Spectra of Surface Waves

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This document represents notes that I have collected over the past decade describing surface wave spectra. When I decided to put these notes into a convenient form for my own use, it seemed that this might be useful to others. It is not claimed to be thorough and carefully checked, nor is it polished as a journal paper would be. There are some references that you may find useful. If you know of material that should be included, please let me know. If you find errors, I should be happy to be told of them.

In the first Section I give some conventional definitions, just so we have a common notation. In the second Section I discuss the generation of surface waves. Finally, in the third Section I present some models that are used to describe "equilibrium" spectra.
ABSTRACT

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1.0 SURFACE WAVE DISPLACEMENT SPECTRA

We consider a "large" ocean of rectangular area $A_e$ (periodic boundary conditions) with a plane surface at $z=0$ ($z>0$ is up!). The vertical displacement from equilibrium is $\zeta(x,t)$. This is expressed as a Fourier expansion,

$$\zeta(x,t) = \frac{i}{2} \sum_k \left[ a_k e^{i(k \cdot x - \omega t)} - C.C. \right].$$

(1.1)

Here $x = (x,y)$ and

$$\omega(k) = [k(g+\gamma k^2)]^{1/2}$$

(1.2)

is the linear wave dispersion relation. We use mks units, so $g=9.8 \text{m/s}^2$ and $\gamma = \sigma/\rho$ is $7.5 \times 10^{-5} \text{m}^3/\text{s}^2$ ($\sigma=7.5 \times 10^{-2} \text{n/m}$ represents a nominal value for the surface tension of uncontaminated water).

For linear waves the $a_k(t)$ are constant in time.

The point of writing (1.1) as done is that $k$ is in the direction of wave propagation.

The power spectrum or displacement, with direction of propogation accounted for, is

$$\psi(k) = \frac{A_e}{(2\pi)^2} \left[ \frac{1}{2} | a_k^2 | \right]$$

(1.3)
\[ \langle \zeta^2 \rangle = \int d^2 k \psi(k) = \sum_k \frac{1}{2} \langle | a_k |^2 \rangle . \]  

(1.4)

The symbol "\( \langle \ldots \rangle \)" represents an average over an ensemble of oceans.

The spectrum \( \psi \) is often expressed in the form

\[ \psi(k) = S(k)G(k,\theta), \]  

(1.5)

where \( k = (k,\theta) \) with \( \theta = 0 \) corresponding to the wind direction. By convention,

\[ \frac{\pi}{2} \int G d\theta = 1 . \]  

(1.6)

The spectrum is also expressed in terms of frequency using (1.2) and the relation

\[ S(k) k dk = S_f(\omega) d\omega . \]  

(1.7)

For linear waves the energy/unit area is

\[ E(k) = \frac{\rho \omega^2}{k} \psi(k) , \]  

(1.8)

and the action density is

\[ F(k) = E(k)/\omega . \]  

(1.9)
To describe the evolution of the action spectrum $F$ in the presence of a large scale surface current $U_s(x,t)$ the radiative transport equation is frequently used:

$$\left[\frac{\partial}{\partial t} + \mathbf{x} \cdot \nabla \mathbf{x} + \mathbf{k} \cdot \nabla \mathbf{k}\right] F(\mathbf{k}, \mathbf{x}, t) = S_t = S_{in} + S_w + S_v . \quad (1.10)$$

Here the ray trajectories are calculated from the equations

$$\begin{align*}
x &= \nabla_k H \\
k &= -\nabla_x H \\
H &= \omega(k) + \mathbf{k} \cdot \mathbf{U}_s .
\end{align*} \quad (1.11)$$

On the right-hand side of (1.10), $S_v$ represents the decay rate due to viscosity, $S_w$ the wind growth rate, and $S_{in}$ the effects of nonlinear wave-wave interaction.
2.0 GROWTH AND DECAY OF SURFACE WAVES

In this section we discuss the right-hand side of (1.10). The decay rate due to viscosity is [see, for example, Phillips, 1977]

\[ S_v = -4\nu k^2 \bar{F} . \]  

(2.1)

We shall use here the nominal value of \( 1.1 \times 10^{-6} \text{m}^2/\text{s} \) for the kinematic viscosity \( \nu \). Surface contaminants may require modification of this value.

The growth due to the wind stress is described by the term \( S_w \). The wind stress on the surface is

\[ \tau_w = \rho_a u_*^2 \]  

(2.2)

where \( \rho_a \) is the density of air and \( u_* \) is the friction velocity. We shall use here the values given by Garratt (1977):

\[ u_* = U(10)[1 + 0.089 U(10)]^{1/4} , \]  

(2.3)

where \( U(10) \) is the wind velocity at 10m above the surface. At a height \( z \) we shall use the logarithmic scaling relation

\[ U(z) = \frac{u_* \ln(z/z_*)}{K} . \]  

(2.4)
This is appropriate in the atmospheric surface boundary layer for conditions of neutral stability (a condition that tends to be valid over oceans; see Panofsky and Dutton, 1984, for a detailed discussion and references). An illustration of wind flow over a Minnesota wheat field (Kaimal, et al. 1976) is shown in Figure 2-1, where the planetary boundary layer height $Z_i = 1250 \text{m}$.

The quantity $K = 0.4$ in (2.4) is the von Karman constant and $Z_0$ is the "surface roughness"

$$Z_0 = 0.014u_*^2 \frac{g}{14} \quad (2.5)$$

(Garratt, 1977). This value is appropriate for the oceans. Over land, terrain topology can lead to very different values (see for example, Panofsky and Dutton, 1984).

Several recent analyses of the wave growth data have used the form implied by the Miles Theory,

$$S_w = \beta(k) F_w \quad (2.6)$$

Empirical models for $\beta$ are deduced. To understand the basis for doing this, we briefly review the Miles Theory. Write
Figure 2-1. Comparison of growth rate models for $\beta$. 
\[ \zeta = e^{ik \cdot x} \xi \]

\[ \phi = e^{ik \cdot x} \overline{\phi}, \quad (2.7) \]

where \( \phi \) is the velocity potential. The linearized Bernoulli equation and the kinematic boundary condition read

\[ \frac{\partial \phi}{\partial t} + \frac{\partial \overline{\phi}}{\partial x} + (g + \nu k^2) \overline{\zeta} = 0, \]

\[ \frac{\partial k}{\partial t} = k \overline{\phi} = 0. \quad (2.8) \]

A linear relation is postulated to relate the pressure variation to the displacement,

\[ \overline{p} = \rho (\alpha + i \mu_g) (\omega^2 k) \overline{\zeta}. \quad (2.9) \]

On replacing \( \frac{\partial}{\partial t} \) by \( -i \Omega \), we obtain the dispersion relation

\[ \Omega \equiv \omega [1 + \frac{1}{2} (\alpha + i \mu_g)]. \quad (2.10) \]

The rate \( \beta \) in (2.6) is seen to be

\[ \beta = \mu_g \omega . \quad (2.11) \]
Experimental techniques to measure wave growth vary, and include wave tanks and field measurements. The growth may be observed directly, using wave staffs, laser slope meters, electromagnetic waves, etc. (Donelan et al. 1985, and, Larson and Wright, 1975). An alternative method is to measure the pressure variation $p$ and displacement $\zeta$. Fourier transform these, and deduce the parameters $(a, \mu_g)$ in (2.9). This has been done, for example, by Snyder, et al. (1981), Hsiao and Shemdii (1983), and Hasselmann et al (1986).

An extensive review of wave growth data published prior to 1980 was made by Plant (1981). This data included wind speeds to 15m/s and frequencies in the range

$$\frac{E}{U(10)} < \omega < 40\pi . \quad (2.12)$$

He suggests the model

$$\beta_p = 0.04\left(\frac{U^*}{V}\right)^2 \omega \cos \phi . \quad (2.13)$$

Here $V = \omega/k$ and $\phi$ is the angle between wind and wave direction. Evidence for the factor "cos $\phi$" is very weak. For higher wind speeds, Amorocho and de Vries (1980) describe some growth rate data.
Hsiao and Shemdin (1983) deduce the model

\[ \beta_s = 1.4 \times 10^{-6} (\mu - 1)^2 \omega, \quad (2.14) \]

\[ \mu = 0.85 \left( \frac{U(10)}{V} \right) \cos \phi. \quad (2.14) \]

This is based on measurements in the range

\[ 1 < \mu < 7, \quad 5 < U(10) < 14 \text{m/s}. \quad (2.15) \]

Donelan and Pierson (1987) proposed a model valid for the capillary range, based on the data of Larson and Wright (1975). This tank data included wavelengths \( \lambda \) in the range

\[ 0.7 < \lambda < 7.0 \text{cm}, \quad (2.16) \]

and

\[ 0.17 < u_* < 1.2 \text{m/s}. \quad (2.16) \]

Donelan and Pierson purpose

\[ \beta_D = 2.3 \times 10^{-4} \left( \frac{U(\lambda/2)}{V} - 1 \right)^2. \quad (2.17) \]

The use of tank data to deduce growth rates on the generally much rougher ocean surface has an unknown validity.
A comparison of the above growth rates is illustrated in Figures (2), (3), and (4). The principal discrepancy is for $\lambda$ in the centimeter range for which (2.13) and (2.14) are not expected to be valid. For long wavelengths the agreement is fair. For waves in the 10's of meter range generation by shorter waves is considered by some to be important.

The term $S_{in}$ in (1.10) describes the effects of non-linear wave-wave interactions. The most elaborate model for this has been given by Hasselmann, who used an assumption of weak interactions and cummulant closure to obtain a Boltzmann-like integral. Some recent calculation using the Hasselmann theory have recently been published by van Gastel (1987 a,b).

The complexity and uncertain accuracy of the Hasselmann theory has led to some simplified models. A very simple phenomenological model was suggested by Hughes (1976), which was generalized by Phillips (1985). When $F$ is sufficiently close to an equilibrium form, $F_{eq}$, these models may be approximated as

$$S_T = -\beta_T (F-F_{eq}) .$$

(2.18)

Estimates for $\beta_T$ were made by Watson (1986) using non-linear wave theory. These are reproduced in Figure 2-5.
Figure 2-2. Comparison of growth rate models for β.
Figure 2-3. Comparison of growth rate models for β.
Figure 2-4. Comparison of growth rate models for $\beta$. 
Figure 2-5. The decay rate constant $\beta_T$ of (2.18) as a function of wind speed wavelengths $\lambda$. 
3.0 EQUILIBRIUM SPECTRAL MODELS

Some time ago Phillips suggested that under conditions of adequate fetch and wind duration wave spectra tend toward an equilibrium state. He argued that there should be an equilibrium range between waves moving at wind speed \([U(10)\omega V(k)]\) and the region of viscous decay. In this domain he proposed that

\[
S = \text{constant} \frac{k^4}{k^4},
\]

the constant being dimensionless (no relevant parameters!) and the power of \(k\) determined by dimensional arguments. Kitaigorodskii (in Phillips and Hasselmann, 1986) has recently reviewed the philosophy of equilibrium spectrum models.

Pierson and Moskowitz (1964) proposed a more elaborate spectrum based on Phillips' ideas:

\[
S = S_p(k) = \left(4 \times 10^{-3}\right) \frac{\exp \left[-0.74(k^*/k)^2\right]}{k^4}, \quad \text{(3.1)}
\]

where

\[
k^* = \frac{g}{U^2(10)}. \quad \text{(3.2)}
\]
The JONSWAP experiment led to the replacement of the Pierson-Moskowitz exponential in (3.1) by the "peaked" exponential $e^{-\Gamma}$, where

$$\Gamma = 0.74 \frac{(k*)^2}{k} - 0.5\exp[-\frac{(\sqrt{k} - 0.9\sqrt{k*})^2}{0.4k*}] . \quad (3.3)$$

Increasing evidence for change led Phillips (1986), Donelan et al. (1985), and others to give up the $k^{-4}$ spectral form by a factor of $\sqrt{k}$. They proposed that

$$S = S_B(k) = \frac{a}{a^{-7/2}k_{\pi}-1/2} e^{-\Gamma}, \quad (3.4)$$

where a reasonable choice for the dimensional constant $a$ is

$$a = 2 \times 10^{-3} . \quad (3.5)$$

Donalen et al. suggest some generalization of (3.4) for limited fetch conditions.

Observation by M. Banner (private communication, see also Donalen and Pierson, 1987) suggest that for $k_1 < k < k_2$, a reasonable model for $S$ is

$$S = S_B(k) = \frac{a^*}{k^4} , \quad (3.6)$$
where $a'$ is chosen to give

$$S_D(k_1) = S_B(k_1)$$  \hspace{1cm} (3.7)

at

$$k_1 = 4m^{-1}.$$  \hspace{1cm} (3.7)

A number of observations suggest that for

$$k > k_2 = 200m^{-1}$$  \hspace{1cm} (3.8)

further models are needed for $S$. Bjerkaas and Riedel (1979) have reviewed the data (particularly that of Mitsuyasu, 1977) and have developed an elaborate model for the range (3.8). A simplified version of this model is

$$S = S_R(k) = \frac{4.4 \times 10^{-5} \frac{3k^2}{(1+kZ)}}{k[1+k^2]} \frac{1}{2} \left[ 1 + \exp \left( k-125u_\ast \right) \right],$$  \hspace{1cm} (3.9)

$$k > k_2.$$  \hspace{1cm} (3.9)

Here

$$p = 3 - 0.4341n(u_\ast)$$  \hspace{1cm} (3.10)
with \( u_* \) in m/s and

\[
k_m = \sqrt{\frac{8 \rho}{\tau}} = 363 \text{ m}^{-1}.
\] (3.11)

The resulting spectrum using \( S_D, S_B, \) and \( S_R \) in the ranges described is illustrated in Figures (6), (7), and (8). Donelan and Pierson (1987) model the regime \( k > k_2 \) without using (3.9), but a version of (3.6).

Models for the approach to equilibrium have been suggested by Hasselmann et al (1976). An example is given for which

\[
U(10) = 0, \ t < 0
\]

\[
= U_*, \ \text{a constant for} \ t > 0.
\] (3.12)

A parameter is defined

\[
\Omega_m = 120 \left( \frac{\gamma \tau}{U_*} \right)^{-\frac{3}{7}}, \ \Omega_m > 1,
\]

\[
= 1, \ \text{if above is less than unity}.
\] (3.13)

Then \( \Gamma \) in (3.4) is modified by replacing \( k_* \) by

\[
k_m = k_* \Omega_m^2.
\] (3.14)
Figure 3-1. Composite spectral model for \( S(k) \).
Figure 3-2. Composite spectral model for $S(k)$. 

Wind speed $= 7$ m/s
Figure 3-3. Composite spectral model for $S(k)$. 

Wind speed = 10m/s
The constant $a$ is replaced by

\[ a = 2 \times 10^{-3} \Omega_m^{2/3} . \quad (3.15) \]

The angle dependence in (1.5) was modelled in the first edition of Phillips' book as

\[
G = \frac{1}{\pi}, \quad |\theta| < \frac{\pi}{2}
\]

\[
= 0, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2} .
\]

Tyler et al. (1974) and Mitsuyasu et al. (1975) recommend the form

\[ G = C \cos\left(\frac{\theta}{2}\right) \]

\[ (3.16) \]

where $C$ is a normalizing constant. Mitsuyasu finds that

\[ s(k) \equiv \frac{5}{12}(\frac{k_*}{k})^4 . \quad (3.17) \]

A more recent review has led Donelan et al. (1985) to suggest replacing (3.16) by

\[ G = \frac{2}{2} \operatorname{sech}^2(\sigma\theta) , \quad (3.18) \]

\[ \sigma = 2.9(\frac{k_*}{k})^{3/2} \quad 1 < \frac{k}{k_*} < 4 \]
= 1.2, otherwise. \hfill (3.19)

(Equation (3.19) represents my simplification of a more elaborate representation.)

The diversity of spectral models and the recent dates on many of the references will convince you that further models can be expected. Comparison of the models described above suggests that changes have tended to be more evolutionary than revolutionary in this field and that the existing models can be useful even if of limited precision.
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