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On the Stability of a Stratified Shear Layer

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On the Stability of a Stratified Shear Layer

The stability of a stratified shear layer is investigated using an exponential density profile and a laminar shear flow with a continuous velocity distribution. It is shown that an exact stability boundary can be obtained from an inhomogeneous inviscid fluid under the action of gravity without the need to impose the Boussinesq approximation. The stability boundary is given by $J - \frac{k^2}{\beta^2} (1 - \beta^{1/4} - \beta^2)$ where $\beta$ is the ratio of the velocity and density gradient scale sizes, $J$ is the Richardson number and $k$ is the perpendicular wavenumber normalized to the velocity gradient scale size; this reduces to the stability boundary derived by Drazin in the limit $\beta = 0$. The solution allows for $c = \beta/2$ where $c$ is the normalized phase velocity.
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ON THE STABILITY OF A STRATIFIED SHEAR LAYER

I. INTRODUCTION

The Kelvin-Helmholtz instability is a widely studied instability driven by velocity shear in neutral fluids as well as in ionized gases\textsuperscript{1-5}. In addition to various examples in neutral fluids, the situation of a shear layer under the action of gravity is also encountered in space (e.g., equatorial spread F, the plasmapause) and laboratory (e.g., laser fusion experiments) plasma phenomena. Recent observations show that large velocity shears exist at the equatorward edge of the diffuse auroral boundary\textsuperscript{7}. Furthermore, the plasmapause boundary is conjectured to be unstable to a ballooning type shear flow instability\textsuperscript{8}. In the case of the equatorial ionosphere, the plasma is confined by a uniform magnetic field and the flow velocity perpendicular to the magnetic field is sheared as a function of the altitude. Gravity is directed opposite to the density gradient so that the system is prone to gravity driven interchange as well as to the shear driven Kelvin-Helmholtz instability. Based on this geometry and physics, linear stability analysis has been performed on a collisional plasma in the equatorial F region of the ionosphere, and a general mode structure equation has been derived\textsuperscript{4,5} which reduces to the general Rayleigh equation dealt with in detail by Drazin\textsuperscript{2}.

In the neutral fluid literature (for example, Drazin\textsuperscript{2}) and by Vinas and Madden\textsuperscript{8}, the Boussinesq approximation is generally assumed in performing the analysis of the general Rayleigh equation; this assumption...
amounts to ignoring all the density gradient terms except the density gradient term that contributes to the buoyancy. In this paper, we relax the Boussinesq approximation and show that the new stability boundary is different than that obtained by Drazin$^2$: the stability boundary is dependent on the density gradient, leading to more stringent restrictions on the Richardson number, and the stability boundary is determined for modes with phase velocity half that of the peak background flow velocity. The following analysis is based on a smooth velocity profile and an exponential density gradient.

II. THEORY

The geometry of the plasma and field configuration used in the analysis is as follows: the magnetic field is uniform and in the $z$ direction ($B = B \hat{z}$), the plasma is inhomogeneous along the $x$ direction, $n = n_0(x)$, gravity is acting along the negative $x$ direction $g = - g \hat{x}$, and the flow velocity along the $y$ direction is sheared in the $x$ direction, $V = - V_0(x) \hat{y}$.

We consider low frequency fluctuations ($\omega \ll \omega_{ce}$ where $\omega_{ce}$ is the electron cyclotron frequency) so that electron inertia is ignored. Two dimensional perturbations are considered. The perturbed quantities vary as $\phi = \phi(x) \exp \left( -i(k_y y - \omega t) \right)$, where $\omega = \omega_r + i \omega_i$. The equation that describes the perturbed electrostatic potential is given as

$$\frac{\partial^2 \phi}{\partial x^2} + p(X) \frac{\partial \phi}{\partial X} + q(X) \phi = 0,$$

where $p(X)$ and $q(X)$ are given by

$$p(X) = \frac{\partial^2 n_0}{\partial X^2},$$

$$q(X) = \frac{\partial n_0}{\partial X}.$$
\[ q(X) = -\kappa^2 + \frac{1}{n} \left[ \frac{\partial^2 U}{\partial x^2} + \frac{\text{sin } n_0}{\partial x} \frac{3U}{\partial x} + \frac{gL_v/V_y^2}{\partial x} \right] \text{sin } n_0, \]  

(3)

where \( V_0 = \frac{\partial y}{\partial x} U(X), c = \omega/k_y V_y, x = x/L_v, \) and \( \kappa = k_y L_v. \) Here \( L_v \) and \( L_n \) are the gradient scale lengths of the velocity and density, respectively. Equation (1) is a general equation that describes Rayleigh-Taylor and Kelvin-Helmholtz instabilities for arbitrary density profiles and drift velocities. This equation is identical to that obtained for counter-streaming neutral fluids in a gravitational field.  

In this paper we consider an exponential profile for the density such that \( n_0(x) = n_0 \exp (\beta x) \) where \( \beta = L_v/L_n \) and represent the drift velocity by a smooth profile \( U(x) = \tanh (x). \) We let \( \beta \) be arbitrary and thus generalize the problem analyzed by Drazin. This is the main objective of this paper. The coefficients \( p(x) \) and \( q(x) \) can be rewritten as

\[ p(x) = \beta \]  

(4)

\[ q(x) = -\kappa^2 + \frac{2\text{sech}^2 x (-2\tanh x + \beta)}{(c - U)} + \frac{J}{(c - U)^2} \]  

(5)

where \( J = (\beta/L_n)/(V_y^2/L_v^2). \) Using the Boussinesq approximation Drazin set \( p = 0 \) and ignored \( \beta \) in the second term in \( q; \) this leads to the condition \( c = 0. \) We relax these assumptions in the following analysis.

Using the transformation \( \phi = t \exp \left[ -\int dX \frac{p(X)}{2} \right] \) we write (1) as

\[ \frac{\partial^2 \phi}{\partial x^2} = Q(X) \phi = 0 \]  

(6)

where \( Q(X) = q - p'/(2) - p^2/c. \) Equations (4) and (5) together with (6) yield

\[ \frac{\partial^2 \phi}{\partial x^2} + \left[ -\kappa^2 - \frac{\beta^2}{4} + \frac{\text{sech}^2 x (-2\tanh x + \beta)}{(c - U)} + \frac{J}{(c - U)^2} \right] \phi = 0. \]  

(7)
With $U$ as the independent variable, (7) can be written as

$$(1 - U^2) \frac{\partial^2 \phi}{\partial U^2} - 2U \frac{\partial \phi}{\partial U} + \left[ \frac{-\alpha^2 + \frac{s^2}{4}}{1 - U^2} \right] + 2 + \frac{2c - \varepsilon}{(U - c)} + \frac{J}{(U - c)^2(1 - U^2)} \phi = 0. \tag{8}$$

Equation (8) is a second order differential equation with four singular points at $U = -c, c, -1,$ and $1$. The new terms in this equation are $-\left(\frac{s^2}{4}\right)/(1 - U^2)$ and $(2c - \varepsilon)/(U - c)$; the former, obtained by retaining the first derivative term in (1), introduces a lower cut-off in the wavenumber, and the latter, obtained by retaining the cross term involving the density inhomogeneity and velocity inhomogeneity, allows $c$ to take the value $s/2$.

We assume a solution to (8) of the form

$$\phi = (U + 1)^{\mu_+} (U - 1)^{\mu_-} (U - c)^{\lambda}, \tag{9}$$

where the exponents $\mu_+$ and $\lambda$ are determined by

$$2\mu_+ = \left[ \alpha^2 - \frac{s^2}{4} - \frac{J}{(1 - c^2)^2} \right]^{1/2}, \tag{10}$$

$$(\alpha^2 - \lambda)(1 - c^2) = -\frac{J}{(1 - c^2)}. \tag{11}$$

Note that for $c = 0$ and $s = 0$ the exponents (10) and (11) reduce to those given by Drazin$^2$. These exponents have to satisfy the condition

$$(\mu_+ + \mu_- + \lambda)(\mu_+ + \mu_- + \lambda + 1) - 2 = 0, \tag{12}$$

as can be seen by matching the coefficients of the constant terms. From (12) we note that $\mu_+ + \mu_- + \lambda = 1$ or $-2$. The latter condition yields a convergent solution with the constraint $2c - s = 0$ or $c = s/2$. 

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Using the definitions for $\mu_+$ and $\lambda$ from (10) and (11) in the condition

$$\mu_+ + \mu_- + \lambda = 1,$$

we obtain

$$J = \hat{k}^2(1 - \beta^2/4 - \hat{k}^2).$$  \hspace{1cm} (14)

From (14) the maximum value of $J$ is given as $J_{\text{max}} = (1 - \beta^2/4)^2/4$, which
occurs at $\hat{k}_{\text{max}} = (1/\sqrt{2})(1 - \beta^2/4)^{1/2}$. So the new stability condition is $J$
$> (1 - \beta^2/4)^2/4$.

From (14) we see that for $\beta = 1$ the critical Richardson number is 9/64,
which means that for $J > 9/64$ the flow is stable; the cutoff wavenumber is
$\hat{k} = \sqrt{3}/2$. In comparison, for a Boussinesq fluid the critical $J$ is 1/4 and
the cutoff $\hat{k}$ is 1. Finally, the stability boundary is not neutral as was
the case with a Boussinesq fluid, but the waves have a phase velocity that
is half that of the peak background flow velocity, i.e., $c = 1/2$ as opposed
to $c = 0$ for a Boussinesq fluid. Finally, for a fluid in a gravity free
field ($J = 0$) the Kelvin-Helmholtz unstable domain is given by $0 < \hat{k}$
$< (1 - \beta^2)^{1/2}$. These conclusions can be clearly seen in Fig. 1, where we
plot $J$ versus $\hat{k}$. The figure shows the stability boundary for various values
of $\beta$. It is worth noting that for $\beta > 2$, that is if the density gradient
scale length is half the velocity gradient scale length or less, the Kelvin-
Helmholtz instability is stable and the system is unstable to the gravity
driven interchange, i.e., only when $J < 0$.

We apply the results to the plasmapause boundary based on the
observations of velocity shears observed by Kelley\textsuperscript{7} and compare our results
with those of Vinas and Madden\textsuperscript{8}. The velocity shear $V_0/L_\gamma$ is estimated to
be 0.17 Hz; when mapped to the plasmapause region\textsuperscript{7} this yields a local
Richardson number of -0.18 [see Ref. 8]. Vinas and Madden\textsuperscript{8} show
that $\delta = 1.5$, which when substituted into (15) yields the critical Richardson number $J < 0.047$ for instability. Thus, our calculations suggest that such strong shears may not drive shear flow instability if steep density gradients exist at the same time.

Figure 1. The stability boundary $J$ versus $k$ for $\delta = 0.0, 0.5, 1.0, 1.5, 2.0, \text{ and } 2.5$. The plasma is unstable (stable) in regions below (above) each curve. The case $\delta = 0$ corresponds to the Boussinesq fluid treated by Drazin. Note that for $\delta \geq 2$, $J < 0$ is required for instability.
III. SUMMARY

In summary, we have shown that the mode equation for a stratified shear layer under the action of gravity can be solved for stability boundary without the need to impose the Boussinesq approximation. For $L_H = L_V \beta = 1$, the critical Richardson number is $9/6^2$ as opposed to $1/4$ and the unstable wavenumber domain is smaller with a cut off for $J = 0$ at $k = \sqrt{3}/2$ rather than at $k = 1$. In addition, the waves do not have a zero phase velocity but a phase velocity that is half the peak background flow velocity. These changes are due to the density gradient terms that were ignored using the Boussinesq approximation. In addition, we have shown that plasmapause boundary may not be unstable to shear flow ballooning instability if strong density gradients exist in conjunction with strong shears at the diffuse auroral boundary at ionospheric heights.

ACKNOWLEDGMENTS

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REFERENCES


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