Optimal Recursive Maximum Likelihood Estimation

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Abstract: In this paper we derive stochastic differential equations for recursive maximum-likelihood estimates for the joint filtering-parameter estimation problem.

Keywords: Maximum likelihood estimation; stochastic differential equations; Hamilton-Jacobi Equation; Nonlinear Filtering

1. INTRODUCTION

In this paper we would like to consider the joint states and parameter estimation problem for the following non-linear stochastic differential system:

\[ dx(t) = f(x(t),0)dt + g(x(t),0)dw(t), \quad 0 \leq t \leq T \]  

with the observation system

\[ dy(t) = h(x(t),0)dt + dv(t), \quad 0 \leq t \leq T. \]  

In the above, \( w(t) \) and \( v(t) \) are standard independent Brownian motions, \( f, g, h \) are at least three continuously differentiable with bounded derivatives with respect to \( x \in \mathbb{R} \) and \( \theta \in \mathbb{R} \) and \( g(x,0) \neq 0 \) for all \( x \neq 0 \).

In addition we assume

\[ E \int_0^T |h(x(t),\theta)|^2 dt < \infty, \]  

and the initial state satisfies

Either 1) \( x(0) = x_0 \in \mathbb{R} \)  

or 2) \( x(0) = x_0 \), a random variable with density \( p_0(x) \geq C_0(h_0; \mathbb{R}), p_0(x) > 0 \).

Let \( x_\theta(t) \) denote the solution of the stochastic differential equation (1.1) starting at \( x_0 \). Then from a result of Kunita [2], we know that \( x_\theta(t) \) is a \( C^2 \)-diffeomorphism, and the inverse map \( x_\theta^{-1} \) satisfies a backward stochastic differential equation.

Let

\[ A(0,t) = \exp \left[ \int_0^t h(x(s),\theta)dy(s) - \frac{1}{2} \int_0^t h^2(x(s),0)ds \right] \]

\[ = \exp \left[ \int_0^t h(x_\theta^{-1}(t),\theta)dy(s) \right] \]

where \( \theta \) denotes a backway stochastic differential (and backward Ito integral respectively).

Let

\[ L(\theta,t) = E(L(\theta,t)|x(t) = x), \]  

where \( E \) denotes expectation with respect to the path space measure of \( x(\cdot) \).

As a criterion, we choose as an estimate

\[ \hat{\theta}(t) = \arg \max L(x,\theta,t), \quad x, \theta \]  

which is a maximum likelihood criterion.

2. STOCHASTIC HAMILTON-JACOBI BELLMAN EQUATION FOR \( L(x,\theta,t) \)

Using the work of Fleming-Mitter [1] and the theory of backway stochastic differential equations [cf. Kunita, loc.cit.] one can show that

\[ S(x,\theta,t) = -\ln L(x,\theta,t) \]  

satisfies the stochastic Bellman Hamilton-Jacobi equation:

\[ dS(x,\theta,t) = \sigma(x,\theta) \frac{\partial S}{\partial x} dx dt + \frac{1}{2} \sigma^2(x,\theta) \frac{\partial^2 S}{\partial x^2} dt \]

\[ + h(x,\theta) \frac{\partial S}{\partial \theta} dt - h(x,\theta) dy(t) \]

where

\[ \sigma(x,\theta) = \frac{1}{2} \sigma(x,\theta)^2 \]

\[ \sigma(x,\theta) = 2\sigma - 2\sigma(x,\theta) \frac{\partial}{\partial x} f(x,\theta) \]

\[ = -\ln P^\theta_{x,t} \]  

where under our assumptions \( P^\theta_{x,t} \), the density corresponding to the \( x(\cdot) \) process exists and is positive for all \( x,t \).

We now define a recursive maximum likelihood estimate. By applying the Generalized Ito Differential Rule [cf. Kunita, loc.cit.], we get
\[ \frac{\partial V}{\partial S} + V^2 \sigma_2(t) + \frac{1}{2} V^2 \sigma(t,t) - V(\theta) d(t,t) = 0 \]  

(2.3)

where

\[ V = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial \theta} \end{pmatrix}, \quad \xi(t) = \begin{pmatrix} \eta(t) \\ \delta(t) \end{pmatrix} \]

which is obtained from the stationarity condition

\[ \frac{\partial V}{\partial S} = 0. \]  

(2.4)

In (2.3) all partial derivatives are computed along \( \xi, \eta \), which is obtained from the stationarity condition (2.4).

Assuming \( V^2 \) is invertible, we obtain a maximum likelihood trajectory for \( \xi(t) \) from (2.2), (2.3) and (2.4) and using \( \frac{\partial V}{\partial S} = V dS \).

A rigorous derivation of these results will be presented elsewhere.

REFERENCES


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