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A new approach for proving temporal properties of concurrent programs is presented. The approach does not use temporal logic. To show that a program satisfies a given temporal property, the property is first decomposed into proof obligations. These obligations are then discharged by devising suitable invariant assertions and variant functions for the program. The approach is quite general - it handles a superset of the properties that can be expressed in linear-time temporal logic.
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ABSTRACT

A new approach for proving temporal properties of concurrent programs is presented. The approach does not use temporal logic. To show that a program satisfies a given temporal property, the property is first decomposed into proof obligations. These obligations are then discharged by devising suitable invariant assertions and variant functions for the program. The approach is quite general—it handles a superset of the properties that can be expressed in linear-time temporal logic.

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1. Introduction

Experience has shown that while it may be possible to understand a sequential program by considering some subset of its executions, this is impossible for concurrent programs. Consequently, over the past 15 years, there has been increasing interest in ways to deduce properties of program behavior from the program text itself. The program text obviously contains all the information needed to decide what executions are possible. Moreover, while the number of possible executions is likely to be intractably large, only a single program text need be analyzed.

An execution of a program can be viewed as a potentially infinite sequence of states called a *history*. In a history, the first state is an initial state of the program and each following state results from executing a single atomic action in the preceding state. In a concurrent or distributed program, a history is the sequence of states that results from interleaving the atomic actions of the processes as they execute.

A *property* defines a set of sequences of states; a program *satisfies* a property if each of its histories is in the set defined by the property. A property can be specified as a predicate on sequences. This allows the essence of the property to be made explicit.

Some examples of properties frequently arising in practice follow.

- *Partial Correctness* includes all sequences of program states such that, if the first state in the sequence satisfies some given precondition and the sequence is finite, then in the final state the program counter denotes the end of the program and some given postcondition is satisfied.

- *Total Correctness*, which is stronger than Partial Correctness, includes all sequences such that if the first state in the sequence satisfies some given precondition, then the sequence is finite and the value of the program counter in the final state denotes the end of the program as well as satisfying some given postcondition.

- *Mutual Exclusion* includes all sequences in which there is no state where the program counters for two or more processes denote control points inside critical sections.

- *Deadlock Freedom* includes all sequences in which there is no state where both (i) some process has no enabled atomic actions and (ii) no subsequent execution by any other process can alter that.

- *First-come First-served* includes all sequences in which processes that request service in one order are not serviced in another order.

- *Starvation Freedom* includes all sequences in which a process with an atomic action that is enabled frequently enough will make progress eventually.
Formulas of temporal logic can be interpreted as predicates on sequences of states, and various formulations of temporal logic have been used for specifying properties of interest to designers of concurrent programs [Lamport 83a] [Lamport 83b] [Manna & Pnueli 81a] [Wolper 83]. While there is not general agreement on the details of such a specification language, there is agreement that temporal logic provides a good basis for such a language and it, or something close to it, is sufficiently expressive.

Temporal logic has also been used in proving properties of concurrent programs [Pnueli 77] [Manna & Pnueli 81b] [Manna & Pnueli 84] [Owicki & Lamport 82]. Here, a program is regarded as defining a collection of temporal logic axioms. The programmer proves a property of interest by using these axioms along with program-independent axioms and inference rules of temporal logic [Manna & Pnueli 83]. Various packagings of the approach avoid the necessity of making temporal inferences by restricting the class of properties that can be proved. Examples include Hoare’s logic for Partial Correctness of sequential programs [Hoare 69] and its extension to concurrent programs [Owicki & Gries 76], GHL (Generalized Hoare Logic) for proving safety properties of concurrent programs [Lamport 80] [Lamport & Schneider 84], and proof lattices for proving liveness properties [Owicki & Lamport 82].

This paper introduces a new approach for proving properties of (concurrent) programs. The approach can handle a broad class of properties, including any property that can be expressed in temporal logic. Using our approach, to prove that a program satisfies some given property, invariance obligations and variance obligations are constructed. Invariance obligations are discharged by finding certain invariance assertions and showing that they are preserved by execution; variance obligations are discharged by finding variant functions and showing that they decrease following certain events. Hoare’s partial correctness logic is used to show that the invariant assertions are preserved by execution and that the variant functions are decreased by execution.

2. Specifying Properties

Our approach is based on specifying properties by using property recognizers, which are similar to Buchi automata [Eilenberg 74]. We are not advocating property recognizers as the basis for a specification language, but we have found them to be a convenient starting point for our verification method. Mechanical procedures exist to translate any temporal logic formula into a corresponding property recognizer [Alpern 86] [Wolper 84], so starting with property recognizers does not constitute a restriction. In fact, property recognizers are more expressive than most temporal logic-based specification languages—there exist properties that can be specified using property recognizers but cannot be specified in (most) temporal logics [Wolper 83].
A property recognizer accepts those sequences of program states that are in the property it specifies. Properties can contain infinite sequences as well as finite ones, so a property recognizer must be able to accept both kinds of sequences. Recall that a finite state-automaton accepts a finite sequence if and only if it halts in an accepting state after reading the final symbol [Hopcroft & Ullman 79]. A Buchi automaton is a finite-state automaton with an acceptance criterion that allows it to accept infinite sequences—it accepts an infinite sequence if and only if it enters an accepting state infinitely often while reading that sequence [Eilenberg 74]. A property recognizer is an automaton that behaves like a standard finite-state automaton for finite input sequences and like a Buchi automaton for infinite input sequences.

An example of a property recognizer, $m_{\text{compl}}$, is given in Figure 2.1. It defines the set of sequences consisting of a (possibly empty) prefix of states in which each state satisfies predicate $\neg P$, immediately followed by either (i) an infinite sequence of states in which $P$ holds for each state, or (ii) a finite sequence of states in which $P$ holds on all except the last state.

Property recognizer $m_{\text{compl}}$ contains three automaton states labeled, $q_0$, $q_1$, and $q_2$. The start state is denoted by an arc with no origin, infinite-accepting states by concentric circles, and finite-accepting states by bullets (•). An infinite sequence is accepted by a property recognizer only if it causes the recognizer to be infinitely often in some infinite-accepting state. A finite sequence is accepted by the property recognizer only if it causes the recognizer to halt (at the end of its input) in some finite-accepting state. In $m_{\text{compl}}$, $q_0$ is the start state, $q_1$ is an infinite-accepting state, and $q_2$ is a finite-accepting state.

Arcs between automaton states are labeled by program state predicates called transition predicates. These define transitions between automaton states based on the next symbol read from the input. For example, the arc labeled $P$ from $q_0$ to $q_1$ in $m_{\text{compl}}$ means that whenever $m_{\text{compl}}$ is in $q_0$ and the next symbol read is a program state satisfying $P$, then a transition to $q_1$ is made. If the next symbol read by a property recognizer satisfies no transition predicate on an arc emanating from the current automaton state, the input is rejected; in this case, we say the transition is undefined for that symbol. This is used in $m_{\text{compl}}$ to ensure that every

![Figure 2.1. $m_{\text{compl}}$](image-url)
finite sequence it accepts ends with a single program satisfying \(-P\); no further transitions are possible from \(q_2\) because there are no arcs emanating from it.

When there is more than one start state or more than one transition is possible from some automaton state for some input symbol, the property recognizer is non-deterministic; otherwise it is deterministic. Thus, \(M_{\text{comp}}\) is deterministic because it has a single start state and disjoint transition predicates label the arcs that emanate from each automaton state.

Formally, a property recognizer \(M\) for a property of a program \(\pi\) is a sextuple \((S, Q, Q_0, Q_{\text{inf}}, Q_{\text{fin}}, \delta)\), where

- \(S\) is the set of program states of \(\pi\),
- \(Q\) is the set of automaton states of \(M\),
- \(Q_0 \subseteq Q\) is the set of start states of \(M\),
- \(Q_{\text{inf}} \subseteq Q\) is the set of infinite-accepting states of \(M\),
- \(Q_{\text{fin}} \subseteq Q\) is the set of finite-accepting states of \(M\), and
- \(\delta : (Q \times S) \rightarrow 2^Q\) is the transition function of \(M\).

Transition predicates are derived from \(\delta\) as follows. \(T_{ij}\), the transition predicate associated with the arc from automaton state \(q_i\) to \(q_j\), is the predicate that holds for all program states \(s\) such that \(q_j \in \delta(q_i, s)\). Thus, \(T_{ij}\) is \(\text{false}\) if no symbol can cause a transition from \(q_i\) to \(q_j\).

In order to formalize when \(M\) accepts a sequence, some definitions are required. For any sequence \(\sigma = s_0s_1\ldots\),

\[
\begin{align*}
\sigma[i] &= s_i \\
\sigma[\ldots i] &= s_0s_1\ldots s_i \\
\sigma[\ldots i] &= s_is_{i-1}\ldots \\
|\sigma| &= \text{the length of } \sigma \text{ (\(\omega\) if } \sigma \text{ is infinite).}
\end{align*}
\]

Transition function \(\delta\) can be extended to handle finite sequences of program states:

\[
\delta^*(q, \sigma) = \begin{cases} 
(q) & \text{if } |\sigma| = 0 \\
(\{q'\} : q' \in \delta(q, \sigma[0]) \land q' \in \delta^*(q'^*, \sigma[1\ldots])) & \text{if } 0 < |\sigma| < \omega
\end{cases}
\]

A run of \(M\) for an input \(\sigma\) is a sequence of automaton states that \(M\) could be in while reading \(\sigma\). Thus, for \(\rho\) to be a run for \(\sigma\), \(\rho[0] \in Q_0\), and \(\forall i : 0 < i < |\sigma| : \rho[i] \in \delta(\rho[i-1], \sigma[i-1])\). Let \(\Gamma_m(\sigma)\) be the set of runs of \(M\) on \(\sigma\). (It is a set because \(M\) might be non-deterministic.)

A finite sequence \(\sigma\) is accepted by \(M\) if and only if \(\delta^*(q_0, \sigma) \cap Q_{\text{fin}} \neq \emptyset\). For an infinite sequence \(\sigma\), define \(INF_m(\sigma)\) to be the set of automaton states that appear infinitely often in any element of \(\Gamma_m(\sigma)\). Then, \(\sigma\) is accepted by \(M\) if and only if \(INF_m(\sigma) \cap Q_{\text{inf}} \neq \emptyset\).

Any set of finite sequences that can be recognized by a non-deterministic finite-state automaton can be recognized by some deterministic finite-state automaton [Hopcroft & Ullman 79]. Unfortunately, Buchi automata, hence property recognizers, do not enjoy this equivalence—there are sets of infinite sequences that can be recognized by non-deterministic
property recognizers but by no deterministic one [Eilenberg 74]. This will ultimately require that we use different techniques for those properties specified by non-deterministic property recognizers from those specified by deterministic ones.

Examples of Property Recognizers

A property recognizer $m_{pc}$ for Partial Correctness is shown in Figure 2.2 and one for Total Correctness, $m_{tc}$, is shown in Figure 2.3. In them, $Pre$ is a transition predicate that holds for states satisfying the given precondition, $Done$ holds for states in which the program counter denotes the end of the program, and $Post$ holds for states satisfying the given postcondition.

![Figure 2.2. $m_{pc}$](image)

![Figure 2.3. $m_{tc}$](image)
A property recognizer for Mutual Exclusion of two processes, $m_{mace}$, is given in Figure 2.4. There, transition predicate $C_{s_{\phi}} \land C_{s_{\psi}}$ holds for any state in which process $\phi$ ($\psi$) is executing in its critical section.

![Figure 2.4. $m_{mace}$](image)

Starvation Freedom for a mutual exclusion protocol is specified by $m_{nsv}$ of Figure 2.5. A process $\phi$ becomes enabled when its state satisfies the predicate $Request_{\phi}$, which characterizes the state of $\phi$ whenever it attempts to enter its critical section, and makes progress when its state satisfies the predicate $Served_{\phi}$, which holds whenever $\phi$ enters its critical section. Notice that $m_{nsv}$ exploits the fact that in a mutual exclusion protocol $\phi$ will make but a single request for each entry into the critical section.

![Figure 2.5. $m_{nsv}$](image)

3. Specifying Programs

A program $\pi$ consists of a predicate $Init_{\pi}$ characterizing its initial states and a collection of atomic actions $A_{\pi}$. Presumably, $Init_{\pi}$ asserts that

- the program counter for each process in $\pi$ denotes the first statement of that process, and
- other program variables have appropriate values according to any initialization in their declarations.

Knowing the atomic actions of a concurrent program is necessary in order to understand its execution, since they define the grain of interleaving of processes. The atomic actions in a process define its control points—the set of values that can be stored in the program counter for that process. We can denote the control points of a program by naming them within
braces in the program text; this results in a control-point annotation. For example, program $\pi_0$ of Figure 3.1 consists of two sequential processes, $\phi$ and $\psi$, each with a single atomic action and two control points. The atomic action in process $\phi$ is called $\alpha_1$ and the control points in $\phi$ are labeled 1 and 2.

Every sequential process $\pi$ has a program counter $pc_{\pi}$. We can use this variable in describing states of the program. For example, $pc_\phi=1 \land pc_\psi=3$ defines the state of $\pi_0$ at its start and $pc_\phi=2 \land pc_\psi=4$ at its finish. The program counter of a sequential process differs from other program variables in that usually only a single process may update it and direct assignments to it are not permitted. Each atomic action, however, changes the value of the program counter. For example, atomic action $\alpha_1$ in $\pi_0$ changes $pc_\phi$ (from 1 to 2) as well as incrementing $x$. The assignment to $pc_\phi$ by $\alpha_1$, though not explicit, can be deduced from the position of $\alpha_1$ in the program text.

By definition, atomic actions are executed indivisibly and to completion, so an atomic action cannot be started unless it will terminate. We therefore assume an atomic action is delayed until the state is one that will permit its termination. Using angle brackets to denote an atomic action, $\alpha_1$ of $\pi_0$ is

$$ (\text{if } pc_\phi = 1 \rightarrow pc_\phi, x := 2, x+1 \phi). \quad (3.1) $$

Here, we use the multiple assignment statement of [Gries 81] and the if of [Dijkstra 76]. The semantics of if require that

$\pi_0$: cobegin
  $\phi$: (1);
  $\alpha_1$: $x := x+1$
  (2);
  //
  $\psi$: (3);
  $\alpha_2$: $x := x+1$
  (4);
  coend

Figure 3.1. Simple Program
Abort if executed in a state where none of the guards \(BO, ..., Bn\) holds. Thus, (3.1) is delayed until the program counter for process \(\phi\) is 1, and then (without interruption) atomically updates the program counter and increments \(x\). An atomic action might be delayed for reasons other than the program counter value. A P operation in process \(\pi\) on a general semaphore \(sem\),

\[\{a\} P(sem) \{b\} \ldots\]
defines an atomic action \(\beta\):

\[\text{if } pc_{\pi} = a \land sem > 0 \text{ then } pc_{\pi} := b, sem := sem - 1\]  

(3.2)

An atomic action is enabled in any state where its execution would not be delayed. Let \(\text{Enabled}(\alpha)\) be the set of states in which \(\alpha\) is enabled. In Figure 3.1,

\[\text{Enabled}(\alpha_1) = pc_{\alpha} = 1\]

and in (3.2),

\[\text{Enabled}(\beta) = pc_{\pi} = a \land sem > 0.\]

We can use \(\text{Enabled}\) to characterize states in which a program \(\pi\) is blocked and can make no further progress because there are and will be no enabled atomic actions:

\[\text{Blocked}_{\pi} = \underset{\alpha : \alpha \neq \beta}{\bigwedge} \neg \text{Enabled}(\alpha)\]

The effects of an atomic action \(\alpha\) can be defined as a relation between the program state before and after it is executed. This relation can be described by a triple \(\{P\} \alpha \{Q\}\), which is valid if executing \(\alpha\) in a state satisfying \(P\) either does not terminate or terminates in a state satisfying \(Q\). \(P\) is called the \textit{precondition} and \(Q\) the \textit{postcondition}.

Programming logics to prove validity of a triple involving a sequential program \(\pi\) are well known [Hoare 69]. One is summarized in Figure 3.2. If the semantics of an atomic action \(\alpha\) is described as a sequential program, then such a logic and the following inference rule can be used to infer triples giving the semantics of \(\alpha\).

\[
\text{( } \text{Rule: } \frac{\{P\} S \{Q\}}{\{P\} \{S\} \{Q\}}
\]

For example, returning to \(\pi_0\) of Figure 3.1, we can establish the validity of \(\{x = 0\} \alpha_1 \{x = 1\}\) as follows:

\(\{x = 0\} pc_{\alpha}, x := 2, x + 1 \{x = 1\}\) (Assignment Axiom)

\(\{x = 0 \land pc_{\alpha} = 1\} pc_{\alpha}, x := 2, x + 1 \{x = 1\}\) (Rule of Consequence)
Skip Axiom: \{P\} \text{skip} \{P\}

Assignment Axiom: \{P\} \quad i := \overline{e} \{P\}

If Rule: \begin{align*}
&\{P \land B_0\} S_1 \{Q\}, \ldots, \{P \land B_n\} S_n \{Q\} \\
&\{P\} \text{ if } B_0 \rightarrow \text{false} \ldots \text{ and } B_n \rightarrow \text{false} \{Q\}
\end{align*}

Do Rule: \begin{align*}
&\{P \land B_0\} S_1 \{P\}, \ldots, \{P \land B_n\} S_n \{P\} \\
&\{P\} \text{ do } B_0 \rightarrow \text{false} \ldots \text{ and } B_n \rightarrow \text{false} \{P \land B_0 \land \ldots \land B_n\}
\end{align*}

Rule of Consequence: \begin{align*}
&P \Rightarrow P', \{P'\} S \{Q\}', \quad Q' = Q \\
&P \quad S \quad \{Q\}
\end{align*}

Conjunction Rule: \begin{align*}
&P \quad S \quad \{Q\}, \quad \{P'\} \quad S \quad \{Q'\} \\
&P \land P' \quad S \quad \{Q \land Q'\}
\end{align*}

Figure 3.2. Partial Correctness Logic

\begin{align*}
&\{x=0\} \text{ if } p c_a = 1 \rightarrow p c_a, x := 2, x+1 \text{ fail } \{x=1\} \quad (\text{If Rule}) \\
&\{x=0\} \quad (\text{if } p c_a = 1 \rightarrow p c_a, x := 2, x+1 \text{ fail}) \quad \{x=1\} \quad (\text{If Rule}) \\
&\{x=0\} \quad \alpha_1 \quad \{x=1\} \quad \text{(definition of } \alpha_1) \\

\text{This type of reasoning, which we employ frequently in the sequel, is facilitated by the following derived rule of inference.}

\text{Atomic Action Rule:} \quad \frac{\{P \land B\} \quad S \quad \{Q\}}{\{P\} \quad (\text{if } B \rightarrow \text{false}) \quad \{Q\}}

4. Verification of Deterministic Properties

The basis for our approach to verifying that a program \(\pi\) satisfies a property \(P\) is the observation that if a property recognizer \(m\) for \(P\) accepts every history of \(\pi\), then \(\pi\) satisfies \(P\). In this section, we consider verification of properties that are specified by deterministic property recognizers; in section 3, we consider non-deterministic property recognizers. Soundness and completeness proofs are given in the Appendix.

Let \(m\) be a deterministic property recognizer for property \(P\). One can think of \(m\) as simulating—in an abstract way—any program that satisfies \(P\). Thus, to show that a program \(\pi\) satisfies \(m\), we demonstrate such a correspondence between \(m\) and \(\pi\). We do this by defining a correspondence invariant \(C_i\) for each automaton state \(q_i\). A correspondence invariant \(C_i\)
for an automaton state $q_i$ is a predicate such that $C_i$ holds on a program state $s$ if and only if there exists a history of $\pi$ containing a program state $s$ and $m$ enters $q_i$ upon reading $s$. Thus, if $m$ is ever in automaton state $q_i$, the last program state it read must satisfy $C_i$. Constraints satisfied by correspondence invariants are defined inductively, as follows.

For the base case, initially, $m$ is in state $q_0$ and $\pi$ is in a state characterized by $Init_\pi$. Suppose that upon reading $s_0$, the first program state of some history of $\pi$, $m$ enters automaton state $q_i$. Thus, $s_0$ satisfies $Init_\pi$ and $T_{0j}$, the transition predicate labeling the edge that connects $q_0$ and $q_j$. Therefore, $C_j$ must satisfy $(Init_\pi \land T_{0j}) \Rightarrow C_j$; for any automaton state $q_j$ entered upon reading the first symbol of any history of $\pi$, we require

$$(\forall j): (Init_\pi \land T_{0j}) \Rightarrow C_j). \quad (4.1)$$

Next we must prove the induction step. Assume that if $m$ enters automaton state $q_i$ upon reading program state $s_k$ in a history of $\pi$ and $0 \leq k < K$, then $s_k$ satisfies $C_i$. Consider the case when $m$ reads $s_K$. Suppose $m$ is in state $q_i$ and that upon reading program state $s_K$, a transition is made to automaton state $q_j$. By the induction hypothesis $s_{K-1}$ satisfies $C_i$ and $s_K$ satisfies transition predicate $T_{ij}$. The appropriate correspondence invariant $C_j$ will hold provided $(C_i) \land (T_{ij} \Rightarrow C_j)$ is valid for any $\alpha$, an atomic action of $\pi$. (If $\alpha$ is not enabled in $s_{K-1}$ then the triple is trivially valid.) Generalizing to handle any atomic action and any automaton state that $m$ might be in when $s_K$ is read, we require:

For all $\alpha$: $\alpha \in A_\pi$:

For all $i$: $q_i \in Q$:

$$(C_i) \land (T_{ij} \Rightarrow C_j)) \quad (4.2)$$

Thus, any collection of predicates satisfying (4.1) and (4.2) are correspondence invariants for $m$ and $\pi$.

In order to establish that $\pi$ satisfies $P$, we must show that every history of $\pi$ is accepted by $m$. There are exactly three ways that $m$ might fail to accept a history $\sigma$ of $\pi$:

1. $m$ attempts an undefined transition when reading $\sigma$.
2. If $\sigma$ is finite, $m$ halts in a non-finite-accepting state.
3. If $\sigma$ is infinite, $m$ never enters an infinite-accepting state after some finite prefix of $\sigma$.

Thus, in order to prove that every history of $\pi$ satisfies $P$, it suffices to show that (1)–(3) are impossible.

Two obligations ensure that (1) is impossible. First, we must show that $m$ can make some transition from its start state upon reading the first program state in a history:
Second, we must show that $m$ can always make a transition upon reading subsequent states in a history. If $m$ is in state $q_i$ then the program state just read by $m$ satisfies a correspondence invariant $C_i$. To avoid an undefined transition, any atomic action $\alpha$ that is then executed must transform the program state so that one of the transition predicates $T_{ij}$ emanating from $q_i$ holds. This is guaranteed by

$$\text{Simulation Basis: } Init_{\pi} \Rightarrow (\textstyle \bigvee_{j:q_j \in Q} T_{0j})$$

(4.3)

We can exploit the fact that $m$ is deterministic to combine and simplify the obligations derived so far. In a deterministic property recognizer, the transition predicates on arcs emanating from any automaton state are disjoint. Thus,

$$\forall i,j,k: q_i, q_j, q_k \in Q \land j \neq k: (T_{ij} \land T_{ik}) = \text{false}.$$  

(4.5)

Using (4.5), we combine (4.1) and (4.3), to obtain

$$\text{Simulation Basis: } Init_{\pi} \Rightarrow (\textstyle \bigvee_{j:q_j \in Q} (T_{0j} \land C_j))$$

(4.6)

and combine (4.2) and (4.4), to obtain

$$\text{Simulation Induction: } \forall \alpha: \alpha \in A_{\pi}:$$

$$\forall i: q_i \in Q:$$

$$\{C_i\} \alpha \{ \textstyle \bigvee_{j:q_j \in Q} (T_{ij} \land C_j) \}$$

(4.7)

To ensure that it is impossible for $m$ to halt in a non-finite-accepting state—(2) above—the correspondence invariant for any non-finite-accepting state must hold only for program states in which subsequent execution by $\pi$ is inevitable. Since $C_i$ holds of the last program state read by $m$, and $\text{Blocked}_m$ holds for all program states of $\pi$ in which subsequent execution is not possible, we require

$$\text{Finite Acceptance: } (\forall i: q_i \in Q - Q_{inf}: C_i \Rightarrow \neg \text{Blocked}_m).$$

(4.8)

Finally, we ensure that (3) is impossible. A set $Q'$ of automaton states is strongly connected if and only if there is a sequence of transitions from any element of $Q'$ to any other without involving an automaton state outside of $Q'$. A reject knot $\kappa$ is a maximal strongly connected subset of $Q$ containing no infinite-accepting states. It may, however, contain finite-accepting states. In order to show that (3) is impossible, we must prove that no run for an infinite history of $\pi$ is restricted to automaton states in $Q - Q_{\kappa}$. We do this by constructing a variant function $\nu$ for each reject knot $\kappa$. 

-11-
A variant function \( v_\varsigma(q,s) = 0 \) is a function from automaton and program states to some well-founded set.\(^1\) For simplicity, assume that this well-founded set is the Natural Numbers. We require that whenever \( v_\varsigma(q,s) = 0 \) for any automata state \( q \) and program state \( s \), either \( q \) is not in \( \kappa \) or else \( q \) is a finite-accepting state and \( s \) is the last state in the history.

**Knob Exit:** \( (\forall i: \; q_i \in \kappa: \; (v_\varsigma(q_i) = 0) \Rightarrow \text{Blocked}_\omega \lor \lnot C_i) \) (4.9)

This means that if \( v_\varsigma(q) = 0 \), either the history is finite and will be accepted by \( m \) or an infinite-accepting state has just been entered since the property recognizer is no longer in \( \kappa \). Finally, to ensure that the variant function does reach 0, we require that it is decreased by every atomic action in \( \pi \) that might be executed:

**Knob Vairance:** For all \( \alpha: \; \alpha \in A_\pi \):

\[
\forall q_i \in \kappa:\; (C_i \land 0 < v_\varsigma(q_i) = V) \Rightarrow \bigwedge_{j: q_j \in \kappa} (((T_{ij} \land C_j) \Rightarrow v_\varsigma(q_j) < V))
\] (4.10)

Note that requiring that \( v_\varsigma(q) \) be decreased by execution of any eligible atomic action does not preclude proving properties under various fairness assumptions. To prove that a property \( P \) holds assuming some fairness property \( F \) holds, a property recognizer for \( F \Rightarrow P \) is constructed and proof obligations are extracted from it. Standard techniques exist to construct a property recognizer for \( F \Rightarrow P \) from property recognizers for \( F \) and \( P \) [Eilenberg 74].

The five proof obligations—Simulation Basis (4.6), Simulation Induction (4.7), Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10)—are of three basic forms. Simulation Basis (4.6), Finite Acceptance (4.8), and Knot Exit (4.9) involve proving that predicate logic formulas are valid. Simulation Induction (4.7) involves proving invariance of some assertions. Knot Variance (4.10) involves proving that certain events cause variant functions to be decreased. Of course, the intellectual challenge in proving that a program satisfies a property lies not in checking the proof obligations, but in devising the correspondence invariants and variant functions. The proof obligations, however, do give insight into forms the correspondence invariants and variant function might take. In particular, the proof obligations define a collection of equations whose unknowns are the correspondence invariants and variant functions. Solving the equations—admittedly a difficult task—would provide the desired correspondence invariants and variant functions.

5. A Detailed Example

To illustrate our verification method, we prove that if program \( \pi_0 \) of Figure 3.1 is started in a state where \( x = 0 \) then it will terminate with \( x = 2 \). This is an instance of Total Correctness.

\(^1\)The program state argument is often left implicit.
For \( \pi_0 \), we have

\[
\text{Init}_{\pi_0} = \ p_{c_d} = 1 \land p_{c_v} = 3
\]

\[
\text{Blocked}_{\pi_0} = \ p_{c_d} = 2 \land p_{c_v} = 4
\]

and \( \Lambda_{\pi_0} = \{\alpha_1, \alpha_2\} \), where

\[
\alpha_1 = (\text{if } p_{c_d} = 1 \rightarrow p_{c_d}, x := 2, x+1 \text{ nil})
\]

\[
\alpha_2 = (\text{if } p_{c_v} = 3 \rightarrow p_{c_v}, x := 4, x+1 \text{ nil})
\]

A property recognizer \( m_c \) for Total Correctness appears in Figure 2.3. For predicates \( Pre, Post, \) and \( Done \) we choose:

\[
Pre = x = 0
\]

\[
Post = x = 2
\]

\[
Done = p_{c_d} = 2 \land p_{c_v} = 4
\]

Thus, \( m_c \) accepts every sequence of states such that if \( x = 0 \) holds for the first state, then the sequence is finite and the final state is one in which \( x = 2 \) and both \( \phi \) and \( \psi \) have terminated.

We first define correspondence invariants for each of the four automaton states of \( m_c \):

\[
C_0 = \text{false}
\]

\[
C_1 = \ p_{c_d} = 1 \Rightarrow (p_{c_v} = 3 \Rightarrow x = 0) \land (p_{c_v} = 4 \Rightarrow (x = 1)) \land
\]

\[
p_{c_d} = 2 \Rightarrow (p_{c_v} = 3 \Rightarrow x = 1) \land p_{c_v} \neq 4) \land
\]

\[
p_{c_v} = 3 \Rightarrow (p_{c_d} = 1 \Rightarrow x = 0) \land (p_{c_d} = 2 \Rightarrow x = 1)) \land
\]

\[
p_{c_v} = 4 \Rightarrow (p_{c_d} = 1 \Rightarrow x = 1) \land p_{c_v} \neq 2))
\]

\[
C_2 = \text{true}
\]

\[
C_3 = \ p_{c_d} = 2 \land p_{c_v} = 4 \land x = 2
\]

To satisfy Simulation Basis (4.6), we must show that

\[
\text{Init}_{\pi_0} \Rightarrow ((\text{false} \land C_0) \lor (Pre \land \neg Done \land C_1) \lor (\neg Pre \land C_2) \lor (Pre \land Done \land Post \land C_3)
\]

is valid. Substituting, we get

\[
(p_{c_d} = 1 \land p_{c_v} = 3)
\]

\[
\Rightarrow (\text{false} \lor (x = 0 \land (p_{c_d} = 2 \land p_{c_v} = 4) \land C_0) \lor (x \neq 0) \lor (x = 0 \land p_{c_d} = 2 \land p_{c_v} = 4 \land x = 2))
\]

which is valid.

To satisfy Simulation Induction (4.7), we must show for each \( \alpha \in \Lambda_{\pi_0} \) that the following triples are valid:
\[\{C_0\} \alpha \{(T_{00} \land C_0) \lor (T_{01} \land C_1) \lor (T_{02} \land C_2) \lor (T_{03} \land C_3)\}\] (5.1)

\[\{C_1\} \alpha \{(T_{10} \land C_0) \lor (T_{11} \land C_1) \lor (T_{12} \land C_2) \lor (T_{13} \land C_3)\}\] (5.2)

\[\{C_2\} \alpha \{(T_{20} \land C_0) \lor (T_{21} \land C_1) \lor (T_{22} \land C_2) \lor (T_{23} \land C_3)\}\] (5.3)

\[\{C_3\} \alpha \{(T_{30} \land C_0) \lor (T_{31} \land C_1) \lor (T_{32} \land C_2) \lor (T_{33} \land C_3)\}\] (5.4)

Since the triples for \(a_2\) are symmetric with those for \(a_1\), we prove only the former.

Triple (5.1) is valid because \(C_0 = false\) and \(\{false\} \alpha \{R\}\) is valid for any \(R\).

Substituting for the transition predicates in (5.2) and simplifying yields

\[\{C_1\} \alpha \{(\neg Done \land C_1) \lor (Done \land Post \land C_3)\}\] (5.5)

From definition (3.1) of \(a_1\) and the Atomic Action Rule, to prove the validity of (5.5), it suffices to demonstrate the validity of

\[\{C_1 \land pc_a = 1\} \land pc_a, x := 2, x+1 \{(\neg Done \land C_1) \lor (Done \land Post \land C_3)\}\]

Expanding and substituting, this is

\[\{(pc_y = 3 \Rightarrow x = 0) \land (pc_y = 4 \Rightarrow x = 1) \land pc_a = 1\}\]

\[pc_a, x := 2, x+1\]

\[\{(\neg (pc_a = 2 \land pc_y = 4) \land C_1) \lor (pc_a = 2 \land pc_y = 4 \land x = 2)\}\]

and follows from the Assignment Axiom and Rule of Consequence.

Triple (5.3) simplifies to \(\{true\} \alpha \{true\}\) because \(C_2 = T_{22} = true\) and is valid.

Triple (5.4) simplifies to \(\{C_3\} \alpha \{false\}\) because \(T_{30}, T_{31}, T_{32},\) and \(T_{33}\) are all \(false\)—those transitions are not possible in \(m_c\). From definition of \(a_1\) (3.1) and the Atomic Action Rule, to prove (5.4) it suffices to show validity of

\[\{C_3 \land pc_a = 1\} \land pc_a, x := 2, x+1 \{false\}\]

Since \(\{C_3 \land pc_a = 1\} = false\), this reduces to \(\{false\} \land pc_a, x := 2, x+1 \{false\}\) which is valid.

To satisfy Finite Acceptance (4.8), since \(Q_{fa} = \{q_2, q_3\}\) we must prove that

\(C_0 = \neg Blocked_{\eta_q} \land (C_1 = \neg Blocked_{\eta_q})\).

Substituting and simplifying, we get

\(\{false \Rightarrow (\neg Blocked_{\eta_q})\} \land (C_1 \Rightarrow (pc_a \neq 2 \lor pc_y \neq 4))\),

which is valid.

The final two obligations concern reject knots. There is a single reject knot \(\kappa = \{q_1\}\) in \(m_c\). Define
\[ v_\pi(q_1) = (2 - pc_\phi) + (4 - pc_\psi). \]

Knot Exit (4.9) requires that
\[ (v_\pi(q_1) = 0) \Rightarrow Blocked_{\pi_0}. \]

This is valid because
\[ (v_\pi(q_1) = 0) \Rightarrow (pc_\phi = 2 \land pc_\psi = 4) \]
\[ = Blocked_{\pi_0}. \]

To satisfy Knot Variance (4.10), we must establish the validity of 2 triples:
\[
\begin{align*}
(C_1 \land 0 < v_\pi(q_1) = V) & \land ((\neg Done \land C_1) \Rightarrow v_\pi(q_1) < V) \quad (5.6) \\
(C_1 \land 0 < v_\pi(q_1) = V) & \land ((\neg Done \land C_1) \Rightarrow v_\pi(q_1) < V) \quad (5.7)
\end{align*}
\]

We give details only for the first; the second is similar. Using definition (3.1) of \( \alpha_1 \), the Atomic Action Rule, and the Rule of Consequence, to prove (5.6) it suffices to prove
\[
(C_1 \land 0 < v_\pi(q_1) = V \land pc_\phi = 1) \land pc_\phi, \ x := 2, \ x + 1 \ (v_\pi(q_1) < V).
\]

This is valid because changing \( pc_\phi \) from 1 to 2 decreases \( v_\pi \).

6. Property Outlines

A property outline provides a compact representation of the correspondence invariants and the Simulation Induction (4.7) obligations for a given property recognizer and program. Property outlines play much the same role in our approach to verification as proof outlines do for verifying Partial Correctness using Hoare's partial correctness logic—they make it easy to do verification informally and make it easy to present a proof. In fact, proof outlines and property outlines are closely related, as we show in section 6.4.

6.1. Proof Outlines

A proof outline for a concurrent program \( \pi \) is the text of \( \pi \) annotated with an assertion \( P^{cp} \) at each control point \( cp \). Each assertion is a first-order predicate logic formula involving the program variables and program counters of \( \pi \).\(^2\) A proof outline is valid provided:

Proof Outline Validity: Executing any enabled atomic action in a state where the assertions associated with the control points denoted by program counters hold produces a state in which the assertions associated with the control points denoted by program counters still hold.

Proving validity of a proof outline for a concurrent program can be reduced to proving

\(^2\)The conjunct \( pc_\phi = cp \) is often left implicit and omitted from \( P^{cp} \) in a proof outline for process \( \phi \).
the validity of a collection of triples [Owicki & Gries 76]. This is done as follows, where \( \text{pre}(\alpha) \) is the assertion immediately preceding \( \alpha \) in the proof outline and \( \text{post}(\alpha) \) is the assertion immediately following it.

**Sequential Correctness:** For each atomic action \( \alpha \) in the proof outline, prove

\[
\{\text{pre}(\alpha)\} \alpha \{\text{post}(\alpha)\}.
\]

**Interference Freedom:** For each atomic action \( \alpha \) in the proof outline and every assertion \( R \) in a process different from the one containing \( \alpha \), prove:

\[
\{\text{pre}(\alpha) \land R\} \alpha \{R\}.
\]

### 6.2. Property Outlines

A property outline for property recognizer \( m \) and program \( \pi \) is obtained by adding information about correspondence invariants to a control-point annotation for \( \pi \). For each control point \( cp \), we specify for every automaton state \( q \) of \( m \) what must hold when the program counter denotes \( cp \) if the property recognizer is in a state \( q \). This is done by placing a property assertion at each control point in a control-point annotation for \( \pi \).

A property assertion has the form

\[
\hat{P}: q_0 \rightarrow P_0 \land q_1 \rightarrow P_1 \land \ldots \land q_n \rightarrow P_n,
\]

where \( \hat{P} \) is a label, \( q_0, q_1, \ldots, q_n \) are the automaton states of \( m \), and \( P_0, P_1, \ldots, P_n \) are first-order predicate logic formulas involving the program variables of \( \pi \) (possibly including program counters). \( \hat{P} \) holds in an automaton state \( q_i \) and program state \( s \) if \( s \) satisfies \( P_i \). A property outline for \( \pi \) and \( m \) is valid provided:

**Property Outline Validity:** Executing any enabled atomic action in an automaton state \( q \) and program state \( s \) where the property assertions associated with the control points denoted by program counters hold produces a program state \( s' \) that causes the property recognizer to make a transition to an automaton state \( q' \) in which the property assertions associated with the control points denoted by program counters still hold.

Figure 6.1 is a valid property outline for \( m_r \) (Total Correctness) and \( \pi_0 \) (of Figure 3.1).

We can exploit the similarity in the definition of validity for proof outlines and for property outlines in developing a procedure to prove validity of a property outline. Define a property triple

\[
\{\hat{P}: q_0 \rightarrow P_0 \land \ldots \land q_n \rightarrow P_n\} \alpha \{\hat{Q}: q_0 \rightarrow Q_0 \land \ldots \land q_n \rightarrow Q_n\},
\]

3If an atomic action like "!" or "@" of CSP spans more than one process, then a third obligation, variously called satisfaction or cooperation, must also be satisfied. Our results for property outlines can also be generalized along these lines.
to be valid if execution of \( \alpha \) in an automaton state \( q_i \) and program state satisfying \( P_i \) either does not terminate or terminates in a program state \( s \) such that (i) \( s \) causes the property recognizer to make a transition to automaton state \( q_j \) and (ii) \( s \) satisfies \( Q_j \). Note that (6.1) cannot be a partial correctness logic triple because it contains property assertions in its pre- and postcondition. However, the interpretation of (6.1) is quite similar to the interpretation of a partial correctness logic triple. In fact, if we can show how to establish the validity of a property triple like (6.1) and one like

\[
\{ \hat{\rho} \land \hat{\kappa} \} \land \{ \hat{\kappa} \},
\]

where \( \hat{\rho} \) and \( \hat{\kappa} \) are property assertions, then we have solved the problem of establishing the validity of a property outline. This is because we can then use Sequential Correctness and Interference Freedom to reduce the problem to showing that a collection of property triples are valid. The soundness of this approach for establishing property outline validity is based on the same argument as for proof outline validity.

Based on the interpretation of property assertions, note that:

\[
((q_0 \rightarrow P_0 \lor \ldots \lor q_n \rightarrow P_n) \land (q_0 \rightarrow R_0 \lor \ldots \lor q_n \rightarrow R_n)) = (q_0 \rightarrow P_0 \land R_0 \lor \ldots \lor q_n \rightarrow P_n \land R_n)
\]
Thus, it suffices to be able to prove the validity of property triples like (6.1) since using (6.3), those like (6.2) can always be transformed to be like (6.1). We therefore turn to the problem of proving validity of property triples.

To prove the validity of (6.1), it suffices to prove the following partial correctness logic triples.

\[\{P_0\} \alpha ((T_{00} \land Q_0) \lor \ldots \lor (T_{0n} \land Q_n))\]  
(6.4)

\[\{P_1\} \alpha ((T_{10} \land Q_0) \lor \ldots \lor (T_{1n} \land Q_n))\]  
(6.5)

\[\ldots\]

\[\{P_n\} \alpha ((T_{n0} \land Q_0) \lor \ldots \lor (T_{mn} \land Q_n))\]  
(6.6)

The first, (6.4), establishes that execution of \(\alpha\) in a state satisfying \(P_0\) either does not terminate or terminates in a state satisfying \(T_{0j} \land Q_j\), for some \(j\). From this, we conclude that execution of \(\alpha\) in a state satisfying \(P_n\) with \(m\) in automaton state \(q_0\) either does not terminate or terminates in a state \(s'\) satisfying \(T_{nj} \land Q_j\), and \(m\) makes a transition to automaton state \(q_j\) upon reading this (next) symbol in the history being generated by \(\pi\). Thus, \(\hat{Q}\) holds for the case that \(m\) is started in \(q_0\). Repeating this argument for the remaining triples, we find that no matter what automaton state \(m\) is in when \(\alpha\) is executed, \(\hat{Q}\) will hold if \(\alpha\) terminates. Thus, (6.4)–(6.6) together imply that executing \(\alpha\) in a state satisfying \(\hat{P}\) either does not terminate or terminates in a state satisfying \(\hat{Q}\), hence \(\{\hat{P}\} \alpha (\hat{Q})\).

We illustrate this approach for proving validity of a property outline, on the one in Figure 6.1. There are two Sequential Correctness obligations:

\[\{1\} \alpha_1 \{2\}\]  
(6.7)

\[\{3\} \alpha_2 \{4\}\]  
(6.8)

And, there are four Interference Freedom obligations:

\[\{1 \land 3\} \alpha_1 \{3\}\]  
(6.9)

\[\{1 \land 4\} \alpha_1 \{4\}\]  
(6.10)

\[\{3 \land 1\} \alpha_2 \{1\}\]  
(6.11)

\[\{3 \land 2\} \alpha_2 \{2\}\]  
(6.12)

The details for only one of these property triples will be given; the remaining ones are left to the energetic reader. Property triple (6.7) is:
Decomposing this into partial correctness logic triples we get:

\[
\begin{align*}
\{1: q_0 &\rightarrow \text{false} \mid q_1 &\rightarrow (pc_v = 3 \Rightarrow x = 0) \land (pc_v = 4 \Rightarrow x = 1) \mid q_2 &\rightarrow \text{true} \mid q_3 &\rightarrow \text{false}\} \\
\{2: q_0 &\rightarrow \text{false} \mid q_1 &\rightarrow (pc_v = 3 \Rightarrow x = 1) \land pc_v \neq 4 \mid q_2 &\rightarrow \text{true} \mid q_3 &\rightarrow pc_v = 2 \land pc_v = 4 \land x = 2\}
\end{align*}
\]

Triples (6.13) and (6.16) follow trivially because the precondition of each is \textit{false}; (6.14) follows from the Assignment Axiom; and (6.15) follows because the postcondition is \textit{true}.

6.3. Proof Obligations and Property Outlines

The proof obligations of section 4 are based on using correspondence invariants that link program states and property recognizer states. Therefore, to show that \( \pi \) satisfies \( m \) using a property outline \( PO \) for \( m \) and \( \pi \), we must be able to extract from \( PO \) the correspondence invariant for each automaton state of \( m \). Doing this turns out to be trivial, due to the way property assertions are defined. Each property assertion in a property outline contains a piece of every correspondence invariant. These pieces are labeled by the automaton state to which they correspond (by the "\( q \sim \)"") and are exactly the part of the correspondence invariant that must hold whenever a program counter denotes the control point to which the property assertion is attached.

Given a program \( \pi \) consisting of a set of processes \( PROC_\pi \), let \( CP_\phi \) be the set of control points in process \( \phi \) for \( \phi \in PROC_\pi \). Suppose the property assertion attached to control point \( cp \) in a valid property outline for \( \pi \) and \( m \) is of the form \((q_0 \rightarrow P_0) \mid q_1 \rightarrow P_1 \mid \ldots \mid q_n \rightarrow P_n\). Then, choose

\[
C_i = \bigwedge_{\phi \in PROC_\pi} \bigwedge_{cp \in CP_\phi} (pc_\phi = cp \Rightarrow P_i)
\]

as the correspondence invariant for automaton state \( q_i \). This choice eliminates the need to demonstrate Simulation Induction (4.7)—this obligation is subsumed by having established validity of the property outline, as we now show.
Consider an atomic action from a process \( \phi \),

\[ \alpha : \left( \text{if } pc_0 = cp \rightarrow \bar{x}, \text{pc}_0 := \bar{z}, cp' \text{ fi} \right) \]

where \( \bar{x} \) is a vector of the program variables changed by executing \( \alpha \) and \( \bar{z} \) is a vector of expressions whose values are assigned to those variables. Simulation Induction (4.7) requires that we prove, for each automaton state \( q_i \),

\[ \{C_i\} \models \{((T_{i0} \land C_0) \lor \cdots \lor (T_{in} \land C_n))\} \]

According to the Atomic Action Rule and Rule of Consequence, this is implied by

\[ \{C_i \land pc_0 = cp\} \bar{x}, \text{pc}_0 := \bar{z}, cp' \{pc_0 = cp' \land ((T_{i0} \land C_0) \lor \cdots \lor (T_{in} \land C_n))\}. \tag{6.18} \]

The precondition and postcondition of (6.18) can be simplified because

\( (C_i \land pc_0 = cp) = (C_i^{-\phi} \land pc_0 = cp \land C_i^{\phi}) \), where

\( C_i^{\phi} = \bigwedge_{\psi \in \text{PROC}_i} \psi' \land \text{cp}_0 = cp \land \psi \land C_i^{\phi} \)

so we have

\[ \{C_i^{\phi} \land pc_0 = cp\} \bar{x}, \text{pc}_0 := \bar{z}, cp' \{pc_0 = cp' \land ((T_{i0} \land P_{i0}^{\phi}) \lor \cdots \lor (T_{in} \land P_{in}^{\phi}))(C_i^{\phi})\}. \tag{6.19} \]

Therefore, due to the Conjunction Rule and the fact that transition predicates are disjoint, it suffices to prove

\[ \{P_i^{\phi} \land pc_0 = cp\} \bar{x}, \text{pc}_0 := \bar{z}, cp' \{pc_0 = cp' \land ((T_{i0} \land P_{i0}^{\phi}) \lor \cdots \lor (T_{in} \land P_{in}^{\phi})))\}, \tag{6.20} \]

\[ \{P_i^{\phi} \land C_i^{\phi} \land pc_0 = cp\} \bar{x}, \text{pc}_0 := \bar{z}, cp' \{pc_0 = cp' \land ((T_{i0} \land P_{i0}^{\phi}) \lor \cdots \lor (T_{in} \land P_{in}^{\phi})))\}. \tag{6.21} \]

Notice, (6.20) is exactly what was proved in the Sequential Correctness step of establishing validity of \( PO \). Now we prove (6.21). Using the Conjunction Rule and the definition of \( C_i^{\phi} \), it suffices to prove:

For all \( \psi : \psi \neq \phi \land \psi \in \text{PROC}_i \):

For all \( c : c \in \text{CP}_i \):

\[ \{P_i^{\phi} \land P_i^{\phi'} \land pc_0 = cp\} \bar{x}, \text{pc}_0 := \bar{z}, cp' \{pc_0 = cp' \land ((T_{i0} \land P_{i0}^{\phi}) \lor \cdots \lor (T_{in} \land P_{in}^{\phi})))\} \]

And, these triples are exactly what was proved in the Interference Freedom step of establishing validity of \( PO \).

Thus, given a valid property outline for \( m \) and \( \pi \), in order to prove that \( \pi \) satisfies \( m \), extract the correspondence invariants from the property outline and prove Simulation Basis (4.6), Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10)—Simulation Induction (4.7) follows immediately from validity of the property outline.
6.4. Proof Outlines Revisited

Proof outlines for partial correctness logic can be formulated as property outlines. Let $PO_{pc}$ be a valid proof outline for a concurrent program $\pi$ where assertion $P^c$ is associated with each control point $cp$ in $\pi$. A valid property outline $PO_{prop}$ that embodies the information in $PO_{pc}$ is one in which each control point $cp$ has associated with it a property assertion $q_0 = P^c$. $PO_{prop}$ is for $m_{rev}$ (given in Figure 6.2) and $\pi$.

![Figure 6.2. m_{rev}](image)

Validity of $PO_{prop}$ follows from the partial correctness logic triples for Sequential Correctness and Interference Freedom used to establish validity of $PO_{pc}$.

7. Mutual Exclusion Example

Solving the mutual exclusion problem involves devising protocols to ensure that two or more processes do not execute in critical sections at the same time. A good solution to the mutual exclusion protocol must not only satisfy this Mutual Exclusion property, but should ensure that a process attempting to enter a critical section eventually does so, assuming no process remains forever in its critical section—Starvation Freedom. We might also require that a protocol satisfy First-come First-served, which asserts that requests to enter a critical section are not served out-of-order.

In this section, we prove that a program $crits$ based on the two-process mutual exclusion protocol in [Peterson 81] satisfies Mutual Exclusion, Starvation Freedom, and First-come First-served. The interested reader might wish to compare our proofs with the operational proofs for Mutual Exclusion and Starvation Freedom in [Peterson 81] and the temporal logic proofs for those properties in [Pnueli 86] and for First-come First-served in [Pnueli & Manna 83].

A control-point annotation for the program is given in Figure 7.1. Assume that initially $active_\phi = active_\psi = false$, since neither $\phi$ nor $\psi$ is initially executing in its critical section, and that $turn$ is initialized to $\phi$ or $\psi$. Thus,

$$Init_{crits} = pc_\phi = 1 \land pc_\psi = 2 \land \neg active_\phi \land \neg active_\psi \land (turn = \phi \lor turn = \psi)$$

$$Blocked_{crits} = active_\phi \land active_\psi \land turn = \phi \land turn = \psi.$$
crits: cobegin

\( \phi: \begin{cases} \{1: \text{do true} \} & \text{non critical section;} \\
\{2: \text{active}_\phi := \text{true} \} & \\
\{3: \text{turn} := \psi \} & \\
\{4: (\text{If } \neg \text{active}_\psi \lor \text{turn} = \phi \text{ skip } \}) & \\
\{5: \text{critical section;} \} & \\
\{6: \text{active}_\phi := \text{false} \} & \end{cases} \od \\
// \psi: \begin{cases} \{8: \text{do true} \} & \text{non critical section;} \\
\{9: \text{active}_\psi := \text{true} \} & \\
\{10: \text{turn} := \phi \} & \\
\{11: (\text{If } \neg \text{active}_\phi \lor \text{turn} = \psi \text{ skip } \}) & \\
\{12: \text{critical section;} \} & \\
\{13: \text{active}_\psi := \text{false} \} & \end{cases} \od \\
\text{coend} \)

Figure 7.1. Peterson's Protocol

7.1. Mutual Exclusion

A property outline for process \( \phi \) of crits and property recognizer \( m_{\max} \) (see Figure 2.4) appears in in Figure 7.2; the property outline for \( \psi \) is symmetric. The only non-trivial part of showing that Figure 7.2 is a valid property outline is showing Interference Freedom—in particular, showing that execution of \( \psi \) cannot invalidate the property assertion at control point...
\[ \phi: \{1: q_0 \text{— true} \} \]
\[ \text{do } true \rightarrow \{2: q_0 \text{— true} \} \]
\[ \text{non critical section; } \]
\[ \{3: q_0 \text{— true} \} \]
\[ \text{active}_{\phi} := \text{true}; \]
\[ \{4: q_0 \text{— active}_{\phi} \} \]
\[ \text{turn} := \phi; \]
\[ \{5: q_0 \text{— active}_{\phi} \} \]
\[ (\text{if } -\text{active}_{\psi} \vee \text{turn} = \phi - \text{skip } ); \]
\[ \{6: q_0 \text{— active}_{\phi} \land (\text{turn} = \phi \lor -\text{active}_{\psi} \lor \text{pc}_{\psi} = 11) \} \]
\[ \text{critical section; } \]
\[ \{7: q_0 \text{— true} \} \]
\[ \text{active}_{\phi} := \text{false} \]
\[ \text{od} \]

Figure 7.2. Mutual Exclusion Property Outline

6, since this is the only property assertion in \( \phi \) that mentions variables altered by execution of \( \psi \). Execution of \( \text{active}_{\psi} := \text{true} \) by \( \psi \) (at control point 10) makes \( \text{pc}_{\psi} = 11 \) true, and execution of \( \text{turn} := \phi \) by \( \psi \) (at control point 11) makes \( \text{turn} = \phi \) true. Thus, the property assertion is not interfered with.

To prove that \( \pi \) satisfies the property accepted by \( m_{\text{main}} \), we must first define \( C_{\phi} \) and \( C_{\psi} \) in terms of the program state:

\[ C_{\phi} = 6 \leq \text{pc}_{\phi} \leq 7 \]
\[ C_{\psi} = 13 \leq \text{pc}_{\psi} \leq 14 \]

Next, we must prove Simulation Basis (4.6), Simulation Induction (4.7), Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10). We can use (6.17) to extract from the property outline a correspondence invariant for automaton state \( q_0 \):
\[ C_0 = (pc_a=4 \Rightarrow \text{active}_a) \land (pc_a=5 \Rightarrow \text{active}_a) \land \\
(pc_a=6 \Rightarrow (\text{active}_a \land (\text{turn} = \phi \lor \neg \text{active}_a \lor pc_a=11)) \land \\
(pc_a=11 \Rightarrow \text{active}_a) \land (pc_a=12 \Rightarrow \text{active}_a) \land \\
(pc_a=13 \Rightarrow (\text{active}_a \land (\text{turn} = \psi \lor \neg \text{active}_a \lor pc_a=4))) \]

Simulation Basis requires that we prove

\[ \text{Init} \Rightarrow (\neg (C_{s_a} \land C_{s_b}) \land C_0) \quad (7.1) \]

Substituting and simplifying, we find that (7.1) is valid. Simulation Induction (4.7) follows because the property outline of Figure 7.2 is a valid. Finite Acceptance, Knot Exit, and Knot Variance are vacuously satisfied because the single automaton state of \( m_{\text{mul}} \) is both a finite-accepting and infinite-accepting state.

### 7.2. Starvation Freedom

In Peterson’s mutual exclusion protocol, process \( \phi \) makes a request to enter its critical section by reaching control point \( 5 \); its request is serviced when it reaches control point \( 6 \). Thus, to use property recognizer \( m_{\text{narr}} \) (Figure 2.5) to show Starvation Freedom for \( \phi \), we choose transition predicates:

- \( \text{Request}_\phi = pc_a=5 \)
- \( \text{Served}_\phi = pc_a=6 \)

A valid property outline for the protocol and \( m_{\text{narr}} \) is given in Figure 7.3. Proving Sequential Correctness and Interference Freedom is simple and is omitted here.

We extract correspondence invariants from the property outline using (6.17):

\[ C_0 = (pc_a \neq 5 \Rightarrow (\text{turn} = \phi \lor \text{turn} = \psi)) \land (pc_a=4 \Rightarrow \text{active}_a) \land (pc_a=5 \Rightarrow \text{false}) \]

\[ C_1 = (pc_a \neq 5 \Rightarrow \text{false}) \land (pc_a=5 \Rightarrow (\text{active}_a \land (\text{turn} = \phi \lor \text{turn} = \psi))) \land \\
(pc_a=12 \Rightarrow (pc_a=5 \land \text{turn} = \psi)) \land (pc_a=12 \Rightarrow pc_a=5) \]

To prove Simulation Basis (4.6) we show that

\[ \text{Init} \Rightarrow (\neg \text{Request}_\phi \land C_0) \lor (\text{Request}_\phi \land C_1) \]

is valid. This simplifies to

\[ pc_a=1 \land pc_\psi=8 \land \neg \text{active}_a \land \neg \text{active}_\psi \land (\text{turn} = \phi \lor \text{turn} = \psi) \]

\[ \Rightarrow (pc_a \neq 5 \lor C_0) \lor (pc_a = 5 \lor C_1) \]

which is valid.

Next, we prove Finite Acceptance (4.8). There is only one non-finite-accepting state in \( m_{\text{narr}}, q_1 \). Thus, Finite Acceptance (4.8) requires that we show that

\[ C_1 \Rightarrow \neg \text{Blocked} \]

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crits: cobegin

\( \phi \): \{1: q_0 \sim (\text{turn} = \varnothing \lor \text{turn} = \psi) \mid q_1 \sim \text{false} \}

\text{do true} \rightarrow \{2: q_0 \sim (\text{turn} = \varnothing \lor \text{turn} = \psi) \mid q_1 \sim \text{false} \}

\text{non critical section};

\{3: q_0 \sim (\text{turn} = \varnothing \lor \text{turn} = \psi) \mid q_1 \sim \text{false} \}

active_0 := \text{true};

\{4: q_0 \sim \text{active}_0 \land (\text{turn} = \varnothing \lor \text{turn} = \psi) \mid q_1 \sim \text{false} \}

\text{turn} := \psi;

\{5: q_0 \sim \text{false} \mid q_1 \sim \text{active}_0 \land (\text{turn} = \varnothing \lor \text{turn} = \psi) \}

(\text{if } \text{active}_0 \lor \text{turn} = \varnothing \text{ then skipf });

\{6: q_0 \sim (\text{turn} = \varnothing \lor \text{turn} = \psi) \mid q_1 \sim \text{false} \}

\text{critical section};

\{7: q_0 \sim (\text{turn} = \varnothing \lor \text{turn} = \psi) \mid q_1 \sim \text{false} \}

active_0 := \text{false}

do

dataout

//

\( \psi \): \{8: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \land \text{turn} = \psi \}

\text{do true} \rightarrow \{9: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \land \text{turn} = \psi \}

\text{non critical section};

\{10: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \land \text{turn} = \psi \}

active_0 := \text{true};

\{11: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \land \text{turn} = \psi \}

\text{turn} := \varnothing;

\{12: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \}

(\text{if } \text{active}_0 \lor \text{turn} = \psi \text{ then skipf });

\{13: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \land \text{turn} = \psi \}

\text{critical section};

\{14: q_0 \sim \text{true} \mid q_1 \sim \text{pc}_0 = 5 \land \text{turn} = \psi \}

active_0 := \text{false}

do

doend

dataout

Figure 7.3. Starvation Freedom Property Outline

is valid. It is.
There is one reject knot \( k = \{q_1\} \) in \( m_{non} \). Choose the following as a variant function \( v_k \) for the knot.

\[
v_k(q) =
\begin{cases}
0, & \text{if } pc_0 \neq 5 \\
1, & \text{if } pc_0 = 5 \land pc_y = 12 \land \text{turn} = \phi \\
2 + ((11 - pc_y) \mod 6), & \text{if } pc_0 = 5 \land \text{turn} = \psi
\end{cases}
\]

To satisfy Knot Exit (4.9), we must prove

\[
(v_k(q_1) = 0) \Rightarrow (\text{Blocked}_{crit} \lor \neg C_1).
\]

This follows because \( v_k(q_1) = 0 \Rightarrow pc_0 \neq 5 \) and \( pc_0 \neq 5 \Rightarrow \neg C_1 \).

To satisfy Knot Variance (4.10), we must show that for every atomic action \( \alpha \):

\[
(C_1 \land 0 < v_k(q_1) = \forall) \Rightarrow ((\neg \text{Served}_a \land C_1) \Rightarrow v_k(q_1) < \forall)
\]

Since \( v_k(q_1) = 1 \Rightarrow (pc_0 = 5 \land pc_y = 12 \land \text{turn} = \phi) \), and \( pc_0 = 5 \land C_1 \Rightarrow \text{active}_a \), it suffices to prove

\[
(\text{active}_a \land pc_0 = 5 \land pc_y = 12 \land \text{turn} = \phi) \Rightarrow ((\neg \text{Served}_a \land C_1) \Rightarrow v_k(q_1) < 1)
\]

for each atomic action \( \alpha \). Only the atomic actions at control points 5 and 12 are potentially enabled in the precondition of (7.3), and from \( \text{active}_a \land \text{turn} = \phi \), we conclude that the one at 12 is not enabled. Since \( \neg \text{Served} \) is false after the atomic action at control point 5 is executed, the postcondition of (7.3) is true and the triple is valid.

Next, we show that (7.2) is valid if \( v_k(q_1) = 2 \). From \( v_k(q_1) = 2 \), we infer \( pc_0 = 5 \land \text{turn} = \psi \) and since \( 2 + ((11 - 11) \mod 6) = 2, pc_y = 11 \). Thus, if suffices to show that

\[
(pc_0 = 5 \land \text{turn} = \psi \land pc_y = 11) \Rightarrow ((\neg \text{Served}_a \land C_1) \Rightarrow v_k(q_1) < 2)
\]

is valid. Only the atomic action at control points 5 and 11 are enabled in the precondition of (7.4), so they are the only ones for which (7.4) is not trivially valid. Executing the atomic action at control point 5 makes \( pc_0 = 6 \), hence the postcondition of (7.4) is true and the triple valid; executing the atomic action at control point 11 makes \( pc_y = 12 \land \text{turn} = \phi \), which decreases \( v_k(q_1) \) to 1.

Finally, we show that (7.2) is valid if \( v_k(q_1) > 2 \). If \( v_k(q_1) > 2 \) then the atomic action at control point 5, as well as an action at 9, 10, or 12 - 14 must be enabled. As already argued, executing the atomic action at 5 decreases \( v_k(q_1) \) to 0. Executing an atomic action at 9, 10, 12, 14 also decreases \( v_k(q_1) \), since by reaching the next control point, the value of \( 2 + ((11 - pc_y) \mod 6) \) is decreased. Execution starting from 13 causes the value of \( 2 + ((11 - pc_y) \mod 6) \) to be decreased provided control point 14 is reached. Thus, our proof of Starvation Freedom is correct only if \( \psi \) is guaranteed to exit its critical section after entering it.
7.3. First-come First-served

A property recognizer for First-come First-served for crits is given in Figure 7.4. Transition predicates \( \text{Request}_0 \) and \( \text{Served}_0 \) are as defined above for Starvation Freedom; the remaining two transition predicates used in \( m_{fcs} \) are:

\[
\begin{align*}
\text{Request}_0 &= pc_0 = 12 \\
\text{Served}_0 &= pc_0 = 13
\end{align*}
\]

A property outline for crits and \( m_{fcs} \) appears in Figure 7.5. Showing that the Property Outline is valid is straightforward; we do not give the details here. Informally, the correspondence invariants characterize states as follows.

- \( C_0 \): either \( \phi \) does not have a pending request or \( \psi \) has a prior request pending.
- \( C_1 \): \( \phi \) has a pending request and \( \psi \) does not.
- \( C_2 \): both \( \phi \) and \( \psi \) have pending requests and the one from \( \phi \) was prior to the one from \( \psi \).

Simulation Basis (4.6) follows trivially. The remaining obligations—Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10)—are vacuously true because every automaton state in \( m_{fcs} \) is both finite-accepting and infinite-accepting. Thus, the proof is completed.

8. Non-deterministic Property Recognizers

The proof obligations of section 4 concern properties specified by deterministic property recognizers. We now address the problem of proving that every history of a program \( \pi \) is accepted by some given non-deterministic property recognizer \( m_{ND} \). Two approaches are discussed. In the first, proof obligations are extracted directly from \( m_{ND} \). In the second, a deterministic property recognizer \( m_D \) is constructed that accepts every history of \( \pi \) accepted by \( m_{ND} \), but not necessarily every sequence of states accepted by \( m_{ND} \). Then, proof obligations are extracted from \( m_D \). The relative completeness result of the Appendix establishes
criu: cobegin
   \phi: (1: q_0\text{--true} \mid q_1\text{--false} \mid q_2\text{--false})
   do \text{true} - (2: q_0\text{--true} \mid q_1\text{--false} \mid q_2\text{--false})
      non critical section;
      (3: q_0\text{--true} \mid q_1\text{--false} \mid q_2\text{--false})
      active_{\phi} := \text{true};
      (4: q_0\text{--active}_{\phi} \mid q_1\text{--false} \mid q_2\text{--false})
      turn := \psi;
      (5: q_0\text{--Request}_{\phi} \land \text{turn} = \psi \land \text{active}_{\phi} \mid q_1\text{--Request}_{\phi} \land \text{active}_{\phi} \mid q_2\text{--Request}_{\phi} \land \text{turn} = \phi \land \text{active}_{\phi})
      (\text{if} - \text{active}_{\phi} \lor \text{turn} = \phi - \text{skip} \phi);
      (6: q_0\text{--true} \mid q_1\text{--false} \mid q_2\text{--false})
      critical section;
      (7: q_0\text{--true} \mid q_1\text{--false} \mid q_2\text{--false})
      active_{\phi} := \text{false}
    od
   \psi: (8: q_0\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_1\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_2\text{--false})
   do \text{true} - (9: q_0\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_1\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_2\text{--false})
      non critical section;
      (10: q_0\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_1\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_2\text{--false})
      active_{\psi} := \text{true};
      (11: q_0\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_1\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_2\text{--false})
      turn := \phi;
      (12: q_0\text{--Request}_{\psi} \Rightarrow (\text{turn} = \psi \land \text{active}_{\psi}) \mid q_1\text{--false} \mid q_2\text{--active}_{\psi} \land \text{turn} = \phi)
      (\text{if} - \text{active}_{\psi} \lor \text{turn} = \psi - \text{skip} \psi);
      (13: q_0\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_1\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_2\text{--false})
      critical section;
      (14: q_0\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_1\text{--Request}_{\psi} \Rightarrow \text{active}_{\psi} \mid q_2\text{--false})
      active_{\psi} := \text{false}
    od
coend

Figure 7.5. First-come First-served Property Outline
that the second approach always works, provided the program has a finite state space; however, the first approach is often simpler and more convenient.

8.1. Extracting Proof Obligations

The proof obligations of section 4 are based on two assumptions that hold for deterministic property recognizers:

(1) There is a single start state.

(2) Disjoint transition predicates label arcs emanating from each automaton state.

These assumptions need not hold for non-deterministic property recognizers. However, given a non-deterministic property recognizer that does not satisfy assumption (1), it is easy to construct one that does. Thus, in adapting the proof obligations developed in section 4 for use with properties specified by non-deterministic property recognizers, we need only be concerned with assumption (2).

Assumption (2) is used in section 4 to combine the constraints on correspondence invariants with the proof obligations that prevent undefined transitions. In particular, (4.1) is merged with (4.3) to form Simulation Basis (4.6), and (4.2) is merged with (4.4) to form Simulation Induction (4.7). Since this merging is not possible when transition predicates are not disjoint, the reasoning of section 4 dictates that for a given program \( \pi \) and non-deterministic property recognizer \( m_{ND} \), showing (4.1), (4.2), (4.3), (4.4), Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10), ensures that every history of \( \pi \) is accepted by \( m_{ND} \).

Unfortunately, these proof obligations may be too strong—not all programs that satisfy \( m_{ND} \) will satisfy (4.1), (4.2), (4.3), (4.4), (4.8), (4.9), and (4.10) because these obligations ensure that for any history of the program, every run of \( m_{ND} \) is accepting. Recall, a property recognizer accepts an infinite sequence provided a single run is accepting. With a deterministic property recognizer, each input results in only a single run, so ensuring that every run is accepting is equivalent to ensuring that the single run is. With a non-deterministic property recognizer, there may be multiple runs. Thus, for non-deterministic property recognizers, the proof obligations are more restrictive than necessary.

8.2. Refining Non-deterministic Recognizers

Non-deterministic property recognizers can specify properties that cannot be specified by deterministic ones [Eilenberg 74]. However, each program \( \pi \) (with a finite state space) that satisfies a property \( P_{ND} \), accepted by a non-deterministic property recognizer \( m_{ND} \), must also satisfy a property \( P_D \), where \( P_D \subseteq P_{ND} \) and \( P_D \) is specified by a deterministic property recogni-
Thus, to prove that \( \pi \) satisfies a property \( ND \) specified by \( m_{ND} \), it suffices to construct \( m_D \) and prove that \( \pi \) satisfies it. We call \( m_D \) a deterministic refinement of \( m_{ND} \).

The construction of \( m_D \) involves repeatedly modifying \( m_{ND} \), using the techniques described below, so that it becomes progressively more deterministic. Clearly, valid modifications must never cause the resulting property recognizer to accept sequences not accepted by the original one; they can, however, cause fewer sequences to be accepted. Satisfying the proof obligations for the deterministic refinement ensures that all histories of the program are accepted by the original property recognizer \( m_{ND} \).

Modifications for obtaining a deterministic refinement fall into two classes: those that result in an automaton that accepts the same sequences as the original; and those that result in an automaton that accepts fewer sequences than the original. The second class of modifications is needed because some non-deterministic property recognizers do not have deterministic equivalents.

By removing transitions from \( m_{ND} \), the resulting property recognizer is more deterministic and can accept no sequence that would not have been accepted by \( m_{ND} \). Thus, this form of modification is one way towards constructing a deterministic refinement.

Pruning: Delete transitions in the property recognizer.

Frequently, Pruning is performed by strengthening transition predicates based on knowledge of the program state. This form of Pruning is illustrated in Figure 8.1.

![Pruning](image)

**Before**

**After**

Figure 8.1. Pruning

Here, transitions from \( q_0 \) to itself under program states that satisfy \( P \) have been pruned.

A second modification that makes a property recognizer more deterministic is to combine automaton states.

Combining: Combine states if it does not permit additional sequences to be accepted.

When combining two states \( q' \) and \( q'' \), all transitions into \( q' \) and \( q'' \) terminate at a new state.

*The proof of this appears in the Appendix as part of the completeness result.*
q. If a non-deterministic choice selected between \( q \) and \( q' \) in the original property recognizer, then that choice is no longer non-deterministic in the resulting one. Two states \( q' \) and \( q'' \) can be combined provided:

Combining Congruent States. If two states \( q' \) and \( q'' \) are congruent then they can be combined and the resultant property recognizer will accept the same set of sequences. Two states \( q' \) and \( q'' \) are congruent if and only if

C1: neither or both are finite-accepting,

C2: neither or both are infinite-accepting,

C3: if there is a transition from \( q' \) to \( q \) under program state \( s \) then there also is a transition from \( q'' \) to some state congruent to \( q \) under program state \( s \).

An example of this is illustrated in Figure 8.2. There, \( q_2 \) and \( q_3 \) are combined.

![Diagram of before and after combining congruent states](image)

**Figure 8.2.** Combining

When C1 or C2 of Combining Congruent States does not hold, it is sometimes possible to promote a non-accepting state to being an accepting state without changing the set of sequences accepted by the property recognizer.

Finite-accepting Promotion. A non-finite-accepting state \( q \) can be promoted to being finite-accepting if for every run that ends in \( q \) there is another run on the same input that ends in a finite-accepting state.

Infinite-accepting Promotion. A non-infinite-accepting state \( q \) can be promoted to being infinite-accepting if for every run that contains \( q \) infinitely-often there is a run (perhaps the same one) on the same input that contains some infinite-accepting state infinitely
Finally, an automaton state may serve many roles. By splitting such a state into several copies, we can separate these roles and then use Pruning to remove transitions or Combining to combine some of the copies with other automaton states.

**Splitting:** Replicate an automaton state and all transitions into and out of it.

Splitting does not change the set of sequences accepted by a property recognizer, but it does put the recognizer into a form where Pruning and/or Combining can be used to move towards a deterministic refinement. Splitting is illustrated in Figure 8.3.

It is not always necessary to construct the actual deterministic refinement of a given non-deterministic property recognizer. Rather, it suffices to use Pruning, Combining, and Splitting to obtain a non-deterministic property recognizer for a property that is also accepted by some deterministic property recognizer. We can then apply one of the known (automatic) procedures to produce a deterministic property recognizer that is equivalent to the given non-deterministic one [Landweber 69].

![Before and After Diagram](image)

---

*Such procedures also indicate if there is no deterministic property recognizer for the given non-deterministic one. Then additional Pruning, Combining, and Splitting must be done.*
9. Discussion

We have shown how to decompose a property into proof obligations. Since properties and proof obligations can be formalized using temporal logic, our approach describes how to break up the task of showing that a program satisfies one temporal formula—the property—into showing that the program satisfies a number of simpler temporal formulas—the proof obligations. Simulation Basis (4.6), Finite Acceptance (4.8), and Knot Exit (4.9) are temporal formulas because they are predicate logic formulas. The remaining two proof obligations, Simulation Induction (4.7) and Knot Variance (4.10), can be formulated in temporal logic, as

Temporal Simulation Induction: For all $i$: \( q_i \in Q \):
\[
\square (C_i \Rightarrow \bigcirc (\bigvee_{j:q_j \in Q} (T_{ij} \land C_j))),
\]
(9.1)

Temporal Knot Variance: For all reject knots $k$ and all $q_i \in k$:
\[
\square ((C_i \land 0 < v_x(q_i) = V) \Rightarrow \bigcirc (\bigwedge_{j:q_j \in k} ((T_{ij} \land C_j) \Rightarrow v_x(q_j) < V)))
\]
(9.2)

where $\square$ denotes the temporal operator "henceforth" and $\bigcirc$ denotes "next".

Other investigations into decomposing temporal properties include [Barringer et al. 84], [Gerth 84], [Jones 83], [Misra et al. 82], [Nguyen et al. 85] and [Stark 84]. Most of that work is concerned with decomposing various classes of global temporal properties of a system into local properties of the system components, resulting in so-called compositional proof systems. The work in [Gerth 84] is most similar to ours in that the primitive formulas into which temporal properties are decomposed resemble triples. That work, however, is concerned only with finite sequences (both as properties and programs) and therefore does not address the problem we are most concerned with.

We chose to express the proof obligations as triples rather than as temporal logic formulas because we believe that people have less trouble understanding and manipulating triples. Moreover, the relation between triples and the program text is always clear—when a proof obligation formulated as a triple cannot be proved, there is little question where in the program to start looking. This is not the case for formulas of temporal logic, because they do not explicitly mention the program. Finally, we hope to integrate our approach with methods to develop a program and its proof of correctness hand-in-hand, as discussed in [Dijkstra 76] [Gries 81]. These methods are formulated in terms of triples, so it made sense for us to remain in that framework.

Considering our proof obligations from a temporal viewpoint does offer some insights. Temporal Knot Variance (9.2) requires that execution of every atomic action cause the value of a variant function to decrease, thereby ensuring progress is made towards accepting the history. Without making assumptions about fairness, this is the only way to ensure that all infinite histories leave a reject knot because an atomic action that does not decrease any
variant function can be repeated indefinitely, resulting in a history that is not accepted by the property recognizer. Thus, while we would be happy to establish

$$\Box((C_i \land 0 < v_i(q_i) = V) \Rightarrow \diamond (\bigwedge_{j: q_j \leq x} ((T_{ij} \land C_j) \Rightarrow v_j(q_j) < V))).$$

(9.3)

(where $\Box$ denotes eventually), without making fairness assumptions, we are forced to demonstrate

$$\Box((C_i \land 0 < v_i(q_i) = V) \Rightarrow \Box (\bigwedge_{j: q_j < x} ((T_{ij} \land C_j) \Rightarrow v_j(q_j) < V))).$$

(9.4)

However, if we can make assumptions about fairness, then we need not prove (9.4), in order to establish (9.3). Instead, it suffices to prove that certain helpful processes that do decrease the variant function are eventually executed and that executing other processes does not increase the variant function. This method is formalized as temporal logic inference rules in [Manna & Pnueli 84]—one rule for each type of fairness (e.g. weak fairness, strong fairness)—and can be adapted to our approach by replacing Knot Variance (4.10) with the hypotheses of the appropriate inference rule. These hypothesis are easily formulated as predicate logic formulas and triples. This, then, provides a second way in our approach to prove a property $P$ under a fairness assumption $F$. The first (section 4), was to construct the property recognizer for $F \Rightarrow P$ and show that the proof obligations it defines are satisfied; the second, is to construct a property recognizer for $P$ and extract proof obligations from it, except with the Knot Variance (4.10) obligation replaced by the hypotheses from the appropriate temporal logic inference rule.

One difference between our approach and most temporal logic verification methods is the treatment of terminating executions. We handle terminating executions by explicitly dealing with finite sequences of program states; it is inconvenient to deal with finite sequences using temporal logics that include a "next" operator, so finite sequences are usually extended to be infinite sequences. Unfortunately, this extension can cause problems because the infinite sequence might not satisfy a property that the original (finite) one did. For example, a common way to extend a finite sequence to an infinite one is by replicating the last state. A property like "the value of the program counter changes between two successive states", though true of a finite sequence, does not hold for an infinite sequence obtained by replicating the last state of a finite sequence. Other ways to extend finite sequences have similar problems.

Another, related, approach to verifying that a program satisfies a property is model checking [Clarke et al. 83] [Emerson & Lei 83] [Lichtenstein & Pnueli 85], where a program $\pi$ is viewed as specifying a Kripke structure $K_{\pi}$. $K_{\pi}$ is a model for $P$ if and only if $\pi$ satisfies $P$. Thus, to determine if $\pi$ satisfies $P$ it suffices to check whether $K_{\pi}$ is a model for $P$, and this amounts to checking each state in the state space to see which sub-formulas of $P$ hold in...
that state. Determining whether if $K_n$ is a model for $P$ requires time linear in both the length of $P$ and the size of the program state space.

Recently, [Vardi & Wolper 85] observed that $K_n$ can be viewed as a Buchi automaton that accepts exactly the histories of $\pi$. From this automaton and one that recognizes sequences satisfying $\neg P$, a Buchi automaton $m_{\pi, \neg P}$ can be constructed that accepts all histories of $\pi$ not satisfying $P$. The decision procedure for the emptiness problem for $m_{\pi, \neg P}$ can then be used to determine if $\pi$ satisfies $P$; the decision procedure is exponential in the length of $P$ and linear in the size of the program state space.

The drawback to both these methods is that they require time linear in the size of the state space. (The fact that the second method is exponential in the length of $P$ is inconsequential due to the relative size of the program state space.) They are practical only for those applications where the program state space is of a manageable size. In our approach, rather than check every state in the state space, the state space is partitioned into equivalence classes defined by the correspondence invariants. The number of correspondence invariants is exponential in the length of $P$, since there is one for each state in $m_P$; the number of proof obligations is linear in the size of the program. Thus, with our method, the number of proof obligations incurred for a deterministic property is exponential in the length of $P$ and linear in the size of the program. Since the size of the program is likely to be substantially smaller than the size of the state space, our approach is rather attractive. Even for non-deterministic properties, the number of proof obligations incurred with our approach is bounded by the size of the state space (see Appendix). Thus, our approach is comparable to the model checking approaches for this case.

Of course, verification is only necessary if synthesis is not possible. Techniques to synthesize the synchronization portion of a finite-state concurrent program from a propositional temporal logic specification are given in [Clarke & Emerson 81] and [Manna & Wolper 84]. The latter technique is most closely related to the work of this paper, since it is based on linear time temporal logic. In it, a model graph for a property $P$ is constructed and then converted into a program. This model graph is just a property recognizer. Restriction to propositional specifications is not a problem for synchronizers, but is not sufficient for specifying many properties of programs, e.g. the relation between the program's input and output.

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*Recall, Buchi automata are special cases of property recognizers.*

*We assume that the cost of deciding the validity of a Hoare triple is constant. This is reasonable for purposes of comparison because in the model checking approach the ability to decide the validity of an implication in constant time follows from the restriction to propositional temporal logic.*
10. Conclusions

A new approach to proving temporal properties of concurrent programs was described. The approach is based on specifying properties using automata, called property recognizers. Property recognizers are quite expressive—any linear-time temporal logic formula can be formulated as a property recognizer. Proof obligation for a property are extracted directly from the recognizer for that property. The proof obligations are predicate logic formulas and triples. Thus, temporal inference is not necessary for proving temporal properties. In fact, the same techniques that work for proving total correctness of sequential programs [Hoare 69] [Dijkstra 76] can be used for proving arbitrary temporal properties of concurrent ones. When proving total correctness of a loop in a sequential program, a loop invariant and variant function must be devised and checked. When our method is used to prove that some arbitrary temporal property holds for a concurrent program, correspondence invariants and variant functions must be devised and checked.

Our approach was illustrated on some standard examples: incrementing x by 2 in parallel [Owicki & Gries 76] and Mutual Exclusion, Starvation Freedom, and First-come, First-served for Peterson's solution to the critical section problem [Peterson 81]. Property outlines were proposed as a succinct way to represent a program and its correspondence invariants for a given property recognizer.

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References


[Lamport & Schneider 84] Lamport, L. and F.B. Schneider. The 'Hoare Logic' of CSP, and All That. ACM Transactions on Programming Languages and Systems 6, 2 (April 1984), 281-296.


Appendix: Soundness and Relative Completeness

The soundness and relative completeness of our approach is shown below. We first show that the proof obligations of section 4 for deterministic property recognizers are sound. We then show that they are complete relative to predicate logic and Hoare's partial correctness logic. Since partial correctness logic is known to be complete relative to predicate logic, our proof obligations are complete relative to predicate logic. Next, we show that the proof obligations of section 8 for non-deterministic property recognizers are also sound, and finally that they are complete relative to our approach for deterministic properties.

Deterministic Property Recognizers

Soundness Theorem: If for a program $\pi$ and deterministic property recognizer $m_\pi$ for property $P$ there are correspondence invariants and variant functions such that Simulation Basis (4.6), Simulation Induction (4.7), Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10) are valid, then $\pi$ satisfies $P$.

Proof. Assume that the proof obligations are valid for some correspondence invariants and variant functions and that $\sigma$ is a history of $\pi$. We must show that $\sigma$ satisfies $P$.

By induction on $n$,
\[ \delta^*(q_0, \sigma[..n]) = q_i \Rightarrow C_i(\sigma[n]) \]
due to Simulation Basis (4.6) and Simulation Induction (4.7). A similar inductive argument shows that $m_\pi$ cannot attempt an undefined transition when reading $\sigma[n]$.

We now show that if $\sigma$ is finite then it is accepted by $m_\pi$. Without loss of generality, let $\sigma[n]$ be the final state of $\sigma$. We must show $\delta^*(q_0, \sigma[..n]) \in Q_{fin}$. Due to Finite Acceptance (4.8), if $\delta^*(q_0, \sigma[..n])$ is a non-finite-accepting state, then $\pi$ cannot be blocked in $\sigma[n]$ and this contradicts the assumption that $\sigma[n]$ is the final state of $\sigma$. Thus, we conclude that $\delta^*(q_0, \sigma[..n])$ is a finite-accepting state, and, by definition, $m_\pi$ accepts $\sigma$, hence $\sigma$ satisfies $P$.

Finally, we show that if $\sigma$ is infinite then it is accepted by $m_\pi$. By Knot Exit (4.9) and Knot Variance (4.10), if $m_\pi$ enters a reject knot $\kappa$ upon reading $\sigma[n]$, then it must exit $\kappa$ before reading the $n + \nu_\kappa(\delta^*(q_0, \sigma[..n]), \sigma[n])^{th}$ symbol of $\sigma$. By the definition of a reject knot, $m_\pi$ cannot reenter $\kappa$ after exiting it without first entering an infinite-accepting state.
Since there are finitely many reject knots and \( k \) is infinite, \( m_p \) must enter an infinite-accepting state infinitely often. Thus, by definition, \( m_p \) will accept \( \sigma \), hence \( \sigma \) satisfies \( P \). \( \square \)

Relative Completeness Theorem: If a program \( \pi \) satisfies a property \( P \) that is accepted by a deterministic property recognizer \( m_p \), then there exist correspondence invariants and variant functions, for which Simulation Basis (4.6), Simulation Induction (4.7), Finite Acceptance (4.8), Knot Exit (4.9), and Knot Variance (4.10) are valid.

Proof. Assume \( m_p \) accepts every history of \( \pi \). We must show that (4.6)-(4.10) for \( \pi \) and \( m_p \) are valid.

Choose correspondence invariants and variant functions as follows. Let \( H_{\pi} \) be the set of histories of \( \pi \). First, for each automaton state \( q_i \), define
\[
C_i(s) = (\exists \sigma, n: \sigma \in H_{\pi}, 0 \leq n: s = \sigma[n] \wedge \delta^*(q_0, \sigma[\ldots n]) = q_i).
\]
Thus, \( C_i(s) \) holds for a program state \( s \) if and only if there is some history of \( \pi \) in which \( s \) caused \( m_p \) to make a transition to \( q_i \). Next, for each reject knot \( \kappa \) and each \( q_i \in \kappa \), define
\[
\nu_\kappa(q_i, s) = \begin{cases} 0, & \text{if } \text{Blocked}_\kappa(s) \wedge C_i(s) \\ 1 + \max(\exists \sigma, n: \sigma \in H, 0 \leq n: s = \sigma[n] \wedge \delta^*(q_0, \sigma[\ldots n]) = q_i) \\ \wedge \neg \text{Blocked}_\kappa(s[n+\nu]) \wedge (\forall j: 0 \leq j \leq \nu: \delta^*(q_0, \sigma[\ldots n+j]) \in \kappa) \\ \text{if } \neg \text{Blocked}_\kappa(s) \wedge C_i(s) \end{cases}
\]
Thus, \( \nu_\kappa(q_i, s) \) is the maximum number of atomic actions \( \pi \) can execute when in state \( s \) and \( m_p \) is in \( q_i \) before \( m_p \) will halt or leave \( \kappa \).

It remains is to prove that (4.6)-(4.10) are valid with these correspondence invariants and variant functions. We consider each proof obligation in turn.

Simulation Basis (4.6). Since \( \pi \) satisfies \( P \), every initial state of \( \pi \) must satisfy some transition predicate \( T_0 \). By construction, this initial state will also satisfy \( C_j \). Thus, (4.6) is valid.

Simulation Induction (4.7). Consider any program history \( \sigma \) and suppose \( \delta^*(q_0, \sigma[\ldots n]) = q_i \) for some \( n \). By construction, \( C_i(\sigma[n]) \). Consider an atomic action \( \alpha \) from \( A_\pi \) that terminates in a state \( s' \) when started in state \( \sigma[n] \). Clearly, \( \sigma s' \) is the prefix of some history \( \sigma' \) of \( \pi \). Since \( m_p \) accepts every history of \( \pi \), \( m_p \) must accept \( \sigma' \), so there must exist an automaton state \( q_j \) such that \( \sigma'[n+1] \) satisfies \( T_{ij} \). By construction, \( C_j(\sigma'[n+1]) \). So, we have shown \( C_i(\alpha) \wedge (T_{ij} \wedge C_j) \) is valid for any atomic action that terminates when started in a state satisfying \( C_i \). Since \( C_i(\alpha) \wedge (T_{ij} \wedge C_j) \) is valid for any atomic action \( \alpha \) that does not terminate when started in a state satisfying \( C_i \), we have shown that (4.7) is valid.

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Finite Acceptance (4.3). Consider any program state \( \sigma[n] \) in some history \( \sigma \) of \( \pi \). Suppose \( \delta^*(q_0, \sigma[\ldots n]) = q_j \). Thus, by construction \( C_j(\sigma[n]) \). If \( q_j \notin Q-Q_{in} \), then \( \sigma[n] \) also satisfies \(-Blocked_\pi \). Otherwise, \( \sigma[n] \) would have to be the final state of \( \sigma \), which would cause \( m_p \) to reject \( \sigma \), contradicting the assumption that every history of \( \pi \) is accepted by \( m_p \). Thus, \( C_j \Rightarrow -Blocked_\pi \) is valid, so (4.8) is valid.

Knot Exit (4.9). The proof that (4.9) is valid is trivial, by construction of \( \nu_\pi \).

Knot Variance (4.10). If \( \alpha \) does not terminate when started in a state satisfying some correspondence invariant \( C_i \) for an automaton state \( q_i \in \kappa \), then
\[
\{ C_i \land \nu_\pi(q_i) = V \} \alpha \{ \land \forall j: q_j \in \kappa \}
\]
(10.1)
is trivially valid.

Suppose \( \alpha \) does terminate and terminates in state \( s' \) when started in state \( s \). Thus, there must exist a history \( \sigma_1 \) and an integer \( n_1 \) such that \( \sigma_1[n_1] = s \) and \( \delta^*(q_0, \sigma_1[\ldots n_1]) = q_i \). There also must exist a history \( \sigma_2 \) and an integer \( n_2 \) such that \( \sigma_2[n_2] = s' \), \( \delta^*(q_0, \sigma_2[\ldots n_2]) = q_j \), \( \forall j: 0 \leq j \leq \nu_\pi(q_j, s') \): \( \delta^*(q_0, \sigma_2[\ldots n_2+j]) = q_i \), and \(-Blocked_\pi(\sigma_2[\nu_\pi(q_j, s')]) \). Let \( \sigma = (q_0, \sigma[0..n_1] \sigma_1[n_1+1..n_2]) \). Since \( \alpha \) terminates in \( s' \) when started in state \( s \), \( \sigma \) is a history of \( \pi \). By the construction of \( \nu_\pi \), we conclude \( \nu_\pi(q_j, s') = 1 \leq \nu_\pi(q_i, s) \). So, (10.1) is valid.

Non-deterministic Property Recognizers

The Soundness Theorem for non-deterministic property recognizers shows that constructing a deterministic refinement suffices for proving the non-deterministic property of interest. The Soundness Theorem for deterministic property recognizers, then allows us to conclude that satisfying the proof obligations extracted from this deterministic refinement are sufficient. Completeness for non-deterministic property recognizers involves showing that if a program \( \pi \) satisfies a property specified by a non-deterministic property recognizer \( m_{ND} \), then it is always possible to construct a deterministic refinement of \( m_{ND} \) by using Combining, Pruning, and Splitting.

Soundness Theorem: If a non-deterministic property recognizer \( m_{ND} \) for a property \( ND \) can be refined to a deterministic property recognizer \( m_D \) for a property \( D \) by using Pruning, Splitting, or Combining, then if program \( \pi \) satisfies \( D \), it will also satisfy \( ND \).

Proof. Suppose \( m_D \) can be obtained from \( m_{ND} \) using a single refinement step. If Splitting is used, then \( m_D \) and \( m_{ND} \) accept exactly the same sequences. If Combining is used, then by the definition of Combining \( m_D \) and \( m_{ND} \) accept exactly the same sequences. Finally, if Pruning is used, then \( m_{ND} \) accepts every sequence accepted by \( m_D \) because Pruning can only result in a refinement that rejects more sequences than the original. Thus, if \( \pi \) satisfies property \( D \), it must also satisfy \( ND \). The theorem then follows by induction of the number of refinement steps.
steps needed to obtain \( m_D \) from \( m_{ND} \). □

Relative Completeness Theorem: If program \( \pi \) has a finite state space and satisfies some property \( ND \) that is accepted by a non-deterministic property recognizer \( m_{ND} \), then there exists a deterministic refinement \( m_D \) of \( m_{ND} \) that \( \pi \) satisfies.

Proof. First, we construct a deterministic property recognizer \( m_\pi \) that accepts \( H_\pi \), the histories of \( \pi \). Define \( m_\pi \) to be \( \langle S_\pi, S_\pi \rightarrow \{\text{start}\}, \{\text{start}\}, S_\pi, \text{Blocked}_\pi, \delta_\pi \rangle \), where \( S_\pi \) is the set of program states of \( \pi \) and

\[
\delta_\pi(\text{start}, s) = s \text{ iff } s \text{ satisfies } \text{Init}_\pi, \text{ and}
\delta_\pi(s, s') = s' \text{ iff there is an atomic action of } \pi \text{ enabled in } s \text{ that terminates in } s'.
\]

Clearly, \( m_\pi \) accepts exactly the histories of \( \pi \).

We can use \( m_\pi \) to refine \( m_{ND} = \langle S_{\pi}, Q, Q_0, Q_{inf}, Q_{fin}, \delta_{ND} \rangle \). Let \( m_{ND \times \pi} \) be the property recognizer \( \langle S_{\pi}, Q \times (S_{\pi} \rightarrow \{\text{start}\}), Q_0 \times \{\text{start}\}, Q_{inf} \times S_{\pi}, Q_{fin} \times \text{Blocked}_\pi, \delta_{ND \times \pi} \rangle \), where

\[
(q', s') \in \delta_{ND \times \pi}((q, s), s') \text{ iff } q' \in \delta_{ND}(q, s) \text{ and } \delta_{\pi}(s, s') = s'.
\]

Note that \( m_{ND \times \pi} \) can be obtained by Splitting each state of \( m_{ND} \), into one copy for each state of \( m_{\pi} \) and then using Pruning.

\( m_{ND \times \pi} \) accepts exactly those sequences that are histories of \( \pi \) (hence, accepted by \( m_{\pi} \)) and accepted by \( m_{ND} \). Since \( \pi \) satisfies \( ND \), every history of \( \pi \) is accepted by \( m_{ND} \). Thus, \( m_{ND \times \pi} \) recognizes the same set of sequences as \( m_{\pi} \). We can now use Combining to obtain \( m_{\pi} \) from \( m_{ND \times \pi} \)—all states of the same second component are combined together. Since \( m_{\pi} \) is deterministic and accepts every history of \( \pi \), we have shown how to obtain a deterministic refinement for \( m_{ND} \). □