A Diffusion on a Fractal State Space

by

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A Diffusion Defined on A Fractal State Space

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Abstract: We define a fractal in the plane known as the Vicsek Snowflake by constructing a skeletal lattice graph and then rescaling spatial dimensions to give a sequence of lattices that converges to a fractal. By defining a simple random walk on the skeletal lattice and then rescaling both time and space, we define a sequence of random walks on the approximating lattices that converge weakly to a limiting process on the snowflake. We show that this limit has continuous sample paths and the strong Markov property, and that it is the unique diffusion limit of random walk on the snowflake in a natural sense. We show that this diffusion has a scaling property reminiscent of Brownian motion, and we introduce a coupling argument to show that the diffusion has transition densities with respect to Hausdorff measure on the snowflake.

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1. Introduction:

Construct a figure in the unit square by the following recursive procedure. Let $G_0$ denote the unit square. Construct $G_1$ by deleting from $G_0$ four squares, each with edge length $\frac{1}{3}$, centered along the four edges of $G_0$. $G_1$ will consist of five squares with edge $\frac{1}{3}$ whose corners overlap. At stage $n$, $G_n$ will consist of $5^n$ squares, each with edge length $3^{-n}$. To construct $G_{n+1}$ from $G_n$, take each square $S$ composing $G_n$, and delete the four square centered along the edges of $S$ with edges of length $3^{-n-1}$. $G_{n+1}$ then consists of $5^{n+1}$ squares with edges of length $3^{-n-1}$.
Take $G = \cap_{n=1}^{\infty} G_n$. Some easy topology shows that $G$ is a closed connected set, with Lebesgue measure 0. In fact, it is not hard to show that $G$ has finite Hausdorff $\log_3 5$-dimensional measure. In the spirit of Mandelbrot, $G$ is a fractal with starter polygon $G_1$. Extensive treatments of such fractal sets have been given by various authors. (See, for example Hutchinson[12] or Barnsley and Demko[4]).

A number of authors have treated the problem of constructing diffusions on nested fractals. Particular attention has been paid to diffusions on the Sierpinski gasket, a fractal constructed from a unit equilateral triangle by successively deleting "middle" triangles. Goldstein[10] and Kusuoka[13] constructed a Brownian motion on the Sierpinski gasket, using a decimation-invariance property. Barlow and Perkins[3] have studied this Brownian motion comprehensively. Brownian motion on the Sierpinski gasket is broadly similar to the natural diffusion on the Vicsek snowflake, and the results of these authors are generally similar to those in the present work. I fully acknowledge the priority of their results. More recently, Lindstrom[14] has constructed a Brownian motion on any fractal set satisfying a general set of nesting axioms from a sequence of random walks, provided that the distribution of the random walk satisfies a non-degeneracy condition.

The first objective of this paper is to construct a diffusion on the Vicsek snowflake, starting from an non-degenerate random walk model. In an important respect, the problem of defining diffusions on the snowflake is more complicated than defining diffusions on the Sierpinski gasket. On the snowflake, one can define a variety of random walk models that are symmetric under the natural symmetries of the square. A natural question is whether one can construct a diffusion for any such model. Another is whether the diffusion on the fractal is unique such diffusion are unique. The snowflake seems to be the simplest nested fractal where such questions arise. For the snowflake, the answer is that if the random walk is not degenerate then the a unique diffusion limit exists independent of the underlying random walk model. The corresponding problem for general nested fractals remains unsolved at the time of this writing.
2. Constructing the Diffusion:

Consider the following system of transformations:

\[ M_1 : x \rightarrow 3^{-1} \cdot x \quad M_4 : x \rightarrow (3^{-1} \cdot x) + (2, 0) \]

\[ M_2 : x \rightarrow 3^{-1} \cdot x + (1, 1) \quad M_5 : x \rightarrow (3^{-1} \cdot x) + (0, 2) \]

\[ M_3 : x \rightarrow 3^{-1} \cdot x + (2, 2) \]

By inspection, \( M_1, \ldots, M_5 \) are strict contractions, with fixed points \((0, 0)\), \((1, 1)\), \((3, 3)\), \((3, 0)\) and \((0, 3)\), respectively. For bounded subsets \( A \) of \( \mathbb{R}^2 \), define \( M(A) = \bigcup_{i=1}^{5} M_i(A) \). It is well-known to geometers that the transformation \( M \) has a unique compact invariant set, whose Hausdorff dimension may easily be computed as \( \log_3 5 \). (See, for example, Barnsley and Demko [4], Dubins and Freedman [6] or Hutchinson [12].) We will call this set the \textit{bounded Vicsek snowflake}, and denote it by \( \Gamma_b \).

For future reference, we establish the following definition. Let \( M_1, \ldots, M_n \) be some sequence of the transformations \( M_1, \ldots, M_5 \). Let \( S = M_1 \circ \ldots \circ M_n(\Gamma_b) \). We will call \( S \) a \textit{square} of \( \Gamma_b \). We will also need an unbounded version of our state space. Let \( \Gamma = \bigcup_{n=0}^{\infty} \Gamma_n \). \( \Gamma \) also has Hausdorff dimension \( \log_3 5 \), and has the property that \( \frac{1}{3} \Gamma = \Gamma \). We will call \( \Gamma \) the \textit{unbounded Vicsek snowflake}.

We wish to construct a diffusion process on \( \Gamma \) and study its basic properties. We shall do this by defining random walks on a sequence of lattices that approximate \( \Gamma \), which we now construct.

In the unit square, let \( U \) denote the complete graph on the corners of \([0, 1]^2\). Let \( V(U) \) and \( E(U) \) denote the vertices and edges of \( U \), respectively.

Define a new graph \( U_0 \) with vertex set \( V(U_0) \) and edge set \( E(U_0) \) by taking

\[ V(U_0) = V(U) \cup [V(U) + (1, 1)] \cup [V(U) + (0, 2)] \cup [V(U) + (2, 0)] \cup [V(U) + (2, 2)] \]

\[ E(U_0) = E(U) \cup [E(U) + (1, 1)] \cup [E(U) + (0, 2)] \cup [E(U) + (2, 0)] \cup [E(U) + (2, 2)] \]

where the arithmetic is done componentwise. We call \( U_0 \) the \textit{unit snowflake lattice}. Note that \( U_0 \) lies in the square \([0, 3]^2\); we will call the points \((0, 0), (0, 3), (3, 3)\), and \((0, 3)\) the \textit{outer corners} of \( U_0 \).
Inductively, we construct a sequence of graphs, using same procedure that yielded $U_0$ from $U$. That is, if $n > 0$, let

$$
\mathcal{V}(U_n) = \mathcal{V}(U_{n-1}) \cup [\mathcal{V}(U_{n-1}) + (1, 1)] \cup [\mathcal{V}(U_{n-1}) + (0, 2)] \cup [\mathcal{V}(U_{n-1}) + (2, 0)]
$$

$$
\mathcal{E}(U_n) = \mathcal{E}(U_{n-1}) \cup [\mathcal{E}(U_{n-1}) + (1, 1)] \cup [\mathcal{E}(U_{n-1}) + (0, 2)] \cup [\mathcal{E}(U_{n-1}) + (2, 0)]
$$

As with $U_0$, say that $(0, 0), (0, 3^n+1), (3^n+1, 3^n+1), \text{ and } (0, 3^n+1)$ are the outer corners of $U_n$.

Let $G = G_1 = \bigcup_{n=0}^{\infty} U_n$. Then $G$ is an infinite graph, with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)$, which we shall call the unbounded snowflake lattice, or, simply, the snowflake lattice. Let $0$ denote the point $(0,0)$, and let $1$ denote $(3,3)$.

We mention two key properties of $G_1$. First, if we let $\Gamma_\infty = \bigcup_{n=1}^{\infty} 3^{-n} \cdot \mathcal{V}(G)$, then $\Gamma_\infty$ is dense in $\Gamma$. Thus, we may have some hope that a suitably scaled sequence of random walks will converge to some process on $\Gamma_\infty$. Second, $\mathcal{V}(G) \subset \mathcal{V}(G)$, and $G \setminus 3 \cdot G$ contains no infinite connected component. We call this the "branching", "nesting" or self-similarity property of $G$, property of random walks on the lattices $G_n$.

Let $x$ be a vertex of $G$, and suppose our particle is at $x$ at time $n$. Let $N_x$ be the number of points adjacent to $x$ in $G$. Suppose our random walk is at $x$ at time $n$. Then at time $n+1$, we choose a vertex adjacent to $y$ according to the following distribution:

If $N_x = 3$:

$$
P[X_{n+1} = y] = p \quad \text{if } x \text{ and } y \text{ are diagonally adjacent}
$$

$$
P[X_{n+1} = y] = (1 - p)/2 \quad \text{if } x \text{ and } y \text{ are vertically or horizontally adjacent}
$$

If $N_x = 6$:
\[ P[X_{n+1} = y] = \frac{p}{2} \quad \text{if } x \text{ and } y \text{ are diagonally adjacent} \]
\[ P[X_{n+1} = y] = \frac{(1-p)}{4} \quad \text{if } x \text{ and } y \text{ are vertically or horizontally adjacent} \]

Here, \(0 \leq p < 1\) is a fixed but arbitrary parameter. This defines a random walk on \(G\), which we denote by \(X^p\). Thus, if \(X_0^p = (0,0)\) then \(P[X_1^p = (1,1)] = p\) while \(P[X_1^p = (1,0)] = \frac{(1-p)}{2}\). If \(X_1^p = (1,1)\), then \(P[X_2^p = (0,0)] = \frac{p}{2}\) and \(P[X_2^p = (1,0)] = \frac{(1-p)}{4} = P[X_2^p = (2,1)]\).

We begin by studying the special case where \(p = \frac{(1-p)}{2} = \frac{1}{3}\), and \(X_0^p = 0\). This will define a simple random walk on the graph \(G\), starting from 0. We call this discrete-time Markov chain, which we will call the random walk on the snowflake lattice. For convenience, we will write \(X^{1/3} = X\). o

Let \(T_1^i\) and \(T_2^j\) be the sequences of times between visits by \(X_n\) to distinct points of \(3 \cdot G\) and \(3^2 \cdot G\), respectively. Since \(3^2 \cdot G \subset 3 \cdot G\), \(T_2^j = \sum_{a(j)} T_1^k\). By the nesting property of the lattice, the distribution of \(b(j) - a(j)\) is the same as the distribution of \(T_1^k\), and the Markov property of \(X_n\) shows that \(b(j) - a(j)\) and \(T_1^k, k = a(j), \ldots, b(j)\) are independent, are independent random variables, equal in distribution to \(T\). Thus, for each \(j\), \(T_j^2\) has the distribution of the second generation of a branching process with offspring distribution equal to that of \(T\).

Similarly, let \(T_1^n\) be the times between visits to distinct points of \(3^n \cdot G\). A similar argument shows that for each \(i\), \(T_1^n\) has the distribution of the \(n\)th generation of a branching process, again with offspring distribution equal to that of \(T\).

Let \(f(u)\) be the generating function of \(T\). By direct calculation, we can show that
\[
f(u) = \frac{\frac{u^3}{(3-2u)(12-12u+u^2)}}{2.4}
\]

(See Section 4 for the details of the computation.) Differentiating \(f\) shows that \(E(T) = 15\) and \(\text{Var}(T) = 114\), so the branching process \(\{T^n\}\) is obviously supercritical. Since \(T\) is always strictly greater than 0, \(\{T^n\}\) has
extinction probability 0. Since this branching property of the hitting times of $X_n$ plays a key role in the remainder of this section, we will review some standard theory of branching processes.

**Theorem 2.1.** Let $Z_n$ be a branching process, with $Z_0 = 1$ and let $f$ be the generating function of the offspring distribution. Suppose that $1 < f'(1-) = m < \infty$ and $f''(1-) < \infty$. Let $W_n = Z_n m^{-n}$. Then there exists a random variable $W$ with $EW = 1$ such that $W_n \to W$ a.s. and in $L^2$ $P[W = 0] = P[Z_n = 0$ for some $n]$. If $\phi(u) = e^{-uw}$, then satisfies Abel's functional equation $\phi(u) = f(\phi(u/m))$.

**Proof:** This is Theorems 1 and 2, and equation (5) in Athreya and Ney [2].

In particular, $W_n \to W$ in distribution.

The theorem implies that $15^{-n}T^n$ converges in distribution to some random variable $W$, with $EW = 1$. As $T^n$ can never be 0, $W$ is strictly positive almost surely, and $\phi(\lambda) = Ee^{-\lambda W}$ satisfies $\phi(\lambda) = f(\phi(15^{-1}\lambda))$.

For $m = 0, 1, \ldots$, define a stochastic process on $\Gamma$ by $Y_m(t) = 3^{-m}X([15^m t])$. ([x] denotes the greatest integer less than or equal to $x$). Observe that for each $n$, $Y_n(t)$ is a random walk on $3^{-n}G$. Let $D[0, \infty]$ be the set of functions $\omega : \mathbb{R}^+ \to \Gamma$ that are right continuous and have left hand limits for all $t$.

**Theorem 2.2.** The sequence of processes $\{Y_n(t)\}_{t \geq 0}$ is tight in $D[0, \infty]$. If $Y_n(t)$ is a subsequence of $\{Y_n(t)\}$ converging weakly to a process $Y_t$, then $Y_t$ has continuous sample paths.

The theorem follows from an estimate of the moments of the displacements $\|Y_n(t) - Y_n(s)\|$, which we state as a lemma.

**Lemma 2.3.** For $n = 1, 2, \ldots, 0 \leq s < t < \infty, \gamma > 0$,

$$E\|Y_n(t) - Y_n(s)\|^\gamma \leq 2\sqrt{2} \cdot [3^{-n\gamma} + C \cdot (t - s)\gamma]$$

(2.5)
where $\rho = \log_{15} 3$ and $C$ is a constant independent of $n$.

Proof: Let $Q = \{ q \in Q : q = p \cdot 15^{-n}, p, q \in \mathbb{Z} \}$. Since $Y_n(t)$ jumps only at points in $Q$, it will suffice to estimate $E[|Y_n(q) - Y_n(r)|]$ for arbitrary $q, r \in Q$. We make the following displacement estimate for random walk $\{X_n\}$ on the snowflake lattice:

$$P[\|X_n - X_m\| > 2\sqrt{2} \cdot 3^k] \leq P[T_k < n - m] \quad (2.6)$$

To establish this estimate, we observe that $2\sqrt{2} \cdot 3^k$ is the diameter of two diagonally adjoining squares in the $3^k \cdot G$. If $\|X_n - X_m\| > 2\sqrt{2} \cdot 3^k$, then between epochs $m$ and $n$, $X_k$ must visit two distinct points of $3^k \cdot G$.

To use estimate (6), we write

$$E[|Y_n(q) - Y_n(r)|]$$

$$\leq (2\sqrt{2})^\gamma \cdot 3^{-\gamma n} P[\|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{-n}]$$

$$+ (2\sqrt{2})^\gamma \cdot \sum_{i=-n}^{\infty} 3^{3\gamma n} P[\|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{i+1}] \quad (7)$$

$$\leq (2\sqrt{2})^\gamma \cdot 3^{-\gamma n} P[\|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{-n}]$$

$$+ (2\sqrt{2})^\gamma \cdot (1 - 3^{-\gamma n}) \cdot 3^{3\gamma n} P[\|Y_n(q) - Y_n(r)\| > 2\sqrt{2} \cdot 3^i] \quad (8)$$

$$\leq (2\sqrt{2})^\gamma \cdot 3^{-\gamma n} P[\|X(15^n \cdot q) - X(15^n \cdot r)\| \leq 2\sqrt{2}]$$

$$+ (2\sqrt{2})^\gamma \cdot (1 - 3^{-\gamma n}) \cdot \sum_{k=0}^{\infty} 3^{(k-\gamma n)} P[\|X(15^n \cdot q) - X(15^n \cdot r)\| > 2\sqrt{2} \cdot 3^k] \quad (9)$$

$$\leq (2\sqrt{2})^\gamma \cdot \left[ 3^{-\gamma n} + (1 - 3^{-\gamma n}) \cdot \sum_{k=0}^{\infty} 3^{(k-\gamma n)} P[15^k \cdot T_k \leq 15^{-k+n} \cdot (q - r)] \right] \quad (10)$$

Inequality (8) comes from substituting

$$P[2\sqrt{2} \cdot 3^i < \|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{i+1}] =$$

$$P[2\sqrt{2} \cdot 3^i < \|Y_n(q) - Y_n(r)\|] - P[2\sqrt{2} \cdot 3^{i+1} < \|Y_n(q) - Y_n(r)\|]$$

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into the estimate and rearranging the terms. (9) follows from the definition of \( Y_n(q) \), and (10) is from estimate (6).

To estimate \( P \left[ 15^k \cdot T_k \leq 15^{-k} \cdot 15^n \cdot (q - r) \right] \) we first estimate the Laplace transforms of \( \{ T_k \} \). Let \( \phi_k \) and \( \phi \) be the Laplace transforms of \( T_k \) and \( W \), respectively. As \( T_k \) has the distributions of a branching process and \( 15^{-k} T_k \rightarrow W \), it is not hard to show that \( \phi_k \uparrow \phi \). So, it will suffice to estimate \( \phi \).

To estimate \( \phi(15^k \cdot (q - r)^{-1}) \), let \( h(u) = -\log(\phi(u)) \) be the cumulant generating function of \( W \). \( h \) satisfies the functional equation \( h(u) = \log(f(\exp(h(15^{-1} \cdot u)))) \), where \( f \) is the generating function of \( T = T^1 \).

Let \( 1 < s < 15 \). Since \( h \) is non-decreasing, we have,

\[
\frac{h(su)}{h(u)} \leq \frac{h(15u)}{h(u)} = h(u)^{-1} \log(f(\exp(h(15^{-1} \cdot u))))
\]

\[
= \left( \frac{1}{\log(\phi(u))} \right) \cdot \log \left( \frac{\phi(u)^3}{(3 - 2\phi(u))(12 - 12\phi(u) + \phi(u)^2)} \right) \tag{2.11}
\]

\[
= 3 - \frac{\log((3 - 2\phi(u))(12 - 12\phi(u) + \phi(u)^2))}{\log(\phi(u))}
\]

The second term goes to 0 as \( u \rightarrow \infty \). Since \( h \) is monotone increasing,

\[
\limsup_{u \rightarrow \infty} h(su)/h(u) \leq 3, \quad 1 \leq s \leq 15. \tag{2.12}
\]

This shows that \( h \) is a function of dominated variation, which implies the existence of constants \( C_1 \) and \( C_2 \), such that \( C_1 u^\rho \leq h(u) \leq C_2 u^\rho \), where \( \rho = \log_{15} 3 \). (See Feller[8], de Haan and Stadtmüller[11]). Restating this in terms of \( \phi \) shows that \( \phi(t) \leq \exp(-C_1 t^\rho) \).

We now apply a standard technical result on Laplace transforms.

**Lemma 2.4.** Let \( U \) be the distribution function of a random variable, let \( \psi(u) \) be its Laplace transform, and let \( a > 0 \). Then, for any \( t > 0 \), \( U(t) \leq e^{-a} \cdot \psi(t^{-1}) \).

**Proof:** This is proved in the Corollary to Theorem XIII.5.1 in Feller [9].
Applying these two lemmas to our series gives

$$
\sum_{k=0}^{\infty} 3^{(k-n)} \gamma_1 \left[ 15^k \cdot T_k \leq 15^{-n} \cdot 15^n \cdot (q-r) \right]
$$

$$
\leq e^{1/15} \cdot \sum_{k=0}^{\infty} 3^{(k-n)} \gamma_1 \phi(15^k \cdot (q-r)^{-1})
$$

(2.13)

$$
\leq e^{1/15} \cdot \sum_{k=-\infty}^{\infty} 3^{(k-n)} \gamma_1 \phi(15^k \cdot (q-r)^{-1})
$$

(2.14)

$$
\leq e^{1/15} \cdot \left[ \sum_{m=1}^{\infty} 3^{-m\gamma} + \sum_{m=0}^{\infty} 3^{m\gamma} \exp (-15^{m\rho} \cdot C_1 (q-r)^{-\rho}) \right]
$$

(2.15)

$$
< \infty
$$

To complete the proof of the estimate, let $j(t) = [\log_{15} t] + 1$ for $t > 0$ and let $F(t) = e^{1/15} \cdot j(t) \cdot \sum_{t=-\infty}^{\infty} 3^t \phi(15^t)$. Substitute inequality (2.14) into the left-hand term in (2.10) and let $m = k - n$, to get

$$
E\|Y_n(q) - Y_n(r)\|^* \leq (2\sqrt{2})^7 \cdot \left[ 3^{-\gamma n} + (1 - 3^{-\gamma}) \cdot e^{1/15} \cdot \sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m \cdot (q-r)^{-1}) \right]
$$

(2.16)

$$
\leq (2\sqrt{2})^7 \cdot \left[ 3^{-\gamma n} + 3^{j(q-r)}(1 - 3^{-\gamma}) \cdot e^{1/15} \cdot \sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m) \right]
$$

(2.17)

From (15), $\sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m) = C < \infty$. Since $j(q-r) \leq \log_{15} (q-r) + 1$, $3^{j(q-r)} \cdot e^{1/15} \leq (q-r)^{\rho n}$. Substituting these expressions into (2.17) gives

$$
E\|Y_n(q) - Y_n(r)\|^* \leq \left[ 3^{-\gamma n} + (q-r)^{\rho n} \cdot (1 - 3^{-\gamma}) \cdot \sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m) \right]
$$

(2.18)

which proves the lemma.

Using our estimate, we can deduce both weak convergence of $Y_n(t)$ and sample continuity of $Y_t$. Weak convergence of $\{Y_n(t)\}$ follows from the standard result on convergence of stochastic processes.

**Theorem 2.5.** Suppose that

$$(X_n(t_1), \ldots, X_n(t_k)) \rightarrow_D (X(t_1), \ldots, X(t_k))$$

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holds whenever \( t_1 < \ldots < t_k \) are points where \( X(t) \) is almost surely continuous, and that

\[
E \left[ \|X_n(t) - X_n(t_1)\|^\gamma \|X_n(t_2) - X_n(t)\|^\gamma \right] \leq (F(t_2) - F(t_1))^{2\alpha}
\]

for \( t_1 \leq t \leq t_2, n \geq 1, \gamma > 0, \alpha > \frac{1}{2}, \) and \( F \) a nondecreasing continuous function. Then \( X_n \rightarrow_D X \).

**Proof:** This is Theorem 15.6 in Billingsley [5].

**Proof of the main theorem:** To show that there exists a weakly convergent sequence \( \{Y_n(t)\} \), it suffices to show that \( \{Y_n\} \) is tight. Let \( s \leq u \leq t \) and consider

\[
E \left[ \|Y_n(u) - Y_n(s)\|^{\gamma} \|Y_n(t) - Y_n(u)\|^{\gamma} \right]
\]

(2.19)

If \( |s - t| < 15^{-n} \) then

\[
E \left[ \|Y_n(u) - Y_n(s)\|^{\gamma} \|Y_n(t) - Y_n(u)\|^{\gamma} \right] = 0
\]

(2.20)

for any \( s \leq u \leq t \), because \( Y_n(t) \) jumps only at integral multiples of \( 15^{-n} \), and if \( |s - t| < 15^{-n} \) then there can be at most one such multiple in \([s, t]\). So, suppose that \( |s - t| \geq 15^{-n} \). Apply Hölder's inequality and the monotonicity of \( E\|Y_n(u) - Y_n(s)\|^{\gamma} \) to see that

\[
E \left[ \|Y_n(u) - Y_n(s)\|^{\gamma} \|Y_n(t) - Y_n(u)\|^{\gamma} \right] \leq E\|Y_n(t) - Y_n(s)\|^{2\gamma}
\]

\[
\leq (2\sqrt{2})^{2\gamma} \left( 3^{-n2\gamma} + (1 - 3^{-2\gamma}) \cdot (t - s)^{2\gamma} \right)
\]

(2.21)

But, since \( |s - t| \geq 15^{-n}, 3^{-n2\gamma} \leq (s - t)^{2\gamma}. \) Substitute this into (2.21) to get

\[
E \left[ \|Y_n(u) - Y_n(s)\|^{\gamma} \|Y_n(t) - Y_n(u)\|^{\gamma} \right] \leq 2 \cdot (2\sqrt{2})^{2\gamma} \cdot (t - s)^{2\gamma}
\]

(2.22)

This suffices to prove tightness.
Choose a subsequence $n'$ so that $j \ Y_{n'}(t) \rightarrow D Y_t$. To show that the sample paths of $Y_t$ are continuous with probability 1, let $n \rightarrow \infty$ in estimate (2.6), giving

$$E\|Y_1 - Y_s\| \leq C \cdot (t-s)^\gamma$$

(2.23)

Kolmogorov's criterion for sample path continuity applies, which completes the proof of our main theorem.

Although we have written our proofs for the special case where $Y_0 = 0$, the proof will also work for $Y_0 = x$ for an arbitrary $x \in \Gamma_\infty$, with a minor modification of the sequences $Y_n(t)$.

For $n = 1, 2, \ldots$ and $m < n$ let $S_{n,k}^m$ be the time of the $k$th visit of the random walk $Y_n(t)$ to $G_m$ and let $T_{n,k}^m = S_{n,m,k}^m - S_{n,k-1}^m$ be the $k$th interarrival time for $S_{n,k}^m$. Previously, we observed that $Y_n(S_{n,k}^m) = 1, 2, \ldots$ is a random walk on $G_m$, for all $m < n$. We can extend this property to $Y_t$.

**Proposition 2.6.** Let $Y_n' \rightarrow Y$. $T_{n',m,k}$ converges in distribution to a random variable $T_{m,k}$ for every $m > 0$; furthermore, the sequence $\{T_{m,k}\}_{k=1}^\infty$ is independent and identically distributed for all $m$. $Y_n'(S_{n',m,k})$ converges weakly to $Y(S_{m,k})$ where $\{Y(S_{m,k})\}_{k=1}^\infty$ is a random walk on the lattice $G_m$.

**Proof:** We have already shown that $T_{n',k}^m$ is distributed as the $n' - m$ generation of a branching process with offspring generating function $f(u)$. Applying the theorem about branching processes cited at the beginning of this section shows that $15 - n' - m T_{n',m,k} \rightarrow D 15 - m W$ also. For each $n'$, and $m \{T_{n',k}^m\}_{k=1}^\infty$ is a sequence of independent random variables. Thus $\{T_{k}^m\}_{k=1}^\infty$ is also an independent sequence.

For any $n'$, the sequence $Y_n'(S_{n',k}^m)$ is a random walk on $G_m$. As for $\{Y(S_{m,k})\}_{k=1}^\infty$, $Y(S_{k}^m) \in G_m, k = 1, 2, \ldots$ with probability 1. It is straightforward, although tedious, to show that for sites $x_1, \ldots, x_\nu \in G_m$, $\{\omega : Y(S_{k}^m, \omega) = x_i, i = 1, \ldots, \nu\}$ is a continuity set for the distribution of $Y_t$. The second statement of the proposition follows.
A consequence of this proposition is

**Corollary 2.7.** The sample paths of $Y_t$ are uniformly continuous in probability.

**Proof:** The corollary follows from Skorokhod's lemma, upon observing that for $u > v$

$$P \left[ \|Y_u - Y_v\| \leq 2\sqrt{2} \cdot 3^{-n} \right| Y_s, 0 \leq s < u \right] \geq P \left[ T_{n,k(u)} < u-v \right| Y_s, 0 \leq s < u \right] \geq 1 - 15^{-n} / (u-v) \quad (2.24)$$

where $k(v)$ is the smallest $k$ such that $S_{n,k} \geq v$. Inequality (24) follows from the fundamental estimate, (25) uses the fact that $\{T_{n,k}\}$ is an i. i. d. sequence of random variables, and (26) is Markov's inequality.

To show that $Y_t$ is a diffusion, we must show that it has the strong Markov property. We consider those stopping times of $Y_t$ that are also stopping times of the embedded random walks $Y(S^T)$, in an appropriate sense. Let $C$ be the class of stopping times $T$ of $Y_t$ such that

i. For some $n$, $Y_T \in G_n$ with probability 1.

ii. If $m \geq n$ and $T = \sum_{k=1}^{N_m} T_{m,k}$, then $N_m$ is a stopping time for the random walk $Y(S_{m,k})$.

Clearly $S_{n,k} \in C$ for any $n$ and $k$, so the class $C$ is not vacuous.

**Proposition 2.8.** Let $T \in C$. Then $Y_t$ satisfies the strong Markov property with respect to $T$. That is for any bounded, measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, $g : \mathbb{R}^k \to \mathbb{R}$, and $s_1, \ldots, s_d, t_1, \ldots, t_k$,

$$E \left[ g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) f(Y(t_1 + T), \ldots, Y(t_k + T)) \right] = E \left[ g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) E_f^T[f(Y(t_1), \ldots, Y(t_k))] \right] \quad (2.27)$$

**Proof:** By the monotone class theorem, it suffices to prove the results for $g(x_1, \ldots, x_d)$ and $f(y_1, \ldots, y_k)$.

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bounded and continuous. For \( n = 1, 2, \ldots \) let

\[
Y_{n,t} = \sum_i Y(S_{n,i}) I[S_{n,i} \leq t < S_{n,i+1}]
\]  

\[(2.28)\]

Obviously, \( \sup_t |Y_t - Y_{n,t}| \leq \sqrt{2} \cdot 3^{-n} \). Thus, \( Y_{n,t} \to Y_t \) uniformly with probability 1, as \( n \to \infty \). By the strong Markov property of random walks on \( G_n \) and the independence of \( T_{n,k} \),

\[
E[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) f(Y_n(T + t_1), \ldots, Y_n(T + t_k))]
\]  

\[(2.28)\]

Since \( T \in G_1 \) almost surely,

\[
E[f(Y_n(T + t_1), \ldots, Y_n(T + t_k))|Y_n,T] = \sum_p E^p[f(Y_n(t_1), \ldots, Y_n(t_k)) \cdot P[Y_n,T = x_p]].
\]

\[(2.29)\]

The distribution of \( Y_{n,T} \) is fixed, so, \( x_p \) and \( P[Y_{n,T} = x_p] \) are fixed as \( n \) varies. Since \( f \) and \( g \) are bounded and continuous,

\[
g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) \to g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) \quad \text{a.s.}
\]

\[(2.29)\]

\[
f(Y_n(T + t_1), \ldots, Y_n(T + t_k)) \to f(Y(T + t_1), \ldots, Y(T + t_k)) \quad \text{a.s.}
\]

\[(2.30)\]

\[
f(Y_n(t_1), \ldots, Y_n(t_k)) \to f(Y(t_1), \ldots, Y(t_k)) \quad \text{a.s.}
\]

\[(2.31)\]

As \( n \to \infty \), the dominated convergence theorem gives

\[
E[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) f(Y_n(t_1 + T), \ldots, Y_n(t_k + T))]
\]  

\[
- E[g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) f(Y(t_1 + T), \ldots, Y(t_k + T))]
\]  

\[(2.32)\]

and

\[
E[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) E^Y_T[f(Y_n(t_1), \ldots, Y_n(t_k))]]
\]  

\[
- E[g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) E^Y_T[f(Y(t_1), \ldots, Y(t_k))]]
\]

\[(2.33)\]

which proves the proposition.

We next use this limited form of the strong Markov property to show that the laws \( P^x \), \( x \in \Gamma_\infty \) are uniformly weakly continuous in \( x \). We do this by studying escape times, the time required for the process to reach \( G_k \),
starting from $G_n$ for $n > k$. Clearly, escape times are stopping times in the class $\mathcal{C}$, so the limited Markov property in the preceding proposition applies to them. We begin by proving a lemma for Markov chains.

**Lemma 2.9.** Let $P$ be an $N \times N$ stochastic matrix. Let $Q$ be the space of measures on $\{1, \ldots, N\}$, with the total variation norm. Then, the transformation $\mu \mapsto \mu P$ is a strict contraction on $Q$ if and only if no rows of $P$ are mutually singular.

**Proof:** Let $\lambda$ and $\mu$ be two distinct measures in $Q$. Without loss of generality, suppose that $\lambda_j - \mu_j > 0, j = 1, \ldots, M, \lambda_j - \mu_j \leq 0, j = M+1, \ldots, N$. Let $\nu = \sum_{j=1}^{M}(\lambda_j - \mu_j) = \sum_{j=M+1}^{N}(\mu_j - \lambda_j)$. As $\mu$ and $\lambda$ are distinct, $\nu > 0$. Let $\nu \alpha_j = (\lambda_j - \mu_j), j = 1, \ldots, M$ and $\nu \beta_j = \nu^{-1}(\mu_j - \lambda_j), j = M+1, \ldots, N$. Then,

$$d(\lambda P, \mu P) = \frac{1}{2} \sum_i \left| \sum_j (\lambda_j - \mu_j)P_{j,i} \right|$$

$$= \frac{1}{2} \sum_i \left| \sum_{j=1}^{M}(\lambda_j - \mu_j)P_{j,i} - \sum_{k=M+1}^{N}(\mu_k - \lambda_k)P_{k,i} \right|$$

$$= \nu \frac{1}{2} \sum_i \left| \sum_{j=1}^{M} \alpha_j P_{j,i} - \sum_{k=M+1}^{N} \beta_k P_{k,i} \right|$$

$$\leq \nu \sum_{j=1}^{M} \sum_{k=M+1}^{N} \alpha_j \beta_k \sum_i \frac{1}{2} |P_{j,i} - P_{k,i}|$$

$$\leq \nu \max_{j,k} d(P_j, P_k)$$

Inequality (2.37) above follows because $f(x, y) = |x - y|$ is convex in both $x$ and $y$; the other relationships are straightforward. If $\max_{j,k} d(P_j, P_k) < 1, d(\mu P, \lambda P) < \nu = d(\mu, \lambda)$. If $d(P_i, P_j) = 1$ for some $i$ and $j$, then take $\mu = \delta_i$, $\lambda = \delta_j$ to get $d(\mu, \lambda) = 1 = d(\mu P, \lambda P)$.
Let \( x \in G_j \). Let \( Y_t^x \) denote the continuous process constructed at the beginning of this section, starting from \( x \). For \( k < j \), let \( T_k = \inf \{ t : Y_t^x \in G_k \} \). Then \( T_k \) is a stopping time such that \( Y(x, T_k) \in G_k \) almost surely. By the strong Markov property of the embedded Markov chains \( Y(T_k, \cdot) \{ Y(x, T_k), k < j \} \) form a discrete time Markov chain.

With probability 1, \( Y(x, T_k) \) will be one of the four corners of the square \( S_k \) on level \( k \) enclosing \( x \). Number these four corners 1, 2, 3 and 4, starting with 1 at the Northeast corner and proceeding clockwise around the square. If \( Y(x, T_k) = i, i = 1, 2, 3, 4 \), then the transition probabilities \( P[Y(x, T_{k-1}) = j | Y(x, T_k) = i] \) will depend on which of the five level-\( k \) squares of \( S_{k-1} \) \( x \) belongs to. Number the four outside squares \( I, II, III, \) and \( IV, \) starting in the Northeast corner and numbering clockwise around the outer squares. Let \( V \) denote the center square of \( S_{k-1} \). By using the generating functions given in display number, it is not hard to calculate:

\[
\begin{align*}
p^{(I)} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1/12 & 1/12 & 1/12 \\ 1/2 & 1/6 & 1/6 & 1/6 \\ 3/4 & 1/12 & 1/12 & 1/12 \end{bmatrix}; & p^{(II)} &= \begin{bmatrix} 1/12 & 3/4 & 1/12 & 1/12 \\ 0 & 1 & 0 & 0 \\ 1/12 & 3/4 & 1/12 & 1/12 \\ 0 & 1/6 & 1/6 & 1/6 \end{bmatrix} \\
p^{(III)} &= \begin{bmatrix} 1/6 & 1/6 & 1/2 & 1/6 \\ 1/12 & 1/12 & 3/4 & 1/12 \\ 0 & 0 & 1 & 0 \\ 1/12 & 1/12 & 3/4 & 1/12 \end{bmatrix}; & p^{(IV)} &= \begin{bmatrix} 1/12 & 1/12 & 1/12 & 3/4 \\ 1/6 & 1/6 & 1/6 & 1/2 \\ 1/12 & 1/12 & 1/12 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
p^{(V)} &= \begin{bmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{bmatrix}
\end{align*}
\]

Direct calculation shows that

\[
\sup_{i,j} d(P^{(I)}(i), P^{(I)}(j)) = \frac{1}{2} \alpha = I, II, III, IV \text{ and } \sup_{i,j} d(P^{(V)}(i), P^{(V)}(j)) = \frac{1}{6} \tag{2.39}
\]

Let \( BC(\Gamma) \) denote the bounded, real-valued continuous functions on \( \Gamma \). For \( g \in BC(\Gamma), x \in \Gamma, t > 0, \) let \( P_t g(x) = E^x g(Y_t) \).
Theorem 2.10. For all $t > 0$, $P_t : BC(\Gamma) \rightarrow BC(\Gamma)$.

Proof: Fix $t > 0$. For $g \in BC(\Gamma)$, we must show that $P_t g(\cdot) \in BC(\Gamma)$. To do this, it will suffice to show that $P_t g(x), x \in G_\infty$ is uniformly continuous and observe that a uniformly continuous function on $G_\infty$ has a unique uniformly continuous extension to $\Gamma$.

Let $x, y \in G_\infty$, choose $M$ sufficiently large that $x, y \in G_M$, and suppose $x$ and $y$ lie within the same square on level $m$ of $G_\infty$, say the square $S_m$. A fortiori, $x$ and $y$ lie in the same square $S_n$ on level $n$ of $G_\infty$ for all $n < m$.

Let $\varepsilon > 0$, and let $g$ be a continuous function on $\Gamma$ with $\|g\| \leq 1$. Then

$$|E^x g(Y_t) - E^y g(Y_t)|$$

$$\leq |E^x g(Y_t) - E^x g(Y_{t+\tau})| + |E^x g(Y_{t+\tau}) - E^x g(Y_{t+\tau})|$$

$$+ |E^x g(Y_t) - E^x g(Y_{t+\tau})|.$$  

(2.40)

We estimate the terms separately.

Note that

$$E^x g(Y(t + T_n)) = \sum_{i=1}^{4} \lambda^x_i E^g(Y_n(t)); \quad E^y g(Y(t + T_n)) = \sum_{i=1}^{4} \lambda^y_i E^g(Y_n(t))$$

(2.41)

where $\lambda^x$ and $\lambda^y$ are the escape distributions of $Y_n(t)$ on the corners of $S_n$ starting from $x$ and $y$, respectively.

Then,

$$|E^x g(Y_n(t + \tau_k)) - E^y g(Y_n(t + \tau_k))| \leq \sum_{i=1}^{4} |\lambda^x_i - \lambda^y_i| E^x g(Y_n(t)) \leq \sum_{i=1}^{4} |\lambda^x_i - \lambda^y_i|$$

(2.42)

Choose $\delta > 0$ such that $|u - v| < \delta$ implies $|g(u) - g(v)| < \frac{\varepsilon}{2}$, and let $A_h = \{ \omega : \sup_{1 \leq t < h} \| Y_t - Y_{t+h} \| > \delta \}$.

We have shown that the paths of $Y_t$ are uniformly continuous in probability, thus, choose $h > 0$ so that $P[A_h] \leq \frac{\varepsilon}{4}$. Then,

$$|E^x g(Y_t) - E^x g(Y_{t+\tau})| \leq E^x |g(Y_t) - g(Y_{t+\tau})|$$

(2.43)
\[ E^\varepsilon[g(Y_t) - g(Y_{t+T})1\{T \geq h\} \cup [A_t])] \leq E^\varepsilon[g(Y_t) - g(Y_{t+T})1\{T < h, A_h\}] \]  
\[ + \leq 2 \cdot \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{2} + \varepsilon \]  
\[ = 3\varepsilon \]  
(2.44)  

Similarly, we can estimate \( |E^\varepsilon g(Y(t)) - E^\varepsilon g(Y(t + T))| \leq 3\varepsilon \) for suitable \( n \).

To complete the estimate, choose \( n \) sufficiently large so that \( P^\varepsilon[T_n \geq \eta] < \frac{1}{2} \). Then, if we apply this to the preceding inequalities, we get

\[ |E^\varepsilon g(Y_n(t)) - E^\varepsilon g(Y_n(t))| < 3\varepsilon + \varepsilon + 3\varepsilon = 7\varepsilon \]  
(2.48)

If we choose \( k \) and \( m \) as in the preceding paragraphs, and let \( |x - y| < 3^{-m} \), then there must exist some \( z \in \Gamma_\infty \) such that \( x \) and \( z \) and \( y \) and \( z \) each lie within a common square of level \( m \) of \( \Gamma \). So,

\[ |E^\varepsilon g(Y_n(t)) - E^\nu g(Y_n(t))| \leq |E^\varepsilon g(Y_n(t)) - E^\varepsilon g(Y_n(t))| + |E^\varepsilon g(Y_n(t)) - E^\varepsilon g(Y_n(t))| \]  
\[ < 14\varepsilon \]  
(2.49)

Let \( n \to \infty \). Then if \( |x - y| < 3^{-m} \), \( |E^\varepsilon g(Y_t) - E^\nu g(Y_t)| < 14\varepsilon \).

This shows that \( E^\varepsilon g(Y_t) \) is a uniformly continuous function of \( x \), for all \( x \in \Gamma_\infty \), all \( t \), and for \( g \) an arbitrary uniformly continuous function.

Proposition 2.11. \( Y_t \) is a Markov process.

Proof: Fix \( t > 0 \) and consider \( S_n, [15^m t] \) where \([x]\) denotes the greatest integer less than \( x \). Then, using our earlier notation,

\[ S_n, [15^m t] = \sum_{j=1}^{[15^m t]} T_{n,j} = \frac{[15^m t]}{15^n} \cdot \left( 15^n \cdot 1 \right) \sum_{j=1}^{[15^m t]} T_{n,j} \]  
(2.50)
As \( n \to \infty \),
\[
\frac{1}{15^n} \cdot \frac{I_{[15^n]}[\sum_{j=1}^{[15^n]} T_{n,j}] - 1}{15^n} \to 1 \text{ a.s.,}
\]
\[
\frac{[15^n]}{15^n} - t
\]
(2.51)
The first limit follows by applying the law of large numbers to the i. i. d. sequence \( T_{n,j} \). The second is simple analysis. Thus, \( S_{n,[15^n]} - t \) almost surely, as \( n \to \infty \).

If \( g(x_1, \ldots, x_d) \) and \( f(y_1, \ldots, y_k) \) are bounded continuous functions and \( s_1 < \ldots < s_d < t < t_1 < \ldots < t_k \), then, by Proposition 1.8, for any \( n \),
\[
E[g(Y(s_1 \land S_{n,[15^n]}), \ldots, Y(s_d \land S_{n,[15^n]})) f(Y(t_1 + S_{n,[15^n]}), \ldots, Y(t_k + S_{n,[15^n]}))]
\]
\[
= E[g(Y(s_1 \land S_{n,[15^n]}), \ldots, Y(s_d \land S_{n,[15^n]})) E[f(Y(t_1), \ldots, Y(t_k))] | Y(S_{n,[15^n]})]
\]
Let \( n \to \infty \), apply the continuity of \( f \) and \( g \), Theorem 1.10, and the sample continuity of \( Y_t \). We get
\[
E[g(Y_{s_1}, \ldots, Y_{s_d}) f(Y_{t_1+t}, \ldots, Y_{t_k+t})] = E[g(Y_{s_1}, \ldots, Y_{s_d}) E^Y_{f(Y_t, \ldots, Y_t)}]
\]
(2.53)
This proves the proposition.

To show that \( Y_t \) is a strong Markov process, we will apply the following theorem:

**Theorem 2.12.** Let \( X_t \) be a process satisfying

i. For all \( f \in BC(\Gamma) \), and all \( t > 0, 0 \leq f \leq 1 \) implies \( 0 \leq Ef(X_t) \leq 1 \).

ii. For all \( s, t > 0 \), \( Ef(X_{s+t}) = E_s^X(Ef(X_t)) \).

iii. For all \( f \in BC(\Gamma), E^x f(X_t) - f(x) \to 0 \) uniformly in \( x \) as \( t \to 0 \). Let \( F^x_t = \sigma(X_s : s \leq t) \) and \( F^x_{t+} = \cap_{s > t} F^x_s \). Then if \( T \) is an \( F^x_{t+} \) stopping time, \( E^\mu[\theta_T \eta[F^x_{T+}]] = E^X(T)[\eta] \), for any measure \( \mu \) and any bounded measurable \( \eta \).

**Proof:** This theorem is given in Williams [17], as Theorem III.15.3.

To apply the theorem, it remains to show that \( E^x f(Y_{t}) - f(x) \to 0 \) uniformly in \( x \) as \( t \to 0 \) for any bounded continuous \( f \).
Consider the operator $\tilde{P}_t : BC(\Gamma) \to BC(\Gamma_\infty)$ defined by $\tilde{P}_t g(x) = E^x g(Y_t), x \in \Gamma_\infty$. Since $E^x g(Y_t)$ is uniformly continuous for $x \in \Gamma_\infty$, we can extend $E^x g(Y_t)$ to a uniformly continuous function defined for all $x \in \Gamma_\infty$ and all $t > 0$. Let $P_t g$ denote this extension of $\tilde{P}_t g$. To apply the theorem from Williams [17] stated earlier, we need only prove the following

Proposition 2.13. For all $f \in BC(\Gamma)$, and all $t > 0, 0 \leq f \leq 1$ implies $0 \leq P_t f \leq 1$, and $P_t f(x) - f(x) \to 0$ uniformly in $x$ as $t \to 0$.

Proof: Let $f \in BC(\Gamma)$. If $0 \leq f \leq 1$, then $0 \leq E^x f(Y_t) \leq 1$ (a.s.), since conditional expectation is a positive operator. Since $\Gamma, BC(\Gamma)$, and $[0, \infty)$ are all separable, we can modify $P$ on a single null set $N$ to get $0 \leq E^x f(Y_t) \leq 1$.

We apply the following lemma.

Lemma 2.14. Let $P_t : BC(\Gamma) \to BC(\Gamma)$ be a substochastic Markov semigroup. If $P_t f(x) \to f(x)$ as $t \downarrow 0$ for all $x \in \Gamma$ and $f \in BC(\Gamma)$, then $P_t f(x) - f(x) \to 0$ uniformly in $x$ as $t \to 0$, for all $f \in BC(\Gamma)$.

Proof: This paraphrases formula III.6.2.iv. in Williams [17].

Let $\epsilon > 0$. Let $x \in \Gamma_\infty$, and let $g \in BC(\Gamma)$. Given $\epsilon > 0$, choose $\delta$ such that $|y - x| < \delta$ implies that $|g(x) - g(y)| < \delta$. Then

$$
|P_t g(x) - g(x)| \leq E^x (g(Y_t) - g(x)) 1[|Y_t - x| \leq \delta] + |E^x (g(Y_t) - g(x)) 1[|Y_t - x| > \delta]|
$$

(2.54)

$$
\leq \frac{\delta}{2} P^x [|Y_t - x| \leq \delta] + 2 \cdot P^x [|Y_t - x| > \delta]
$$

$$
\leq \frac{\delta}{2} + \frac{\delta}{2} = \epsilon
$$

This completes the proof of the proposition.
In terms of Williams [17], this makes $P_t$ an FD semigroup and $Y_t$ an FD process. Applying the theorem that we stated earlier in this section to $Y_t$, we conclude

**Corollary 2.15.** $Y_t$ has the strong Markov property.

Henceforth, we will refer to $Y_t$ as the snowflake diffusion.

In fact we can show that the snowflake diffusion is the unique diffusion limit for any of the random walks $X^p$, within a constant change of time scale.

By Proposition 2.6, random walks on all the lattices $G_n$ are embedded in the snowflake diffusion. We also note in passing that for $Y_0 = 0$, $3Y_t \equiv_D Y_{15t}$; we will return to this point in the next section at greater length. These two facts form the basis for the proof of the following proposition.

**Proposition 2.16.** Within a non-random change in time scale, $Y_t$ is the unique limit in distribution of the processes $Y_n(t)$.

**Proof.** For each $n$, define the sequence $T^+_n, T^+_n, \ldots$ as in proposition 2.6. For each $k$, $T^+_n$ is the sum of $k$ independent random variables equal to $15^{-n}W$ in distribution. If $\text{Var}(W) = \sigma^2$, then $E T^+_n = k \cdot 15^{-n}$ and $\text{Var}(T^+_n) = k\sigma^2$.

For arbitrary $n$ and $t$,

$$Y_n(t) = 3^{-n}X([15^n t]) = D 3^{-n}Y(T^n([15^n t])) = Y(15^{-n}T^n([15^n t])). \quad (2.55)$$

Note that $t - 15^{-n} \leq 15^{-n} \cdot [15^n t] \leq t + 15^{-n}$. Now, $T^+_n$ is the sum of $k$ independent random variables, with distribution $15^{-m} \cdot W$. Thus,

$$E(15^{-m} \cdot T([15^n t])) = 15^{-m} \cdot E(T([15^n t])) = 15^{-m} \cdot [15^{-m} \cdot t] \rightarrow t \quad (2.56)$$

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\[ \sigma^2(15^{-m} \cdot T([15^{m}t])) = 15^{-2m} \cdot \sigma^2(T([15^{m}t])) = 15^{-2m} \cdot 15^{-m} \cdot \sigma^2 = 15^{-m} \cdot \sigma^2 \to 0 \]  

(2.57)

as \( n \to \infty \), where \( \sigma^2 = \sigma^2(W) \). Thus, as \( n \to \infty \), \( Y_n(t) \to \rho Y_t \).

By the same argument, for any \( t_1 < \ldots < t_k \) and any convergent subsequence \( Y_n(t) \) converges in distribution.

Then by the same analysis as in the preceding paragraph,

\[ (Y_n(t_1), \ldots, Y_n(t_k)) \to \mathcal{D} (Y(t_1), \ldots, Y(t_m)) \]  

(2.58)

Thus all limits of \( \{Y_n(t)\}_{n=1}^{\infty} \) have the same finite dimensional distributions. Since the finite dimensional distributions of a stochastic process determine the process, it follows that \( Y_t \) is the unique limit of the processes \( \{Y_n(t)\}_{n=1}^{\infty} \).

Now we consider again the general model for random walk proposed at the beginning of this section. Let \( X_p \) be random walk on the snowflake lattice with parameter \( p \). Let \( p_0 = p \) and for \( n = 1, 2, \ldots \), let \( p_n \) be the probability that \( X_p \) starts from 0 and reaches \((3^n, 3^n)\) before either \((3^n, 0)\) or \((0, 3^n)\). It is not hard to see that in general \( p_n \neq p_{n-1} \). We can find a recurrence relationship by calculating computing \( p_1 \) as a function of \( p_0 \). Observe that we can identify vertices of \( U_1 \) that are reflections of one another across diagonals of the square. Thus, we can identify the following sets of vertices

\[
\begin{align*}
a &= \{(0,0)\} \quad b = \{(0,1), (1,0)\} \\
c &= \{(1,1)\} \\
d &= \{(1,2), (2,1)\} \quad e = \{(0,2), (1,3), (2,0), (3,1)\} \\
f &= \{(2,2)\} \\
g &= \{(2,3), (3,2)\}
\end{align*}
\]  

(2.59)

For each \( i \in \{a, \ldots, g\} \), let \( q_i \) be the probability that \( X_n \) starts from vertex \( i \) and reaches \((3,3)\) before either
Using the Markov property of $X^P$, we get the following system of equations for $q_i$

\[
\begin{align*}
q_a &= (1-p)q_b + pq_c \\
q_b &= \frac{1}{2}(1-p)q_a + pq_b + \frac{1}{2}(1-p)q_c \\
q_c &= \frac{1}{2}p q_a + \frac{1}{2}(1-p)q_b + \frac{1}{2}(1-p)q_d + \frac{1}{2}pq_e \\
q_d &= \frac{1}{2}(1-p)q_c + \frac{1}{2}pq_d + \frac{1}{2}(1-p)q_e + \frac{1}{2}(1-p)q_f \\
q_e &= \frac{1}{2}(1-p)q_d + \frac{1}{2}pq_c \\
q_f &= \frac{1}{2}pq_e + \frac{1}{2}(1-p)q_d + \frac{1}{2}(1-p)q_g + \frac{1}{2}pq_f \\
q_g &= \frac{1}{2}(1-p)q_f + pq_g + \frac{1}{2}(1-p)
\end{align*}
\]

(2.60)

Applying Cramer's rule, we get

\[
q_a = \frac{(\frac{1}{2})^5 \cdot (1-p)^3 \cdot (1+p)^3}{(\frac{1}{2})^5 \cdot (1-p)^3 \cdot (1+p)^3 \cdot (4-3p)} = \frac{1}{4-3p}
\]

(2.61)

Formula (2.61) gives us a recurrence for $p_1$ in terms of $p_0$. Furthermore, we see that $p = \frac{1}{3}$ is a fixed point for this recurrence formula, and that for any $p_0 < 1$, $p_n \to \frac{1}{3}$ as $n \to \infty$.

For $n = 1, 2, \ldots$, let $T^{n,p} = T(n,p)$ be the number of steps $X^P$ requires to pass from 0 to some other vertex in $3^n \cdot G$. In analogy with the branching property of $X^{1/3}$, it is not hard to see that $T^{n,p} = \sum B T_k^{n-1,p}$, where $T_k^{n-1,p} = 1, 2, \ldots$ is a sequence of random variables distributed as $T^{n-1,p}$, $B$ is distributed as $T(1, p_{n-1})$ and $B$ and $T_k^{n-1,p} = 1, 2, \ldots$ are mutually independent.

Let $m_n = ET(1, p_n)$, and let $M_n = \prod_{i=1}^n m_i$. Then, $ET^{n,p} = M_n$, and an argument analogous to that for branching processes shows that $T^{n,p} \cdot m_n^{-1}$ is a positive martingale converging a.s. and in distribution to some random variable $\hat{W}$.

Lemma 2.17. Within a constant scaling factor, $\hat{W} = D W$.

Proof: Let

\[
f_n(u) = Eu^{T(1,p)\lambda}, \quad g_n(\lambda) = E \exp (-\lambda T(n,p) \cdot m_n^{-1})
\]

(2.62)

Then from the branching property of $T^{n,p}$ outlined in the preceding two paragraphs

\[
g_n(\lambda) = f_n (g_{n-1}(\lambda \cdot m_n^{-1}))
\]

(2.63)
Since $T_{n}^{\cdot} \cdot M_{n}^{-1} \rightarrow_{D} \tilde{W}$, $g_{n}(\lambda) \rightarrow g(\lambda) = \mathbb{E}\exp(-\lambda \tilde{W})$ as $n \rightarrow \infty$. $p_{n} \rightarrow 1/3$, so $f_{n}(u) \rightarrow f(u)$, and $m_{n} \rightarrow 15$. Substituting these limits in (2.63) gives

$$g(\lambda) = f(g(\lambda \cdot 15^{-1})),$$

(2.64)

the same functional equation satisfied by $W$. It is well known that the solution to Abel's functional equation is unique, except for a constant multiplier of the argument. (See, for example, Seneta [16], Theorem 3.1.)

This implies that $\tilde{W} =_{D} W$ within a constant change of scale.

Let $Y_{\cdot}^{n}(t) = 3^{-n}X^{\cdot}([M_{n}t])$. Arguments analogous to those in Theorem 2.2, Lemma 2.3, Propositions 2.6, 2.8, 2.11, and 2.13 show that we can find a diffusion $\tilde{Y}_{\cdot}$ on $\Gamma$ and a subsequence $n'$ such that $Y_{n'}^{n} \rightarrow \tilde{Y}$ weakly.

Let $\{\tilde{T}_{n}\}$ be the sequence of times when $\tilde{Y}_{\cdot}$ visits distinct vertices of $G_{1}$. Note that $\tilde{T} =_{D} \tilde{W}$. As $p_{n} \rightarrow 1/3$, $\{\tilde{Y}(\tilde{T}_{n})\} =_{D} \{X^{1/3}\}$

Theorem 2.18. $\tilde{Y}_{\cdot} =_{D} Y_{\cdot}$, within a constant change of time scale.

Proof: As a consequence of Lemma 2.17, we can choose a time scale for $\tilde{Y}_{\cdot}$ so that $3\tilde{Y} =_{D} \tilde{Y}_{15\cdot}$. The proof now proceeds as in Proposition 2.16. Let

$$Y_{n}(t) = 3^{-m}\tilde{Y}(\tilde{T}([15^{m}t])) = \tilde{Y}(15^{-m} \cdot \tilde{T}([15^{m}t])).$$

(2.65)

As $\tilde{Y}_{\cdot}$ has continuous sample paths, $Y_{n}(t) \rightarrow \tilde{Y}_{\cdot}$ almost surely. On the other hand, $Y_{m}(t) =_{D} Y_{m}(t)$, and $Y_{m}(t) \rightarrow_{D} Y_{\cdot}$. Thus, $\tilde{Y}_{\cdot} =_{D} Y_{\cdot}$. This proves the theorem.
3. Scaling Properties:

In this chapter, we will study some of the detailed sample path properties of the snowflake diffusion. Let \( n \geq 0 \), let \( x, y \) be points on \( G_n \cap [0,3] \) such that \( x, y \in S \) for some square \( S \) on level \( n \). Let \( T_y = \inf\{t : Y_t = y\} \) be the first visit of the diffusion on \( \Gamma_n \) to \( y \); set \( T_y = \infty \) if this set is empty.

We begin by proving two technical lemmas.

**Lemma 3.1.** \( E^x T_y < 369 \cdot 3^{-n} \).

**Proof:** Suppose \( x, y \in S \). There exists \( m \geq n \) such that \( x, y \in G_m \). We will establish our estimate by induction on \( m \).

Suppose that \( m = n \). Then \( x, y \) are necessarily corners of \( S \). Let \( T_1, T_2, ... \) be the times between successive visits to \( G_n \). Then, \( T_y = \sum_{i=1}^{N(y)} T_i \) where

\[
N_y = \sum_{i=1}^{M} \left( \sum_{j=1}^{R_{i,j}} + 1 \right)
\]

(3.1)

\( M \) is the number of corners of \( S \) that \( Y_t \) visits before \( T_y \), \( N_i \) is the number of excursions to \( G_n \setminus S \) between the time \( Y_t \) hits the \( i-1^{st} \) and \( i^{th} \) distinct corners of \( S \), and \( R_{i,j} \) the number of points in \( G_n \setminus S \) that \( Y_t \) visits on the \( j^{th} \) such excursion.

The strong Markov property shows that \( M \) has a geometric distribution with parameter \( \frac{1}{2} \). For \( i = 1, 2, ... \), \( N_i \) either is identically \( 0 \) or else has a geometric distribution with parameter \( \frac{1}{2} \). To estimate \( E R_{i,j} \), note that each excursion outside of \( S \) is a random walk on a finite graph, and \( R_{i,j} \) is the number of steps the walk takes to return to its starting point. It is well-known that this expected time is equal to twice the total number of edges in the graph divided by the degree of the starting vertex; see Gobel and Jagers [9] for this and other general results about random walks on finite graphs. Thus, \( E R_{i,j} \) is proportional to the number.
of the edges in the graph cut out of $G_n$ by $S$. This number, in turn, is less than the number of edges in $G_m$.

It is not hard to compute that $G_n$ has $6 \cdot 5^{n+1}$ edges, so, $ER_{i,j} \leq 20 \cdot 5^n$

Recall that $ET_i = 15^{-n}$, and apply Wald's identity.

$$E^xT_y = EM \cdot (EN(i) \cdot ER_{i,j} + 1) \cdot ET$$

$$\leq 3 \cdot (2 \cdot 20 \cdot 5^n + 1) \cdot 15^{-n}$$

$$\leq 41 \cdot 3^{-(n-1)}$$

We now proceed by induction. Suppose that if $x, y \in G_m$ with $|x - y| \leq 3^{-n}$, then $E^xT_y \leq 82 \cdot \sum_{i=1}^{m} 3^{-i+1}$.

Let $x, y \in G_{m+1}, |x - y| \leq 3^{-n}$. There exist points $v$ and $w$ in $S \cap G_n$ such that $x$ and $v$ and $y$ and $w$ are adjoining corners of squares on $G_{m+1}$. ( $x$ and $v$ may be identical, and so forth.) The strong Markov property gives

$$E^xT_y \leq E^xT_v + E^vT_w + E^wT_y$$

$$\leq 41 \cdot 3^{-m} + 82 \cdot \sum_{i=1}^{m-1} 3^{-i} + 41 \cdot 3^{-m}$$

$$= 82 \cdot \sum_{n}^{m+1} 3^{-i+1}$$

completing the inductive step.

Finally, if $x, y \in S \cap -\infty$, then

$$E^xT_y \leq 82 \cdot \sum_{n}^{\infty} 3^{-i+1} = 369 \cdot 3^{-n}$$

(2.4)

This completes the proof of the lemma.

Corollary 2.2. If $x, y \in G_{\infty} \cap [0, 3]$, such that $|x - y| < 3^{-n}$, then $E^xT_y \leq 738 \cdot 3^{-n}$.

Proof: If $|x - y| < 3^{-n}$, then either they lie in a single square $S$ or level $n$, or else they lie in two such adjoining squares. In either case, the preceding lemma gives the desired bound.

Recall that $3 \cdot G_n = G_{n+1}$ This fact and the uniqueness of the snowflake diffusion limit imply the following
Lemma 3.3. Let \( x \in \Gamma \). Then \( 3Y^x_t =_D Y^{3x}_{15t} \).

Proof: Let \( x \in \Gamma_\infty \). Using the notation of section 2, consider the sequence of processes \( 3 \cdot Y_n(x,t), n = 1^\infty \).

We have shown that \( Y_n(x,t) \rightarrow Y^x_t \) as \( n \rightarrow \infty \), so, trivially, \( 3 \cdot Y_n(x,t) \rightarrow 3 \cdot Y^x_t \)

On the other hand,

\[
3 \cdot Y_n(x,t) = 3 \cdot 3^{-n} X(3^n x, [15^n t]) = 3^{-n+1} X(3^{n-1} 3x, [15^{n-1} (15t)]) = Y_{n-1}(3x, 15t)
\] (3.4)

Thus, \( 3 \cdot Y_n(x,t) \rightarrow Y^{3x}_{15t} \). Since both of these limits are unique, it follows that \( 3Y^x_t =_D Y^{3x}_{15t} \), for \( x \in \Gamma_\infty \).

For general \( x \) in \( \Gamma \) choose a sequence \( x_n \) in \( \Gamma_\infty \) converging to \( x \). Then, for each \( n \), \( 3Y^{x_n}_t =_D Y^{3x_n}_{15t} \). As \( Y^x \) has the Feller property,

\[
3Y^{x_n}_t =_D 3Y^x_t \quad \text{and} \quad Y^{3x_n}_{15t} =_D Y^{3x}_{15t} \quad n \rightarrow \infty
\] (3.5)

Therefore, \( 3Y^x_t =_D Y^{3x}_{15t} \).

Corollary 3.4. \( 3Y^0_t =_D Y^0_{15t} \).

For starting points in \( G_\infty \) other than 0, a result similar to the preceding corollary also holds. Let \( x \in \Gamma_\infty \).

and define \( M_x : y \rightarrow 3^{-1} \cdot (y - x) + x \). Then \( M_x x = x \).

Corollary 3.5. Let \( x \in G_\infty \). Then there exists a stopping time \( T_x > 0 \) such that \( M_x Y^x_t =_D Y^x_{15t}, 0 \leq t \leq T_x \).

Proof: Suppose that \( x \in G_\infty \). Then there exists either one or two squares \( S \) on level \( n \) such that \( x \) is a corner of \( S \). However, if \( x \) is a corner of \( S \), then there is some square \( S' \) on level \( n + 1 \) contained in \( S \) such that \( x \) is also a corner of \( S' \). Clearly, \( M_x : S \rightarrow S' \). Let \( U' \) be the union of any squares \( S' \) on level \( n + 1 \) adjoining \( x \).

Then, the proof of the fractal scaling law will continue to work if we set \( T_x \) to be the first time the process started from \( x \) exits \( U' \) and \( T_x \) satisfies the requirements given in the statement of the corollary.
We shall call the scaling property stated in either corollary the unbounded fractal scaling law.

If we restrict our attention to $\Gamma_b$, then our scaling properties are not quite so tidy. However, a closely related property does apply, which arises from the invariance of $\Gamma_b$ under a family of contractions. Define

$$N_1 : x \rightarrow 3 \cdot x$$
$$N_2 : x \rightarrow -3 \cdot x + (6, 6)$$
$$N_3 : x \rightarrow 3 \cdot x - (6, 6)$$
$$N_4 : x \rightarrow 3 \cdot x - (1 \frac{1}{2}, 4 \frac{1}{2})$$
$$N_5 : x \rightarrow 3 \cdot x - (4 \frac{1}{2}, 1 \frac{1}{2})$$

and

$$N(x) = \begin{cases} N_1(x), & x \in [0, 1]^2 \\ N_2(x), & x \in [1, 2]^2 \\ N_3(x), & x \in [2, 3]^2 \\ N_4(x), & x \in [0, 1] \times [2, 3] \\ N_5(x), & x \in [2, 3] \times [0, 1] \end{cases} \quad (3.6)$$

Note that $N$ is a continuous mapping of $\Gamma_b$ onto $\Gamma_b$.

Lemma. Let $x \in \Gamma$. Then, $N(Y^x_t) =_D Y^{N(x)}_{15t}$.

Proof: Inspection will verify that if $X_n$ is a random walk on $G_k$ starting from $x$, then $N(X_k^n)$ is a random walk on $G_{k-1}$ starting from $N(x)$, for any $k$. We have shown in Section 2 that $X_k^{Y^x_t} \rightarrow Y_t$ weakly. Then

$$N(X_k^{Y^x_t}) = N(Y^x_t) \rightarrow Y_{15t} \quad (3.7)$$

and since $N$ is continuous, $N(X_k^{Y^x_t}) \rightarrow N(Y_t)$.

We shall call this scaling property the bounded fractal scaling law. As an important consequence, note that for any $x \in \Gamma_b$ and any measurable set $A$, $P^{N(x)}[Y^x_{15t} \in A] = P^x[Y_t \in N^{-1}(A)]$.

Our next step is to show that two independent copies of the snowflake diffusion restricted to $\Gamma_b$ meet in finite time with probability 1. First we prove two technical lemmas. For $x, y \in \Gamma$, let $d(x, y)$ denote the Euclidean distance between $x$ and $y$. Let $D[0, \infty]$ denote the space of functions $\omega : \mathbb{R}^+ \rightarrow \Gamma$ which are right continuous and have left limits for all $t > 0$.  

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Lemma 3.7. For any \( t, \epsilon > 0 \) and any compact set \( K \), let

\[
H(t, \epsilon) = \{ \omega : \inf_{0 < u < t + \epsilon} d(\omega(u), K) > \epsilon \},
\]

Then \( H(t, \epsilon) \) is an open subset in the topology \( T \) on \( D[0, \infty) \) defined by convergence in the Skorokhod metric on compact intervals \([0, p]\).

Proof: Let \( \omega, \nu \in D[0, \infty] \). For any \( p > 0 \), let \( \rho_p(\omega, \nu) \) equal the infimum of those \( \epsilon > 0 \) for which there exists a continuous, increasing function \( \lambda : [0, p] \rightarrow [0, p] \) such that

1. \( \sup \{ 0 < t < p : |\lambda(t) - t| \leq \epsilon \} \)
2. \( \sup \{ 0 < t < p : d(\mu(t), \nu(t)) \leq \epsilon \} \)

It is easy to see that \( \rho_p \) is a pseudo-metric and that the topology \( T \) is induced by the family of pseudo-metrics \( \{ \rho_p \}_{p > 0} \).

Suppose \( \omega \in H(t, \epsilon) \). Let \( a = \inf \{ d(\omega(u), K), 0 < u < t + \epsilon \} \) By hypothesis. \( a - \epsilon = \delta > 0 \). Choose \( \nu \in D[0, \infty] \) with \( \rho_t(\omega, \nu) < \frac{\delta}{2} \). There exists a strictly increasing continuous function \( \lambda : [0, t] \rightarrow [0, t] \) satisfying i. and ii. By the triangle inequality, for any \( u \in [0, t] \),

\[
d(\nu(u), K) \geq d(\omega(\lambda(u)), K) - d(\omega(\lambda(u)), \nu(u)) > \epsilon + \delta - \frac{\delta}{2} > \epsilon;
\]

since \( \lambda : [0, t] \rightarrow [0, t] \), \( d(\omega(\lambda(u)), K) > \epsilon + \delta \), for \( 0 \leq u \leq t \). Thus, \( \nu \in H(t, \epsilon) \), so \( H \) is open.

Let \( T = \inf \{ t : \omega(t) \in K \} \). For every \( \epsilon > 0 \), \( H(t, \epsilon) \subset \{ \omega : T(\omega) > t \} \). Let \( C[0, \infty] \) denote the continuous functions from \( \mathbb{R}^+ \) to \( \Gamma_k \).

Lemma 3.8. \( \{ \omega \in C[0, \infty] : T(\omega) > t \} = \cup_{k=1}^{\infty} H(t, k^{-1}) \)

Proof: Suppose \( \omega \) is continuous, and that for every \( k > 0 \), there exists \( s_k \in [0, t] \) such that \( d(\omega(s_k), K) \leq k^{-1} \).

As \( [0, t] \) is compact, there exists a convergent subsequence \( s_k^* \) with \( s_k^* \rightarrow s \in [0, t] \). As \( \omega \) is continuous.
\( \omega(x_k) = \omega(x) \). However, \( d(\omega(x_k), K) = 0 \), so \( \omega(s) \in K \), and \( T(\omega) \leq s < t \). Thus, if \( T(\omega) > t \) then
\[ \inf \{ d(\omega(s), K), 0 \leq s \leq t \} > k^{-1}, \text{ for some } k. \]

Using the continuity of \( \omega \) again, there must be some \( m \) such that \( d(\omega(s), K) > k^{-1}, t \leq s < t + m^{-1} \). If not, there would exist a sequence \( x_n \rightarrow t \) such that \( d(\omega(x_n), K) \leq k^{-1} \), forcing \( d(\omega(t), K) \leq k^{-1} \). Choose \( N = \min \{ k, m \} \) to give \( \omega \in H_{N-1} \).

To restate the previous result

**Corollary 3.9.** \( T : C[0, \infty) \rightarrow [0, \infty) \) is a lower semicontinuous function.

**Proof:** Since \( \bigcup_{k=1}^{\infty} H_{k-1} \) is an open set, this follows by definition.

We now proceed to prove the main result of this section.

**Theorem 3.10.** Let \( Y \) and \( Y' \) be two independent copies of the diffusion on \( \Gamma_b \). Let \( T_M = \inf \{ u : Y_u = Y'_u \} \).

Then \( T_M < \infty \) a. s.

**Proof:** In Aldous [1] it is shown that if \( X \) and \( X' \) are two independent copies of a continuous time random walk on a finite graph \( H \), then there exists some constant \( D \) such that \( E T_M \leq D \max_{i,j} E^{ij} T_j \). where the maximum is taken over all pairs of states \( i, j \). By Markov's inequality, it follows that \( P[T_M > t] \leq t^{-1} \cdot D \max_{i,j} E^{ij} T_j \).

Let \( Y_n(t) \) be a random walk in continuous time on \( G_n \). We have shown in the preceding section that for any pair of vertices \( x \) and \( y \), such that \( ||x - y|| < 3^{-i} \) then \( E^{ij} T_j \leq 738 \cdot 3^{-i} \). If we regard \( Y_n(t) \) as a sequence of processes on \( \Gamma_b \), then \( P[T_M^{(n)} > t] \leq t^{-1} \cdot 738 D \) for all \( n \).

Consider the set \( \Delta = \{(x, x) : x \in \Gamma_b \} \). Clearly, \( T_M \) is the first hitting time on \( \Delta \) for the process \( (Y, Y') \).
As before, let $H_t = \{ \omega : \inf d(\omega(u), \Delta) > \epsilon, 0 < u < t + \epsilon \}$. Since $H_t \subset \{ T_M < t \}$, it follows that

$$P[Y_n(t) \in H_t] \leq P[T_M > t] \leq t^{-1} \cdot 738D \tag{3.9}$$

for all $n$. As $H_t$ is open, Prokhorov's theorem and the Lemma 3.1 show that

$$P[Y_t \in H_d] \leq \liminf_{n \to \infty} P[Y_n(t) \in H_d] \leq t^{-1} \cdot 738D \tag{3.9}$$

As $Y_t$ has a.s. continuous sample paths, the preceding equation and Lemma 3.8 show that

$$P[T_M > t] = \liminf_{n \to \infty} P[Y_t \in H_{n-1}] \leq t^{-1} \cdot 738D \tag{3.10}$$

Letting $t \to \infty$ gives $P[T_M = \infty] = 0$, which is what we proposed to prove.

We next consider invariant measures for $Y_t$. For the moment, restrict the random walks $Y_n(t)$ to $\Gamma_k$. (Or, equivalently, restrict them to $[0,3]^3$). Since each $Y_n$ is now a random walk on a finite graph, it has a unique reversible stationary measure, which we shall call $\mu_n$. Regard $Y_n$ as a $\Gamma_k$-valued stochastic process, and $\mu_n$ as a measure on $\Gamma_k$. Since $\Gamma_k$ is compact, $\{\mu_n\}$ has a weakly convergent subsequence $\{\mu_n'\}$.

Proposition. $Y_t$ has a stationary distribution $\mu$, and $\mu_n' \to_D \mu$.

Proof: To show the proposition, we apply weak convergence. We have previously shown that

$$E \left[ \|Y_n(u) - Y_n(s)\|^p \|Y_n(t) - Y_n(u)\|^{p'} \right] \leq D(t - s)^{\rho \gamma} \tag{3.11}$$

for any $\rho, \gamma > 0$ and any $s \leq u \leq t$. Since $\Gamma_k$ is compact, the sequence $\mu_n$ is weakly precompact. Let $n'$ be a sequence of integers such that $\mu_{n'}$ converges, say to $\mu$. If we let $Y_{n'}^\mu$ be the stationary version of $Y_{n'}$, then standard results on weak convergence (See, again, Theorem 15.6 in Billingsley) show that $Y_{n'}^\mu$ converges weakly to a process $Y^\mu$, where $Y^\mu$ is a version of $Y$ with stationary distribution $\mu$. 

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Theorem 3.11. The distribution of $Y_t$ converges $\mu$ in total variation norm

Proof: Let $\mu_t(y, A) = P^y_t[Y_t \in A]$. Let $Y'_t$ be an independent stationary diffusion on $\Gamma_b$ Couple $Y_t$ to $Y'_t$ by letting $Y$ and $Y'$ move independently prior to $T_M$ but specifying that they move identically afterwards. Then for any measurable $A$,

$$|\mu_t(A) - \mu(A)| \leq P[Y_t \neq Y'_t] \leq t^{-1} \cdot 738D \quad (3.12)$$

and the inequality is uniform over all measurable $A$. The theorem follows.

Corollary 3.12. $\mu$ is the unique stationary distribution for $Y_t$.

Theorem 3.13. $\mu$ is normalized Hausdorff $\log_3 3$-dimensional measure, restricted to $\Gamma_b$.

Proof: Since $\mu$ is a stationary measure, we apply the preceding theorem and the bounded fractal scaling law to get

$$P^0_t[Y_t \in N^{-1}(A)] = \mu(N^{-1}(A)) \quad (3.13)$$

$$P^0_t[Y_t \in N^{-1}(A)] = P^0_t[Y_{1t} \in A] - \mu(A) \quad (3.14)$$

as $t \to \infty$. So, $\mu$ satisfies the equation $\mu(A) = \mu(N^{-1}(A))$. Theorem 4.4.1 in Hutchinson [12] shows that there is a unique measure on $\Gamma_b$ that satisfies this equation. Since $\mu$-dimensional Hausdorff measure restricted to $\Gamma_b$ also satisfies this equation, it follows that $\mu$ is $\mu$-dimensional Hausdorff measure on $\Gamma_b$.

Now consider $Y_t$ on $\Gamma$. If we let $\mu$ be Hausdorff $\log_3 5$-dimensional measure restricted to $\Gamma$, we have the following

Corollary 3.14. $\mu$ is an invariant measure for $Y_t$.

Proof: Normalize $\mu$ so that $\mu(\Gamma_b) = 1$, and let $f$ be a continuous function on $\Gamma$ with compact support.
Choose \( N \) sufficiently large so that \( \Gamma \cap [0,3^N] \) contains the support of \( f \). Then for \( n > N \), \( \mu \) is an invariant measure for \( Y_t^{(n)} \), the diffusion process restricted to \( \Gamma \cap [0,3^N] \). Thus, as \( n \to \infty \),

\[
\int f d\mu = \int \mathbb{E}^x[f(Y_t^{(n)})]d\mu(dx) - \int \mathbb{E}^x[f(Y_t)]d\mu(dx)
\]

(3.15)

As the continuous functions with compact support are dense in \( L^1 \), the result follows.

We have shown in Theorem 3.11 that \( \mu_t \) converges to Hausdorff log_3 5-dimensional measure in total variation norm for \( Y_t \) restricted to \( \Gamma_b \). This, together with the scaling laws, implies that \( Y_t \) has a transition density with respect to Hausdorff measure.

Recall the transformations \( N_1, ..., N_5 \) and \( N \) defined in (3.6). Let \( \mu_t(\cdot) = P^0[Y_t \in \cdot] \) and let \( \mu(\cdot) \) denote Hausdorff log_3 5-dimensional measure, restricted to \( \Gamma \).

Theorem 3.15. \( \mu_t \ll \mu \) for all \( t > 0 \).

Proof: We begin by proving the theorem for \( Y_t \) restricted to \( \Gamma_b \). Let \( B \subset \Gamma_b \) be a set with \( \mu(B) = 0 \) and suppose that \( \mu_t(B) = q > 0 \). We observe in passing that \( \mu(B) = 0 \) iff \( \mu(N(B)) = 0 \), and that \( N^{-1}(N(B)) \supset B \). Let \( B_\infty = \bigcup_k N^k(B) \). Since \( \mu(B) = 0 \), \( \mu(N^k(B)) = 0 \) for all \( k \), so \( \mu(B_\infty) = 0 \). On the other hand, \( B \subset N^{-k}(B_\infty) \) for all \( k \). By the bounded fractal scaling law,

\[
\mu_{15^t}(B_\infty) = \mu_t(N^{-k}(B_\infty)) \geq \mu_t(B) = q
\]

(3.16)

Thus, \( \liminf_{k \to -\infty} \mu_{15^t}(B_\infty) \geq q \). But this contradicts the fact that \( \mu_t \ll \mu \) in total variation norm as \( t \to \infty \). By contradiction, \( \mu_t(B) = 0 \). Thus, \( \mu_t \ll \mu \).

To show absolute continuity for the unbounded process, let \( Y_t^{(n)} \) denote the diffusion process restricted to \( \Gamma \cup [0,3^n] \). As \( n \to \infty \), \( Y_t^{(n)} \) restricted to \( \Gamma_b \) shows that
$P[y^{(n)} \in \cdot]$ is absolutely continuous with respect to Hausdorff measure for all $n$. Therefore, it follows that $P[y \in \cdot]$ is also absolutely continuous with respect to Hausdorff $\log_5 3$-dimensional measure on $\Gamma$. 
4. Computing Generating Functions

We can determine the generating function of the distribution of $T'$ by elementary calculations. Let $k(u) = E^0 u^\tau$ where $\tau$ is the time required to cross from 0 to (1, 1). $\tau$ has a geometric distribution with $p = 1/3$, so $k(u) = u/(3 - 2u)$, trivially.

Suppose $X_0 = 0$, and let $\sigma$ denote the first hitting time on an outer corner, other than 0. Let $f(u) = E^0 u^\sigma$.

We calculate the generating functions of some hitting times for corner 1, as a preliminary to calculating the generating function of $\sigma$.

Let $\mu$ denote the first hitting time on 1, on the on the set where 1 is the first outer corner of $U_0$ that the walk visits. As the distribution of the random walk is not affected by symmetries of $U_0$, the $g$ depends only on the graph distance between the random walk's starting point and 1. Thus we can identify vertices of $\Gamma_b$ by

$$a = \{(0,1),(1,0),(0,2),(1,3),(2,0),(3,1)\}$$

$$b = \{(1,1),(1,2),(2,1)\}$$

$$c = \{(2,2)\}$$

$$d = \{(2,3),(3,2)\}$$

For $i$ a vertex in $U_0$, let $g_i(u) = E^i u^\mu$. $g$ then satisfies the following system of equations

$$g_a = \frac{1}{3} u g_a + \frac{1}{6} u g_b$$
$$g_b = \frac{1}{3} u g_a + \frac{1}{6} u g_b + \frac{1}{3} u g_c$$
$$g_c = \frac{1}{3} u g_b + \frac{1}{6} u g_d + \frac{1}{6} u$$
$$g_d = \frac{1}{3} u g_c + \frac{1}{6} u g_d + \frac{1}{6} u$$

Direct calculation shows that this system of equations has the solutions:

$$g_a = \frac{u^3}{3(2-u)(18-15u+u^2)}$$
$$g_b = \frac{u^2(3-u)}{3(2-u)(18-15u+u^2)}$$
$$g_c = \frac{u^2(3-u)}{2u(3-2u)}$$
$$g_d = \frac{u^2(3-u)}{(2-u)(18-15u+u^2)}$$

Now, let $\lambda$ be the first hitting time for any corner other than 0. Again we can use symmetry to identify
vertices, in the groups

\[ w = \{(0, 1), (1, 0)\} \]

\[ x = \{(1, 1), (1, 2), (2, 1)\} \]

\[ y = \{(1, 2), (2, 1), (2, 2)\} \]

\[ z = \{(0, 2), (1, 3), (2, 0), (3, 1), (2, 3), (3, 2)\} \] \hspace{1cm} (4.4)

Then \( h_i(u) = E^i u^\lambda \) satisfies

\[
\begin{align*}
h_w &= \frac{1}{3} u h_w + \frac{1}{3} u h_x \\
h_x &= \frac{1}{3} u h_x + \frac{1}{3} u h_y \\
h_y &= \frac{1}{3} u h_w + \frac{1}{3} u h_y + \frac{1}{6} u h_z + \frac{1}{6} u \\
h_z &= \frac{1}{3} u h_y + \frac{1}{3} u h_z + \frac{1}{6} u
\end{align*}
\] \hspace{1cm} (4.5)

with solutions

\[
\begin{align*}
h_w &= \frac{u^3}{(2 - u)(18 - 15u + u^2)} \\
h_x &= \frac{u^2(3 - u)}{(2 - u)(18 - 15u + u^2)} \\
h_y &= \frac{2u(9 - 3u - u^2)}{3(2 - u)(18 - 15u + u^2)} \\
h_z &= \frac{u(36 - 30u + 5u^2)}{3(2 - u)(18 - 15u + u^2)}
\end{align*}
\] \hspace{1cm} (4.6)

Using the generating functions \( g, h, \) and \( k \), we can now calculate \( f(u) \). Start at 0 and condition on \( N \), the number of returns to 0 before hitting a different outer corner. We use the symmetry of \( U_0 \) and the generating functions we have just calculated, to get

\[
E^0 u^\sigma = k(u) \sum_0^\infty h_x(u) [g_e(u) k(u)]^n = k(u) \frac{h_x(u)}{[1 - g_e(u) k(u)]}
\] \hspace{1cm} (4.7)

If we substitute the generating functions we have already calculated, we get

\[
E^0 u^\sigma = \left( \frac{u}{3 - 2u} \right) \left[ \frac{u^3}{1 - \frac{(2 - u)(18 - 15u + u^2)}{2u(3 - 2u)}} \right]
\] \hspace{1cm} (4.8)

\[
= \frac{u^3}{(3 - 2u)(12 - 12u + u^2)} \hspace{1cm} (4.6)
\]

\[
= f(u) \hspace{1cm} (4.9)
\]

The symmetry of \( U_0 \) shows that \( f(u) \) is the generating function for the distribution of the time to cross between any two distinct corners of \( G \).

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References:


We define a fractal in the plane known as the Vicsek Snowflake by constructing a skeletal lattice graph and then rescaling spatial dimensions to give a sequence of lattices that converges to a fractal. By defining a simple random walk on the skeletal lattice and then rescaling both time and space, we define a sequence of random walks on the approximating lattices that converge weakly to a limiting process on the snowflake. We show that this limit has continuous sample paths and the strong Markov property, and that it is the unique diffusion limit of random walk on the snowflake in a natural sense. We show that this...
20. ABSTRACT (continued...)

diffusion limit of random walk on the snowflake in a natural sense. We show that this diffusion has a scaling property reminiscent of Brownian motion, and we introduce a coupling argument to show that the diffusion has transition densities with respect to Hausdorff measure on the snowflake.