THE APPLICATION OF STATISTICAL TURBULENCE THEORY TO
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THE APPLICATION OF STATISTICAL TURBULENCE THEORY TO CONVECTIVE INSTABILITIES

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1 November 1986

Technical Report

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THE APPLICATION OF STATISTICAL TURBULENCE THEORY TO CONVECTIVE INSTABILITIES

Rino, Charles L.

This work was sponsored by the Defense Nuclear Agency under RDT&E RMC Code B3220867663 RW RA 00001 25904D.

Statistical turbulence theories have been used effectively to characterize spectral density functions in fluid turbulence, and in turbulent plasmas, magnetic fields, and passive scalars in turbulent fluids. In this report, we review the applications to convective instabilities of the turbulence theories developed originally and independently by Kraichnan and Kadomtsev, but generalized for ionospheric applications by Sudan. These instabilities are believed to be the dominant structuring mechanism in equatorial spread F, artificial barium clouds, and in high-altitude nuclear plumes at late times.

We have emphasized the ramifications of the cross-field anisotropy unique to convective instabilities, which have not been treated previously. We show that the limiting form of the one-dimensional spectral density functions measured perpendicular to the E x B drift direction is proportional to $k^{-1}$ where $k$ is the wave number.
**CONVERSION TABLE**

Conversion factors for U.S. Customary to metric (SI) units of measuremen

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<th>MULTlPLY TO GET</th>
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<th>TO GET</th>
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<tr>
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<td>torr (mm Hg. 0°C)</td>
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*The bequerel (Bq) is the SI unit of radioactivity, 1 Bq = 1 event/s
**The Gray (Gy) is the SI unit of absorbed radiation.
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SECTION 1
INTRODUCTION

The simplest mathematical model that contains the essential physics of a convectively unstable plasma is given by the coupled, nonlinear differential equations

\[ \frac{\partial \mathbf{N}}{\partial t} + (\mathbf{b} \times \mathbf{\nabla} \phi) \cdot \mathbf{V_N} = K \mathbf{V_N}, \]  

(1)

and

\[ \mathbf{V} \cdot (\mathbf{N} \mathbf{V} \phi) = D \cdot \mathbf{V_N}, \]  

(2)

where \( N(p,t) \) is the electron density in a plane normal to the magnetic field. The vector \( D/B \) is the cross-field drift velocity caused by the combined forces of gravity, neutral winds, and electric fields.

Thus, \( D \) can be thought of as an effective electric field. The potential, \( \phi(p,t) \), which is established by Eq. (2) from the instantaneous electron density configuration, modifies the effective electric field in such a way that, as time evolves, small-scale structures are drawn out of the trailing edge of high-density regions and are thereby enhanced. The structure is ultimately removed by cross-field diffusion at a rate determined by the diffusion coefficient \( K \).

The physics of this two-dimensional model and its potential limitations are discussed in detail in Perkins et al. [1973], Ossakow [1981], and Zalesak et al. [1985]. Here, we shall discuss the applications of statistical turbulence theory to determine the quasi-steady-state spectral density function (SDF) that characterizes the late-time evolution of plasma structure implied by Eqs. (1) and (2). As yet, no viable theory has emerged that can explain the measured kilometer-scale...
spectral characteristics of naturally occurring equatorial F-region ionospheric irregularities that severely affect satellite radio signals [Aarons, 1982]. Yet, it is generally believed that the development and evolution of these irregularities can be modeled by using straightforward modifications of Eqs. (1) and (2) [Zalesak et al., 1982]. Our motivation in undertaking this study was to determine if formal turbulence theory can be modified to explain the results of recent data analyses.

Ott and Farley [1974] first pointed out that Eq. (1) is mathematically identical to the two-dimensional Navier-Stokes (NS) equation when it is written in terms of vorticity,

\[ \mathbf{w} = \nabla \times \mathbf{v}, \]  

where \( \mathbf{v} \) is the incompressible fluid velocity (\( \nabla \cdot \mathbf{v} = 0 \)). If \( \mathbf{v} \) is confined to a plane, it is easily shown that the momentum balance of inertial, pressure, and viscous forces can be written as

\[ \frac{\partial \mathbf{w}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{w} = \nu \nabla^2 \mathbf{w}, \]

where \( \nu \) is the kinematic viscosity. Thus the momentum balance for two-dimensional fluids when expressed in terms of scalar vorticity is identical to the continuity equation for electron density; however, to complete the formal parallel we must identify the electron drift velocity

\[ \mathbf{v}_e = \mathbf{a}_z \times \nabla \psi / B \]

with the fluid velocity \( \mathbf{v} \).

Because \( \mathbf{v} \) is divergence free, it too can be derived from a potential function. If we let \( \mathbf{v} = \mathbf{a}_z \times \nabla \psi \), it follows from Eq. (3) that

\[ \nabla^2 \psi = -\mathbf{w}. \]
Thus, for inertially driven, two-dimensional turbulence, the potential function that generates the velocity structure is related to the scalar vorticity by Poisson's equation. Because of the profound differences between Eqs. (2) and (6), theories that address the turbulent structure of $w$ cannot be applied directly to the convective plasma system described by Eqs. (1) and (2). Indeed, even if one were to accept ordinary turbulence as the source of the structure in the velocity field, the electron density must act as a passive scalar to map the inertially driven velocity structure [Kelley and Ott, 1978]. Ideally, one would proceed from a general plasma theory that accommodates both collisional and inertial forces [Kintner and Seyler, 1985]; however, the collisional limit is the most difficult to treat, and it appears to have some unique properties.

Consider that in Kolmogoroff's theory of fully developed turbulence, locally unstable eddies spontaneously break up into smaller eddies until they are removed by viscous dissipation. In sharp contrast, convectively unstable density irregularities evidently do not spontaneously break up or bifurcate. Rather, early in the nonlinear phase of the structure development, fingerlike striations develop with a comparatively small-size distribution. Evidently the striations continue to elongate without further significant change in their individual shapes [McDonald et al., 1981; Zalesak et al., 1985]. This process has been referred to as "freezing." Clearly, the structure dynamics are vastly different in the collisional and inertial models, and we should expect these differences to manifest themselves in a viable theory.

Formal plasma turbulence theories have been developed by Kraichnan and Kadmotsev. Both theories use the Direct Interaction Approximation, which was developed by Kraichnan [1958, 1959]; his turbulence theory is usually referred to as the DIA. Kadmotsev's method, which was developed more intuitively, is usually referred to as the the Weak Coupling Approximation (WCA) [Kadmotsev, 1965]. Sudan and Pfirsch [1984] have shown that the two methods ultimately give rise to formally identical mathematical models; however, the WCA form suggests a somewhat simpler
procedure for deriving spectral characteristics. Following Sudan and Pfirsch's approach, we shall start with the DIA because it proceeds directly from Eq. (1). We shall then recast the equations in the WCA form and follow Sudan's prescription for deriving the form of one-dimensional spectral-density function. The main difference is that we have attempted to relax the assumption of isotropy, which is critical in all the applications of the method to date [Sudan and Keskinen, 1979; Sudan, 1983; Sudan and Keskinen, 1984]. This is achieved by approximating the anisotropy, which manifests itself in the "frozen" striation distribution, by a one-dimensional transfer of scalar variance in the Fourier domain. This provides an alternate means of obtaining a one-dimensional conservation equation in place of the angle-averaged form that is normally used.

The first step is to transform Eq. (1) into the spatial Fourier domain. We assume that $N(\rho,t)$ and $\phi(\rho,t)$ admit spectral decompositions of the form

$$
\begin{align*}
N(\rho,t) & = \sum \hat{N}(k,t) \exp \{ik.\rho\} \\
\phi(\rho,t) & = \sum \hat{\phi}(k,t) \exp \{ik.\rho\}
\end{align*}
$$

(7)

where

$$
k = (n\Delta k_x, m\Delta k_y)
$$

(8)

for all integer values of $n$ and $m$. Thus, $N$ and $\phi$ are periodic on the lattice bounded by $L_x = 2\pi/\Delta k_x$ and $L_y = 2\pi/\Delta k_y$. As discussed by Kraichnan [1975], this imposes no limitations of practical significance. Also, because $N$ and $\phi$ are real, $\hat{N}$ and $\hat{\phi}$ have the hermitian property

$$
\begin{align*}
\hat{N}(k) & = \hat{N}^*(-k) \\
\hat{\phi}(k) & = \hat{\phi}^*(-k)
\end{align*}
$$

(9)
Substituting Eq. (7) into Eq. (1) and performing some standard Fourier manipulations gives the spectral domain form of Eq. (1), namely,

$$\frac{\dot{\hat{N}}(k)}{\dot{t}} + k^2 \hat{N}(k) = \sum_{A} (\hat{b} \times \hat{k'} \times \hat{k''}/B) \hat{\phi}(k')\hat{N}(k'') \quad .$$  \hspace{1cm} (10)

The symbol A means that only those modes that satisfy the triangle equality

$$k = k' + k'' \quad .$$  \hspace{1cm} (11)

contribute to the summation. We could have included the delta function \(\delta(k - k' - k'')\) in the summation, but the alternate notation is more compact. In the interest of notational efficiency, we have also suppressed the independent time variable \(t\).

Next, we multiply Eq. (10) by \(\hat{N}^*(k)\) to obtain

$$\frac{\dot{\hat{N}}(k)^2}{\dot{t}} + 2k^2 |\hat{N}(k)| = E(k) \quad .$$  \hspace{1cm} (12)

where

$$E(k) = 2 \sum_{A} (\hat{b} \cdot \hat{k'} \times \hat{k''}/B) \hat{\phi}(k')\hat{N}(k'') \quad .$$  \hspace{1cm} (13)

Equations (12) and (13) characterize the temporal evolution of the spectral intensity for a single realization of the process. The nonlinear mixing term \(E(k)\) has the important property

$$\sum E(k) = 0 \quad ,$$  \hspace{1cm} (14)

which does not depend on how \(\hat{\phi}(k)\) is related to \(\hat{N}(k)\). Thus, in Eqs. (1) and (4) the nonlinear interaction of the Fourier modes does not change the scalar variance.
or enstrophy (the name given to the variance of scalar vorticity) in the NS system. This conservation property is the basis for developing a statistical turbulence theory.

A formal statement of the conservation of scalar variance can obtained from Eqs. (12) and (14) as

$$\frac{\partial \Phi}{\partial t} + 2k \sum k^2 |\hat{N}(k)|^2 = 0 \quad .$$

(16)

Thus, the structure evolution is such that the only change in scalar variance is through diffusion. Turbulence theory attempts to characterize the average spectral characteristics in the quasi steady state where the tendency for structure to form is essentially balanced by the structure removal process. In fluid turbulence, energy must be supplied continuously to sustain this steady state. In collisional plasma structuring, a reservoir of structure at low spatial frequencies can sustain structure growth. Thus, in our application of the DIA, we do not include a forcing function.

We envision many realizations of the process with small, randomly irregular changes in the initial conditions, but with similar structure reservoirs at small spatial wave numbers to sustain the growth of small-scale structure at late times. We expect the ensemble average of the spectral intensities from each realization, formally

$$\phi(k) = \langle |\hat{N}(k)| \rangle \quad ,$$

(17)

to yield an identifiable SDF, although it is well known that Eqs. (1) and (2) will not support a true steady state. If we attempt to evaluate Eq. (17) by using Eqs. (12) and (13), we encounter the third-order moment $\langle \phi(k') \hat{N}(k''') \hat{N}^2(k) \rangle$. Iterating the procedure will always introduce a moment of higher order. Thus, irrespective of the mathematical subtleties associated with statistical stationarity, the formal moment
equations are indeterminant. Direct interaction is a closure approximation in which the third-order moments are evaluated in terms of fourth-order moments containing the temporal response function of Eq. (1) to small changes in $\phi$. The fourth-order moments are evaluated by using the well-known relation for Gaussian processed, although the Gaussian assumption itself is not critical to the development.
SECTION 2

THE DIA APPROXIMATION

The Direct Interaction approximation is discussed in considerable
detail in Leslie [1973]. For our purposes here, we have simply followed
the step-by-step procedure he presents in his Chapter 4.3. The computa-
tional details are omitted. As a preliminary, we first consider the
response of Eq. (10) to a small perturbation in \( \hat{\phi} \). To terms that are
first order in \( \delta \hat{\phi} \) and \( \delta \hat{N} \), we have

\[
\delta \hat{N}(k;t) = \sum_{\mathbf{b}, \mathbf{k} \neq \mathbf{k}'} \int_{-\infty}^{t} G(k;t,t') \delta \hat{\phi}(k;t') \hat{N}(k';t') \, dt'.
\]

where \( G(k;t,t') \) is the solution to

\[
\left[ \frac{\partial}{\partial t} + \mathbf{k} \mathbf{k}' \right] G(k;t,t') - \sum_{\mathbf{b}} (\mathbf{b} \cdot \mathbf{k} \times \mathbf{k}') / B \hat{\phi}(k';t) G(k';t,t') = \delta(t-t').
\]

To apply the DIA, we first expand \( \hat{N} \), \( \hat{\phi} \), and \( \hat{G} \) in series of the form

\[
\hat{N} = \hat{N}^{(0)} + \varepsilon \hat{N}^{(1)},
\]

where \( \varepsilon \) is the order of the nonlinear terms in Eqs. (10) and (19). By
substituting these expansions into Eqs. (10) and (19) and equating terms
of equal order, we obtain equations for the first-order terms \( \hat{N}^{(1)} \) and
\( \hat{G}^{(1)} \) in terms of the zeroth-order terms. Consistent with this approxi-
mation, we let

\[
\hat{\phi}(k) = W(k) \hat{N}(k),
\]
where \( W(k) = W(-k) \) to preserve the hermitian property of \( \hat{\phi}(k) \). This is tantamount to linearizing the potential equation, Eq. (2), but within the context of the DIA, this is a local approximation that preserves the essential nonlinearity in the problem.

The lowest order term in the expansion of \( \hat{N} \) is assumed a Gaussian variable, and the corresponding term in the expansion of \( G \) is assumed exact. After considerable algebraic manipulations, we obtain the following equations for the time-dependent SDF:

\[
\frac{1}{2} \frac{\partial S_N(k,t,t')}{\partial t} + i \mathbf{k}^2 S_N(k,t,t') = \frac{A}{4} (\hat{b} \cdot \mathbf{k} \times \mathbf{k}' / B)^2 f(k',k'') \times \int_{-\infty}^{t'} G(k',t,t'') S_N(k',t,t'') S_N(k''',t',t'''') dt''' \\
- \int_{-\infty}^{t} G(k',t,t'') S_N(k,t,t'') S_N(k''',t',t'''') dt''' \]  

(22)

where

\[
f(k',k'') = \frac{1}{2} |W(k') - W(k'')|^{2} ,
\]

(23)

\[
\left[ \frac{\partial}{\partial t} + i \mathbf{k}^2 \right] G(k,t,t') = -\frac{A}{4} (\hat{b} \cdot \mathbf{k} \times \mathbf{k}' / B)^2 f(k',k'') \times \int_{t'}^{t} G(k''',t,t''') S_N(k'',t,t''') G(k,t'',t''') dt''' + \delta(t - t') \]

(24)

and

\[
S_N(k,t,t') = \langle N(k,t) N^{*}(k,t') \rangle .
\]

(25)

As the DIA equations are written here, they accommodate both the NS and convective plasma equations. For NS, it follows from Eq. (6) that
which is exact. For a convectively unstable plasma, it follows from Eq. (2), essentially by dropping the nonlinear term, that

$$ W(k) = -k^{-2} $$

(26)

Because Eq. (26) is isotropic, the functions in Eqs. (22) and (24) depend only on the magnitude of $k$ and the equations can be reduced to a one-dimensional form by integrating over all directions. When Eq. (27) applies, however, the basic equations admit no obvious simplification.

Thus, we transform the complete DIA equations to their more compact WCA form. We first assume that the time dependences are functions only of the corresponding time differences, whereby Eqs. (22), (24), and (25) can be transformed to the temporal frequency domain. After some straightforward manipulations, we obtain the equivalent WCA equations

$$(w - iKk^2) S_N(k;w) = -\sum \left( \hat{b} \cdot k' \times k''/B \right)^2 f(k';k'')$$

(28)

and

$$\Gamma(k;w) = \sum \left( \hat{b} \cdot k' \times k''/B \right)^2 f(k';k'')$$

(29)
The new function \( \Gamma(k;w) \) is related to \( G(k;w) \) by the equation

\[
G(k;w) = \frac{2\pi i}{[w - iK^2 + \Gamma(k;w)]} .
\] (30)

Eq. (28) can be written the equivalent form

\[
|w + iK^2 + \Gamma(k;w)|^2 S_N(k;w) = \sum_{\Delta}(b.k' \times k''/B)^2 f(k',k'')
\]

\[
\times \int S_N(k'';w')S_N(k''';w - w') \, dw' .
\] (31)

which isolates the nonlinear damping function \( \Gamma(k;w) \). The WCA equations are usually presented in this form [Eqs. (18) and (19) in Sudan and Keskinen, (1979)].

In the general forms presented here, neither the DIA nor the WCA equations can be solved, but we can establish an important property of Eqs. (22) and (28). Working from Eq. (28), we define the argument of the summation as \( R(k, k', k'') \), which has the property

\[
R(k,k',k'') + R(k',k,k'') + R(k,k'',k') = 0 .
\] (32)

It follows from Eq. (32) that the summation of the right-hand side of Eq. (28) over all \( k \), allowing for the triangle equality constraint, is zero. Thus, the conservation property of the basic equations is preserved in the DIA or WCA equations for the spectral density function itself. This property is used to determine the spectral characteristics.
SECTION 3
DETERMINATION OF SDF CHARACTERISTICS

Sudan and Keskinen [1979] have argued that $S_N(k;w)$ will be highly peaked about $w = 0$ with a characteristic width that depends on $\Gamma(k;w)$. They have postulated a functional relation of the form

$$S(k;w) = \phi(k)g(w/\Gamma_k), \quad (33)$$

where $\Gamma_k$ is the direction averaged value of $\Gamma(k;w)$. For the moment, let us consider only the fact that we have effectively assumed that the time or temporal frequency dependence of $S_N(k;w)$ is a multiplicative function. With this assumption alone, we can rewrite Eq. (28) as

$$\frac{\partial \phi(k)}{\partial t} + k \phi^2 \phi(k) = 2 \phi(k) \sum_{k'}(\hat{b} \times \hat{k'}) \frac{A}{B} f(k';k''') \phi(k') \phi(k'')$$

$$- \phi(k) \Gamma(k) \quad (34)$$

where

$$\phi(k) = 2 \pi \int \int_{-\infty}^{\infty} \frac{g(w'/\Gamma_k g(w''/\Gamma_k')}{[w + i k^2 + \Gamma(k;w)]} \frac{dw \, dw'}{2\pi 2\pi}, \quad (35)$$

and $\Gamma(k)$ denotes the value of $-i\Gamma(k;w)$ integrated over all $w$. Both $\phi(k)$ and $\Gamma(k)$ are purely real.

At this point, the usual approach is to assume that $\phi(k)$ is independent of the direction of $k$, as would be strictly true when Eq. (26) holds. Indeed, this same property would then apply to $\Gamma(k)$ and $\phi(k)$. The remaining term in Eq. (34) can be evaluated and a conservation
relation formulated in terms of the magnitude $k$. One then characterizes the transfer of scalar variance from large to small wave numbers. It happens, however, that the second term in Eq. (34) dominates this process, and a differential equation of the form

$$\frac{d}{dt} \int_{0}^{k} \phi(k') dk' + \int_{0}^{k} \Gamma(k') \phi(k') dk' = \int_{0}^{k} k'^2 \phi(k') dk' , \quad (36)$$

where $a$ is a small positive number results. The lower limit of the nonlinear damping term is nonzero because arbitrarily large structures are essentially background inhomogeneities.

In the stationary state, the time derivative is replaced by a constant. If $K = 0$, it follows that $\Gamma(k) \phi(k)$ must be a function, say $Y(k)$, with the property

$$Y(k) = aY(ak) , \quad (37)$$

so that the integral over $ak$ to $k$ is zero. The solution to Eq. (37) is proportional to $k^{-1}$. To determine the form of $\phi(k)$, we must establish a relation between $\Gamma(k)$ and $\phi(k)$. Sudan and Keskinen [1979] have derived an approximate form by using the Gaussian form

$$g(w/\Gamma_k) = |1/[2\pi \Gamma_k^2]|^{1/2} \exp[-w^2/\Gamma_k^2] \quad (38)$$

for $g(w/\Gamma_k)$. With appropriate assumptions, they were able to derive the approximate relation (for two dimensions)

$$\Gamma_k = D r_i^{1/2} \Gamma_k \quad (39)$$

where $I_k$ denotes the angle averaged value of $\phi_N(k)$. It follows that $I_k$ is proportional to $k^{-2}$. This would seem to imply that in the "cascade" range, the one-dimensional spectral density function is proportional to $k^{-1}$, which seems to be close to the measured value. As we have noted,
however, this approach is inconsistent with the anisotropy of the collisional instability.

We suggest applying the argument in a different way. Because the problem is approximately one dimensional, with most of the small-scale structure characterized by Fourier modes along D, it seems reasonable to replace the angle averaged quantities by integrals over the direction orthogonal to D, which we take to lie along the x axis. We obtain a relation identical to Eq. (36) with \( \theta(k) \) replaced by \( \theta(k_y) \). It follows that in the cascade range,

\[
\theta(k_y) \Gamma(k_y) = k_y^{1/2}
\]

(40)

and

\[
\Gamma(k_y) = k_o v_{x,y}^{1/2} \theta^{1/2}(k_y)
\]

(41)

from which it follows that \( \theta(k_y) = k_y^{-1} \); however, because of the one-dimensional approximation, this result should be regarded as a limiting form.
SECTION 4
DISCUSSION

We have shown that a consistent application of statistical turbulence theory to convective plasma instabilities must accommodate the anisotropy of the structure, which is unique to the collisional limit. To get some handle on how this anisotropy might affect the theory, we have followed the general prescription suggested by Sudan, but attempted to relax the isotropy assumption by going to the opposite extreme of a one-dimensional transfer of scalar variance. For well-developed structures transverse to the dominant convection direction, the limiting form of the one-dimensional spectral density function is $k^{-1}$. Along the convection direction, the one-dimensional spectrum should be "shape-dominated" and, therefore, $k^{-2}$ at long wavelengths, giving way to a much steeper power law where diffusion is effective. Indeed, in the diffusion range the spectrum should be isotropic, at least if classical cross-field diffusion is the dominant mechanism for structure removal.

Ott and Farley [1974] first pointed out that a $k^{-1}$ one-dimensional spectral density function for density structure is the form that results from the purely dimensional arguments of turbulence theory; however, they questioned the existence of a truly inertial-like subrange because it implied no dependence on the initial gradient; moreover, the available data then seemed to imply a $k^{-2}$ form. In our development, we have not addressed the question of the constants or other factors that determine the level of the spectrum in addition to its shape. The dependence on a length scale comes from the $a$ in Eq. (36). For fluid turbulence, $a$ is related to the Kolmogoroff constant, but the theory as it is applied here does not determine its value nor does it identify an actual physical scale that $a$ can be identified with, although an outer scale or maximum size for transfer of scalar variance is reasonable.
We also note that Batchelor [1959], in treating the structure of passive scalar additives in turbulent fluids, showed that in the presence of a linear velocity shear the average one-dimensional spectrum for a conserved scalar governed by Eq. (1) is also $k^{-1}$. In fluid turbulence, this comes about when the viscous cutoff for velocity turbulence occurs at a larger scale than the diffusive cutoff for the scalar additive. This is referred to as the viscous convective subrange [Tennekes and Lumley, 1972]. The velocity shear is a consequence of the turbulent dissipation mechanism. To apply Batchelor's theory to convective instabilities, the presence of the velocity shears must be established independently. The direct application of the turbulence theory is more satisfying.

There is yet another turbulence theory argument that gives rise to limiting spectral forms that have power-law slopes less than two. These are discussed in Kintner and Sevler [1985], and they involve a dual cascade, which can occur in two dimensional fluids. Indeed, the fact that Eq. (4) conserves enstrophy introduces a second conserved quantity because energy is conserved as well. Kraichnan [1967] showed that under these conditions two types of cascade can occur. At scale sizes above a stirring scale where the structure is initiated, there is an upward cascade of enstrophy and scaling arguments give a $k^{-1}$ form for the energy spectrum. Below the stirring scale, there is a downward cascade of energy and the conventional $k^{-5/3}$ spectrum results. To the extent that density is purely a passive scalar in this process and is related to velocity via Eqs. (21) and (27), its spectrum assumes the same two-component spectral form. The corresponding forms that apply when Eqs. (21) and (26), which are exact, hold are given in Table I of Kintner and Sevler, [1985]. The problem here is that the conditions under which Eq. (27) can be used give rise to anisotropy, in any case, would preclude the possibility of density behaving as a passive scalar as well. Indeed, Sudans's theory has yet to be generalized to accommodate more than a single conserved quantity. More important, the experimental
data, although showing two-component forms with fairly shallow low-frequency indices, have not shown the two-component form predicted by the dual cascade theory.

To summarize, the two-dimensional spectral density function for convective instabilities as modeled by Eqs. (1) and (2) is anisotropic at the longest wavelengths with a power-law index approaching -1 in the direction perpendicular to the convection direction and steeping to -2 or greater along the convection direction. Beyond the freezing scale where diffusion becomes effective, the spectrum would be more nearly isotropic. The ramifications of this model for propagation models are probably not severe because the slope change occurs only over a limited scale-size regime. As a means of testing this hypothesis, however, the propagation issues should be pursued.
SECTION 5

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