation is to change the integration variable* from true anomaly, \( v \), to eccentric anomaly, \( E \). Instead of (225) we have

\[
\frac{d\theta}{dt} = -2J_0 ;
\]  

(226)

the integral of this is evidently secular, rather than short-periodic, but for convenience we use the notation appropriate to short-period perturbations and write (with the most useful integration constant)

\[
\delta \rho = -2J_0 E .
\]  

(227)

The secular perturbation in \( \rho \), that has just emerged, is dealt with in the usual way by choice of a suitable value for \( n' \), not compelled to be equal to \( n' \). With

\[
n_T^2 a_T^3 = u_T = u(1 - J_0),
\]  

(228)

we naturally take

\[
\pi = n_T = n'/(1 - J_0)
\]  

(229)

exactly, the formula being compatible with (211). (Equation (229) is in the spirit of (115), though not just a particular case of the earlier equation, which is only valid for \( k > 0 \).) Then (209) and (229) give

\[
\frac{df}{dt} = n - \pi J_0 n(3a/r - 1) ,
\]  

(230)

and hence

\[
f = \pi t + J_0 (3E - M) .
\]  

(231)

* The use of \( E \) as integration variable leads to a (finite) solution of the general problem when \( \xi < 0 \). The analysis is more complicated now, however, as \( v \) has to be replaced by \( E \) in occurrences of \( \sin u \), induced by the factor \( P_{-2-1} \) (\( \sin \beta \)) of \( U_1 \), as well as in the basic (negative) power of \( 1 + e \cos v \) that arises.

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In combination with (227) we may now write

$$\delta L = J_0 (E - M) = J_0 e \sin E$$

(using Kepler's equation). Using (219), we finally get

$$\delta M = M - M = J_0 e^{-1} (q \sin v + e^2 \sin E) .$$

We introduce $M_T$ via $E_T$, since from the equation

$$r = a(1 - e \cos E)$$

we get (with $r = r_T$, and both $a = a_T$ and $e = e_T$ known)

$$E - E_T = J_0 e^{-1} q^{-1} \sin v .$$

Then subtraction of the versions of Kepler's equation for $M$ and $M_T$, with the aid of $e = e_T$ again, leads to

$$M - M_T = J_0 e^{-1} (q \sin v + e^2 \sin E) ,$$

so that (233) yields

$$M_T = M .$$

Thus four of the $\zeta_T$ are the same as the corresponding $\zeta$, the only differences being for the elements $a$ and $e$. Further, we can apply (120) and (122) to the $\zeta_T$, getting

$$\delta r = J_0 a (e^2 q^{-1} \sin v \sin E - 1)$$

and

$$\delta \omega = J_0 eq^{-2} \sin v (2 + e \cos v) .$$

In view of (207), this completes the analysis for $i = 0$. 
8.2 The trivial case \( f = 1 \)

From (4),

\[
U_1 = -\omega J_1 \frac{R}{r^2} \sin \beta .
\]

(240)

This transforms to

\[
U_1 = U_1' = -\omega A_{1,1} \frac{D}{{p^2}} \cos u'.
\]

(241)

by (15), in the general analysis, where

\[
A_{1,1} = J_1 \frac{R}{p} s .
\]

(242)

by (14), and \( \sin \beta = s \sin u = s \cos u' \). As noted in Ref. 4, (240) implies that

\[
\frac{\mu}{\kappa} U_1 = \nu \left[ r^2 + z^2 - 2rz \cos \left( \frac{\pi}{2} - s \right) \right]^{-1} \circ 0(J_1^2)
\]

(243)

where \( z = -J_1 R \), so the overall potential is the same (to first order) as for a central force towards the point at distance \( z \), and axially 'north', from the nominal centre of 'unperturbed' attraction. (The precise representation of this configuration requires that for each \( f > 0 \), \( J_1 \) has a specific value, given by \( -(-J_1)^k \).)

We have three essentially equivalent parameters (\( J_1 \), \( A_{1,1} \), and \( z \)), and our formulae can be expressed in terms of any one of these, but it is more convenient to use a fourth parameter, \( \lambda \), defined by

\[
\lambda = \frac{z}{p} = -s^{-1} A_{1,1} = -J_1 R/p
\]

(244)

Then the general formulae, taken with \( \lambda = 1 \), lead to the following: (131) gives

\[
\delta r = 0 ;
\]

(245)

(140), with \( \kappa = 0 \) and \( J = 0 \), gives

\[
\delta b = \lambda \sigma ;
\]

(246)
and (182), with $k = 1$, gives

$$\delta w = -\lambda s \cos u.$$  \hfill (247)

Though (247) is taken from a special-case formula (because $j = -k + 1$), the general formula, (156), actually gives the same result because the second term of this formula, which is responsible for the zero denominator, does not arise. Our formulae are consistent with (132), (142) and (157), which give

$$(N_1, r, N_1, b, N_1, \omega) = (0, 1, 1),$$  \hfill (248)

as seen also from Table 6.

Expressions for the $\delta \xi$ can be written down easily enough, following the general analysis, but it is of more interest to obtain formulae relating the $\xi$ (mean elements relative to the nominal attraction centre) to the $\xi_T$ (unchanging osculating elements relative to the 'true' centre). This can be done via the $\xi$ (varying osculating elements) and derived quantities, since no conventional definitions are involved in relating the $\xi$ to the $\xi_T$.

We start with (243), which may be taken to express $u/r_T$. It leads to

$$r - r_T = \lambda p \sin \beta,$$  \hfill (249)

so that, in view of (245),

$$\mathcal{F} = r_T + \lambda p s \sin u.$$  \hfill (250)

Now this is true for all $u$; but $\mathcal{F} - a_T$, $\mathcal{F} - e_T$, and $\mathcal{F} - M_T$ must all be independent of $u$, whilst defining $\mathcal{F} - r_T$ via (120). This is only possible if

$$\mathcal{F} = a_T,$$  \hfill (251)

$$\mathcal{F} = e_T - \lambda q^2 s \sin \omega,$$  \hfill (252)

and

$$\mathcal{F} = M_T + \lambda e^{-1} q^3 s \cos \omega.$$  \hfill (253)
Thus we have established three of the desired relationships; (251) could also be obtained, more directly, by identifying two different expressions for \( V^2 \) (cf (210)).

We can proceed in a similar way to get the relationships for \( \tilde{I} \) and \( \tilde{\Omega} \). Some geometrical visualization is needed, and we may regard the difference between \( b \) and \( b_T \) as validly defined, independently of the precise location of the 'mean orbital plane' which is involved in defining the coordinate \( b \). This difference is given by a projection of the displacement \( z \) perpendicular to the (mean) orbital plane, such that

\[
b - b_T = cz/r = \lambda op/r . \tag{254}\]

Then from (246),

\[
\tilde{U} = b_T + \lambda e \cos \nu \tag{255}
\]

(where \( \tilde{U} \) is actually zero, by definition, but this is not relevant to the argument). As with (250), this is true for all \( \nu \), whilst \( \tilde{U} - b_T \) may be expressed in terms of \( \tilde{I} - I_T \) and \( \tilde{\Omega} - \Omega_T \) via (121). It follows that

\[
\tilde{I} = I_T + \lambda e \sin \omega \tag{256}
\]

and

\[
\tilde{\Omega} = \tilde{\Omega}_T + \lambda e^{-1} \cos \omega . \tag{257}
\]

This only leaves the relationship between \( \tilde{\omega} \) and \( \omega_T \) to be established. It was not obvious how to proceed, analogously to (249) and (254), via a formula for \( \tilde{\omega} - \omega_T \), so the procedure adopted was based on formulae for \( V - V_T \) and \( U - U_T \). The first of these comes easily from (252) and (253); thus

\[
V = V_T + \frac{\lambda e}{2} \left\{ e^2 \cos (u - \nu) + 4e \cos u + (2 - e^2) \cos \omega \right\} . \tag{258}
\]

For the other relationship, we need the special formula (that can be derived for the given geometry)

\[
c(u - u_T) = \sin u \cos u (1 - I_T) - (1 - e^2 \sin^2 u) (\tilde{\Omega} - \tilde{\Omega}_T) . \tag{259}
\]
From this, using (256) and (257) together with the expressions (omitted here) for $\delta_1$ and $\delta_2$, we get

$$u - u_T = \frac{1}{2} a^{-1} \left[ e s^2 \cos (u + v) + 2 \cos u + e(1 + c^2) \cos w \right]. \tag{260}$$

If we introduce also the (omitted) expression for $\delta u$, (260) gives

$$U = u_T + \frac{1}{2} a^{-1} \left[ e s^2 \cos (u + v) + 4 s^2 \cos u + e(1 + c^2) \cos w \right]. \tag{261}$$

From (258) and (261) we have, finally,

$$\omega = \omega_T + \frac{1}{2} a^{-1} s^{-1} (s^2 - s^2 e^2) \cos w. \tag{262}$$

It is worth remarking, in conclusion, that (262) can be used to infer the formula for $\omega - \omega_T$ that seemed less obvious, intuitively, than the formula for $b - b_T$. We find that

$$\omega = \omega_T + (\lambda p/\rho) \cos u. \tag{263}$$

Now that the missing formula is available, it is much easier to visualize its geometrical interpretation, especially for polar orbits ($s = 1$).

### 8.3 The Case $l = 2$

This time we start by noting, from Table 6 or the underlying formulae, that

$$(N_2, r, N_2, d, N_2, \omega) = (2, 2, 3, \cdots), \tag{264}$$

so that there are altogether seven terms in the coordinate perturbations. As in previous papers, we simplify the coordinate expressions by using the notation $K$ and $h$, where

$$K = \frac{1}{2} J_2 (R/p)^2 \tag{265}$$

and

$$h = 1 - \frac{1}{2} e. \tag{266}$$
Then Tables 1 and 2 give

\[ A_{2,0} = -\frac{1}{2}K_h \quad A_{2,1} = -K_c s \quad A_{2,2} = \frac{1}{2}K_r \]  \hspace{1cm} (267)

and Table 4 gives

\[ B_{1,0} = B_{2,0} = 1 \quad B_{2,1} = B_{2,-1} = \frac{1}{2}e \]  \hspace{1cm} (268)

The two terms of \( \delta r \) are given by (131) with \((k, j) = (2, 0)\) and \((0, 0)\). We get, immediately,

\[ \delta r = \frac{1}{2}K_p (f \cos 2u - 2h) \]  \hspace{1cm} (269)

confirming equation (188) of Part 1. (The single-term variable part of this formula has been given by other authors, of course; the best-known derivation was probably that of Kozal\textsuperscript{15}, but King-Hele and Gilmore established the result somewhat earlier, in equation (A-59) of Ref 16.)

The two terms of \( \delta b \) are given by (140) with \((k, j) = (1, 1)\) and \((1, -1)\), since \((1, 0)\) corresponds to an 'excluded term'. We get

\[ \delta b = \frac{1}{2}K_c s \{ \sin \{u + v\} - 3 \sin \omega \} \]  \hspace{1cm} (270)

confirming equation (189) of Part 1.

For \( \delta w \), we might have expected five terms after the exclusion of \((k, j) = (0, 0)\). But \((2, 1)\) is an example of the specific null term given in general by \((2, t - 1)\), whilst the terms for \((0, 1)\) and \((0, -1)\), being identical, are combined. None of the terms is given by the general formula, (156): the term associated with \((2, 0)\) is given by (192); the term associated with \((2, -1)\) is given by (182), and the pair of terms associated with \((0, \pm 1)\) are each given by either* (182) or (184). Overall, we get

\[ \delta w = \frac{1}{4}K \{ f \sin 2u + 4ef \sin (u + \omega) + 8eh \sin v \} \]  \hspace{1cm} (271)

confirming equation (190) of Part 1.

* In writing down specific formulae for \( \delta r \) and \( \delta w \) when \( k \) is even and \( \delta \) zero, we must always remember to double the coefficient of each term with \( j = 0 \), when our intention is to cover the corresponding term with \(-j \) (cf the footnote of section 6.1).
The long-term motion, for \( \ell = 2 \), comes entirely from the secular rates of change, \( \dot{\chi} \) and \( \dot{\omega} \), given (from section 5) by \( -K_n c \) and \( \frac{1}{2} K_n (4 - 5f) \) respectively. They are quoted here, only to remind the reader of the additional 'carry-over' terms in \( \delta r \), \( \delta \theta \) and \( \delta \dot{\omega} \) that they induce. In Part 1, these terms are included in equations (188)-(190); the intervening equations, (191)-(193), refer to the velocity-coordinate perturbations, namely, \( \delta \dot{r} \), \( \delta \dot{\theta} \) and \( \delta \dot{\omega} \).

Finally, of course, since \( J_2 \) for the Earth is of order \( \sqrt{J_2} \) for \( \ell > 2 \), the perturbations of order \( J_2^2 \) have to be taken into account for Earth satellites. Part 1 gives a detailed analysis of these perturbations, and the resulting formulae constitute the principal results of that Report: equations (320), (343), and (359) of Ref 1 give the contributions to \( \delta r \), \( \delta \theta \) and \( \delta \dot{\omega} \), respectively, whilst the long-term effects are covered by equations (297) to (300).

8.4 The case \( \ell = 3 \)

From Table 6,

\[
\begin{pmatrix} \varphi_3,r, \varphi_3,\theta, \varphi_3,\phi \end{pmatrix} = (5, 6, 9),
\]

so that there are 20 terms, in total, in the coordinate perturbations. As in Part 1 we write

\[
H = \frac{1}{2} J_3 (R/p)^3.
\]

Tables 1 and 2 give

\[
A_{3,0} = \frac{1}{3} H c (2 - 5f), \quad A_{3,1} = -\frac{1}{2} H s (4 - 5f),
\]

\[
A_{3,2} = -\frac{1}{9} H c, \quad A_{3,3} = \frac{1}{8} H f;
\]

also Table 4 gives, in addition to quantities we already have from (268),

\[
B_{3,0} = 1 + 4e^2, \quad B_{3,1} = e, \quad B_{3,2} = -4e^2.
\]
We start with $\delta r$, separating (for convenience) the effects for $k = 3$ and $k = 1$. For $k = 3$, all the a-priori values of $j$ must be included, namely, 1, 0 and -1; for $k = 1$, on the other hand, we exclude $j = 0$. Then (131) gives, corresponding to the two values of $k$,

$$\delta r = \frac{1}{2}H_p \sin (3u + v) + 15 \sin 3u + 20 \sin (2u + \omega)$$

(277)

and

$$\delta r = \frac{1}{2}H_p (4 - 5) \sin (u + v) - 3 \sin \omega$$

(278)

these conform with equation (408) of Part 1. (The total $\delta r$ is, of course, given by adding the two contributions.)

For $\delta b$, the effects are for $k = 2$ and $k = 0$, and the a-priori values of $j$ are the five with $|j| \leq 2$. For $k = 2$ we exclude $j = -1$, and for $k = 0$ we exclude $j = 1$; for $k = 0$ we also lose a term on combining the terms with $j = 1$. Then (140) gives, corresponding to the two values of $k$,

$$\delta b = - \frac{1}{2} \cos (2u + v) + 15 \cos (2u + v)$$

$$+ 20(2 - 5) \cos 2u - 30 \cos 2\omega$$

(279)

and

$$\delta b = - \frac{1}{2} \cos (2 - 5) \cos 2 \cos - 3(2 + 5)$$

(280)

these conform with equation (411) of Part 1.

For $\delta w$, the effects are for $k = 3$ and $k = 1$, with the same a-priori $j$ values as for $\delta b$. For $k = 3$, all five values yield terms, but only three of them come from the general (156); for $j = -1$ we use (192) and for $j = -2$ we use (182). For $k = 1$, the term with $j = -1$ is excluded, and the only general term is for $j = 2$; the terms for $j = 0$ and $-2$ come from (192), (182) and (184) respectively. Corresponding to the two values for $k$, we get

* In contradistinction to the previous footnote, and as noted in general after (141) in section 6, the two terms do not have the same numerical coefficient.
\[
\delta w = - \frac{1}{2} H_s f [2e^2 \cos (3u + 2v) + 11e \cos (3u + v) \\
+ 4(5 + e^2) \cos 3u + 25e \cos (2u + w) + 50e^2 \cos (u + 2w)]
\]  

\[(281)\]

and

\[
\delta w = \frac{1}{2} H_s (h - 5f) [e^2 \cos (u + 2v) - 2e \cos (u + v) \\
- 2(18 + 7e^2) \cos u + 9e^2 \cos (v - \omega)]
\]

\[(282)\]

these conform with equation (413) of Part 1.

Expressions for the long-period rates of change of the mean elements can be written down from the formulae of section 5. The results agree with (from Ref 1) (373), (376), (384), (389) and (399), for \( \varpi, \Omega, \varpi, \omega \) and \( \Omega \), respectively.

8.5 The (new) case \( I = 4 \)

From Table 6,

\[
(N_4, r, N_4, b, N_4, w) = (11, 11, 15)
\]

\[(283)\]

so that there are altogether 37 terms in the coordinate perturbations, which we obtain first. To simplify our expressions, we define

\[
G = \frac{1}{16} J_4 (R/p)^2
\]

\[(284)\]

Then Tables 1 and 2 give

\[
A_{4,0} = 480 (6 - 40r + 35f^2), \quad A_{4,1} = 480 G\cos (4 - 7f)
\]

\[(285)\]

\[
A_{4,2} = -320 Gf (6 - 7f), \quad A_{4,3} = -1120 Gcsf, \quad A_{4,4} = 550 Gf^2
\]

\[(286)\]

also Table 4 gives, in addition to quantities we already have from (276),

\[
B_{4,0} = 1 + i \epsilon^2, \quad B_{4,1} = \frac{1}{2} (1 + i \epsilon^2), \quad B_{4,2} = \frac{1}{2} \epsilon^2, \quad B_{4,3} = \frac{1}{2} \epsilon^3
\]

\[(287)\]
We start with $6r$, as usual, separating the effects for $k = 4$, $k = 2$ and $k = 0$. The a-priori values of $j$ are the five values for which $|j| \leq 2$. All values apply when $k = 4$; for $k = 2$ we exclude $j = -1$; and for $k = 0$ we exclude $j = \pm 1$, whilst the terms for $j = \pm 2$ are identical. Then (131) gives, corresponding to the three values of $k$,

$$
6r = -2Gpf^2 \left[ 6e^2 \cos 2(2u + v) + 35e \cos (4u + v) + 28(2 + e^2) \cos 4u \\
+ 105e \cos (3u + \omega) + 70e^2 \cos 2(u + \omega) \right] \quad (288)
$$

$$
6r = -8Gpf(6 - 7f) \left[ 2e^2 \cos 2(u + v) + 15e \cos (2u + v) \\
+ 20(2 + e^2) \cos 2u - 30e^2 \cos 2\omega \right] \quad (289)
$$

and

$$
6r = -24Gp(8 - 40f + 35f^2) \left[ e^2 \cos 2v - 3(2 + e^2) \right] \quad (290)
$$

(The dominant (e-free) terms of (288) and (289) were originally given in equation (A108) of Ref. 16.)

For $6b$, the effects are for $k = 3$ and $k = 1$, the a-priori values of $j$ being the seven with $|j| \leq 3$. For $k = 3$ we exclude $j = -2$, and for $k = 1$ we exclude $j = 0$ and $j = -2$. Then (140) gives, corresponding to the two values of $k$,

$$
6b = -4Gosf \left[ 4e^3 \sin (u + v) + 35e^2 \sin (3u + 2v) \\
+ 28e(4 + e^2) \sin (3u + v) + 70(2 + 3e^2) \sin 3u \\
+ 140e(4 + e^2) \sin (2u + \omega) - 140e^3 \sin 3\omega \right] \quad (291)
$$

and

$$
6b = -4Gosf(4 - 7f) \left[ 4e^3 \sin (u + 3v) + 45e^2 \sin (u + 2v) \\
+ 60e(4 + e^2) \sin (u + v) - 180e(4 + e^2) \sin \omega \\
- 20e^3 \sin (v - \omega) \right] \quad (292)
$$

For $6w$, the effects are again for $k = 4$, 2 and 0, with the same a-priori $j$ values as for $6b$. For $k = 4$, all seven $j$ values yield terms, of which five come from the general (156), for $j = -2$ we use (192) and for $j = 3$ we use (182). For $k = 2$, the term with $j = -2$ is excluded, whilst
for \( j = 3 \), (156) gives another example of a 'specifically null' term (as in section 8.3); there are non-null general terms for \( j = 2 \) and \( j = 1 \); and the terms for \( j = 0, -1 \) and -3 come from (192), (182) and (184) respectively. Finally, for \( k = 0 \) the term with \( j = 0 \) is excluded; the other terms come in pairs, being 'general' for \( j = \pm 3 \), from (192) and (194) for \( j = \pm 2 \), and from (182) and (184) for \( j = \pm 1 \). Corresponding to the three values of \( k \), we get

\[
6w = -Gf^2[4e^3 \sin (4u + 3\nu) + 31e^2 \sin 2(2u + \nu) + 4e(21 + 5e^2) (x) \\
\times \sin (4u + \nu) + 28(3 - 4e^2) \sin 4u + 80e(7 + e^2) \sin (3u + \omega) \\
+ 175e^2 \sin 2(u + \omega) + 140e^3 \sin (u + 3\omega)] , \\
\]

(293)

and

\[
6w = 4Gf(6 - 7f) \{2e^2 \sin 2(u + \nu) + 4e(5 + 2e^2) \sin (2u + \nu) \\
+ 5(8 - 7e^2) \sin 2u - 80e(5 + e^2) \sin (u + \omega) \\
- 40e^3 \sin (\nu - 2\omega) \} \\
\]

(294)

It only remains to give the expressions for the \( \dot{t} \) (secular and long-period) from section 5. They may also be derived (as a check) from the author's early Ref 4; also, the version of Kepler's third law, given here as (303), checks with equation (156) of Ref 3.

There are, of course, no secular rates of change in \( \pi \) or \( T \). Their long-period rates are given by (108) and (109), with just \( k = 2 \). Thus

\[
\ddot{\pi} = -480 Gneq^2f(6 - 7f) \sin 2\omega \\
\]

(296)

and

\[
\ddot{T} = 480 Gneq^2os(6 - 7f) \sin 2\omega . \\
\]

(297)

The secular rate of change of \( \Pi \), given by (110) with \( k = 0 \), is

\[
\ddot{\Pi} = 480 Gneq(4 - 7f)(2 + 3e^2) , \\
\]

(298)
and the long-period rate (given just with \( k = 2 \)) is

\[
\ddot{\Omega} = -960 \, G n e^2 q (3 - 7f) \cos 2\omega .
\] (299)

For the variation of \( \omega \), we work with \( \dot{\Omega} \), given by the second term of (111). Thus the secular rate is given by

\[
\dot{\Omega} = -120 \, G n (8 - 40f + 35f^2) (4 + 3e^2) \cos 2\omega .
\] (300)

so that from (298) and (300) the secular rate for \( \omega \) is

\[
\dot{\omega} = -120 \, G n \left[ 4(16 - 62f + 49f^2) + 9e^2 (8 - 28f + 21f^2) \right] .
\] (301)

Similarly, the long-period rate for \( \psi \) is given by

\[
\dot{\psi} = -240 \, G n f (6 - 7f)(2 + 5e^2) \cos 2\omega ,
\] (302)

from which the rate for \( \psi \) is at once available.

Finally, for the variation of \( \Omega \), we deal with the secular perturbation by the modification of Kepler’s third law given by (116). This gives

\[
\frac{m^2}{a^3} = u \left[ 1 + 288 \, G n q^2 (8 - 40f + 35f^2) \right] .
\] (303)

based on the perturbation rate (residual to the mean motion)

\[
\dot{\mathcal{H}} = -144 \, G n q^3 f (8 - 40f + 35f^2)
\] (304)

given by (113). Again, the long-period rate, from (113), is given by

\[
\ddot{\mathcal{H}} = 480 \, G n q^3 f (6 - 7f) \cos 2\omega ;
\] (305)

this checks with (302) and the long-period rate for \( \mathcal{L} \), which from (114) is

\[
\ddot{\mathcal{L}} = -1680 \, G n e^2 q f (6 - 7f) \cos 2\omega .
\] (306)
For completeness in implementing the (first-order) effects of J₄ (and any other Jₖ), it is necessary to incorporate terms relating to what Part 1 describes as the difference between mean and semi-mean elements. The effects induced by the secular variation are included by adding \( (\alpha/n)m \), \( (\omega/n)m \) and \( (\Omega/n)m \) to \( \alpha \), \( \omega \) and \( \Omega \), respectively, where \( \alpha \), \( \omega \) and \( \Omega \) are given by (298), (301) and (304), and where \( m = \nu - M \) as in Part 1. The effects induced by the long-period variation, on the other hand, are allowed for via additional terms in the expressions for \( \delta r \), \( \delta b \) and \( \delta w \). Using (120)-(122), we find that these additional terms are given by

\[
\delta r = 480 Ge^{m}\{(6 - 7f) \sin (\mu + \omega) \}, \quad (307)
\]

\[
\delta b = 240 Ge^{m}\cos \left\{ 3(4 - 7f) \cos (\nu - \omega) - 7f \cos (\nu + 2\omega) \right\} \quad (308)
\]

and

\[
\delta w = 240 Ge^{m}\{(6 - 7f) e\cos 2\omega + 4 e\cos (\nu + \omega) - 4 e\cos 2\omega \} \quad (309)
\]

9 CONCLUSIONS

The main function of Part 1 of the present trilogy of Reports was to provide details of a new theory of satellite motion, largely based on the use of a particular system of spherical-polar coordinates in the representation of the short-period components of the orbital perturbations. The emphasis was on the derivation of the second-order perturbations due to the zonal harmonic J₂, but the first-order perturbations due to J₃ were derived as well. The latter derivation has now been extended to an arbitrary zonal harmonic, Jₖ (where \( k \) is positive), with the development of general formulae of which those for J₃ were just a particular case.

The main formulae, which (in their generality) are believed to be entirely novel, are those for the perturbations in coordinates. The general terms of these formulae are given by the summations in (131), (140) and (156), for the perturbations in \( r \), \( b \) and \( w \), respectively. Terms that would have a zero denominator are excluded from these summations, as a consequence of the optimal definition of mean elements, except that replacement terms are needed for the perturbations in \( w \); the formulae for the replacement terms are (182), (184), (192) and (194).
The formulae for coordinate perturbations are complemented by the formulae, given in section 5, for the rates of change of the mean elements. In principle, the integration of the rate-of-change expressions is immediate, leading to the secular and long-period perturbations in the elements. In practice, however, there are complications, as was indicated in Part 1. One of these complications results from the fact that the expressions really arise as rates of change with respect to true anomaly, rather than time, and this leads to additional effects that are short-periodic in nature. However, the difficulty can easily be dealt with via the concept of semi-mean elements; the matter was fully discussed in Part 1, and has been touched on here in the context of the derivation of the appropriate perturbation terms for $i = 4$ (section 8.5). The other complications arise in the long-term evolution of the mean elements, the chief source of difficulty being the well-known singularities in the standard set of elements. A preliminary consideration of these difficulties was included in Part 1, but a full analysis is held over to Part 3, which will also give some numerical results.

The main limitation of the theory presented by the trilogy is apparent from its overall title - the gravitational field is assumed to be axi-symmetric, i.e. represented by zonal harmonics alone. For a complete field, with the tesseral harmonics included, the author has already published some general formulae (in terms of cylindrical coordinates rather than spherical coordinates, though that is a minor detail), but they apply only to near-circular orbits. The formulae were originally given in Ref 10, then in Ref 5, and finally as equations (92) - (94) of Ref 9.

In the formulae referred to, the inclination functions involve an additional suffix, $m$, to cover the longitude-dependent harmonics. For $m = 0$, the functions reduce to the $A_k^m(i)$ and $A^m_k$ of the present paper, whilst the formulae themselves are then equivalent to truncated versions of the present equations (131), (140) and (156). Since we now have one set of formulae that relate to all inclination functions, though the formulae are truncated in regard to eccentricity, and another set of formulae that are valid for any eccentricity, though only relating to inclination functions for which $m = 0$, an obvious goal is the derivation of formulae that are 'general' in both respects. There is a fundamental difficulty, however, arising from the rotation of the gravitating primary, which we are able to neglect in the trilogy because it is assumed to take place about the axis of symmetry.
The root of the trouble is that the disturbing function \( U \) is no longer time-independent when the rotation of an arbitrary primary is allowed for. This nullifies our key constant, \( a' \), and leads ipso facto to the important phenomenon of resonance\(^{13} \). It will not be easy to develop a unified theory that covers resonant effects by the same formulae as non-resonant ones. However, a starting point is obviously the generation of the formulae referred to (in the preceding paragraph) as being 'general in both respects'; Appendix A gives an outline of what is involved.
Appendix A

EXTENSION TO THE GENERAL GRAVITATIONAL FIELD

Extending the theory of this Report to the tesseral harmonics (sectorial included) is an easier matter than might have been expected, so long as the rotation of the gravitating body is neglected; i.e., we suppose the sidereal angle, \(\nu\), to be fixed. We assume the potential to be described by the usual harmonic coefficients, \(C_{lm}\) and \(S_{lm}\), where \(-C_{0,0}\) can be identified with the zonal coefficient \(J_1\) and \(S_{1,0}\) is taken as zero. For convenience, we introduce the polar equivalents, \(J_{lm}\) and \(\lambda_{lm}\), where \(\lambda\) is longitude and

\[
(C_{lm}, S_{lm}) = J_{lm} (\cos \lambda_{lm}, \sin \lambda_{lm}). \tag{A-1}
\]

(Note: \(\lambda_{lm}\) is not uniquely defined if \(m > 1\), and if \(m = 0\) we set \(J_{l,0} = -J_{l,0}\), so \(J_{l,0}\) must be allowed to be negative.) It is usual, in practice, to work with normalized versions of \(C_{lm}\) and \(S_{lm}\) (and hence \(J_{lm}\)), but this is an irrelevant complication here. The potential due to \(J_{lm}\), generalizing equation (4) of the main text, is given by

\[
U_{lm} = \frac{2}{\pi} J_{lm} \left(\frac{R}{r}\right)^{l+2} P_0^m (\sin \theta) \cos m(\lambda - \lambda_{lm}). \tag{A-2}
\]

The expansion of \(U_{lm}\), in terms of the orbital elements, is customarily based on the family of inclination functions, \(F_{imp}(i)\), such that, generalizing equation (8),

\[
P_0^m (\sin \theta) \exp (i m \lambda) = \sum_{p=0}^{1} F_{imp}(i) \exp i[(l - 2p)u + m(n - \nu)]. \tag{A-3}
\]

As already indicated in the main text of the Report, however, we prefer to use the index \(k = l - 2p\), rather than \(p\). This index only takes values that are of the same parity as \(l\), but in the extension to \(m > 0\) we have to allow negative values of \(k\), so that its range is now from \(-l\) to \(+l\); as compensation, we no longer require the factor \(u_k\) introduced at equation (8). Further, we prefer the inclination functions to be real for all values of the indices, so we define, as an unnormalized equivalent of the \(\bar{F}_{km}(i)\) in Ref 9,

\[
F_{km}^*(i) = i^{-2k} F_{imp}(i). \tag{A-4}
\]

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The $U_{lm}$ expansion will involve $\Omega$, as well as the other elements, to reflect the abandonment of axial symmetry, but for convenience we work with $\Omega'$, defined by

$$\Omega' = \Omega - \nu - \frac{1}{2} k.$$  \hfill (A-5)

Then $U_{lm}$ can be decomposed into $\sum_k U_{lm}^k$, where

$$U_{lm}^k = \frac{\mu}{r} (R/r)^{k} J_{lm} F_{lm}^k(1) \cos \{k \nu' + m(\Omega' - \lambda_{lm})\} ;$$  \hfill (A-6)

this is compatible with equation (71) of Ref 9, in which $Y$ and $X$ were the negatives of the present $\nu'$ and $\Omega'$. (Compatibility with two other papers can be obtained by noting that $F_{lm}^k$ here is $\nu^{l+k}$ times the unnormalized equivalent of the $F_{lm}^{imp}$ used in Ref 13, whilst $F_{lm}^{imp}$ in Ref 17 is identical with $F_{lm}^{imp}$ introduced at (A-3) here.)

Next we introduce quantities $A_{lmk}$ that directly generalize the $A_{lm}$ of the main text, defining

$$A_{lmk} = - J_{lm} (R/p)^{k} F_{lm}^k(1).$$  \hfill (A-7)

We also generalize $C_{jm}^k$ and $S_{jm}^k$, by defining

$$C_{jm}^k = \cos \{j \nu' + k \nu' + m(\Omega' - \lambda_{lm})\} ,$$

$$S_{jm}^k = \sin \{j \nu' + k \nu' + m(\Omega' - \lambda_{lm})\} ,$$  \hfill (A-8)

and henceforth we will omit the superfices.

Then (A-6) to (A-8) give

$$U_{lm}^k = - \frac{\mu}{p} (p/r)^{k+1} A_{lmk} C_0 ,$$  \hfill (A-9)

which is a straight generalization of equation (15) of the main text. When $m \leq k$, we can also generalize the preceding equation (14) by writing

$$A_{lmk} = - J_{lm} (R/p)^{k} a_{lmk} s^{k-m} (1 + c)^m A_{lm}^k(1) ,$$  \hfill (A-10)
where \( A^k_{lm}(1) \) is from the family of 'normalized' inclination functions introduced in Ref 17 to generalize the \( A^k_{lm}(1) \), and \( a_{lmk} \) generalizes \( a_{lm} \); the formula for \( a_{lmk} \) can be inferred from equation (11) ibid. It then follows from (A-7) and (A-10) that

\[
P_{lm}^k(1) = a_{lmk} s^{k-m} (1 + c)^m A_{lm}^k(1) \quad (A-11)
\]

It was assumed, in (A-10), that \( m \leq k \). When \( 0 \leq k < m \), a different generalization became necessary in Ref 17, leading to

\[
A_{lmk} = -J_{lm} (R/p)^t a_{lmk} s^{m-k} (1 + c)^k A_{lm}^k(1) \quad (A-12)
\]

where now the formula for \( a_{lmk} \) can be inferred from equation (13) ibid; also (A-7) and (A-12) lead to

\[
P_{lm}^k(1) = a_{lmk} s^{m-k} (1 + c)^k A_{lm}^k(1) \quad (A-13)
\]

For \( k = m \), (A-12) and (A-13) are consistent with (A-10) and (A-11), respectively, but otherwise the dual definitions of \( A_{lm}^k(1) \) and \( a_{lmk} \) are distinct. A further complication is that an extension of (A-10) and (A-11) to negative \( k \) is not generally available; (A-12) and (A-13) still operate for \( k < 0 \), with \( |k| \leq m \), but there is a marked lack of symmetry between the forms of \( A_{lm}^k(1) \) and \( a_{lmk} \) for \( k < 0 \) in relation to \( k > 0 \). The difficulty for negative \( k \) is not too serious, however, as \( P_{lm}^k(1) \) can then be derived from

\[
P_{lm}^k(1) = (-1)^m P_{lm}^{-k} (\pm 1) \quad (A-14)
\]

By appeal to (A-14) as required, the \( A_{lmk} \) can always be obtained. There are advantages in the adoption of a different 'normalization' for the \( A_{lm}^k(1) \), however, such that they constitute a fully unified family of functions, defined for all \( k \) and symmetric in regard to the sign of \( k \). The constants \( a_{lmk} \) must then also be redefined, to preserve the \( A_{lmk} \) unchanged, and the connecting form as is based on (A-12) rather than (A-10). We introduce a variable sign into (A-12), to make the \( a_{lmk} \) always positive; expressed symmetrically in regard to the sign of \( k \), the connecting formula is then

\[
A_{lmk} = (-)^{(l+k-2)} J_{lm} (R/p)^t a_{lmk} s^m \left( \frac{1 + c}{1 - c} \right)^k A_{lm}^k(1) \quad (A-15)
\]
The new $A_{lm}^k(i)$ are defined for $l \geq m$, as before, but for all $k$ now, not just for $|k| \leq i$ (quite apart from the problem with $k < 0$). The recurrence relation for fixed $m$ and $k$ is

\[(l - 1)(l^2 - m^2)A_{lm}^k - (2l - 1)(l(l - 1)c - mk) A_{l-1,m}^k + \delta[(l - 1)^2 - k^2] A_{l-2,m}^k = 0 , \quad (A-16)\]

which is slightly simpler than the relation in Ref 17; the starting values for this are

\[A_{lm}^k(1) = 1 \quad \text{and} \quad A_{m+1,m}^k(1) = (m + 1)c - k , \quad (A-17)\]

though the second of (A-17) can be dispensed with if we define $A_{m-1,m}^k(1)$ to be zero. A recurrence relation for fixed $l$ and $m$ is also available, viz

\[(l - k)(1 + c)A_{lm}^{k+1} - 2(m - kc)A_{lm}^k + (l + k)(1 - c)A_{lm}^{k-1} = 0 , \quad (A-18)\]

in which the symmetry (in regard to the sign of $k$) is obvious; expressions for $A_{lm}^k(i)$, with $|k| < i$, can be generated 'from either end' by use of just one starting value from the pair

\[A_{lm}^k(1) = [\delta(c + 1)]^{l-m} \left(\frac{2l}{l+m}\right) . \quad (A-19)\]

There is also a recurrence relation for fixed $l$ and $k$, but instead of giving it we note that the set of 15 relations, each involving $A_{lm}^k(i)$ and two 'adjacent' functions from a three-dimensional table, can all be generated from various subsets of just three relations; one of the simplest such subsets consists of (A-18) and the following pair of relations:

\[(l + k)A_{l-1,m}^k - (m + 1c)A_{lm}^k + \delta[(1 + c)A_{l-1,m}^{k+1} = 0 \quad (A-20)\]

and

\[2(l - m + 1)A_{lm}^k - (m + k)(1 - c)A_{lm}^k - (l - k)(1 + c)A_{lm}^{k+1} = 0 . \quad (A-21)\]

(Note in proof: see Refs 19 and 20 for computational aspects of these relations.)
We will not enlarge on the advantages of the redefinition of \( A_{km}(1) \) and \( \sigma_{km} \), but two disadvantages must be mentioned. First, since the factors in \( s \) and \( c \) in (A-15) can be combined as \( (1 + c)^{m+k}(1 - c)^{m-k} \), we see that a negative power of either \( 1 + c \) or \( 1 - c \) appears whenever \( m < |k| \), and this has to be cancelled by a corresponding positive power that is present in the new \( A_{km}(1) \). Second, the use of (A-16) and (A-17) to compute \( A_{km}(1) \) under these circumstances \( (m < |k|) \) is inefficient, since the recurrence process has to work through the unwanted functions with \( m < k < |k| \).

The \( A_{km}(1) \) are, like the \( A_{k}(1) \) of the main text, defined regardless of parity. The constants \( \sigma_{km} \) (and hence the quantities \( A_{km} \)) are only defined for \( l \) and \( k \) of the same parity, however, and (as redefined) their only property to be stated here is that of complete symmetry, so that

\[
a_{km}^k = a_{km}^{-k} \quad (> 0).
\]  

But, just as in the main text, we require another set of constants, \( \alpha_{km} \), and quantities, \( A_{km} \), defined when \( l \) and \( k \) are of opposite parity, to allow the formula for \( \delta b \) to be expressed. The connecting formula corresponding to (A-15) is

\[
A_{km} = (-)^{(l+k-1)} J_{km} (R/p)^{l} \alpha_{km} \sigma^{m} \left( \frac{1 + c}{1 - c} \right)^{l-k} A_{km}^{l} \quad (A-23).
\]

The \( \alpha_{km} \) are available at once from the \( \alpha_{km} \), since (cf (26) of the main text, which, because of the redefinition, is not being directly generalized)

\[
\alpha_{km} = \alpha_{l-1,m,k} \quad (A-24).
\]

Tables and further properties of the redefined inclination functions, and the associated constants, will be given in a separate paper.

By making use of the quantities \( A_{km} \) and \( A_{km} \), we find no difficulty on extending the theory, largely because the treatment of \( (p/r)^{l+k} \), in (A-9), via the \( B_{ij} \), goes through unchanged from the main text. Further, the energy-based exact quantity, \( a' \), is still available, following the assumption that the attracting body does not rotate. Thus, equations (65), (76), (88) and (105), for \( \delta a \), \( \delta a/\delta v \), \( \delta v/\delta v \) and \( \delta v/\delta v \), respectively, are unchanged apart from the appearance of \( A_{km} \) in place of \( A_{km} \). Equation (82), for \( \delta a/\delta v \), requires a
corresponding change, such that the derivative \( A'_{\lambda m k} \) replaces \( A_{\lambda m} \). This just
leaves (79), for \( \frac{d}{dv} \), for which a slightly more complicated expression is now
required, to reflect the fact that \( p c^2 \) is no longer an invariant. We have, in
fact,

\[
\frac{d(p c^2)}{dv} = 2 m p c A_{\lambda m k} \sum B_{\lambda j} S_j. \tag{A-25}
\]

From this, using the version of \( \frac{d}{dv} \) corresponding to equation (79), we get

\[
\frac{d}{dv} = s^{-1}(k c - m) A_{\lambda m k} \sum B_{\lambda j} S_j; \tag{A-26}
\]

in comparison with equation (79), we see that the only additional change is the
replacement of \( k c \) by \( k c - z \).

Six of the seven formulae that define \( \delta r \), \( \delta b \) and \( \delta w \) completely, for
the zonal harmonics, are immediately applicable to the zonal harmonics, so long
as \( A_{\lambda m k} \) replaces \( A_{\lambda m} \) and the trigonometric argument includes the term
\( m(\lambda' - \lambda m) \). These six are (131) and (156), for the general \( \delta r \) and \( \delta w \), and
(182), (184), (192) and (194), the four special formulae for \( \delta w \). In the
seventh formula, (140) for \( \delta b \), \( A_{\lambda k} \) must be replaced by \( (\lambda + m)A_{\lambda m k} \), in
addition to the inclusion of the extra term in the trigonometric argument. (It
is, perhaps, surprising that the change to (140) is as slight as this, but it
would have been even less if \( A_{\lambda k} \) and \( A_{\lambda m k} \) had been defined to include the
factors \( \lambda \) and \( \lambda + m \) respectively; the reason for excluding these factors
was, essentially, to give a degree of homogeneity to (26) and (A-24.)

Finally, of course, the numbers of terms in \( \delta r \), \( \delta b \) and \( \delta w \), for a
given \( J_{\lambda m} \), are greater for \( m > 0 \) than for \( m = 0 \), to reflect the distinction
between positive and negative \( k \). These numbers are otherwise independent of
\( m \), however, in consequence of which we write the formulae as follows:

\[
N'_{\delta r} = 2\lambda^2 - 3\lambda + 1, \tag{A-27}
\]

\[
N'_{\delta b} = 2\lambda^2 - 3\lambda + 2 \tag{A-28}
\]

and

\[
N'_{\delta w} = \begin{cases} 2\lambda^2 & \text{for odd } \lambda \\ 2(\lambda^2 - 1) & \text{for even } \lambda \end{cases} \tag{A-29}
\]
Appendix B

THE QUANTITIES $B_{ij}$, AND $B_{0,1}$ IN PARTICULAR

When $j \geq 0$, $B_{ij}$ may be expressed in terms of the hypergeometric function as:

$$B_{ij} = \frac{(i-j)!}{(i+1)!} (\sqrt{2})^j F\left(\frac{j-\frac{1}{2} + 1}{2}, \frac{j-\frac{1}{2} + 2}{2}; j + 1; e^2\right), \quad (B-1)$$

where (B-1) applies for all $i$. This result is proved, in terms of the function $B_i^j(e)$, in Appendix E of Ref 4; it is also quoted, in terms of the equivalent Hansen function, at equation (32) of Ref 8. For $(0 \leq j < i)$, (B-1) gives a polynomial in $e^2$; for $j \geq i > 0$ it gives zero; and for $i \leq 0$ it gives a power series in $e^2$, which can be transformed into a closed expression involving $q \left(-\sqrt{1-e^2}\right)$ and perhaps $\beta \left(-e/(1+q)\right)$. All this is consistent with Table 4.

Equation (B-1) breaks down when $j < 0$, owing to the eventual occurrence of a zero denominator when the hypergeometric function is expanded. Since

$$B_{ij} = B_{i,-j}, \quad (B-2)$$

however, this break-down is of no account. The justification of (B-2) comes from the Hansen-function equivalence and the relation

$$x_m^{i,j} = x_m^{i,-j}, \quad (B-3)$$

which follows immediately from the definition of Hansen's functions by equation (35) of the main text.

The quantity $B_{0,1}$ is of particular interest, being the simplest of the $B_{ij}$ that involve $\beta$. Once it is known, the other such $B_{ij}$ can be progressively derived using the recurrence relations of the main text.

From (B-1) we have

$$B_{0,1} = -\frac{1}{2} e F(1, 1; 2; e^2). \quad (B-4)$$
But as a particular case of the hypergeometric relation proved as equation (9.2) of Ref. 18, we have

\[ F(1, 1\frac{1}{2}; 2; e^2) = 2q^{-1}F(\frac{1}{2}, 1; \frac{1}{2}; q^2) - 2F(1, 1\frac{1}{2}; 1\frac{1}{2}; q^2) \]  
(B-5)

Also, it is immediate (from the expansion) that

\[ F(a, b; a; q^2) = (1 - q^2)^{-b}, \]  
(B-6)

independently of \( a \), so that (B-5) gives

\[ F(1, 1\frac{1}{2}; 2; e^2) = 2/q(1 + q). \]  
(B-7)

Finally, (B-4) and (B-7) give

\[ B_{0, 1} = -8q^{-1}, \]  
(B-8)

in conformity with the entry in Table 4.
<table>
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<tr>
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</tr>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>c(1 - \frac{1}{3} t)</td>
<td>1 - \frac{1}{3} t</td>
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<tr>
<td>4</td>
<td>1 - 5t + \Psi f^2</td>
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<td>c(1 - 7t + \Psi f^2)</td>
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<tr>
<td>6</td>
<td>1 - \Psi f + \Psi f^2 - 3\Psi f^3</td>
<td>c(1 - \frac{1}{3} t + \Psi f^2)</td>
<td>1 - 3t + \Psi f^2</td>
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<td>1 - \Psi f</td>
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Table 2

THE CONSTANTS $\alpha_{ik}$ AND $\alpha_{jk}$

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Table 3

THE FUNCTIONS $B_j^i(z)$

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</tr>
<tr>
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<td>$1 + \frac{1}{2}z^2$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$1 + \frac{1}{3}z^2$</td>
<td>$1 + \frac{1}{4}z^2$</td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>5</td>
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<td>$1 + \frac{1}{4}z^2$</td>
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<td>1</td>
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<td>$1 + \frac{1}{4}z^2 + \frac{1}{5}z^3$</td>
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<tr>
<td>7</td>
<td>$1 + \frac{1}{3}z^2 + \frac{1}{4}z^3 + \frac{1}{5}z^4$</td>
<td>$1 + \frac{1}{4}z^2 + \frac{1}{5}z^3 + \frac{1}{6}z^4$</td>
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### Table 4

**The Quantities $b_{ij}$**

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<th>3</th>
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<tbody>
<tr>
<td>-3</td>
<td>$-\frac{1}{3}e^{-7}(4 + e^2)$</td>
<td>$\frac{1}{6}e^{-7}(2 + 3e^2)$</td>
<td>$\frac{1}{2}e^{-7}(4 + e^2)$</td>
<td>$\frac{1}{4}e^{-7}$</td>
<td>$\frac{1}{4}e^{-7}$</td>
<td>$-\frac{1}{4}e^{-7}$</td>
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<tr>
<td>-2</td>
<td>$-\frac{4}{3}e^{-5}$</td>
<td>$\frac{1}{3}e^{-5}(2 + e^2)$</td>
<td>$-\frac{1}{2}e^{-5}$</td>
<td>$\frac{1}{4}e^{-5}$</td>
<td>$\frac{1}{4}e^{-5}$</td>
<td>$-\frac{1}{2}e^{-5}(3 + 9e + 8e^2)$</td>
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</tr>
<tr>
<td>-1</td>
<td>$-e^{-3}$</td>
<td>$e^{-3}$</td>
<td>$-e^{-3}$</td>
<td>$\frac{1}{2}e^{-3}(1 + 2e)$</td>
<td>$\frac{1}{2}e^{-3}(1 + 2e)$</td>
<td>$-\frac{1}{2}e^{-3}(3 + 3e)$</td>
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<tr>
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<td>$e^{-1}$</td>
<td>$-e^{-1}$</td>
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</tr>
<tr>
<td>3</td>
<td>$e$</td>
<td>$\frac{1}{2}(2 + e^2)$</td>
<td>$e$</td>
<td>$\frac{1}{2}e^2$</td>
<td>$\frac{1}{2}e^2$</td>
<td>0</td>
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</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2}e(1 + e^2)$</td>
<td>$\frac{1}{2}(2 + 3e^2)$</td>
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Table 5
THE QUANTITIES $E_{1k}$

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<td>$3e^q^5$</td>
<td>$-\frac{1}{4}e^q^5(4 + 3e^2)$</td>
<td>$5e^q^5$</td>
<td>$-\frac{1}{4}e^2e^q^5$</td>
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<td>$3e^q^{-3}$</td>
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<tr>
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<td>0</td>
<td>$-q^{-1}$</td>
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<tr>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>$e$</td>
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</tr>
<tr>
<td>2</td>
<td>$3e$</td>
<td>$\frac{1}{2}(1 + 2e^2)$</td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2}e(4 + e^2)$</td>
<td>$1 + 4e^2$</td>
<td>$\frac{1}{2}e(2 + 3e^2)$</td>
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</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2}e(4 + 3e^2)$</td>
<td>$\frac{1}{2}(4 + 27e^2 + 4e^4)$</td>
<td>$\frac{1}{2}e(2 + 5e^2)$</td>
<td>$\frac{1}{2}e^2(3 + 4e^2)$</td>
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Table 6
THE NUMBER OF TERMS IN $\delta_r$, $\delta_b$ AND $\delta_w$

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<th>$N_{L,w}$</th>
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LIST OF SYMBOLS

(Usage for the main text only)

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<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>$a$</td>
<td>semi-major axis</td>
</tr>
<tr>
<td>$a'$</td>
<td>energy-based fixed value of $a$</td>
</tr>
<tr>
<td>$A_k^N(1)$</td>
<td>'normalized' function of inclination</td>
</tr>
<tr>
<td>$A_k$</td>
<td>quantity, based on $A_k^N(1)$, defined by (14)</td>
</tr>
<tr>
<td>$A_k$</td>
<td>similar to $A_k^N$, but defined by (17)</td>
</tr>
<tr>
<td>$A_{ik}$</td>
<td>derivative of $A_k$ with respect to $i$</td>
</tr>
<tr>
<td>$A_{ik}^+$</td>
<td>$\frac{ks^{-1} A_k + e^{-1} A_k'}{A_k}$</td>
</tr>
<tr>
<td>$b$</td>
<td>latitude-like spherical coordinate of $(r, b, w)$</td>
</tr>
<tr>
<td>$B_j^N(e)$</td>
<td>normalized function of eccentricity</td>
</tr>
<tr>
<td>$B_j$</td>
<td>quantity related to $B_j^N(e)$, defined by (32)</td>
</tr>
<tr>
<td>$B_j$</td>
<td>derivative of $B_j$ with respect to $e$</td>
</tr>
<tr>
<td>$c$</td>
<td>$\cos i$</td>
</tr>
<tr>
<td>$C_j^N$ (or $C_j$)</td>
<td>$\cos (jv + ku)$ (different meaning in Part 1)</td>
</tr>
<tr>
<td>$d$</td>
<td>shorthand for $k + j - 1$ etc in sections 7.4 and 7.5</td>
</tr>
<tr>
<td>$D_k^N(1)$</td>
<td>inclination function, quoted from Ref 4</td>
</tr>
<tr>
<td>$e$</td>
<td>eccentricity</td>
</tr>
<tr>
<td>$E$</td>
<td>eccentric anomaly (only required in section 8)</td>
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### LIST OF SYMBOLS (continued)

<table>
<thead>
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<th>Description</th>
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<tr>
<td>( E_1(e) )</td>
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</tr>
<tr>
<td>( E_{4J} )</td>
<td>quantity related to ( E_1(e) ), defined by (48)</td>
</tr>
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<td>( f )</td>
<td>( \sin^2 \iota )</td>
</tr>
<tr>
<td>( G )</td>
<td>( \frac{\pi}{4} J_4 (R/p)^4 ) (in section 8.5)</td>
</tr>
<tr>
<td>( h )</td>
<td>angular momentum (but ( 1 - \frac{1}{4} f ) in section 8.3)</td>
</tr>
<tr>
<td>( H )</td>
<td>( \frac{1}{4} J_3 (R/p)^3 ) (in section 8.4)</td>
</tr>
<tr>
<td>( i )</td>
<td>inclination</td>
</tr>
<tr>
<td>( J )</td>
<td>index associated with multiples of ( v )</td>
</tr>
<tr>
<td>( J_k )</td>
<td>zonal harmonic coefficient for the Earth</td>
</tr>
<tr>
<td>( k )</td>
<td>index associated with multiples of ( u )</td>
</tr>
<tr>
<td>( K )</td>
<td>( \frac{1}{4} J_2 (R/p)^2 ) (in section 8.3)</td>
</tr>
<tr>
<td>( l )</td>
<td>index of ( J_k )</td>
</tr>
<tr>
<td>( L )</td>
<td>quantity such that ( L = \dot{M} + \dot{\varphi} = n + \dot{\varphi} )</td>
</tr>
<tr>
<td>( m )</td>
<td>( v - M ) in section 8.5 (and Part 1); otherwise an arbitrary integer</td>
</tr>
<tr>
<td>( M )</td>
<td>mean anomaly</td>
</tr>
<tr>
<td>( n )</td>
<td>mean motion</td>
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</table>
We will require derivatives of the inclination functions. It is evident from (10) that

\[
\frac{d}{dt}[A_k^l(t)] = -\frac{(l-k)(l+k+1)}{2(k+1)} s A_k^{l-1}(t),
\]

from this and (14) it follows that the (partial) derivative of \(A_{lk}\) with respect to \(l\) is given by

\[
A_{lk}' = J_k(n/p)A_{lk} s^{k-1} \left\{ k \cdot A_k^l + \frac{(l-k)(l+k+1)}{2(k+1)} f A_k^{l-1} \right\},
\]

where \(f = s^2\). The quantity in (curly) brackets is the \(D_k^l(1)\) of Ref. 4. We will also require, finally, the particular combinations of \(A_{lk}\) and \(A_{lk}'\) denoted by \(A_{lk}^s\) and \(A_{lk}'^s\), and given by

\[
A_{lk}^s = k s^{l-1} A_{lk} + c^{-1} A_{lk}^s,
\]

the \(s^{-1}\) and \(c^{-1}\) factors do not imply singularities, as they must always cancel via \(A_{lk}\) and \(A_{lk}'\) respectively.

The \(A_k^l(1)\) and \(A_{lk}\) (and hence the \(A_{lk}'\)) may be computed with the aid of recurrence relations. A fixed \(k\) was stipulated in Ref. 4 for the formula

\[
(k + l) A_k^l(1) - (2k - 1) c A_k^l(1) = (l - k - 1) A_k^{l+2}(1),
\]

valid for \(l \geq k + 2\) with the starting values \(A_k^0(1) = 1\) and \(A_k^1(1) = 0\); (21) is even valid for \(l = k + 1\), if an arbitrary (but finite) \(A_k^l(1)\) is assumed. However, it is usually more useful to stipulate a fixed \(k\), the required formula was given by Merson\(^{11}\), being

\[
A_k^l(1) = c A_k^{l+1}(1) - \frac{(l - k - 1)(l + k + 2)}{4(k+1)(k+2)} f A_k^{l+2}(1),
\]

valid for \(l \geq 2\) \(k \geq 0\) with the starters \(A_k^0(1) = 1\) and \(A_k^1(1) = 0\),

(22) is also valid for \(k = l = 1\), with an arbitrary (finite) \(A_k^1(1)\). Either of the two preceding 'pure' three-term recurrence relations, (21) or (22), can be used with just one 'mixed' such relation to generate all the relations connecting the \(A_k^l(1)\), perhaps the simplest 'mixed' relation (with neither \(l\) nor \(k\) fixed).
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$N_{l,r}$, $N_{l,b}$, $N_{l,w}$</td>
<td>number of terms (for given $l$) in $\delta r$, $\delta b$, $\delta w$</td>
</tr>
<tr>
<td>$p$</td>
<td>parameter (semi-latus rectum) of orbit</td>
</tr>
<tr>
<td>$P_2^k()$</td>
<td>Legendre polynomial (of argument supplied)</td>
</tr>
<tr>
<td>$P_2^k()$</td>
<td>Legendre associated function</td>
</tr>
<tr>
<td>$q$</td>
<td>$\sqrt{1 - e^2}$</td>
</tr>
<tr>
<td>$Q_2^k(e)$</td>
<td>normalized eccentricity function quoted from Ref 3</td>
</tr>
<tr>
<td>$r$</td>
<td>radius-vector coordinate of $(r, b, w)$</td>
</tr>
<tr>
<td>$R$</td>
<td>Earth's equatorial radius</td>
</tr>
<tr>
<td>$R_j$ (for $R_{2k}$)</td>
<td>quantity defined by (125) (different in Part 1)</td>
</tr>
<tr>
<td>$s$</td>
<td>$\sin i$</td>
</tr>
<tr>
<td>$S_j^k$ (or $S_j$)</td>
<td>$\sin (jv + ku')$ (different meaning in Part 1)</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T_j$ (for $T_{2k}$)</td>
<td>quantity defined by (137)</td>
</tr>
<tr>
<td>$u$</td>
<td>argument of latitude, $v + \omega$</td>
</tr>
<tr>
<td>$u'$</td>
<td>modifier $u = v + \omega - \frac{j\omega}{1}$</td>
</tr>
<tr>
<td>$U_k$</td>
<td>potential due to $J_k$</td>
</tr>
<tr>
<td>$u_k^k$</td>
<td>component of $U_k$</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS (continued)

\( v \)
true anomaly

\( V \)
orbital speed (used only in section 8)

\( V_j, 1, V_j, 0, V_j, -1 \)
quantities introduced at (149)

\( w \)
longitude-like spherical coordinate of \((r, b, w)\)

\( W_0, W_{i,0}, W_{i,-1} \)
quantities introduced at (153)

\( x, y \)
general unknown quantities (different in Part 1)

\( x^{ij}_n \)
generic Hansen function (of eccentricity)

\( z \)
\(-J_1 R\) (section 8.2)

\( a_k \)
fixed constant, defined by (12)

\( a_k \)
fixed constant, defined by (16)

\( \theta \)
geocentric latitude (declination); \( \phi/(1 + q) \) in section 3

\( \delta \)
symbol for pure short-period perturbation \( (\delta_p \text{ in Part 1}) \)

\( \xi \)
generic orbital element (osculating)

\( \bar{\xi} \)
mean element corresponding to \( \xi \)

\( \bar{\xi} \)
semi-mean element

\( \dot{\xi} \)
rate of change of \( \xi \) due to \( U_{ik} \)

\(-1)^{\frac{1}{a}}\)
LIST OF SYMBOLS (concluded)

\( k \)
index related to \( k \), but of opposite parity

\( \lambda \)
\(-J_1 R/\rho \) (section 8.2)

\( \mu \)
Earth's gravitational constant

\( \rho \)
quantity such that \( \dot{\rho} = \dot{\sigma} + q\dot{\psi} \)

\( \sigma \)
modified mean anomaly at epoch

\( \sum \)
summation (different use of \( \Sigma \) in Part 1)

\( u_k \)
1 if \( k = 0 \), 2 if \( k > 0 \)

\( \psi \)
quantity such that \( \dot{\psi} = \dot{\omega} + c\dot{\Omega} \)

\( \omega \)
argument of perigee

\( \omega' \)
\( \omega - \frac{1}{2} \dot{\omega} \)

\( \Omega \)
right ascension of the node

\( f \)
\( \int n \, dt \)
# REFERENCES

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<td>R.H. Gooding</td>
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### Title
Satellite motion in an axi-symmetric gravitational field

### Part 2: Perturbations due to an arbitrary $J_2$

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### Descriptors (Keywords)
- Astrodynamics
- Celestial mechanics
- Geopotential
- Gravitational effects
- Orbital elements
- Orbit calculations
- Satellite perturbations
- Zonal harmonics

### Report continues the presentation of the untruncated orbital theory begun in Technical Report 88068. The effects of the general zonal harmonics, $J_l$, are now covered, the main results being a trio of formulae for perturbations in the spherical-polar coordinates introduced in the previous paper. The formulae are only first-order in $J_2$, but, in conjunction with the second-order results for $J_2$ published in Part 1, the complete set of formulae may be regarded as constituting a second-order theory, the Earth's $J_2$ being much larger than $J_2$ for $l > 2$.

The mean elements of the theory are defined in such a way that, for each $J_l$, the coordinate-perturbation formulae have their simplest possible form, with no occurrence of zero denominators. The general formulae are used in a rederivation of the results for $J_2$, given in Part 1, and in a derivation of results for $J_4$.

Numerical comparisons with reference orbits are held over to a later report (Part 3).