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Peter O'Brien and John L. Wyatt, Jr.

Abstract

This paper extends the results of Penfield and Rubinstein on signal delay in RC tree networks. Both estimates and bounds are derived for the step response of "leaky" RC trees and meshes, a class of networks that is appropriate for modelling interconnect in digital bipolar circuits. This paper is intended to serve as a tutorial as well as a research report. Therefore existing work is explained in some detail along with the derivation of new results.
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Signal Delay in Leaky RC Mesh Models for Bipolar Interconnect

Peter O'Brien and John L. Wyatt, Jr.

This paper extends the results of Penfield and Rubinstein on signal delay in RC tree networks. Both estimates and bounds are derived for the step response of "leaky" RC trees and meshes, a class of networks that is appropriate for modelling interconnect in digital bipolar circuits. This paper is intended to serve as a tutorial as well as a research report. Therefore existing work is explained in some detail along with the derivation of new results.

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I. Introduction

Simple closed-form bounds for signal propagation delay in linear RC tree models for interconnect were derived in [1]. The theory has since been extended to include RC mesh models as well [2]. The goal of this report is to further extend the theory to allow for RC networks with grounded resistors (i.e., "leaky" RC trees and meshes -- see section II for a precise definition of these terms). Both delay estimates and delay bounds are developed.

The practical motivation behind allowing resistive paths to ground in RC mesh circuits is an attempt to more accurately model interconnect on bipolar chips. The base of a bipolar transistor loading the interconnect offers a (nonlinear) resistive path to ground. A corresponding gate electrode of an MOS transistor would not offer such a resistive path to ground, and the original work of Penfield et. al. relied upon the absence of such paths.

Fig. 1: Lumped linear RC tree model for MOS logic with fanout
Consider the RC tree in Fig. 1. The voltage source $e(t)$ represents the output of some logic gate. The output of this gate fans out through the interconnect to the inputs of two other logic gates at nodes 4 and 5. Since nodes 4 and 5 represent the gate terminals of MOS transistors, there is no need for a resistive path to ground in the model.

The analogous situation in bipolar logic is illustrated in Fig. 2. Here nodes 4 and 5 represent the base terminals of bipolar transistors, so we need a resistive path to ground in the model. The resulting circuit is not an RC mesh, but would be one except for the resistive paths to ground. This is the type of network discussed in this report. The organization of this report can be seen in the table of contents.
II. Classes of Networks Considered

A lumped linear RC mesh is any network consisting of 2-terminal linear resistors and capacitors driven by independent sources, such that one side of every capacitor and every source is grounded, no resistor is grounded, and between any two nodes other than ground there exists a path consisting entirely of resistors. An RC mesh is called an RC tree if it does not contain a closed path of resistors.

However, in this paper we shall consider more general types of circuits: leaky RC meshes and trees. A leaky RC mesh is any network meeting all of the requirements of an RC mesh given above, except for the restriction that there be no grounded resistors. So for every leaky RC mesh, there is an underlying RC mesh which is obtained by open-circuiting all of the grounded resistors in the original leaky RC mesh. A network is a leaky RC tree if it is a leaky RC mesh whose underlying RC mesh is an RC tree (i.e., there are no resistor loops in the underlying RC mesh).

III. Network Matrices

In this section, we define certain network matrices to be used later in the report. Well-known facts about these matrices will be presented. We shall motivate these definitions by the example circuit in Fig. 3. Note that this example circuit would be an RC mesh without the grounded resistor $R_6$. 
The ground node and the node connected to the independent source are not numbered. The remaining nodes are numbered in any order from 1 to N, where N is the total number of capacitors (in Fig. 3, N=4).

a.) The resistance matrix \( R \):

![Resistor subnetwork R extracted from the circuit in Fig. 3 (driven with current sources)](Image of resistor subnetwork)
We isolate the resistor subnetwork $R$ containing all the resistors and assign reference directions to the capacitor currents $i_1, \ldots, i_N$ as shown in Fig. 4 (note that positive current is defined to be flowing out of the capacitor and into the resistor subnetwork). Let the node connected to the independent source serve as the datum node of $R$. Define the matrix $R$ as follows:

With the datum grounded, the node voltages with respect to ground are given in terms of the capacitor currents by the resistance matrix $R$, as shown below:

$$
\begin{bmatrix}
    v_1(t) \\
    v_2(t) \\
    \vdots \\
    v_N(t)
\end{bmatrix} = \begin{bmatrix}
    R_{11} & R_{12} & \cdots & R_{1N} \\
    R_{21} & R_{22} & \cdots & R_{2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    R_{N1} & R_{N2} & \cdots & R_{NN}
\end{bmatrix} \begin{bmatrix}
    i_1(t) \\
    i_2(t) \\
    \vdots \\
    i_N(t)
\end{bmatrix}
$$

We can introduce vector notation to write (1) in a more compact form. Let $v$ denote the column vector of node voltages $(v_1, v_2, \ldots, v_n)^T$. Let $i$ denote the column vector of capacitor currents $(i_1, i_2, \ldots, i_N)^T$. Then (1) becomes:

$$v(t) = R \ i(t)$$

So with the datum grounded, the numerical value of any matrix entry $R_{ij}$, in ohms, is equal to the potential in volts we would observe at node $i$ if a 1-amp current were injected into node $j$ and all external nodes of $R$ other than $j$ and the datum were open-circuited.
In the special case of an RC tree the matrix entries can be read off by inspection, since in that case $R_{ij}$ is simply the sum of the resistances along the route obtained by intersecting the path from node 0 to datum with the path from node 0 to datum. So, the topological definition of $R_{ij}$ for an RC tree given in [1] is consistent with the more general interpretation of $R_{ij}$ as an element of the resistance matrix $R$ describing the resistor subnetwork $R$.

From circuit theory, the resistance matrix $R$ has certain well-known properties independent of the particular resistor subnetwork being described by $R$:

(i) $R$ is symmetric (i.e., $R_{ij} = R_{ji}$ for all $i,j$): This is a consequence of $R$ being a "reciprocal" network. [3; Chapter 16]

(ii) $R$ is positive-definite (assuming all the resistors in $R$ are positive): This is true because the power dissipated in $R$, given by the quadratic form $i^T R i$ is positive for any nonzero choice of currents $i$.

These properties of $R$ are quite standard and play only a minor role in the theory. But $R$ possesses more structure than symmetry and positive-definiteness, and this additional structure, described in Section VI and proved in Appendix A, is the foundation for the theory developed in this report.
b.) The conductance matrix $G$:

\[
\begin{bmatrix}
G_{11} & G_{12} & \cdots & G_{1N} \\
G_{21} & G_{22} & \cdots & G_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
G_{N1} & G_{N2} & \cdots & G_{NN}
\end{bmatrix}
\]

or, using the vector shorthand:

\[
\mathbf{i}(t) = G \mathbf{v}(t)
\]

So with the datum grounded, the numerical value of any matrix entry $G_{ij}$, in mhos, is equal to the current in amps we would observe flowing into
node 1 if a 1-volt source were applied to node 1 and all external nodes of R other than 1 were grounded.

If R is the resistance matrix for the same resistor subnetwork described by G, then \( G = R^{-1} \). Like R, G is also positive-definite symmetric (for similar reasons as R), and G possesses additional structure described in Appendix A which is crucial to the results in Section VI.

c.) The capacitance matrix C:

This matrix is not as central to the theory as either the R or the G matrix. It is simply notational shorthand for writing down the capacitors in the RC network under consideration. We define

\[ C = \text{diag}(C_1, C_2, \ldots, C_N) \]

So C is a "diagonal" matrix, (i.e., has off-diagonal elements equal to zero), and the diagonal elements are equal to the capacitor values consistent with the assigned node numbers.

One simple property to note about C is that its inverse, \( C^{-1} \), is also diagonal: \( C^{-1} = \text{diag}(C_1^{-1}, C_2^{-1}, \ldots, C_N^{-1}) \). (Of course, \( C^{-1} \) doesn't exist in the degenerate case where some individual capacitors are equal to zero.)

IV. Network Differential Equations in Canonical Form

In order to write down the network differential equations, we must first consider what happens when the datum, e(t), is not grounded. In the previous section we wrote the node voltages in terms of the capacitor currents when the datum was grounded:
\begin{equation}
\dot{v}(t) = R \dot{i}(t) \quad (\text{datum} \ e(t) = 0)
\end{equation}

When \( e(t) \) is not zero, then by superposition we can write:

\begin{equation}
\dot{v}(t) = R \dot{i}(t) + x e(t)
\end{equation}

where \( x = (x_1, x_2, \ldots, x_N)^T \) is a column vector of dimensionless numbers (each \( x_i \) satisfying the inequality: \( 0 < x_i \leq 1 \)). Referring back to Fig. 4, each \( x_i \) is numerically equal to the potential (in volts) produced at node \( i \) due solely to a 1-volt source at the datum, while open-circuiting all of the external current sources.

**Example:** In Fig. 3, if \( R_6 = \infty \) (open-circuit), the circuit is an RC mesh and \( x = (1 \ 1 \ 1 \ 1)^T \).

**Example:** In Fig. 3, if \( R_6 \) is some finite resistance, \( x \) requires some computation. The result is:

\[
x = \frac{1}{R_1 + R_6 + (R_2 + R_3)} \begin{bmatrix}
R_6 + (R_2 + R_3) \\
R_6 + \frac{R_3 (R_4 + R_5)}{R_2 + R_3 + R_4 + R_5} \\
R_6 + \frac{R_4 (R_5)}{R_2 + R_3 + R_4 + R_5} \\
R_6
\end{bmatrix}
\]

The general computational problem of calculating \( x \) is discussed in Section VIII.
To begin the formulation of the network differential equations, we start by rewriting equation (6),

\[ x \dot{e}(t) - v(t) = -R i(t) \]  

(7).

The constitutive relations for the capacitors can be neatly written as:

\[ i(t) = -C \ddot{v}(t) \]  

(8),

where the minus sign arises from our sign convention for capacitor currents. Substituting (3) into (7) we have,

\[ x \dot{e}(t) - v(t) = [R C] \ddot{v}(t) \]  

(9).

The system of equations in (9) is the canonical form for leaky RC meshes. The equations in (9) are also the most general: valid for all time and for any possible \( e(t) \). At this point, it is worthwhile specializing the system in (9) somewhat by considering only a certain form for \( e(t) \).

Since the voltage \( e(t) \) models the output of a logic gate undergoing a logic transition, typically \( e(t) \) reaches and remains at its final value, \( e(\rightarrow) \), for some time interval \( t > T \).
Example:

![Graph of e(t) with final value e(\infty) = 2 and T = 2]

Example: \( e(t) = "unit step at the origin" \)

![Graph of e(t) with final value e(\infty) = 1 and T = 0]

For an \( e(t) \) of this form, we can replace \( e(t) \) by \( e(\infty) \) for \( t \geq T \).

So (9) becomes:

\[
xe(\infty) - v(t) = [RC]\dot{v}(t),
\]

valid for a "saturating" \( e(t) \) for \( t \geq T \).

Now we can identify \( xe(\infty) \) as the final equilibrium voltage distribution.
At equilibrium $v(t) = 0$, so (10) becomes:

$$x_e(\infty) - v_{eq} = [RC]0 = 0 \implies v_{eq} = x_e(\infty)$$  \hspace{1cm} (11)

Finally, since estimates for and bounds on the step response will be derived later in this report, the canonical equations for a step input will be given. Using (10) and (11),

Network Differential Equations for a Step Input:

$$v_{eq} - v(t) = [RC] \dot{v}(t); \ t \geq 0$$  \hspace{1cm} (12)

\section*{V. Delay Estimation}

In this section, a good first-order estimate of the delay through a leaky RC mesh is developed. This delay estimate is called the "Elmore Delay" in the literature [4], also [1], [5-7]. We will denote it here by $T_{D_i}$. By convention, we call the output node of interest "node 1." So the subscript on $T_{D_i}$ simply indicates that the delay estimate depends on which node in the RC network we have chosen as our output node. Finally, since $T_{D_i}$ is just a single number, the use of $T_{D_i}$ alone takes no account of the threshold voltage of the device driven by node 1. A simple procedure for accounting for the logic threshold is presented.

\textbf{a.) Definition of $T_{D_i}$:}

We define $T_{D_i}$ in terms of the response at the output node 1 to a step transition at the input. We allow for a non-zero initial
equilibrium at the time of the step input, and denote the initial equilibrium voltage distribution with the vector \( v_o = (v_{1_o}, v_{2_o}, \ldots, v_{N_o})^T \). Using the same notation as the previous section, we denote the final equilibrium voltage distribution with the vector \( v_{eq} = (v_{1_{eq}}, v_{2_{eq}}, \ldots, v_{N_{eq}})^T \). The step transition can be either a step increase \( (v_{eq} > v_o) \) or a step decrease \( (v_{eq} < v_o) \). The amplitude of the step input is equal to \( \frac{1}{x_i} (v_{eq} - v_i) \).

**Definition:**

\[
T_{D_i} = \int_0^\infty \frac{v_i(t) - v_i}{v_{eq_i} - v_{i_o}} \, dt,
\]

where \( v_i(t) \) is the response at node \( i \) to a step transition at the input at time \( t=0 \) of amplitude \( \frac{1}{x_i} (v_{eq_i} - v_{i_o}) \).

Graphically, \( T_{D_i} \) is shown below:

![Graph showing the definition of \( T_{D_i} \)](image-url)
b.) **Computation of** \( T_{D_1} \):

Rewriting the canonical matrix equation for a step input, we have:

\[
v_{eq} - v(t) = \begin{bmatrix} R \end{bmatrix} \hat{y}(t); \quad t > 0
\]

(14).

Taking the \( i^{th} \) row of this matrix equation,

\[
v_{eq} - v_i(t) = \sum_{k} R_{ik} C_k \hat{v}_k(t); \quad t > 0
\]

(15)

(where \( \sum \) is taken over all capacitor nodes in the RC network).

Dividing by \( v_{eq} - v_i \) and then integrating from \( t=0 \) to \( t=\infty \), we obtain

\[
T_{D_1} = \int_{0}^{\infty} \frac{v_{eq} - v_i(t)}{v_{eq} - v_i(0)} dt = \frac{\sum_{k} R_{ik} C_k \int_{0}^{\infty} \hat{v}_k(t) dt}{v_{eq} - v_i(0)}
\]

(16).

So in order to compute \( T_{D_1} \) we need to know:

1) The initial and final equilibrium voltage distributions, \( v_{o} \) and \( v_{eq} \). Note that this is really equivalent to knowing the single vector \( x \), since for a step input we have \( v_{o} = xe(0^-) \) and \( v_{eq} = xe(0^+) \).

In fact, we can express \( T_{D_1} \) in terms of \( x \) instead of \( v_{o} \) and \( v_{eq} \):

\[
T_{D_1} = \frac{1}{\sum_{k} R_{ik} C_k x_k}
\]

(17).
2) The \( i^{th} \) row (or column) of the resistance matrix \( R \).

3) The capacitance matrix \( C \) (this requires no computation).

Note that when we are dealing with an RC mesh (i.e., no resistive paths to ground) starting from a zero initial equilibrium, then \( v_0 = 0 \) and \( v_i = v_k \) for all \( k \neq i \). Hence \( T_{D_i} \) reduces to \( \sum_k R_{ik} C_k \), which is consistent with the definition given in earlier works [1], [2].

c.) Alternate Interpretation of \( T_{D_i} \):

We have already defined \( T_{D_i} \) in terms of the response observed at node \( i \), \( v_i(t) \), to a step input applied at time \( t=0 \), when the RC network is at some (possibly non-zero) equilibrium. Alternately, we can interpret \( T_{D_i} \) in terms of \( \dot{v}_i(t) \). Note that \( \dot{v}_i(t) \), in addition to being the time derivative of \( v_i(t) \), is also the response observed at node \( i \) to an impulse input applied at time \( t=0 \), when the RC network is at zero equilibrium.

We will show in the next section that the step response \( v_i(t) \) is monotone, which means that \( \dot{v}_i(t) \) can be interpreted as a density function for a continuous random variable, when properly normalized:

\[
(\dot{v}_i(t))_{\text{normalized}} = \left( \frac{\dot{v}_i(t)}{\int_0^t \dot{v}_i(t) \, dt} \right) = \left( \frac{\dot{v}_i(t)}{v_i - v_{i0}} \right)_{\text{eq}}
\]  

(18).

\( T_{D_i} \) is the first moment (or "center of mass") of this normalized zero-state impulse response, as shown below:

Integrating by parts,

\[
\int_0^t \left( \frac{\dot{v}_i(t)}{v_i - v_{i0}} \right) dt = \int_0^t \frac{v_i - v_i(t)}{v_{i\text{eq}} - v_{i0}} \, dt - \left. \left[ t \left( \frac{v_i - v_i(t)}{v_{i\text{eq}} - v_{i0}} \right) \right] \right|_0^t = T_{D_i}
\]  

(19).
where we have used the fact that \( \lim_{t \to \infty} t(v_i - v_i(t)) = 0 \) because \( v_i(t) - v_i \) exponentially as \( t \to \infty \).

d.) **Incorporating Logic Threshold:**

As mentioned earlier, the use of \( T_D \) alone takes no account of the threshold voltage of the device driven by node \( \text{Node} \). To do so we need an estimate for the entire step response voltage waveform \( v_i(t) \). A reasonable estimate is

\[
\dot{v}_i(t) = v_i(t) + (v_i - v_i) e^{-t/T_D}
\]

since, like the exact but unknown response:

1. \( v_i(t) \) starts at \( v_i \).
2. \( v_i(t) \) is monotone.
3. \( v_i(t) \) asymptotes to \( v_i \).
4. \[
\int_0^{\infty} \frac{v_i(t) - v_i}{v_i - v_i} dt = T_D.
\]

In the next section we will show that \( v_i(t) \) always lies between the best currently available upper and lower bounds on the exact step response \( v_i(t) \).

We can invert this estimate of the entire waveform in order to estimate the time to reach a given logic threshold voltage \( v_T \):

\[
v_T = v_i(t) + (v_i - v_i) e^{-t/T_D} \quad \Rightarrow \quad t = T_D \ln \left( \frac{v_i - v_i}{v_i - v_T} \right)
\]

**Example:** If the logic threshold of the device driven by node \( \text{Node} \) is

\[
v_T = v_i + \frac{1}{2} (v_i - v_i),
\]
then instead of using \( T_D \) as a delay estimate,
we use the somewhat lower estimate, $T_{D_i}$ in $2 \approx 0.69 T_{D_i}$.

Example: If the logic threshold of the device driven by node $i$ is $v_T = v_{i_0} + (1 - \frac{1}{e}) (v_{i_{eq}} - v_{i_0}) \approx v_{i_0} + 0.63 (v_{i_{eq}} - v_{i_0})$, then we use exactly $T_{D_i}$ as a delay estimate.

The estimate (20) is not generally exact: for one thing, it contains only a single natural frequency: \(-\frac{1}{T_{D_i}}\). To better appreciate the nature of the approximation involved, Horowitz [7] has noted that, in the case of no grounded resistors, a sufficient condition for (20) to be exact is that the time derivatives of all the capacitor voltages are equal: i.e., $\dot{v}_k(t) = \dot{v}_i(t)$ for all $k$. In the case where we do have grounded resistors, the estimate (20) is exact when the normalized time derivatives of all the capacitor voltages are equal:

$$\frac{\dot{v}_k(t)}{v_{i_{eq}} - v_{i_0}} = \frac{\dot{v}_i(t)}{v_{i_{eq}} - v_{i_0}}$$

for all $k \neq i$. (22)

We can see this by going back to the $i$th row of the original system of equations for the node voltages:

$$v_{i_{eq}} - v_i(t) = \sum_{k} R_{ik} C_k \dot{v}_k(t)$$

$$= \sum_{k} R_{ik} C_k \frac{v_{k_{eq}} - v_{k_0}}{v_{i_{eq}} - v_{i_0}} \dot{v}_i(t) \quad \text{; using the approximation in (22)}$$

$$= \left[ \begin{array}{c} \sum_{k} R_{ik} C_k (v_{k_{eq}} - v_{k_0}) \\ v_{i_{eq}} - v_{i_0} \end{array} \right] \dot{v}_i(t)$$

$$= T_{D_i} \dot{v}_i(t)$$

(23).
The solution to the first-order differential equation given by (23) is
\[ v_i(t) = v_i(t). \]

Another view of the nature of the approximation (20) will emerge from consideration of the step response bounds, which we consider next.

\textbf{VI. Optimal Control Method for Determining Bounds on the Step Response}

The purpose of this section is to clearly derive step response bounds for leaky RC meshes using an optimal control point-of-view. The bounds derived will be analogous to those in [1]. Recent bound-tightening results such as slew-rate limitations [8], and (in the case of RC trees) spatial convexity [9] will not be considered here. These recent improvements are omitted because they do not add to the basic understanding of the optimal control approach. We do not yet know exactly to what extent these bound-tightening results can be applied to leaky RC meshes. It is clear, however, that at least the simplest of the slew-rate limitations applies and always gives somewhat tighter bounds than those developed here.

a.) Preliminary Results:

\textbf{Fact 1:} For any three nodes \( i, j, k \) of a leaky RC mesh,

\[ R_{ij} R_{kj} > R_{ij} R_{kji} \quad (24), \]

where the resistances in (24) are elements of the resistance matrix \( R \) in (1).
The proof of Fact 1 depends on certain properties of the conductance matrix $G$. These properties, as well as the proof of Fact 1, are given in Appendix A.

**Fact 2:** For any node $j$ of a leaky RC mesh, the observed response, $v_j(t)$, to a step transition at the input applied at time $t=0$, when the network is at some (possibly non-zero) equilibrium, is monotone:

$$
\dot{v}_j(t) > 0, \forall t > 0 ; \text{if } v_j > v_{j_0}
$$

$$
\dot{v}_j(t) < 0, \forall t > 0 ; \text{if } v_j < v_{j_0}
$$

(25).

The proof of Fact 2 is also given in Appendix A.

**Fact 3:** For any two nodes $i$, $k$ of a leaky RC mesh:

On an "up" transition ($v_{i_{eq}} > v_{o}$) we have,

$$
R_{ii}[v_{k_{eq}} - v_k(t)] > R_{ki}[v_{i_{eq}} - v_i(t)]
$$

(26)

$$
R_{ki}[v_{k_{eq}} - v_k(t)] < R_{kk}[v_{i_{eq}} - v_i(t)]
$$

(27).

On a "down" step transition ($v_{eq} < v_{o}$) the sense of each inequality is reversed,

$$
R_{ii}[v_{k_{eq}} - v_k(t)] \leq R_{ki}[v_{i_{eq}} - v_i(t)]
$$

(28)

$$
R_{ki}[v_{k_{eq}} - v_k(t)] \geq R_{kk}[v_{i_{eq}} - v_i(t)]
$$

(29).
The proof of Fact 3, which uses both Fact 1 and Fact 2, is presented here in the body of the report.

**Proof of Fact 3:**

The system of differential equations for the node voltages is:

\[ v_i - v(t) = [RC]\dot{v}(t) \]  

Taking the \( k^{th} \) row \( \Rightarrow v_k - v(t) = \sum_j R_{kj} C_j \dot{v}_j(t) \)  

Taking the \( i^{th} \) row \( \Rightarrow v_i - v_i(t) = \sum_j R_{ij} C_j \dot{v}_j(t) \)

Using (31) and (32), we compute:

\[ R_{ii} [v_k - v_k(t)] - R_{ki} [v_i - v_i(t)] = \sum_j (R_{ii} R_{kj} - R_{ki} R_{ij}) C_j \dot{v}_j(t) \]

By applying Facts 1 and 2, (26) and (28) now follow directly from (33).

By interchanging the subscripts \( i \leftrightarrow k \) and using the fact that \( R \) is symmetric, (27) and (29) follow from (26) and (28) respectively.

b.) **A New State Variable:**

At this point, we will introduce a new state variable, \( f_i \), that will be part of the reduced order model in the formulation of the optimal control problem. We also will prove a lemma concerning \( f_i \). The result of this lemma will serve as the state constraints for the optimal control problem. The presentation of the optimal control problem itself is delayed until later (part (c) of Section VI).

The new state variable is defined as follows:
where (as usual): \[ v_i(t) \] is the response at node \( i \) to a step input at time \( t=0 \).
\[ v_i \] is the initial equilibrium voltage at node \( i \).
\[ v_i \] is the final equilibrium voltage at node \( i \).

This definition is illustrated below:

\[ f_i(t) \triangleq \int_0^t \frac{v_i(t') - v_i(t')}{v_i - v_i} \, dt' \]  

Fig. 7: Definition of \( f_i \)

Note the similarity of Fig. 7 to Fig. 6 (where we defined \( T_{D_i} \)). In fact, we can verify directly from these illustrations the initial condition:

\[ f_i(0) = T_{D_i} \]  

Due to normalization in the denominator of (34), the behavior of \( f_i(t) \) is the same for "up" steps and "down" steps of arbitrary magnitude at the input. Starting at \( T_{D_i} \) when \( t=0 \), \( f_i(t) \) will decrease monotonically towards...
Lemma: For any choice of output node \( j \) in a leaky RC mesh,

\[
T_{R_i} \left( \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \leq f_i(t) \leq T_p \left( \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right)
\]  \hspace{1cm} (36)

where:

\[
T_{R_i} \triangleq \frac{1}{R_{ii}} \sum_k R_{ik} C_k
\]  \hspace{1cm} (37)

\[
T_p \triangleq \sum_k R_{kk} C_k
\]  \hspace{1cm} (38)

Proof of Lemma:

Starting with the \( i \)th row of the canonical matrix differential equation,

\[
v_{i_{eq}} - v_i(t') = \sum_k R_{ik} C_k \frac{\dot{v}_k(t')}{v_{i_{eq}} - v_{i_o}} \]  \hspace{1cm} (39)

we divide by \( v_{i_{eq}} - v_{i_o} \), and then integrate from \( t'=t \) to \( t'=\infty \) to obtain

\[
f_i(t):
\]

\[
f_i(t) = \int_t^\infty \frac{v_{i_{eq}} - v_i(t')}{{v_{i_{eq}} - v_{i_o}}} dt' = \sum_k R_{ik} C_k \frac{\int_t^\infty \frac{\dot{v}_k(t')}{v_{i_{eq}} - v_{i_o}} dt'}{{v_{i_{eq}} - v_{i_o}}}
\]

\[
= \sum_k R_{ik} C_k (v_{i_{eq}} - v_{i_o}) \]  \hspace{1cm} (40)

Using (26) from Fact 3 on an "up" step [or (28) on a "down" step] in (40) gives:

\[
f_i(t) = \frac{R_{ik} C_k}{R_{ii}} \left( v_{i_{eq}} - v_i(t) \right) \]  \hspace{1cm} (41)

\[
= \left( \frac{1}{R_{ii}} \sum_k R_{ik} C_k \right) \left( \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right)
\]
where we have used the fact that $R_{ki} = R_{ik}$, by symmetry of $R$.

Using (27) from Fact 3 on an "up" step [or (29) on a "down" step] in (40) gives:

$$f_i(t) \leq \sum_k R_{ik} C_k \left( \frac{R_{kk}}{R_{ki}} \frac{(v_i - v_i(t))}{v_{eq} - v_{eq}} \right)$$

$$= \left( \sum_k R_{kk} C_k \right) \left( \frac{v_i - v_i(t)}{v_{eq} - v_{eq}} \right)$$

(42),

where, again, we have used the fact that $R_{ki} = R_{ik}$. Together, (41) and (42) constitute (36). This proves the lemma.

c.) The Optimal Control Approach:

The step response bounds, as in [1], can be derived by "ad hoc" manipulation of (25) and (36), but the methodology is somewhat obscure. A clearer view emerges from recasting these calculations into the form of a linear minimum - (and maximum -) time optimal control problem with state constraints. In addition, this approach has the advantage that it becomes much clearer how to incorporate additional information in order to obtain tighter bounds.

The optimal control approach is based on a reduced order model of the voltage transition process at a selected output node $i$. We regard this process as being described by the following second order model:
where:

(i) \( g_i(t) \) is shorthand for the \textbf{normalized} "distance to go" for the step response at node \( i \):

\[
\dot{g}_i(t) = \frac{v_i - v_i(t)}{v_{eq_i} - v_{eq_o}}
\]

i.e., \( g_i(t) \) is defined as in (44).

(ii) \( f_i(t) \) is defined as in (34).

\[
\begin{align*}
\text{[Note that } f_i(t) &= \int_{v_i}^{v_i(t)} \frac{v_i - v_i(t')}{v_{eq_i} - v_i} dt' = \int_{v_i}^{g_i(t')} g_i(t') dt',
\] \\
\text{so by the fundamental theorem of calculus we have } \dot{f}_i(t) &= -g_i(t), \text{ as in (43).]}
\]

(iii) The input \( u(t) \) is introduced to represent the unknown waveform

\[
\dot{g}_i(t) = -\left( \frac{\dot{v}_i(t)}{v_i - v_{eq_i}} \right).
\]

In addition to the \textbf{dynamics} in (43) we also have,

\textbf{initial conditions} (see (35)):

\[
\begin{bmatrix} f_i(0) \\ g_i(0) \end{bmatrix} = \begin{bmatrix} T_{D_i} \\ 1 \end{bmatrix}
\]

\textbf{state constraints} (see (36)):

\[
T_{R_i} \leq g_i(t) \leq T_F g_i(t)
\]

and the \textbf{input constraint} (see (25)):

\[
u(t) \leq 0
\]
To arrive at upper and lower bounds on $g_i(t)$ we pose the following optimal control problem: What is the minimum time $t_{\min}(g_i^*)$ and the maximum time $t_{\max}(g_i^*)$ required for a solution of (43) to pass through a given "target" value $g_i^*$ (with $0 < g_i^* < 1$) and still satisfy the constraints given by (45), (46), and (47)? Once the maximum (minimum) time $t_{\max}(g_i^*)$ ($t_{\min}(g_i^*)$) is known as a function of $g_i^*$, then the inverse function gives an upper (lower) bound on $g_i^*(t)$. [An upper (lower) bound on $g_i^*(t)$ can then be easily converted to a lower (upper) bound on the voltage step response $v_i(t)$.]

The above problem is called a "state constrained linear optimal control problem" and the Pontryagin's maximum principle could be used to find the optimal $u(t)$ and the resulting optimal state-space trajectory $(f_i(t), g_i(t))$. However, the problem can be solved in a much simpler way as follows. Regard any trajectory in the $f_i$-$g_i$ plane as a function of $f_i$, i.e., $g_i(f_i)$. From (43) we have

$$\frac{dt}{df_i} = -\frac{1}{g_i}$$  \hspace{1cm} (48),

and the time required by the trajectory to pass through two points $(f_i(0), g_i(0))$ and $(f_i(T), g_i(T))$ is

$$T = \int_0^T dt = \frac{f_i(T)}{f_i(0)} - \frac{1}{g_i(f_i)} df_i = \left[ \frac{f_i(0)}{g_i(f_i)} - \frac{1}{g_i(f_i)} \right]$$ \hspace{1cm} (49).
The trajectory corresponding to a voltage transition at node \( i \) starts at \((f_i(0), g_i(0)) = (0, 1)\) and because \( \dot{f}_i(t) < 0 \) and \( \dot{g}_i(t) < 0 \), it moves down and to the left towards the origin (see Fig. 8). The trajectory is confined to the region between \( g_i = \frac{1}{T_p} f_i \) and \( g_i = \frac{1}{T_R} f_i \) as stated by the state constraints (46). From (49), it can be seen that for \( g_i(t) \) to change from 1 to a particular value \( g_i^* \), the lowest (highest) possible trajectory (i.e., with the minimum (maximum) possible \( g_i(t) \)) over the widest (narrowest) integration interval on the \( f_i \)-axis requires maximum (minimum) time and therefore is the slowest (fastest) trajectory. We can
then calculate $t_{\text{max}}(g_i^*)$ and $t_{\text{min}}(g_i^*)$ for the optimal trajectories, using (49).

**Solution to the maximum-time problem:**

There are two different algebraic forms for $t_{\text{max}}(g_i^*)$ over the range $0 < g_i^* < 1$. This is a result of the fact that the maximum-time state space trajectory assumes a more complex form as $g_i^*$ is decreased below $\frac{T_{D_i}}{T_p}$.

**Case #1** \( \frac{T_{D_i}}{T_p} \leq g_i^* < 1 \):

![Maximum-Time Trajectory Diagram](image)

Fig. 9: Maximum-Time Trajectory (for $\frac{T_{D_i}}{T_p} \leq g_i^* < 1$) using (49),
\[ t_{\text{max}}(g_i^*) = \int_{T_R g_i}^{T_D} \frac{1}{g_i} \, dt_i = \frac{1}{g_i} \int_{T_R g_i}^{T_D} \, dt_i = \frac{1}{g_i} \left[ T_D - T_R g_i^* \right] \]

\[ = \frac{T_D}{g_i} - \frac{T_R}{g_i} \]  

(50).

**Case 2** \((0 < g_i^* \leq \frac{T_D}{T_p})\):

![Diagram of maximum-time trajectory](image)

**Fig. 10:** Maximum-Time Trajectory (for \(0 < g_i^* \leq \frac{T_D}{T_p}\))

using (49),
\[
\begin{align*}
t_{\text{max}}(g_i^*) & = \int_{\theta_i^*}^{T_{p_i}} g_i^* \frac{1}{\theta_i} \, df_i + \int_{\theta_i^*}^{T_{p_i}} g_i^* \frac{1}{\theta_i} \, df_i \\
& = \frac{1}{g_i} \left[ T_{p_i} g_i^* - T_{R_i} g_i^* \right] + T_{p_i} \left[ \ln(T_{p_i}) - \ln(T_{p_i} g_i^*) \right] \\
& = T_{p_i} - T_{p_i} \ln \left( \frac{T_{p_i}}{T_{p_i}} \right) + T_{p_i} \ln \left( \frac{T_{p_i}}{T_{p_i} g_i^*} \right)
\end{align*}
\]

Solution to the minimum-time problem:

There are three different algebraic forms for \( t_{\text{min}}(g_i^*) \) over the range \( 0 < g_i < 1 \). This is a result of the fact that the minimum-time state space trajectory assumes a progressively more complex form as \( g_i^* \) is decreased below \( \frac{T_{D_i}}{T_p} \) and \( \frac{T_{R_i}}{T_p} \).
Case #1 \[ \left( \frac{T_{D_i}}{T_p} \leq g_i^* \leq 1 \right) : \]

Fig. 11: Minimum-Time Trajectory (for \( \frac{T_{D_i}}{T_p} \leq g_i^* \leq 1 \))

Using (49), we see that there is no area (along the \( f_i \)-axis) under the minimum-time trajectory, so:

\[
\tau_{\text{min}}(g_i^*) = 0
\] (52).
Case #2 \( \left( \frac{T_{R_i}}{T_p} \leq \frac{q_i^*}{T_p} \leq \frac{T_{D_i}}{T_p} \right) \):

\[ t_{\text{min}}(q_i^*) = \int_{T_p q_i^*}^{T_{D_i}} \frac{1}{(l)} \, df_i = \int_{T_p q_i^*}^{T_{D_i}} \, df_i = \frac{T_{D_i} - T_p q_i^*}{T_p q_i^*} \quad (53). \]
Case #3 \((0 < g_i^* \leq \frac{T_{R_i}}{T_p})\):

Using (49),

\[
\begin{align*}
t_{\text{min}}(g_i^*)& = \int_{T_{p}g_i^*}^{T_{R_i}} \frac{1}{f_i} df_i + \int_{T_{R_i}}^{T_{D_i}} \frac{1}{(1)} df_i \\
& = T_{R_i} \int_{T_{p}g_i^*}^{T_{R_i}} \frac{1}{f_i} df_i + \int_{T_{R_i}}^{T_{D_i}} df_i \\
& = T_{R_i} \left[ \ln(T_{R_i}) - \ln(T_{p}g_i^*) \right] + [T_{D_i} - T_{R_i}] \\
& = \frac{T_{R_i} \ln \left( \frac{T_{R_i}}{T_{p}g_i^*} \right)}{T_{D_i} - T_{R_i}} + T_{D_i} - T_{R_i} \quad \text{(54)}.
\end{align*}
\]

Fig. 13: Minimum-Time Trajectory (for \(0 < g_i^* \leq \frac{T_{R_i}}{T_p}\)
The results in (50) - (54) can be summarized in the following graph of both $t_{\text{max}}(g_i^*)$ and $t_{\text{min}}(g_i^*)$ vs. $g_i^*$ (over the entire range $0 < g_i^* < 1$):

![Graph showing $t_{\text{min}}(g_i^*)$ and $t_{\text{max}}(g_i^*)$ vs. $g_i^*$](image)

Fig. 14: Maximum and Minimum Times as a Function of $g_i^*$

Since $\dot{g}_i(t) \leq 0$, at any instant of time $t$ the actual value of $g_i(t)$ is less than or equal to the value of $g_i^*$ that is reached via the maximum-time trajectory in time $t$, and is greater than or equal to the value of $g_i^*$ that is reached via the minimum-time trajectory in time $t$. For this reason, the inverse functions of $t_{\text{max}}(g_i^*)$ and $t_{\text{min}}(g_i^*)$
are upper and lower bounds respectively on \( g_i(t) \).

\[
g_i(t) \leq g_i(t) \leq \bar{g}_i(t)
\]

where \[
\begin{align*}
g_i(t) &\triangleq t_{\min}^{-1}(t) \\
\bar{g}_i(t) &\triangleq t_{\max}^{-1}(t)
\end{align*}
\]

These bounds are plotted below:

Fig. 15: Bounds on \( g_i(t) \)

\[
\bar{g}_i(t) = \begin{cases} 
1 & ; 0 \leq t \leq T_{D_i} - T_{R_i} \\
\frac{T_{D_i}}{t + T_{R_i}} & ; T_{D_i} - T_{R_i} \leq t \leq T_P - T_{R_i} \\
\frac{T_{D_i} e^{(T_P - T_{R_i})/T_P} - t/T_P}{T_P} & ; t > T_P - T_{R_i}
\end{cases}
\]

(56).
Finally, the bounds on $g_i(t)$ can be easily converted to bounds on the voltage step response $v_i(t)$ as follows:

$$
0 \leq t \leq T_{D_i} - T_{R_i}
$$

$$
\frac{T_{D_i} - t}{T_p} ;
$$

$$
T_{R_i} \left( \frac{T_{D_i} - T_{R_i}}{T_{R_i}} \right) / T_{R_i} = \frac{T}{T_{R_i}} ;
$$

$$
t \geq T_{D_i} - T_{R_i}
$$

(57).

$$
g_i(t) = \left\{ \begin{array}{l}
\frac{T_{D_i} - t}{T_p} ;
\end{array} \right.
$$

$$
= \left\{ \begin{array}{l}
\frac{T_{R_i}}{T_p} \left( \frac{T_{D_i} - T_{R_i}}{T_{R_i}} \right) / T_{R_i} ;
\end{array} \right.
$$

$$
t \geq T_{D_i} - T_{R_i}
$$

(58).

$$
\left( g_i(t) = \frac{v_{i_{eq}} - v_{i_0}}{v_{i_{eq}} - v_{i_0}} \right) \leq g_i(t)
$$

$$
\left( v_{i_{eq}} - (v_{i_0} - g_i(t)) \right)
$$

(59).

$$
\left( v_{i_{eq}} - v_{i_0} \right)
$$

These bounds are plotted below:
Fig. 16: Bounds on $v_1(t)$

Now that bounds for the step response have been derived, it is worthwhile to reconsider the Elmore waveform estimate discussed in Section $v, v_1(t) \triangleq v_1 + (v_1 - v_0) e^{-t/T_{D_i}}$. When plotted in the $f_i - g_i$ plane, the trajectory corresponding to $v_1(t)$ is a straight line from the initial condition $(f_i(0), g_i(0)) = (T_{D_i}, 1)$ to the origin. It is therefore a feasible (lies inside the state-constraint cone: $f_i \leq g_i \leq T_{P_R}$) but not optimal solution to the optimal control problem. Hence the Elmore waveform estimate will always lie between the upper and lower bounds on the exact response $v_1(t)$,

$$v_1(t) \leq \tilde{v}_1(t) \leq \check{v}_1(t)$$

(60).
For any leaky RC mesh it is always true that $T_{R_i} \leq T_{D_i} \leq T_p$ (evaluate the state constraints (46) at time $t=0$; see also Appendix B). The Elmore waveform estimate, $v_i(t)$, represents an effort to approximate the dynamics of a higher order network by one with a single time constant $T_{D_i}$; the estimate is exact only in that case. However, $v_i(t)$ is a very good estimate whenever $T_p - T_{R_i} \ll T_{D_i}$ because this means the state-constraint cone shown in Fig. 8-13 is very narrow, so the actual trajectory in the $f_i - g_i$ plane is well-approximated by the Elmore trajectory of a straight line of slope $\frac{1}{T_{D_i}}$. (In fact, the voltage bounds, $v_i(t)$ and $\bar{v}_i(t)$, are very tight whenever $T_p - T_{R_i} \ll T_{D_i}$ for exactly the same reason. The minimum- and maximum-time trajectories are very close together simply by virtue of the fact that they each must lie inside a very narrow state-constraint cone.)

VII. Bounds for Non-Step Inputs

In the previous section we derived rigorous upper and lower bounds on the step response voltage waveform at any selected output node $i$. A question which naturally arises is: How do we bound the response at node $i$ to an arbitrary non-step input? In this section, we will show a general method for deriving bounds for an arbitrary input in terms of the step response bounds. We will first consider the case where the leaky RC mesh is at zero initial equilibrium (this treatment is essentially the same as the one given in Appendix E of reference [1]). We will then show how these results can be easily adapted to an arbitrary (possibly non-zero) initial equilibrium. Finally, we will carry out the calculations...
explicitly for a particular example: a **saturating ramp** input.

a.) Non-Step Input (Zero Initial Equilibrium):

The method presented here depends on the fact that the RC network is **linear** (and time-invariant). We will first introduce some notation:

- $h_1(t) \triangleq \text{zero-state response at node } i \text{ to a unit impulse input.}$
- $w_1(t) \triangleq \text{zero-state response at node } i \text{ to a unit step input; so } w_1(t) = h_1(t).$

(Here we use the notation $w_1(t)$ rather than $v_1(t)$. The reason for this is that $w_1(t)$ denotes a **more specific** waveform than was denoted by our previous usage of $v_1(t)$. We had previously used $v_1(t)$ to represent the response to a step input of **arbitrary magnitude** starting from some **arbitrary initial equilibrium**. Note also the difference in dimensions: $v_1(t)$ has dimensions of volts, but $w_1(t)$ is **dimensionless** since it represents the **unit** step response.)

- $e(t) \triangleq \text{arbitrary non-step input.}$
- $y_1(t) \triangleq \text{zero-state response at node } i \text{ to the input } e(t).$

Using Laplace transforms,

\[
\begin{align*}
W_1(s) &= H_1(s) \cdot \frac{1}{s} \quad (61), \\
Y_1(s) &= H_1(s) \cdot E(s) \quad (62).
\end{align*}
\]

Combining (61) and (62) we have,

\[
Y_1(s) = \left[ \frac{1}{s} H_1(s) \right] [sE(s)] = [W_1(s)] [sE(s)] \quad (63).
\]

But, $sE(s) = L\left( \frac{de(t)}{dt} \right)$ ; assuming $e(0)=0$ \quad (64).
So converting (63) back to the time domain (and using (64)), we have:

\[
y_i(t) = w_i(t) * \frac{de(t)}{dt} = \int_0^t w_i(t-t') \frac{de(t')}{dt'} \, dt'
\]

(65).

The step response bounds for \( w_i(t) \) are given by:

\[
w_i(t) \leq w_i(t) \leq \bar{w}_i(t)
\]

(66),

where (referring back to (59)) we have:

\[
\begin{align*}
\bar{w}_i(t) &= x_i(1-\bar{g}_i(t)) \\
\bar{w}_i(t) &= x_i(1-\tilde{g}_i(t))
\end{align*}
\]

(67),

since \( w_{i0} = 0 \) and \( \bar{w}_i = x_i \).

In order to obtain bounds on \( y_i(t) \), we note that convolution of \( \frac{de(t)}{dt} \) retains the sense of the inequality when \( \frac{de(t)}{dt} \geq 0 \) \( \forall t \geq 0 \), and reverses the sense of the inequality when \( \frac{de(t)}{dt} \leq 0 \) \( \forall t \geq 0 \), i.e.,

\[
\begin{cases}
\frac{w_i(t) * de(t)}{dt} \leq (y_i(t) = w_i(t) * \frac{de(t)}{dt} ) \leq \frac{-\bar{w}_i(t) * de(t)}{dt} ; \quad \frac{de(t)}{dt} \geq 0 \quad \forall t \geq 0 \\
\frac{-\bar{w}_i(t) * de(t)}{dt} \leq (y_i(t) = w_i(t) * \frac{de(t)}{dt} ) \leq \frac{\bar{w}_i(t) * de(t)}{dt} ; \quad \frac{de(t)}{dt} \leq 0 \quad \forall t \geq 0
\end{cases}
\]

(68).

So, when the arbitrary input \( e(t) \) is monotone, the bounds on \( y_i(t) \) are given directly by (68):
In typical CAD applications, the input $e(t)$ is modelled by a monotone waveform. Hence the bounds given in (69) are sufficient for most cases. However in the rare case where the input $e(t)$ has both positive and negative slopes, we can still bound the response $y_i(t)$ in a similar way. We do so by considering separately the time intervals where $\frac{de(t)}{dt} > 0$ and $\frac{de(t)}{dt} < 0$ when we perform the convolution operation (for details, see Appendix E of [1]).

b.) Non-Step Input (Arbitrary Initial Equilibrium)

Suppose that we desire bounds on the response $y_i(t)$ to a non-step input $e(t)$ when we are starting from some non-zero equilibrium.

Define:

$$ \Delta e(t) \triangleq e(t) - e(0) $$
$$ \Delta y_i(t) \triangleq y_i(t) - y_i(0) $$

(70).
We can derive bounds for the response $\Delta y_i(t)$ to the non-step input $\Delta e(t)$ by the convolution procedure discussed in part (a), since the $(\Delta y, \Delta e)$ system is starting from a zero equilibrium. Just add $y_i(0)$ to these bounds in order to obtain bounds on $y_i(t)$:

$$y_i(t) \leq y_i(t) \leq \bar{y}_i(t)$$

where,

$$y_i(t) = y_i(0) + \Delta y_i(t)$$

$$\bar{y}_i(t) = y_i(0) + \bar{\Delta y}_i(t)$$

(71).

The justification for this is that the $(\Delta y, \Delta e)$ system is described by the same differential equation as the $(y, e)$ system:

$$\Delta y(t) = \dot{y}(t)$$

$$= [RC]^{-1} (xe(t) - y(t)) ; \text{see (9)}$$

$$\Delta = [RC]^{-1} (x(\Delta e(t) + e(0)) - (\Delta y(t) + y(0)))$$

$$= [RC]^{-1} (x \Delta e(t) + xe(0) - \Delta y(t) - y(0))$$

$$\uparrow$$

cancel, since $y(0) = xe(0)$

$$= [RC]^{-1} (x \Delta e(t) - \Delta y(t))$$

(72).

c.) Example: Saturating Ramp Input

The non-step input, $e(t)$, considered in this example will be called a "saturating ramp" since it starts from zero volts at time $t=0$
and rises linearly until it saturates at $V_{\text{sat}}$ volts at time $t = T_{\text{sat}}$ (see Fig. 17).

![Saturating Ramp Input](image)

**Fig. 17: Saturating Ramp Input**

Furthermore, in this example the bounds calculations will be done under the assumption that the ramp rises sharply compared to the characteristic time constants of the RC network being driven. Specifically, we shall assume:

$$0 < T_{\text{sat}} < \min \left\{ \left( T_{D_i} - T_{R_i} \right), \left( T_{P_i} - T_{D_i} \right) \right\}$$

(73).

[There is nothing fundamental about the assumption in (73). Upper and lower bounds for any choice of $T_{\text{sat}}$ can be calculated by the procedure outlined in this section; the only difference being that the convolution integrals in (69) could assume a different form.]

In the following bounds calculations for the zero-state response at node 1 to the saturating ramp input $e(t)$, we will make use of the following abbreviation:
\[ K_i = \frac{V_{sat}}{T_{sat}} x_i \]  

(Note that \( x_i \) is dimensionless, so that \( K_i \) has dimensions of volts/seconds.)

**Upper Bound**: There are four different upper bound formulas depending on the value of time \( t \).

Using (69), we have:

1) for \( 0 < t \leq T_{sat} \)

\[
\bar{y}_i(t) = K_i \int_{0}^{t} \left[ 1 - \frac{T_{D_i} - (t-t')}{T_p} \right] dt'
\]

\[
= K_i [(1 - \frac{t}{T_p})t + \frac{t^2}{2T_p}] 
\]  

(75).

2) for \( T_{sat} < t \leq T_{D_i} - T_{R_i} \)

\[
\bar{y}_i(t) = K_i \int_{0}^{T_{sat}} \left[ 1 - \frac{T_{D_i} - (t-t')}{T_p} \right] dt'
\]

\[
= K_i [(1 - \frac{T_{D_i}}{T_p} - \frac{T_{sat}}{2T_p}) T_{sat} + \frac{T_{sat}}{T_p} t] 
\]  

(76).

3) for \( T_{D_i} - T_{R_i} < t \leq (T_{D_i} - T_{R_i}) + T_{sat} \)

\[
\bar{y}_i(t) = K_i \int_{0}^{T_{D_i} - T_{R_i}} \left[ 1 - \frac{T_{D_i} - (t-t')}{T_p} e^{\frac{(T_{D_i} - T_{R_i})}{T_{R_i} e}} \right] dt'
\]

\[
+ K_i \int_{t-(T_{D_i} - T_{R_i})}^T \left[ 1 - \frac{t}{T_p} \right] dt'
\]
\[
\begin{align*}
&= K_i \left\{ \frac{1}{T_p} (T_{D_i} - t) [t - (T_{D_i} - T_{R_i})] + \frac{1}{2T_p} [t - (T_{D_i} - T_{R_i})]^2 \right. \\
&\quad \left. - \frac{T_{R_i}^2}{2T_p} [1 - e^{-(T_{D_i} - T_{R_i})/T_{R_i}}] + (1 - \frac{T_{D_i}}{T_p} - \frac{T_{sat}}{2T_p}) T_{sat} + \frac{T_{sat}}{T_p} t \right\}
\end{align*}
\]

(77).

4) for \( t \geq (T_{D_i} - T_{R_i}) + T_{sat} \),
\[
\begin{align*}
-\gamma_i(t) &= K_i \int_0^t \left[ 1 - \frac{T_{R_i}}{T_p} e^{-(T_{D_i} - T_{R_i})/T_{R_i}} \right] dt' \\
&= K_i \left\{ T_{sat} - \frac{T_{R_i}^2}{T_p} e^{-(T_{D_i} - T_{R_i} + T_{sat})/T_{R_i}} \left[ 1 - e^{-T_{sat}/T_{R_i}} \right] \right\}
\end{align*}
\]

(78).

**Lower Bound:** There are five different lower bound formulas depending on the value of time \( t \).

Using (69), we have:

1) for \( 0 < t < \frac{T_{D_i} - T_{R_i}}{T_{D_i}} \),
\[
\gamma_i(t) = 0
\]

(79).

2) for \( \frac{T_{D_i} - T_{R_i}}{T_{D_i}} < t < \frac{(T_{D_i} - T_{R_i}) + T_{sat}}{T_{D_i}} \),
\[
\begin{align*}
\gamma_i(t) &= K_i \int_0^t \left[ 1 - \frac{(t-t') + T_{R_i}}{T_{R_i}} \right] dt' \\
&= K_i \left\{ [t - (T_{D_i} - T_{R_i})] - T_{D_i} \ln \left( \frac{t + T_{R_i}}{T_{D_i}} \right) \right\}
\end{align*}
\]

(80).


3) for \( (T_{D_i} - T_{R_i}) + T_{sat} < t < (T_p - T_{R_i}) \),

\[
X_i(t) = K_i \int_0^{T_{sat}} \left[ 1 - \frac{T_{D_i}}{(t-t')+T_{R_i}} \right] dt'
\]

\[
= K_i \left\{ T_{sat} - \frac{T_{D_i}}{T_{R_i}} \ln \left( \frac{t'+T_{R_i}}{t+T_{R_i} - T_{sat}} \right) \right\}
\]

(81).

4) for \( T_p - T_{R_i} < t < (T_{p-R_i}) + T_{sat} \),

\[
X_i(t) = K_i \int_0^{t-(T_p-T_{R_i})} \left[ 1 - \frac{T_{D_i}}{T_p} \right] e^{-\frac{T_{D_i}}{T_p} \left( T_p - (t-t') \right) / T_p} dt'
\]

\[
+ K_i \int_{t-(T_p-T_{R_i})}^{T_{sat}} \left[ 1 - \frac{T_{D_i}}{(t-t')+T_{R_i}} \right] dt'
\]

\[
= K_i \left\{ T_{sat} - T_{D_i} \ln \left( \frac{t+T_{R_i}}{t+T_{R_i} - T_{sat}} \right) - T_{D_i} \left[ 1 - e^{-\frac{(t-(T_p-T_{R_i}))}{T_p}} \right] \right\}
\]

(82).

5) for \( t \geq (T_{p-R_i}) + T_{sat} \),

\[
X_i(t) = K_i \int_0^{T_{sat}} \left[ 1 - \frac{T_{D_i}}{T_p} \right] e^{-\frac{T_{D_i}}{T_p} \left( T_p - (t-t') \right) / T_p} dt'
\]

\[
= K_i \left\{ T_{sat} - T_{D_i} \left[ 1 - e^{-\frac{(T_p-T_{R_i}+T_{sat})}{T_p}} \right] \right\}
\]

(83).
VIII. Additional Computational Burdens Arising from the Presence of Resistive Paths to Ground

The formulas for the time constants $T_{R_1}$, $T_{D_1}$, and $T_p$ indicate that the information required in order to compute bounds for leaky RC meshes is as follows:

1) The $i$th row (or column) of the resistance matrix $R$.
2) The diagonal elements of the resistance matrix $R$.
3) All elements of the vector $x$ (i.e., the equilibrium voltage distribution for the entire network when a 1-volt source is applied at the input).

Items 1) and 2) above do not represent much of an additional burden over non-leaky RC meshes that happen to contain resistor loops. If a non-leaky RC mesh contains resistor loops, it is already difficult to calculate the entries of the $R$ matrix. The presence of grounded resistors does not make this already difficult problem that much harder. If, however, the RC network under consideration is a leaky RC tree, than items 1) and 2) post a clear-cut extra computational burden. The reason for this is that the entries of the $R$ matrix will require some computation: we cannot read them off by inspection as we can with a non-leaky RC tree [1].

Item 3) above represents a clear-cut extra computational burden regardless of what the underlying non-leaky RC mesh looks like. For a non-leaky RC mesh, all equilibrium voltages are equal. If resistive paths to ground are present, then some computation is required. However, as the following lemma shows, the extra computation for the equilibrium
voltage distribution caused by the presence of any number of grounded resistors in the RC network is no worse than the extra computation required to consider only one (in most cases) extra node in the RC network as an output node to be bounded.

Lemma: Assume (as is usually the case) that the RC network under consideration has only one capacitor node connected to the voltage source through a single resistor (see Fig. 18). Label this node as "1", and label the resistor connecting the source to node 1 as "R_s" ("source resistance"). We have the following relation:

\[ x_k = \frac{R_{kl}}{R_s} \quad \text{for nodes } k \text{ in the RC network} \]  

(84),

where \( R_{kl} \) is an element of the resistance matrix \( R \).

Fig. 18: RC network with only one node connected directly to the source.
Proof of Lemma:

Assume the input voltage source is a step of magnitude $v_o$ volts at time $t=0$, and that the network is starting from a zero equilibrium condition. The differential equation for the RC network is,

$$[RC] \dot{v}(t) = v_{eq} - v(t) \tag{85}. $$

Taking the $k^{th}$ row at time $t=0^+$, we have:

$$\sum_j R_{kj} C_j \dot{v}_j(0^+) = v_{eq} - v_k(0^+) \tag{86}. $$

But $v_k(0^+) = 0$ for all $k$, and $\dot{v}_j(0^+) = 0$ for all $j$ except $j=1$. So (86) becomes,

$$R_{kl} C_l \dot{v}_l(0^+) = v_{eq} \tag{87}. $$

We identify $C_l \dot{v}_l(0^+) = \frac{v_o}{R_s}$ as the current flowing into node 1 at time $t=0^+$,

$$R_{kl} \frac{v_o}{R_s} = v_{eq} \tag{88}, $$

$$\Rightarrow x_k = \frac{v_{eq}}{v_o} = \frac{R_{kl}}{R_s} \tag{89}. $$

This proves the lemma.

This lemma shows that computing $x_k$ for all $k$ is equivalent to computing $R_{kl}$ for all $k$ (i.e., the first column (or row) of the resistance matrix $R$). However, the first column (or row) of $R$ is exactly what would be needed if we wanted to bound the response at node 1. Hence, the calculation of the equilibrium voltage distribution is no worse computationally.
than considering one more node as an output node.

Note that in the rare case when there are \( M > 1 \) capacitor nodes connected directly to the source by resistors \( R_{S_1}, \ldots, R_{S_M} \), then a straightforward extension to the preceding lemma holds:

\[
X_k = \sum_{j=1}^{M} \frac{R_{S_j}}{j} \quad \text{nodes } k
\]

This means that calculation of the equilibrium voltage distribution is equivalent to computing the first \( M \) columns (or rows) of \( R \), which is computationally equivalent to considering nodes \( 1, \ldots, M \) as additional output nodes to be bounded.

The computational problem now consists of finding the first row (or column) of \( R \), the \( i^{th} \) row (or column) of \( R \) for each output node \( i \), and the diagonal elements of \( R \). Unfortunately, the \( R \) matrix is "global" in character and is difficult to compute if there are resistor loops or resistive paths to ground. However, the conductance matrix \( G = R^{-1} \) is "local" in character, sparse (i.e., has many zero entries), and can be determined virtually by inspection even when there are resistor loops or grounded resistors present. So one approach is to determine \( G \) directly and then invert it to find \( R \). This approach, however, raises some doubts because it finds all of \( R \) instead of just the entries of \( R \) that are actually needed. An alternate approach is given below.

Alternate Approach:

In this alternate approach, we only calculate the entries of the \( R \) matrix that are needed. If the entry \( R_{ik} \) of \( R \) is desired, the conductance
The resistive subnetwork for the example calculation is shown in Fig. 19. We wish to calculate the entry $R_{ik}$ of the $R$ matrix that describes this resistor subnetwork. The first step is to "personalize" the subnetwork to nodes (1) and (k) by eliminating as many nodes as possible (see Fig. 20).
Note: We will use a prime to distinguish the subnetwork in Fig. 20 from the subnetwork in Fig. 19.

The key point to notice about this node elimination procedure is that,

\[ R'_{ik} = R_{ik} \]  

(91).

Hence, we can determine \( R'_{ik} \) from the much smaller conductance matrix \( G' \) (rather than \( G \)), either by directly inverting \( G' \) or by using cofactors:

\[ R'_{ik} = R_{ik} = R_{32} = \frac{(\text{cof } G_{23})}{(\det G')} \]  

(92).

The conductance matrix \( G' \) can be read off quickly from Fig. 20 as:

\[
G' = \begin{bmatrix}
\frac{11}{6} & -1 & -\frac{1}{3} \\
-1 & \frac{3}{2} & -\frac{1}{2} \\
-\frac{1}{3} & \frac{1}{2} & \frac{11}{6}
\end{bmatrix}
\]  

(93).

Using the cofactor method, we have:

\[
\det G' = \frac{11}{6} + \frac{1}{2} - \frac{1}{2} - \frac{1}{3} - \frac{1}{3} - \frac{1}{2} = \frac{9}{4}
\]  

(94).

\[
\text{cof } G'_{23} = \begin{vmatrix}
\frac{11}{6} & -1 \\
-\frac{1}{3} & \frac{1}{2}
\end{vmatrix} = \frac{5}{4}
\]  

(95).
This result of \( R_{ik} = \frac{5}{9} \) is correct, and can be verified by going back to Fig. 19 without considering Fig. 20 at all.

In summary, this approach suggests a tradeoff. Instead of doing one calculation with the large original conductance matrix \( G \), you can do several calculations with smaller "personalized" conductance matrices \( G' \).

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Appendix A: Proof of Facts 1 and 2

The proof of Fact 1 depends on additional structure of the conductance matrix $G$ alluded to in Section III.

**Lemma:** Assuming all resistors in a given resistor subnetwork $R$ are positive, the conductance matrix $G$ which describes $R$ has the following properties:

1) $G_{nn} > 0$ Vn; (i.e., all diagonal elements are positive)

2) $G_{mn} < 0$ Vm, n; (i.e., all off-diagonal elements are non-positive)

**Proof of Lemma:**

Apply a 1-volt source to node $n$ of the resistor subnetwork $R$.

Ground the datum and all external nodes other than $n$.

The Thevenin equivalent circuit looking into node $n$ is a positive resistor to ground ($v_{\text{thev}} = 0$). Since we are applying a 1-volt source to node $n$, positive current flows into node $n$ $\Rightarrow G_{nn} > 0$.

The Thevenin equivalent circuit looking into any other node $m$ (m $\neq n$) is either a positive resistor to ground ($v_{\text{thev}} = 0$; occurs when node $m$ is not connected to node $n$ through a single resistor), or a positive resistor in series with a positive voltage source ($v_{\text{thev}} > 0$; occurs when node $m$ is connected to node $n$ through a single resistor).

*It is intuitively clear that the open-circuit voltage at node $m$ is non-negative. This fact is a special case of the "voltage minimax theorem" [3; pg. 778].
In either case, since we are applying ground to node $m$, there can be no positive current flowing into node $m$ $\Rightarrow G_{m} < 0$.

**Proof of Fact 1:**

(See Section VI for the statement of Fact 1.)

The proof below is essentially the same as for non-leaky RC meshes given in [2]:

If $i=j$ or $i=k$, (24) is satisfied trivially. If $j=k$, then (24) becomes $R_{ii}R_{kk} > R_{ki}^2$ since $R$ is symmetric, and is true because $R$ is positive-definite and hence the 2x2 principal submatrix of $R$ consisting of elements in rows $i$ or $k$ that also lie in columns $i$ or $k$ must have a positive determinant. Now suppose $i$, $j$, and $k$ are distinct, and consider the resistive subnetwork $\hat{R}$ obtained from the original resistive subnetwork $R$ by open-circuiting all external terminals of $R$ except $1'$, $3'$, and the datum. Taking terminals $1$, $2'$, $3$ respectively of $\hat{R}$, we see that $\hat{R}$ has the following resistance matrix:

$$
\hat{R} = \begin{bmatrix}
R_{ii} & R_{ij} & R_{ik} \\
R_{ji} & R_{jj} & R_{jk} \\
R_{ki} & R_{kj} & R_{kk}
\end{bmatrix}
$$

Now consider $\hat{G} = \hat{R}^{-1}$, the conductance matrix of $\hat{R}$. The off-diagonal elements of $\hat{G}$ are non-positive (by the preceding lemma), so in particular $G_{kj} \leq 0$. Taking the inverse of $R$ using cofactors, we see that

$$
G_{kj} = (\hat{R}^{-1})_{kj} = [(\text{cof } R_{jk})/(\text{det } \hat{R})] \leq 0.
$$

Since $\hat{R}$ is positive-definite
\[ \Rightarrow \det R > 0, \text{ it follows that } \text{cof } R_{jk} \leq 0. \text{ But } \text{cof } R_{jk} = (-1)^{j+k} (R_{ik} R_{kj} - R_{ik} R_{kj}). \text{ This proves (24).} \]

**Proof of Fact 2:**

(See Section VI for the statement of Fact 2.)

Assume that the step transition is "up", i.e., \( v_{eq} > v_0 \). We want to prove that \( \dot{v}_j(t) > 0 \), \( \forall \) nodes \( j \) in the leaky RC mesh. (The argument for a "down" step is entirely analogous.)

Note first that \( \dot{v}_j(0^+) > 0 \) for every capacitor connected to the source through a single resistor, and \( \dot{v}_j(0^+) = 0 \) for all other capacitors. Therefore \( \dot{v}(0^+) \) lies in the "first orthant" of \( \mathbb{R}^N \) (the generalization to \( N \)-dimensions of the "first quadrant" of \( \mathbb{R}^2 \)).

We claim that \( \dot{v}(t) \), once inside the first orthant, can never leave. To see this, first differentiate (12) to obtain the following differential equation for the evolution of \( \dot{v}(t) \):

\[
\dot{v}(t) = -(RC)^{-1} \dot{v}(t) = -[C^{-1}G] \dot{v}(t) \quad (A-2).
\]

The off-diagonal elements of \( G \) are non-positive (by the previous lemma), so the off-diagonal elements of \(-[C^{-1}G]\) are non-negative, i.e., \(-[C^{-1}G]\) is a "quasi-monotone increasing" operator [10]. If \( \dot{v}(t) \) were to exit the first orthant, it would have to do so by passing through the boundary, i.e., one of the axis planes. Thus there would exist an instant \( t^* \) and a capacitor \( k \) such that \( \dot{v}_k(t^*) = 0 \) although \( \dot{v}_j(t^*) > 0 \forall j \neq k \). But from (A-2), this implies:
\[ v_k(t^*) = - \sum_{j \neq k} \frac{C_{kj}}{C_k} \dot{v}_j(t^*) \geq 0 \]  

(A-3).

So in fact \( \dot{v}_k(t) \) does not become negative at \( t = t^* \). In other words, the vector field (A-2) never points out of the first orthant when evaluated on its boundary. (For a "down" step, \( \dot{v}(0^+) \) lies in the negative "first orthant" and can never leave it.)

For a more complete explanation, and consideration of limiting cases, see references [11], [12], and [13].
Appendix B: Term-by-Term Ordering of the Time Constants

We have the following formulas for the time constants,

\[
\begin{align*}
T_{R_i} &= \frac{1}{R_{ii}} \sum_{k} R_{ik}^2 C_k \\
T_{D_i} &= \frac{1}{x_i} \sum_{k} R_{ik} C_k X_k \\
T_P &= \sum_{k} R_{kk} C_k
\end{align*}
\] (B-1).

It is already known that the inequality \( T_{R_i} < T_{D_i} < T_P \) holds for any leaky RC mesh (just evaluate the state constraints (46) at time \( t=0 \)). However, it is not immediately obvious that this inequality also holds term-by-term for each corresponding term in the summation (i.e.,

\[
\frac{1}{R_{ii}} R_{ik}^2 C_k < \frac{1}{x_i} R_{ik} C_k X_k < R_{kk} C_k
\] for each node \( k \)). The goal here is to establish this result. After simplification, the inequality we wish to prove reduces to:

\[
\frac{R_{ik}}{R_{ii}} \frac{x_k}{x_i} < \frac{R_{kk}}{R_{ik}}
\] (B-2).

Proof of (B-2):

Starting from zero initial equilibrium, apply an "up" step of 1 volt at time \( t=0 \) to the input of any leaky RC mesh. Note that \( v_0 = 0 \) and \( v_{eq} = x \). Evaluating (26) at time \( t=0 \), we obtain:
\[ R_{ii}[x_k^0] \geq R_{kk}[x_i^0] \Rightarrow \frac{x_k}{x_i} \geq \frac{R_{ki}}{R_{ii}} \quad (B-3) \]

Evaluating (27) at time \( t=0 \), we obtain:

\[ R_{ki}[x_k^0] \leq R_{kk}[x_i^0] \Rightarrow \frac{x_k}{x_i} \leq \frac{R_{kk}}{R_{ki}} \quad (B-4) \]

The fact that \( R \) is symmetric (so that \( R_{ki} = R_{ik} \)) together with (B-3) and (B-4) proves (B-2).
Appendix C: M-Matrix Systems

The purpose of this appendix is to outline (from a more abstract "systems" point-of-view) the underlying reason why the waveform bounding results presented in [1] and [2] carry over to the leaky RC meshes in this report. Leaky RC meshes are similar to non-leaky RC meshes [2] and non-leaky RC trees [1] in one fundamental respect: They are all linear dynamical systems with an "M-matrix" as the system matrix [14]. In the literature, there are many different equivalent statements about M-matrices that would all serve equally well as definitions. The statement best suited to our purposes is as follows.

Definition: Let M be a non-singular square matrix of real numbers, and let $P = M^{-1}$. Then M is said to be an "M-matrix" if all off-diagonal elements of M are non-positive and all elements of P are non-negative.

In terms of the RC networks in this report, the M-matrix is

$$M = C^{-1}G \quad (C-1),$$

and the non-negative inverse M-matrix is

$$P = RC \quad (C-2),$$

as we can easily verify.

The conductance matrix G has non-positive off-diagonal elements
(by the lemma in Appendix A). Premultiplying by the positive diagonal matrix \( C^{-1} \) does not change this, i.e., \([C^{-1}G]\) has non-positive off-diagonal elements.

The resistance matrix \( R \) is clearly non-negative, assuming that the individual resistors are non-negative. Postmultiplying by the positive diagonal matrix \( C \) does not change this, i.e., \([RC]\) is non-negative.

Hence \([C^{-1}G]\) is indeed an M-matrix, as is \( G \) itself. (In general, the set of M-matrices is invariant under the group action consisting of right and left multiplication by positive diagonal matrices.)

The network differential equations (12) can now be viewed more abstractly as a linear dynamical system with a (negative) M-matrix as the system matrix:

\[
\dot{v}(t) = -Mv(t) + Mv_{eq}
\]

We can successfully bound the step response of the abstract system, (C-3), because both Fact 1 (appropriately modified) and Fact 2 from Section VI apply to the abstract system.

**Fact 1-C:** Let \( P \) be a non-singular \( NXN \) square matrix which is the inverse of an M-matrix. Then \( P \) has the following structure:

\[
P_{ij}P_{kj} \leq P_{ik}P_{ij} \text{ for any } i,j,k \in \{1, \ldots, N\}
\]
Proof of Fact 1-C:

If \( i = j \) or \( i = k \), (C-4) is satisfied trivially. If \( j = k \), (C-4) becomes
\[
P_{ii}^2 P_{kk} \geq P_{ki}^2 P_{ik}
\]
(note that \( P_{ki} \neq P_{ik} \) in general; the inverse M-matrix \( P \) is not necessarily symmetric), which is true because any principal minor of an inverse M-matrix is known to be positive [16; Cor. 1, p. 198] and hence the 2x2 principal submatrix of \( P \) consisting of the elements in rows \( i \) or \( k \) that also lie in columns \( i \) or \( k \) must have a positive determinant.

Suppose \( i, j, \) and \( k \) are distinct, and consider the matrix \( P \) which is the 3x3 principal submatrix of \( P \) consisting of the elements in rows \( i, j, \) or \( k \) that also lie in columns \( i, j, \) or \( k \). Since any principal submatrix of an inverse M-matrix is known to be an inverse M-matrix [15, p. 329], \( P \) is an inverse M-matrix and hence \( (P^{-1})_{kj} \leq 0 \). Taking the inverse of \( P \) using cofactors, we have
\[
0 \geq (P^{-1})_{kj} = [(\text{cof } p_{jk})/\det P], \text{ and } \det P > 0
\]
[16]. Thus \( \text{cof } p_{jk} = P_{ki}^2 P_{ij} - P_{ii}^2 P_{kj} \leq 0 \), proving (C-4).

Note: Although not proven here, Fact 1-C also holds for a slightly more general class of matrices than inverse M-matrices. Going back to the more concrete domain of RC networks, if any individual resistor or capacitor is equal to zero then Fact 1-C still holds for \( P = RC \) even though \( P \) is not invertible in this case. However, \( P \) is contained in the closure of the set of inverse M-matrices. The practical advantage of this is that the bounding technique is still valid for circuits with minor specification errors (e.g., leaving out a capacitor).
Fact 2-C: If \( v(t) \) in the abstract system (C-3) is initially non-decreasing (non-increasing) it will remain non-decreasing (non-increasing) for all time:

\[
\begin{align*}
\dot{v}(0) > 0 & \Rightarrow \dot{v}(t) > 0 \quad \forall t > 0 \\
\dot{v}(0) < 0 & \Rightarrow \dot{v}(t) < 0 \quad \forall t > 0 \\
\end{align*}
\]

(C-5).

Proof of Fact 2-C:

The differential equation for the evolution of \( \dot{v}(t) = -M \dot{v}(t) \) is quasi-monotone increasing (since the off-diagonal elements of \(-M\) are non-negative). Hence the proof is entirely analogous to the proof in Appendix A.
References


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