An Incomplete Lipschitz-Hankel Integral of $K_0$

Part I

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An Incomplete Lipschitz-Hankel Integral of $K_0$, Part I

Incomplete Lipschitz-Hankel integrals
Kampé de Fériet functions
Special functions

An incomplete Lipschitz-Hankel Integral of $K_0$ and related integrals are given in terms of elementary, cylindrical, and Kampé de Fériet functions. Some of the properties of these Kampé de Fériet functions are derived.

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AN INCOMPLETE LIPSCHITZ-HANKEL INTEGRAL OF $K_0$

PART I

INTRODUCTION

An incomplete Lipschitz-Hankel integral of cylindrical functions of order zero, $C_0$, may be defined by

$$C_0(a, z) \equiv \int_0^z e^{at} C_0(t) dt$$

Of interest in applications are the functions $J_0(a, z)$, $I_0(a, z)$, and $N_0(a, z)$ where $J$ denotes the Bessel function of the first kind, $I$ denotes the modified Bessel function, and $N$ denotes the Bessel function of the second kind or Neumann function. $J_0(a, z)$ and $N_0(a, z)$ occur in problems in the theory of diffraction in optical apparatus [1, p. 227]. The function $I_0(a, z)$ plays an important role in the study of oscillating wings in supersonic flow and arises in the study of resonant absorption in media with finite dimensions [1, p. 195].

In this report we are interested in

$$K_0(a, z) \equiv \int_0^z e^{at} K_0(t) dt \quad (1)$$

where $K$ denotes the MacDonald function or Bessel function of imaginary argument. We shall show that $K_0(a, z)$ can be written in closed form in terms of elementary functions, $K_0$, $K_1$, and Kampé de Fériet double hypergeometric functions. As an application it shall be shown that $K_0(\gamma, z)$ occurs when the statistical distribution of the maxima of a random function is applied to the amplitude of a sine wave in order to calculate the distribution of its ordinate. This latter distribution is of interest in the study of the scattered coherent reflected field from the sea surface [2].

Moreover we derive formulas for several integrals that are not readily available, and we exhibit some of the properties of the Kampé de Fériet functions associated with $K_0(a, z)$.

PRELIMINARY DEFINITIONS

The Pochhammer symbol $(a)_n$ is defined for nonnegative integers $n$ as a ratio of gamma functions:

$$(a)_n \equiv \Gamma(a + n)/\Gamma(a) = a(a + 1) \ldots (a + n - 1)$$

$$(a)_0 \equiv 1$$

(2)
Following Srivastava and Panda [3, p. 63] we define the Kampé de Fériet double hypergeometric functions:

\[ \mathbf{F}_{l:m; n; \alpha}^{p+q:k} \left( (a_\alpha) : (b_\beta) ; (c_\gamma) ; (\alpha_\alpha), (\beta_\beta), (\gamma_\gamma) ; x, y \right) \equiv \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_r \prod_{j=1}^{q} (b_j)_s}{\prod_{j=1}^{l} (\alpha_j)_r \prod_{j=1}^{m} (\beta_j)_s} \frac{x^r y^s}{r! s!} \]

where the Pochhammer symbols \((a)_n\) are defined by Eq. (2). For convergence

\[ p + q < l + m + 1, \quad p + k < l + n + 1, \quad |x| < \infty, \quad |y| < \infty, \text{ or} \]

\[ p + q = l + m + 1, \quad p + k = l + n + 1, \text{ and} \]

\[ \begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-k)} < 1 & \text{if } p > l \\ \max \{ |x|, |y| \} < 1 & \text{if } p \leq l \end{cases} \]

As special cases we define

\[ L[\alpha, \beta; \gamma, \delta; x, y] \equiv \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n}{(\gamma)_m (\delta)_n} \frac{x^m y^n}{m! n!} \quad |x| < \infty, \quad |y| < \infty \quad (3) \]

\[ M[\alpha, \beta; \gamma, \delta; x, y] \equiv \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n}{(\gamma)_m (\delta)_n} \frac{x^m y^n}{m! n!} \quad |x| < \infty, \quad |y| < 1 \quad (4) \]

We may then write

\[ L[\alpha, \beta; \gamma, \delta; x, y] = \mathbf{F}_{0:0,0}^{1:1,1} \left[ \begin{array}{c} \alpha; \beta; \\ \gamma, \delta; x, y \end{array} \right] \]

\[ M[\alpha, \beta; \gamma, \delta; x, y] = \mathbf{F}_{1:0,0}^{1:1,1} \left[ \begin{array}{c} \alpha; \beta; \\ \gamma, \delta; x, y \end{array} \right] \]

**SOME ELEMENTARY PROPERTIES OF** \(M[\alpha, \beta; \gamma, \delta; x, y]\)

Substituting [4, p. 266]

\[ \frac{(\alpha)_p}{(\gamma)_p} = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 t^{\rho+\alpha-1}(1 - t)^{\gamma-\alpha-1} dt \]
where $\Re \gamma > \Re \alpha > 0$, and $p = m + n$ into Eq. (4), we deduce an integral representation for $M$:

$$M[\alpha, \beta; \gamma; \delta; x, y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 \, _0F_1[- \delta; x; t] \, t^{\alpha - 1} (1 - t)^{\gamma - \alpha - 1} (1 - yt)^{-\beta} \, dt$$

$$= \frac{\Gamma(\gamma) \Gamma(\delta)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \frac{\delta!}{2} \int_0^1 \, I_{\delta - 1} (2\sqrt{x}) \, t^{\frac{\delta - 1}{2}} (1 - t)^{\alpha - 1} (1 - yt)^{-\beta} \, dt$$

Here we have used the equation

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \, _0F_1[- \nu + 1; \nu + 1; z^2/4]$$

We obtain directly from Eq. (4) the generating relation

$$M[\alpha, \beta; \gamma; \delta; x, y] = \sum_{n=0}^\infty \frac{(\alpha)_n}{(\gamma)_n (\delta)_n} \frac{x^n}{n!} \, _2F_1[n + \alpha, \beta; n + \gamma; y]$$

We now prove the following

**THEOREM:** Suppose $-1 < \Re (\gamma - \alpha - \beta) < 0$, $\arg y < \pi$, $\arg(1 - y) < \pi$. Then for $y = 1$,

$$M[\alpha, \beta; \gamma; \delta; x, y] = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \, _1F_2[\alpha; \gamma - \beta, \delta; x]$$

$$+ \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} (1 - y)^{\gamma - \alpha - \beta} \, _0F_1[- \delta; \gamma, x] + O(1 - y)$$

or

$$M[\alpha, \beta; \gamma; \delta; x, y] = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \, _1F_2[\alpha; \gamma - \beta, \delta; x] + O(1 - y)$$

$$+ \Gamma(\alpha + \beta - \gamma) \frac{\Gamma(\gamma) \Gamma(\delta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\delta!}{2} (1 - y)^{\gamma - \alpha - \beta} \, I_{\delta - 1}(2\sqrt{x})$$

**Proof:** The following result is found in [4, Eq. (9.5.7), p. 249]: for $\alpha + \beta - \gamma \neq 0$, $\pm 1, \pm 2, \ldots$, $|\arg z| < \pi$, $|\arg(1 - z)| < \pi$

$$\, _2F_1[\alpha, \beta; \gamma; z] = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \, _2F_1[\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z]$$

$$+ (1 - z)^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} \, _2F_1[\gamma - \alpha, \gamma - \beta; 1 - \alpha - \beta + \gamma; 1 - z]$$
Hence

\[
\gamma F_1[n + \alpha, \beta; n + \gamma; y] = \frac{\Gamma(n + \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(n + \gamma - \beta)} \cdot \gamma F_1[n + \alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - y] \\
+ (1 - y)^{\gamma - 1} \frac{\Gamma(n + \gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(n + \alpha)\Gamma(\beta)} \cdot \gamma F_1[\gamma - \alpha, n + \gamma - \beta; 1 - \alpha - \beta + \gamma; 1 - y]
\]

Now suppose that \(-1 < \text{Re}(\gamma - \alpha - \beta) < 0\). Then for \(y \to 1\) we have

\[
\gamma F_1[n + \alpha, \beta; n + \gamma; y] = \frac{\Gamma(n + \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(n + \gamma - \beta)} + \frac{\Gamma(n + \gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - y)^{\gamma - 1} + O(1 - y)
\]

Substituting this result into Eq. (6) gives

\[
M[\alpha, \beta; \gamma, \delta, x, y] = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma - \beta)_n} \frac{x^n}{n!} \\
+ \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - y)^{\gamma - 1} \sum_{n=0}^{\infty} \frac{\lambda^n}{(\delta)_n n!} + O(1 - y)
\]

from which we obtain Eq. (7). Then using Eq. (5) we obtain Eq. (8).

Employing series rearrangement we deduce

\[
M[\alpha, \beta; \gamma, \delta, x, t] = \sum_{p=0}^{\infty} \frac{(\alpha)_p (\beta)_p}{(\gamma)_p} \frac{x^p}{p!} \cdot \gamma F_2[-p; \delta, 1 - \beta - p; 1/t]
\]

(9)

Using a general result of Srivastava [3, Eq. (30), p. 145] we find Eq. (9) in a different form, viz.

\[
M[\alpha, \beta; \gamma, \delta, t x, t] = \sum_{p=0}^{\infty} \frac{(\alpha)_p (\beta)_p}{(\gamma)_p} \frac{t^p}{p!} \cdot \gamma F_2[-p; \delta, 1 - \beta - p; x]
\]

From Eq. (9) it follows that

\[
M[\alpha, \beta; \gamma, \delta, x, t] = \sum_{p=0}^{\infty} \frac{(\alpha)_p}{(\gamma)_p (\delta)_p} \frac{x^p}{p!} \cdot \gamma F_0[\beta, -p, 1 - \delta - p; -1]
\]

(10)

Equation (10) may be obtained directly from [3, Eq. (60), p. 194].
We remark that it may be shown that $M[a, l; y, \delta, x, y]$ converges on the unit circle $|y| = 1$ if and only if $\sum_{n=0}^{\infty} k_r n! \frac{x^n}{n!} \sum_{r=0}^{\infty} \frac{(-a)_r}{(r)_r} \frac{y^r}{r!} = 1$ converges on $|y| = 1$.

SOME ELEMENTARY PROPERTIES OF $L[a, \beta; y, \delta; x, y]$

Using series rearrangement we find

$$L[a, \beta; y, \delta; x, t x] = \sum_{n=0}^{\infty} \frac{(a)_n}{(y)_n (\delta)_n} \frac{x^n}{n!} \sum_{r=0}^{\infty} \frac{(-a)_r}{(r)_r} \frac{y^r}{r!} = 1$$

This can also be obtained from [3, Eq. (30), p. 145] in a different form. Using Vandermonde's theorem [5, Eq. (1.7.7), p. 28]

$$\sum_{n=0}^{\infty} k_r n! \frac{x^n}{n!} \sum_{r=0}^{\infty} \frac{(-a)_r}{(r)_r} \frac{y^r}{r!} = 1$$

so that we have a reduction formula for $L$, viz,

$$L[a, \beta; y, \delta; x, x] = \sum_{n=0}^{\infty} \frac{(a + \beta)_n}{(y)_n (\delta)_n} \frac{x^n}{n!} = 1$$

This result can be obtained also by using the following general result of Srivastava [3, Eq. (20), p. 55] applied to Eq. (3):

$$\sum_{m, n=0}^{\infty} c_m n! (\rho) m (\sigma)_n \frac{x^{m+n}}{m! n!} = \sum_{n=0}^{\infty} c_n (\rho + \sigma)_n \frac{x^n}{n!}$$

provided each series is absolutely convergent.

We obtain directly from Eq. (3) the generating relation

$$L[a, \beta; y, \delta; x, y] = \sum_{n=0}^{\infty} \frac{(a)_n}{(y)_n (\delta)_n} \frac{x^n}{n!} \sum_{r=0}^{\infty} \frac{(-a)_r}{(r)_r} \frac{y^r}{r!} = 1$$

Finally, using [3, Eq. (43), p. 150] we obtain

$$L[a, \beta; y, \delta; -x, x \tan^2 \theta] = (\cos^2 \theta)^{\mu} \sum_{n=0}^{\infty} \frac{(\beta)_n (\sin^2 \theta)^n}{n!} \sum_{r=0}^{\infty} \frac{(-a)_r}{(r)_r} \frac{y^r}{r!} = 1$$
A CLOSED FORM FOR \( K_e(a, z) \)

From Eq. (1) we write

\[
K_e(a/\beta, \beta) = \beta \int_0^1 e^{at} K_0(\beta t) \, dt
\]  

(13)

Using [6, p. 89] we find the following formulas:

\[
\int_0^1 s^m K_0(z) \, ds = \frac{K_0(z)}{m+1} \binom{m+1}{1/2, m+3/2, 3/4}  
\]

\[
+ \frac{z K_1(z)}{(m+1)^2} \binom{m+3}{1/2, m+3/2, 3/4} \quad m = 0, 2, 4, \ldots
\]  

(14)

\[
\int_0^1 s^m K_0(z) \, ds = \frac{2^{m+1}}{z^{m+1}} \binom{m+2}{1/2} \binom{m+1}{1/2}
\]

\[
- \frac{(m-1) K_0(z)}{z^2} \binom{m+2}{1, 1-m/2, 3-m/2, 4/2^2}
\]

\[
- \frac{K_1(z)}{z} \binom{1-m/2, 1-m/2, 4/2^2}{1}
\]  

(15)

Integrating term by term we find

\[
\int_0^1 \exp(a t) K_0(\beta t) \, dt = \sum_{n=0}^{\infty} \frac{a^n}{n!} \int_0^1 t^n K_0(\beta t) \, dt = \sum_{n=0}^{\infty} \frac{a^n}{n!} \int_0^1 t^n K_0(\beta t) \, dt
\]

\[
= \sum_{n=0}^{\infty} \frac{a^n}{(2n)!} \int_0^1 t^{2n} K_0(\beta t) \, dt + \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(2n+1)!} \int_0^1 t^{2n+1} K_0(\beta t) \, dt
\]
so that using Eqs. (14) and (15)

\[
\int_0^1 \exp(\alpha t) K_0(\beta t) dt = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} \frac{K_0(\beta)}{2n + 1} \left( \begin{array}{c}
1; \\
\frac{2n + 1}{2}, \\
\frac{2n + 3}{2}, \\
\frac{\beta^2}{4}
\end{array} \right)
\]

\[
+ \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} \frac{\beta K_1(\beta)}{(2n + 1)^2} \left( \begin{array}{c}
1; \\
\frac{2n + 3}{2}, \\
\frac{2n + 3}{2}, \\
\frac{\beta^2}{4}
\end{array} \right)
\]

\[
+ \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n + 1)!} \frac{\Gamma(n + 1) \Gamma(n + 1)}{\beta^{2n+2}}
\]

\[
- \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n + 1)!} \frac{K_0(\beta)}{\beta^2} \left( \begin{array}{c}
1; \\
-n, \\
-1 - n, \\
-4/\beta^2
\end{array} \right)
\]

\[
- \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n + 1)!} \frac{K_1(\beta)}{\beta} \left( \begin{array}{c}
1; \\
-n, \\
-1 - n, \\
-4/\beta^2
\end{array} \right)
\]

(16)

We shall consider each of the above five sums in the order in which they appear. We find

\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n + 1)!} \left( \begin{array}{c}
1; \\
\frac{2n + 1}{2}, \\
\frac{2n + 3}{2}, \\
\frac{\beta^2}{4}
\end{array} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{(\alpha^{2/4})^n}{n!} \left( \begin{array}{c}
1; \\
\frac{1}{2}, \\
\frac{3}{2}, \\
\frac{\beta^2}{4}
\end{array} \right) = L [1/2, 1/2, 3/2, 1/2, 3/2, \alpha^{2/4}, \beta^2/4].
\]

\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n + 1)(2n + 1)!} \left( \begin{array}{c}
1; \\
\frac{2n + 3}{2}, \\
\frac{2n + 3}{2}, \\
\frac{\beta^2}{4}
\end{array} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(1/2)_n}{(3/2)_n(3/2)_n} \frac{(\alpha^{2/4})^n}{n!} \left( \begin{array}{c}
1; \\
\frac{1}{2}, \\
\frac{3}{2}, \\
\frac{\beta^2}{4}
\end{array} \right) = L [1/2, 1/2, 3/2, 1/2, 3/2, \alpha^{2/4}, \beta^2/4]
\]

where in the latter two cases we have used Eq. (12).

\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n + 1)!} \frac{\Gamma(n + 1) \Gamma(n + 1)}{\beta^{2n+2}} \frac{\sin \left( \frac{\alpha(\pi/\beta)}{\sqrt{\beta^2 - \alpha^2}} \right)}{\sqrt{\beta^2 - \alpha^2}} = \sin \left( \frac{\alpha(\pi/\beta)}{\sqrt{\beta^2 - \alpha^2}} \right) \quad |\alpha/\beta| \leq 1, \quad \alpha \neq \pm \beta
\]
where we have used [9, Eq. (9.121-14), p. 1041] the result: \( {}_1F_1[1, 1; 3/2; \sin^2 z] = z / \sin z \cos z \):

\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \cdot \frac{1}{(5/2)^n} \cdot {}_3F_0[1, -1 - n, -n; -4/\beta^2] = \frac{\alpha^3}{3} M \left[ 1, 1; \frac{\alpha^2}{4}, \frac{\alpha^2}{\beta^2} \right],
\]

and finally

\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} \cdot \frac{1}{(5/2)^n} \cdot {}_3F_0[1, -1 - n, -n; -4/\beta^2] = \frac{\alpha^3}{3} M \left[ 1, 1; \frac{\alpha^2}{4}, \frac{\alpha^2}{\beta^2} \right],
\]

where in the latter two cases we have used Eq. (10).

Defining

\[
L_0(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(1/2)_m (1)_n}{(3/2)_m + (3/2)_n} \frac{x^m y^n}{m! n!} = L \left[ 1/2, 1; 3/2, 3/2, x, y \right],
\]

\[
L_1(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(1/2)_m (1)_n}{(1/2)_m + (3/2)_n} \frac{x^m y^n}{m! n!} = L \left[ 1/2, 1; 1/2, 3/2, x, y \right],
\]

\[
M_0(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(1)_n}{(3/2)_m + (3/2)_n} \frac{x^m y^n}{m! n!} = M \left[ 1, 1; 3/2, 1, x, y \right],
\]

\[
M_1(x, y) \equiv \sum_{m, n=0}^{\infty} \frac{(2)_m + (1)_n}{(5/2)_m + (3/2)_n} \frac{x^m y^n}{m! n!} = M \left[ 2, 1; 5/2, 2, x, y \right],
\]

we have from Eq. (16) and the above results

\[
\int_0^1 \exp (\alpha t) K_0(\beta t) dt = K_1(\beta) \left[ \beta L_0(\alpha^2/4, \beta^2/4) - \frac{\alpha}{\beta} M_0(\alpha^2/4, \alpha^2/\beta^2) \right] + K_0(\beta) \left[ L_1(\alpha^2/4, \beta^2/4) - \frac{\alpha^3}{3\beta^2} M_1(\alpha^2/4, \alpha^2/\beta^2) \right] + \frac{\sin^{-1}(\alpha/\beta)}{\sqrt{\beta^2 - \alpha^2}}
\]

(17)
which we may write using Eq. (13)

\[
K_{\alpha}(a, z) = z K_1(z) \left[ z L_0 \left( \frac{a^2 z^2}{4}, \frac{z^2}{4} \right) - a R_0 \left( \frac{a^2 z^2}{4}, a^2 \right) \right] + z K_0(z) \left[ L_1 \left( \frac{a^2 z^2}{4}, \frac{z^2}{4} \right) - \frac{1}{3} a R_1 \left( \frac{a^2 z^2}{4}, a^2 \right) \right] + \frac{e^{-z}}{\sqrt{1 - a^2}} \left( \frac{\alpha}{z} \right)
\]

(18)

We have then given \( K_{\alpha}(a, z) \) in terms of elementary, MacDonald, and Kampé de Fériet functions.

We remark that in view of Eq. (6) and the definitions of \( R_0 \) and \( R_1 \)

\[
M_0(x, y) = \sum_{n=0}^{\infty} \frac{1}{(3/2)_n} \frac{x^n}{n!} \left( \frac{y}{4} \right)^n, F_1[1, n + 1; n + 3/2; x]
\]

\[
M_1(x, y) = \sum_{n=0}^{\infty} \frac{1}{(5/2)_n} \frac{x^n}{n!} \left( \frac{y}{4} \right)^n, F_1[1, n + 2; n + 5/2; x]
\]

Since each of the Gauss hypergeometric functions above is conditionally convergent on the unit circle \(|y| = 1\) except at \( y = 1\) we see that \( M_0(x, y) \) and \( M_1(x, y) \) are conditionally convergent on \(|y| = 1\) except at \( y = 1\). Hence Eq. (17) is valid for \( \alpha/\beta \leq 1, \alpha \neq \pm \beta \) and Eq. (18) is valid for \( \alpha i \leq 1, \alpha \neq \pm 1 \). We shall show shortly that Eq. (17) is valid in the limit even when \( \alpha = \pm \beta \). See Ref. 1 for other representations of \( K_{\alpha}(a, z) \).

In a future report (Part II) it shall be shown that Eq. (18) is easily extended to the entire complex \( \alpha \)-plane in terms of elementary, MacDonald and Kampé de Fériet functions.

**KING'S INTEGRAL**

Using properties of \( L \) and \( M \) we have derived earlier we shall derive (a formula for) King's integral (6, Eq. (12), p. 123):

\[
\int_0^\infty \exp\{ K_0(t) \} dt = a \exp\{ K_0(\alpha) + K_1(\alpha) \} - 1
\]

(19)

that is we shall show Eq. (17) is valid in the limit for \( \alpha = \beta \). Using Eq. (8) we find for \( \alpha = \beta \)

\[
M_0(\alpha^2/4, \alpha^2/\beta^2) = \frac{\pi}{2} \frac{L_0(\alpha)}{\sqrt{1 - \alpha^2/\beta^2}} - \cosh \alpha + O(1 - \alpha^2/\beta^2)
\]

\[
M_1(\alpha^2/4, \alpha^2/\beta^2) = \frac{3}{\alpha} \left[ \frac{\pi}{2} \frac{L_1(\alpha)}{\sqrt{1 - \alpha^2/\beta^2}} - \sinh \alpha + O(1 - \alpha^2/\beta^2) \right]
\]

(Also we remark on top of p. 9)
Substituting these equations into Eq. (17) gives

\[
\int_0^1 \exp(\alpha t) K_0(\beta t) \, dt = \frac{\sin^{-1}(\alpha/\beta) - \frac{\pi}{2} \left[ \alpha K_1(\beta) I_0(\alpha) + (\alpha^2/\beta) K_0(\beta) I_1(\alpha) \right]}{\sqrt{\beta^2 - \alpha^2}} + \frac{\alpha}{\beta} K_1(\beta) \cosh \alpha
\]

\[
+ \frac{\alpha^2}{\beta^2} K_0(\beta) \sinh \alpha + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) + \beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + O(1 - \alpha^2/\beta^2)
\]

Using the reduction formula Eq. (11) for \( L \) we deduce

\[
L_0(x^2/4, x^2/4) = \frac{\sinh x}{x}
\]

\[
L_1(x^2/4, x^2/4) = \cosh x
\]

Now holding \( \beta \) fixed and letting \( \alpha \to \beta \) we obtain after simplification

\[
\int_0^1 \exp(\beta t) K_0(\beta t) \, dt = [K_0(\beta) + K_1(\beta)] \exp \beta + \lim_\alpha \to \beta J(\alpha, \beta)
\]

where \( J(\alpha, \beta) \) is the first term on the right-hand side of Eq. (20). We find however that

\[
\lim_\alpha \to \beta \text{ numerator } J(\alpha, \beta) = \frac{\pi}{2} \left[ 1 - \frac{1}{\beta} K_1(\beta) I_0(\beta) - \beta K_0(\beta) I_1(\beta) \right] = 0
\]

\[
\lim_\alpha \to \beta \text{ denominator } J(\alpha, \beta) \approx 0
\]

so that on applying L'hospital's rule we have

\[
\lim_\alpha \to \beta J(\alpha, \beta) = -\frac{1}{\beta}
\]

Hence

\[
\int_0^1 \exp(\beta t) K_0(\beta t) \, dt = [K_0(\beta) + K_1(\beta)] \exp \beta - 1/\beta
\]

and a simple transformation now gives Eq. (19). We may perform a similar analysis for \( \alpha \to \beta \) to obtain

\[
\int_0^1 \exp(-\beta t) K_0(\beta t) \, dt = [K_0(\beta) - K_1(\beta)] \exp(-\beta) + 1/\beta
\]
A DISTRIBUTION FOR THE ELEVATION OF A SINE WAVE

Consider the random variable \( y = H \sin \theta \), where \( H \) is a random variable with density \( K(H, \epsilon) \), \(|H| < \infty\), and \( \theta \) is a random variable, independent of \( H \), with density

\[
U(\theta) = \begin{cases} 
\pi^{-1} & |\theta| \leq \pi/2 \\
0 & |\theta| > \pi/2 
\end{cases}
\]

Let \( D(y, \epsilon) \) be the density function for \( y \). It is shown in Ref. 2 that

\[
D(y, \epsilon) = \frac{1}{\pi} \int_{-\infty}^{-|y|} \frac{K(H, \epsilon) dH}{\sqrt{H^2 - y^2}} + \frac{1}{\pi} \int_{|y|}^{\infty} \frac{K(H, \epsilon) dH}{\sqrt{H^2 - y^2}} \tag{21}
\]

Rice [7] and Cartwright and Longuet-Higgins [8] have derived an expression for the statistical distribution of the maxima of a random function that may be expressed in the form

\[
K(H, \epsilon) = \frac{\epsilon}{\sigma_H \sqrt{2\pi}} \exp \left( \frac{-H^2}{2\epsilon^2 \sigma_H^2} \right) + \frac{\sqrt{1 - \epsilon^2}}{2\sigma_H \sqrt{2\pi}} H \exp \left( \frac{-H^2}{2\sigma_H^2} \right) \left[ 1 + \text{erf} \left( \frac{\sqrt{2}}{2} \frac{H \sqrt{1 - \epsilon^2}}{\sigma_H} \right) \right] \tag{22}
\]

Here \( \sigma_H \) is the standard deviation of \( H \), and \( 0 < \epsilon < 1 \) is known as the spectral width parameter. It is shown in Ref. 2 that the standard deviation \( \sigma \) of \( y \) is given by

\[
\sigma = \sigma_H \sqrt{\eta}
\]

where \( \eta \) is defined by

\[
\eta \equiv \left[ 1 + \frac{\pi}{2} (1 - \epsilon^2) \right]^{-1/2}
\]

Substituting Eq. (22) into Eq. (21) and using the latter result gives

\[
D(y, \epsilon) = \frac{\epsilon}{2 \pi^{1/2} \eta \sigma} \exp \left( \frac{-y^2}{3\epsilon^2 \eta^2 \sigma^2} \right) K_0 \left( \frac{\sqrt{1 - \epsilon^2}}{8 \epsilon^2 \eta^2 \sigma^2} \right) \exp \left( \frac{-y^2}{4 \eta^2 \sigma^2} \right) \Psi \left( \frac{\sqrt{1 - \epsilon^2}}{\epsilon}, \frac{\eta}{2 \eta \sigma} \right) \tag{23}
\]

where the function \( \Psi(k, u) \) is defined by

\[
\Psi(k, u) \equiv \int_0^\infty \exp(-s^2) \text{erf}(k \sqrt{u^2 + s^2}) ds \tag{24}
\]

For real \( u \) and \( k \) it is shown in Ref. 2 that

\[
\pi^{1/2} \int_0^\infty \exp(-s^2) \text{erf}(k \sqrt{u^2 + s^2}) ds = \tan^{-1} k + \frac{k}{1 + k^2} \int_0^{1/2(1 + \epsilon^2)} \exp \left( \frac{1 - k^2}{1 + k^2} s \right) K_0(s) ds
\]
Using Eqs. (1) and (24) this may be written
\[ \Psi(k, u) = \frac{\tan^{-1}(k)}{\pi^{1/2}} + \frac{1}{\pi^{1/2}} \frac{k}{1 + k^2} K_e \left( \frac{1 - k^2}{1 + k^2}, \frac{1}{2} u^2(1 + k^2) \right) \]

We may then write Eq. (23)
\[ D(y, \varepsilon) = \frac{\varepsilon}{2\pi^{3/2}\eta \sigma} \exp \left[ -\frac{y^2}{8\varepsilon^2\eta^2\sigma^2} \right] K_0 \left( \frac{\varepsilon^2}{8\varepsilon^2\eta^2\sigma^2} \right) \]
\[ + \frac{\sqrt{1 - \varepsilon^2}}{\pi^{3/2}\eta \sigma} \exp \left[ -\frac{\varepsilon^2}{4\eta^2\sigma^2} \right] \left[ \cos^{-1} \varepsilon + \varepsilon \sqrt{1 - \varepsilon^2} K_0(2\varepsilon^2 - 1, y^2/8\varepsilon^2\eta^2\sigma^2) \right] \]

where \( K_e(a, z) \) is given by Eq. (18).

**SOME INTEGRALS RELATED TO \( K_e(a, z) \)**

The following integrals can easily be obtained from Eq. (17):
\[ \int_0^1 \sin (\alpha t) K_0(\beta t) dt = \sinh^{-1}(\alpha/\beta) \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} K_1(\beta) M_0(-\alpha^2/4, -\alpha^2/\beta^2) \]
\[ + \frac{\alpha^2}{3\beta^2} K_0(\beta) M_1(-\alpha^2/4, -\alpha^2/\beta^2) \quad \alpha/\beta \leq 1, \quad \alpha \neq \pm i \beta \]
\[ \int_0^1 \cos (\alpha t) K_0(\beta t) dt = \beta K_1(\beta) L_0(-\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(-\alpha^2/4, \beta^2/4) \]

Further, using the result [2] for \( 0 < |\alpha| < 1, 0 < x \)
\[ K_e(\alpha, x) = \text{sgn} \alpha \left\{ \exp (\alpha x) \left[ \int_0^\infty \frac{\cos (\alpha x) dt}{(1 + t^2)^{1/2} + x^2 t^2} + \int_0^\infty \frac{t \sin (\alpha x) dt}{(1 + t^2)^{1/2} + x^2 t^2} \right] - \frac{\cos^{-1}(|\alpha|)}{\sqrt{1 - \alpha^2}} \right\} \]
we find for \( 0 < \alpha < \beta \)
\[ \int_0^\infty \frac{\cos (\alpha x) dx}{(1 + x^2)\sqrt{\beta^2 + \alpha^2 x^2}} = \cosh \alpha \left\{ \frac{\pi/2}{\sqrt{\beta^2 - \alpha^2}} - \frac{\alpha}{\beta} K_1(\beta) M_0(\alpha^2/4, \alpha^2/\beta^2) - \frac{\alpha^2}{3\beta^2} K_0(\beta) M_1(\alpha^2/4, \alpha^2/\beta^2) \right\} \]
\[ - \sinh \alpha \left\{ \beta K_1(\beta) L_0(\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) \right\} \]
\[\int_0^\infty \frac{x \sin (\alpha x) dx}{(1 + x^2 \sqrt{\beta^2 + \alpha^2 x^2})} = \cosh \alpha \left( \beta K_0(\beta) L_0(\alpha^2/4, \beta^2/4) + K_0(\beta) L_1(\alpha^2/4, \beta^2/4) \right)\]

\[-\sinh \alpha \left( \frac{\pi/2}{\sqrt{\beta^2 - \alpha^2}} - \frac{\alpha}{\beta} K_1(\beta) M_0(\alpha^2/4, \alpha^2/\beta^2) - \frac{\alpha^3}{3\beta^2} K_0(\beta) M_1(\alpha^2/4, \alpha^2/\beta^2) \right)\]

In addition [9, Eq. (3.367), p. 316] we have

\[\int_0^\infty \frac{e^{-\rho t} \sin \theta dt}{(1 + t + \cos \theta) \sqrt{\rho^2 + 2t}} = \exp \left[ 2\rho \cos^2 \frac{\theta}{2} \right] [\theta - \sin \theta K_\epsilon(\cos \theta, \rho)] \quad \text{Re } \rho > 0\]

CONCLUSIONS

The Kampé de Fériet functions have been used to put in closed form the incomplete Lipschitz-Hankel integral \( K_{\epsilon}(\alpha, z) \) and several related integrals that are not readily available and are of interest in mathematical physics and applications. Some of the properties of the Kampé de Fériet functions associated with \( K_{\epsilon}(\alpha, z) \) are derived. These properties are useful in deriving additional results quickly. As an example we have given an elementary derivation of a closed form for King’s integral based on generating function techniques.

In addition, the utility of a closed form for \( K_{\epsilon}(\alpha, z) \) is indicated by deriving a certain density function that is associated with the scattered coherent return from the sea surface.

REFERENCES


