RELIABILITY IMPORTANCE FOR CONTINUUM STRUCTURE FUNCTIONS

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*Research supported by the Air Force Office of Scientific Research, AFSC, USAF, under grant AFOSR-84-0243. The US Government is authorised to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon.

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ABSTRACT

A continuum structure function is a nondecreasing mapping from the unit hypercube to the unit interval. A definition of the reliability importance, \( R_i(\alpha) \) say, of component \( i \) at level \( \alpha \ (0 < \alpha < 1) \) is proposed. Some properties of this function are deduced, in particular conditions under which \( \lim_{\alpha \to 0} R_i(\alpha) = \lim_{\alpha \to 1} R_i(\alpha) = 0 \) and conditions under which \( R_i(\alpha) \) is positive \( (0 < \alpha < 1) \).

KEYWORDS: Continuum structure function; reliability importance; key vector
1. INTRODUCTION

Let $\phi: \{0,1\}^n \rightarrow \{0,1\}$ be a binary coherent structure function and let $h: [0,1]^n \rightarrow [0,1]$ be the corresponding reliability function (see Barlow and Proschan (1975a), Chapters 1 and 2). The reliability importance of component $i$ is defined as

$$I(i) = \frac{\partial h(\beta)}{\partial p_i} = h(1_i, \beta) - h(0_i, \beta)$$

($i=1,2,\ldots,n$), writing $(\beta, \beta) = (p_1, \ldots, p_{i-1}, \beta, p_{i+1}, \ldots, p_n)$ where $p_i = P(X_i = 1)$ and where $X_1, \ldots, X_n$ are independent binary random variables denoting the states of the components of $\phi$. This definition is due to Birnbaum (1969); see Barlow and Proschan (1975b) and Natvig (1979), (1984) for some alternative approaches. Various authors have proposed extensions of this concept to the multistate case (e.g. Barlow and Wu (1978), Griffith (1980), Natvig (1982), Block and Savits (1982)), but a general theory of reliability importance for structure functions on domains other than $\{0,1\}^n$ has yet to be developed. In this paper, we present a definition of reliability importance for continuum structure functions (CSFs), i.e. mappings of the form $\gamma: \Delta \rightarrow [0,1]$, where $\Delta = [0,1]^n$, which are nondecreasing in each argument and which satisfy $\gamma(0) = 0$ and $\gamma(1) = 1$ where $\beta$ denotes $(\beta, \ldots, \beta)$. See Block and Savits (1984) and Baxter (1984), (1986) for further details of CSFs. The reliability importance, $R_i(\alpha)$ say, of component $i$ ($i=1,2,\ldots,n$) will depend on the state $\alpha$ ($0<\alpha<1$) of the system. Our main results are conditions on $\gamma$ under which $\lim_{\alpha \rightarrow 0} R_i(\alpha) = \lim_{\alpha \rightarrow 1} R_i(\alpha) = 0$ and conditions under which $R_i(\alpha)$ is positive.
We shall make frequent use of the following sets:

\[ U_\alpha = \{ x \in \Delta | y(x) > \alpha \} \]
\[ L_\alpha = \{ x \in \Delta | y(x) < \alpha \} \]
\[ P_\alpha = \{ x \in \Delta | y(x) > \alpha \text{ whereas } y(\cdot) < \alpha \text{ for all } \cdot \} \]
\[ K_\alpha = \{ x \in \Delta | y(x) < \alpha \text{ whereas } y(\cdot) > \alpha \text{ for all } \cdot \} \]

where \( \gamma < (>) \) means that \( \gamma < (>) x \) but that \( \gamma \neq x \).

2. \textbf{KEY VECTORS}

The motivation for our definition (below) is most readily understood by observing that one can write

\[ I(i) = P(\phi(X) = 1 | X_i = 1) - P(\phi(X) = 1 | X_i = 0), \]

i.e. \( I(i) \) is the probability that repairing component \( i \) will restore a failed system to the operating state (or, equivalently, that the failure of component \( i \) will cause an operating system to fail). A possible generalisation of \( I(i) \) to the continuum case would be to regard part of the unit interval, say \([0, \alpha) \) \((0<\alpha<1)\), as corresponding to the failure states of the system and to regard \([\alpha, 1]\) as the operating states, in which case one could define the reliability importance of component \( i \) \((i=1, 2, \ldots, n)\) to be

\[ P(y(X) > \alpha | X_i > \alpha) - P(y(X) > \alpha | X_i < \alpha). \]
Consideration of the CSF $\gamma(x_1, x_2) = x_1 x_2$ suggests that this definition is not wholly satisfactory: if $x_1 = x_2 = \beta \in [\alpha, \sqrt{\alpha})$ ($0<\alpha<1$), then neither component is in the failed state even though the system itself should be regarded as failed. This difficulty may be circumvented by replacing $\sim$ by a suitably chosen element of $aU\alpha$; considerations of symmetry indicate that the vector chosen, called the key vector, should also lie on the diagonal of the unit hypercube. Hence, before proceeding to a definition of reliability importance for CSFs, it is convenient to define and study the key vector of $\cap\alpha$.

**Definition**

Let $H = \{\mu | 0 < \mu < 1\}$ be the diagonal of the unit hypercube. We say that the vector $\delta = \delta(\alpha) = H \cap \cap\alpha$ is the key vector of $\cap\alpha$ and we call $\delta$ the key element.

**Lemma 2.1**

The CSF $\gamma$ is right (left)-continuous if and only if each $\cup\alpha(\cap\alpha)$ is closed.

**Proof:** A CSF is right (left)-continuous if and only if it is upper (lower) semicontinuous which is the case if and only if each $\cup\alpha(\cap\alpha)$ is open (Royden (1968), p. 161).

**Theorem 2.2**

For any CSF $\gamma$, the key vector always exists and, if $\gamma$ is continuous, $\gamma(\delta) = \alpha$ for all $\alpha \in (0, 1]$. 

Proof: To show that the key vector exists for any CSF, it is sufficient to show that $H \cap \mathcal{U}_\alpha \neq \emptyset$ for all $\alpha \in (0,1]$.

Let $\gamma$ be an arbitrary CSF. Then $1 \in \mathcal{U}_\alpha$ for all $\alpha \in (0,1]$ by definition, so $\overline{\mathcal{U}}_\alpha \neq \emptyset$ for all $\alpha \in (0,1]$. If $\mathcal{U}_\alpha = \overline{\mathcal{U}}_\alpha$, it is immediate that $H \cap \mathcal{U}_\alpha \neq \emptyset$ since $1 \in H \cap \overline{\mathcal{U}}_\alpha$ for all $\alpha \in (0,1]$. Suppose that $\mathcal{U}_\alpha$ is a proper subset of $\overline{\mathcal{U}}_\alpha$ and consider $\Delta' = \Delta - \mathcal{U}_\alpha$. Since $\overline{\mathcal{U}}_\alpha$ and $\mathcal{U}_\alpha$ are disjoint and $\mathcal{U}_\alpha$ is a proper subset of $\overline{\mathcal{U}}_\alpha$, it follows that $\Delta' = (\overline{\mathcal{U}}_\alpha \cap \mathcal{U}_\alpha) - \mathcal{U}_\alpha = \overline{\mathcal{U}}_\alpha \cap (\overline{\mathcal{U}}_\alpha - \mathcal{U}_\alpha)$ is a separation of $\Delta'$, i.e. $\Delta'$ is a disconnected set. Now suppose that $H \cap \mathcal{U}_\alpha = \emptyset$ for some $\alpha \in (0,1]$. Then, for all $\beta \in H$, $\beta \notin \mathcal{U}_\alpha$, so $H \subseteq \Delta'$. Clearly, $H$ is connected. Since $\Delta'$ is a disconnected set with separation $\overline{\mathcal{U}}_\alpha \cap (\mathcal{U}_\alpha - \mathcal{U}_\alpha)$, $H$ must be properly contained in either $\overline{\mathcal{U}}_\alpha$ or $\mathcal{U}_\alpha - \mathcal{U}_\alpha$. Since $\mathcal{U}_\alpha$ is a proper subset of $\overline{\mathcal{U}}_\alpha$ and since $1 \in \overline{\mathcal{U}}_\alpha$, it is obvious that $1 \notin \mathcal{U}_\alpha$, i.e. $1 \notin \overline{\mathcal{U}}_\alpha - \mathcal{U}_\alpha$, and thus $H \cap (\overline{\mathcal{U}}_\alpha - \mathcal{U}_\alpha) \neq \emptyset$. Further, since $\alpha \in (0,1]$, $\mathcal{U}_\alpha \subseteq \overline{\mathcal{U}}_\alpha$, so $H \cap \overline{\mathcal{U}}_\alpha \neq \emptyset$. This is a contradiction to the assertion that $H$ must be properly contained in either $\overline{\mathcal{U}}_\alpha$ or $\mathcal{U}_\alpha - \mathcal{U}_\alpha$. Thus $H \cap \mathcal{U}_\alpha \neq \emptyset$ for all $\alpha \in (0,1]$ for any CSF, as claimed.

We now show that if $\gamma$ is continuous, then $\gamma(\delta) = \alpha$. Since, by continuity, $\delta \in \mathcal{U}_\alpha \subseteq \overline{\mathcal{U}}_\alpha$, it follows that $\gamma(\delta) \geq \alpha$. Suppose that there exists an $\alpha \in (0,1]$ such that $\gamma(\delta) > \alpha$. Since $\delta \in \mathcal{U}_\alpha$, for any $\varepsilon > 0$ and for all $n$ we have $\delta - 2^{-n}\varepsilon \in H$ whereas $\delta - 2^{-n}\varepsilon \notin \mathcal{U}_\alpha$, so $\delta - 2^{-n}\varepsilon \in \mathcal{U}_\alpha$, i.e. $\delta - 2^{-n}\varepsilon \in L_\alpha$. However, $\lim_{n \to \infty} (\delta - 2^{-n}\varepsilon) \notin \mathcal{U}_\alpha$ since $\mathcal{U}_\alpha$ is not closed. Hence, by Lemma 2.1, $\mathcal{U}$ is not continuous. This is a contradiction and so $\gamma(\delta) = \alpha$ as claimed.

This completes the proof. $\Box$
Since the key vector $\delta$ exists for any CSF and for any $\alpha \in (0,1]$, and since $\Delta$ is symmetric about $H$, we define reliability importance as follows.

**Definition**

The *reliability importance* $R_i(\alpha)$ of component $i$ at level $\alpha \in \text{Im } \gamma - \{0\}$ for the CSF $\gamma$ is defined as

$$R_i(\alpha) = P\{\gamma(X) > \alpha | X_i > \delta\} - P\{\gamma(X) > \alpha | X_i < \delta\}$$

where $X$ is a random vector and where $\delta$ is the key element of $U_\alpha$. ($\text{Im } \gamma$ denotes the image of $\gamma$.)

**Remarks**

1. We may interpret $R_i(\alpha)$ as $P\{\gamma(X) \geq \alpha \text{ iff } X_i \geq \delta\}$.

2. Replacing $\Lambda$ and $\gamma$ by $\{0,1\}^n$ and $\phi$, a binary coherent structure function, respectively, in this definition yields the (Birnbaum) reliability importance of component $i$, and hence the above definition is a direct generalisation of reliability importance in the binary case.

3. **Boundary Behaviour**

   In this section, we derive conditions under which $\lim_{\alpha \to 0} R_i(\alpha) = 0$, i.e. under which component $i$ does not affect the state of the system when the latter is at one of the extremum of its range. The
following notation will subsequently prove useful:

\[ U(\gamma) = \{ x \in A | x \geq \gamma \} \]
\[ L(\gamma) = \{ x \in A | x < \gamma \} \]
\[ f_i(\alpha) = P(\gamma(X) \geq \alpha | X_i > \delta) \]
\[ g_i(\alpha) = P(\gamma(X) \geq \alpha | X_i < \delta) \]

where \( X \) is a random vector so that \( R_i(\alpha) = f_i(\alpha) - g_i(\alpha) \).

Lemma 3.1
Let \( \gamma \) be a continuous CSF and write \( P_\alpha = \{ \gamma_t, t \in T(\alpha) \} \). Then

\[ U_\alpha = \bigcup_{t \in T(\alpha)} U(\gamma_t) \]

Proof: See Block and Savits (1984), Theorem 2.

Proposition 3.2
For any CSF \( \gamma \),

(i) \( \lim_{\alpha \to 0} U_\alpha = A_0 \) where \( A_0 = \{ x \in A | \gamma(x) > 0 \} \)
(ii) \( \lim_{\alpha \to 1} U_\alpha = A_1 \) where \( A_1 = \{ x \in A | \gamma(x) = 1 \} \).

Proof: Since \( \gamma \) is nondecreasing, \( U_\alpha \supset U_\beta \) whenever \( \alpha < \beta \).

(i) For given \( \alpha \in (0,1) \), let \( N \) be a positive integer satisfying \( \frac{1}{N} \leq \alpha \).

Further, let \( \alpha > \alpha_1 > \alpha_2 > \cdots > 0 \) be a refinement of \([0,\alpha)\) where \( \alpha_m = 1/(N+m) \). Then the sequence \( (U_{1/(N+m)})_{m=1}^{\infty} \) is increasing with limit
\[
\lim_{\alpha \to 0} U_\alpha = \lim_{m \to \infty} U_1(N+m) = \bigcup_{m=1}^\infty U_1(N+m). \]
We show that \( \bigcup_{m=1}^\infty U_1(N+m) = A_0. \)

Let \( x \in \bigcup_{m=1}^\infty U_1(N+m); \) then \( x \in U_1(N+m) \) for some \( m \) so that \( \gamma(x) > 1/(N+m) > 0, \)
i.e. \( x \in A_0, \) and hence \( \bigcup_{m=1}^\infty U_1/(N+m) \subseteq A_0. \) Conversely, let \( x \in A_0. \) Then
\( \gamma(x) = \beta \) for some \( \beta > 0 \) and there exists an integer \( N' \) such that
\( \frac{1}{N'} < \beta \) and an integer \( m \) such that \( N+m > N' \) so \( \gamma(x) = \beta > \frac{1}{N'} > 1/(N+m) \)
and \( x \in U_1/(N+m), \) hence \( A_0 = \bigcup_{m=1}^\infty U_1/(N+m). \)

(ii) The proof is similar. \( \Box \)

Theorem 3.3

Suppose that \( \gamma \) is a continuous CSF and that \( X_1, \ldots, X_n \) are independent, absolutely continuous random variables.

(i) If for all \( x \in P_1, y_j = 1 \) for some \( j \neq i, \) then \( \lim_{\alpha \to 1} R_i(\alpha) = 0. \)

(ii) If for all \( x \in K_0, x_j = 0 \) for some \( j \neq i, \) then \( \lim_{\alpha \to 0} R_i(\alpha) = 0. \)

Proof: Since \( \gamma \) is nondecreasing, for any \( \alpha \in (0,1] \)
\( P\{X \in U | X_i = 0\} < f_i(\alpha) < P\{X \in U | X_i = 1\} \)
and
\( P\{X \in U | X_i = 0\} < g_i(\alpha) < P\{X \in U | X_i = 1\}. \)

Thus, if we show that \( \lim_{\alpha \to 1} P\{X \in U | X_i = 1\} = 0 \) under the hypothesis of (i),
then \( \lim_{\alpha \to 1} f_i(\alpha) = \lim_{\alpha \to 1} g_i(\alpha) = 0 \) so that \( \lim_{\alpha \to 1} R_i(\alpha) = 0. \) Further, if we show
that \( \lim_{\alpha \to 0} P(X \in U | X_i = 0) = 1 \) under the hypothesis of (ii), then

\[
\lim_{\alpha \to 0} f_i(\alpha) = \lim_{\alpha \to 0} g_i(\alpha) = 1 \quad \text{so that} \quad \lim_{\alpha \to 0} R_i(\alpha) = 0.
\]

(i) Since, by Proposition 3.2, \( \lim \alpha = A_1 = (\exists \in \Delta \gamma(\gamma) = 1) \), it follows from the continuity of probability measures that

\[
\lim_{\alpha \to 1} P(X \in U | X_i = 1) = 0.
\]

\( P(\exists \in A_1 | X_i = 1) \). We show that \( P(\exists \in A_1 | X_i = 1) = 0 \).

Define \( P_i^j = \{ y \in P_1 | y_j = 1, j \neq i \} \) and write \( P_1^j = \{ y \in \gamma, \tau \in T(j) \} \); by hypothesis, the \( P_i^j \)'s \( j \neq i \) form a partition of \( P_1 \). Define

\[
A_j = \bigcup_{\tau \in T(j)} U(y_\tau), j \neq i.
\]

Then, by Lemma 3.1, \( A_1 = U_1 = \bigcup_{j \neq i} A_j \), so

\[
P(\exists \in A_1 | X_i = 1) = P(\exists \in \bigcup_{j \neq i} A_j | X_i = 1).
\]

By the inclusion-exclusion principle,

\[
P(\exists \in A_1 | X_i = 1) = \sum_{k=1}^{n-1} (-1)^{k-1} \eta_k
\]

where \( \eta_k = \sum_{1 < k_1 < k_2 < \cdots < k_j < n-1} P(X \in A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_j} | X_i = 1) \).

We show that \( \eta_1 = 0 \).

By definition, \( \eta_1 = \sum_{j \neq i} P(X \in A_j | X_i = 1) \).

Let \( z_q = \inf_{\tau \in T(j)} y_\tau, q \neq j, q = 1, 2, \ldots, n \) and \( \xi_j = \min(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \),

and let \( Q_j = [\xi_j, 1] \times \cdots \times [\xi_j, 1] \times \{ 1 \} \times [\xi_j, 1] \times \cdots \times [\xi_j, 1] \) where the subscript \( j \) on \{1\} indicates that this is the \( j \)th term in the product. We claim
that $A_j \subseteq Q_j$. Let $x \in A_j$. Then $x \in U(\chi_\tau)$ for some $\tau \in T(j)$, and hence $x \geq \chi_\tau$ and $y_j = 1$, so that $x_j = 1$ and $x_q \geq z_q$ for $q = 1, 2, \ldots, n$, $q \neq j$. Thus $x_j = 1$ and $x_q \geq \xi_j$ for $q = 1, 2, \ldots, n$, $q \neq j$, from which it follows that $x \in Q_j$. This holds for all $x \in A_j$, so $A_j \subseteq Q_j$. Hence

$$
\pi_1 = \sum_{j \neq i} P(X \in A_j \mid X_i = 1) \\
< \sum_{j \neq i} P(X \in Q_j \mid X_i = 1) \\
= \sum_{j \neq i} P((1, x) \in Q_j) \\
= \sum_{j \neq i} P(X_1 \geq \xi_j, \ldots, X_{j-1} \geq \xi_j, X_j = 1, X_{j+1} \geq \xi_j, \ldots, X_n \geq \xi_j) \\
= \sum_{j \neq i} \prod_{k \neq j} P(X_k \geq \xi_j) P(X_j = 1) \text{ by independence} \\
= 0 \text{ since each } X_j \text{ is absolutely continuous,}
$$

so $\pi_1 = 0$ as claimed.

Since, for any $i : 2$, $\pi_2 < \pi_1 = 0$, we see that $P(X \in A_1 \mid X_i = 1) = 0$ as claimed.

(ii) The proof is similar. \Box
4. A CONDITION FOR POSITIVE RELIABILITY IMPORTANCE

In this section, we derive a condition under which the reliability importance \( R_i(\alpha) \) is positive for \( \alpha \in (0,1) \).

**Lemma 4.1**

Let \( \nu \) be Lebesgue measure on \( \mathbb{R}^n \). Then

(i) \( \nu\{U(y_i)\} = 0 \) if and only if \( y_i = 1 \) for some \( i = 1, 2, \ldots, n \)

(ii) \( \nu\{\bigcup_{t \in T} U(x_t)\} = 0 \) if and only if \( \nu\{U(x_t)\} = 0 \) for all \( t \in T \)

where \( T \) is an index set.

**Proof:**

(i) This is trivial.

(ii) Suppose that \( \nu\{\bigcup_{t \in T} U(y_t)\} = 0 \) and that, conversely, there exists some \( t' \in T \) such that \( \nu\{U(x_{t'})\} \neq 0 \). Then \( \nu\{\bigcup_{t \in T} U(y_t)\} \geq \nu\{U(x_{t'})\} = 0 \), a contradiction.

Suppose, now, that \( \nu\{U(y_t)\} = 0 \) for all \( t \in T \). Let \( \Lambda'' = \Lambda - (0,1)^n \); clearly \( \nu(\Lambda'') = 0 \). Let \( \bar{x} \in \bigcup_{t \in T} U(y_t) \); then \( \bar{x} \in U(y_t) \) for some \( t \in T \).

Since \( \nu\{U(y_t)\} = 0 \) for all \( t \in T \), it follows from (i) that \( y_i = 1 \) for some \( i = 1, 2, \ldots, n \). Thus, by definition of \( U(y_t) \), \( \bar{x} \geq y_t \) implies \( x_i = 1 \), i.e. \( \bar{x} \in \Delta'' \). This holds for all \( \bar{x} \in \bigcup_{t \in T} U(y_t) \), so \( \bigcup_{t \in T} U(y_t) \subset \Lambda'' \) and hence

\( \nu\{\bigcup_{t \in T} U(y_t)\} = 0. \)
A similar argument shows that $\nu(\bigcup_{t \in T} L(y_t)) = 0$ if and only if $\nu(L(y_t)) = 0$ for all $t \in T$. 

Theorem 4.2

The distribution function $F$ is absolutely continuous if and only if $\mu << \nu$ where $\nu$ is Lebesgue measure and where $\mu$ is the induced Lebesgue-Stieltjes measure satisfying $\mu((-\infty, x]) = F(x)$ for all $x \in \mathbb{R}$.

Proof: See Billingsley (1979, p. 367).

Proposition 4.3

Let $\gamma$ be a continuous CSF. If $\nu(U_\alpha) > 0$ for $\alpha \in (0,1)$, then $\delta \in (0,1)$ where $\delta$ is the key element of $U_\alpha$.

Proof: We show that $\delta \notin (0,1)$. Suppose that $\delta = 0$ for $\alpha \in (0,1)$. Since $\gamma$ is continuous, $\tilde{\omega} = \emptyset \in \emptyset U_\alpha \subset U_\alpha$, and so $\gamma(\tilde{\omega}) > \alpha > 0$, a contradiction to the definition of $\gamma$.

Suppose, now, that $\delta = 1$ for $\alpha \in (0,1)$ and let $\Delta" = \Delta - (0,1)^n$. We show that $U_\alpha \subset \Delta"$. It is sufficient to show that for all $\bar{x} \in U_\alpha$, $x_i = 1$ for some $i = 1, 2, \ldots, n$. Suppose, conversely, that there exists a vector $\bar{x} \in U_\alpha$ such that $x_i < 1$ for all $i = 1, 2, \ldots, n$. Then $\bar{z} = \max(x_1, \ldots, x_n) < 1$ and $\bar{z} \in H \cap U_\alpha$, in contradiction to the assumption that $\tilde{\omega} = \emptyset \in \emptyset H \cap U_\alpha$. Hence, for all $\bar{x} \in U_\alpha$, $x_i = 1$ for some $i = 1, 2, \ldots, n$ so that $U_\alpha \subset \Delta$". Since $\nu(\Delta") = 0$, we see that $\nu(U_\alpha) = 0$, a contradiction to the given hypothesis. 

We introduce the following notation for future reference. Let

\[ D_i = D_i(\delta) = [0,1] \times \cdots \times [0,1] \times [0,1] \times [0,1] \times \cdots \times [0,1] \]

\[ E_i = E_i(\delta) = [0,1] \times \cdots \times [0,1] \times [0,1] \times [0,1] \times \cdots \times [0,1] \]

where the subscript \( i \) labels the \( i \)th term in the product.

**Theorem 4.4**

Let \( \gamma \) be a continuous CSF such that \( \nu(U_{\alpha}) > 0 \) for all \( \alpha \in (0,1) \)
where \( \nu \) is Lebesgue measure on \( \mathbb{R}^n \) and suppose that \( X_1, \ldots, X_n \) are independent, absolutely continuous random variables. Then \( R_i(\alpha) = 0 \)
for \( \alpha \in (0,1) \) if and only if \( y_i = 0 \) for every \( \gamma \in \mathcal{P}_\alpha \) for which \( \nu(U(\gamma)) > 0 \).

**Proof:** Define the induced Lebesgue-Stieltjes measure \( \tilde{P}_\gamma = P \circ \chi^{-1} \).
Observe that, since, from Proposition 4.3, the key element \( \delta \in (0,1) \),
and since \( X_i \) is absolutely continuous, it follows from Theorem 4.2 that
\( P\{X_i \geq \delta \} > 0 \) and \( P\{X_i < \delta \} > 0 \). Write \( \mathcal{P}_\alpha = \{\chi_t, t \in T(\alpha)\} \). Then, from
Lemma 3.1, \( U_\alpha = \bigcup_{t \in T(\alpha)} U(\chi_t) \); this is clearly a Borel set and so we can
write

\[ f_i(\alpha) = P_{\chi}(\bigcup_{t \in T(\alpha)} U(\chi_t) \cap D_i)/P\{X_i \geq \delta\} \]

\[ g_i(\alpha) = P_{\chi}(\bigcup_{t \in T(\alpha)} U(\chi_t) \cap E_i)/P\{X_i < \delta\} \]

"If" Define \( P_{\alpha 1} = \{\gamma \in \mathcal{P}_\alpha \mid y_i \neq 1 \text{ for all } i = 1,2,\ldots,n\} \),
\( P_{\alpha 2} = \{\gamma \in \mathcal{P}_\alpha \mid y_i = 1 \text{ for some } i = 1,2,\ldots,n\} \)
and write $P_{a_1} = \{\chi_{\alpha}, \alpha \in L(\alpha)\}$ and $P_{a_2} = \{\chi_{\alpha}, \alpha \in S(\alpha)\}$ for suitable index sets $L(\alpha)$ and $S(\alpha)$. Then, from Lemma 4.1(i), $\forall \{U(\chi_{\alpha})\} > 0$ for all $\alpha \in L(\alpha)$ and $\forall \{U(\chi_{\alpha})\} = 0$ for all $\alpha \in S(\alpha)$, so, from Lemma 4.1(ii),

\[(4.1) \quad \forall \{ \bigcup_{s \in S(\alpha)} U(\chi_s) \} = 0.
\]

Now

\[
P_{\alpha} \left( \bigcup_{\alpha \in L(\alpha)} U(\chi_{\alpha}) \cap D_i \right)
\]

\[
= P_{\alpha} \left( \bigcup_{\alpha \in L(\alpha)} U(\chi_{\alpha}) \right) \cup \bigcup_{s \in S(\alpha)} U(\chi_s) \cap D_i \right)
\]

\[
= P_{\alpha} \left( \bigcup_{\alpha \in L(\alpha)} U(\chi_{\alpha}) \cap D_i \right) + P_{\alpha} \left( \bigcup_{s \in S(\alpha)} U(\chi_s) \cap D_i \right) - P_{\alpha} \left( \bigcup_{\alpha \in L(\alpha)} U(\chi_{\alpha}) \cap D_i \right).
\]

Consider the second term in this sum; clearly

\[
P_{\alpha} \left( \bigcup_{s \in S(\alpha)} U(\chi_s) \cap D_i \right) \leq P_{\alpha} \left( \bigcup_{s \in S(\alpha)} U(\chi_s) \right) = 0
\]

from (4.1) and Theorem 4.2. Similarly, the third term vanishes, and hence

\[
P_{\alpha} \left( \bigcup_{\alpha \in L(\alpha)} U(\chi_{\alpha}) \cap D_i \right) = P_{\alpha} \left( \bigcup_{\alpha \in L(\alpha)} U(\chi_{\alpha}) \cap D_i \right).
\]

Since, by hypothesis, $y_i = 0$ for all $\chi \in P_{a_1}$, $U(\chi)$ must be of the form

\[
[y_1, 1] \times \cdots \times [y_{i-1}, 1] \times [0, 1] \times [y_{i+1}, 1] \times \cdots \times [y_n, 1]
\]
and so

\[
P_X\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} U(x_{\alpha}) \cap D_1 \} = P\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} \{ x_j \geq y_\alpha \} \cap \{ x_i \geq \delta \} \}
\]

= \( P\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} \{ x_i \geq y_\alpha \} \} P\{ x_i \geq \delta \} \) by independence.

Thus,

\[
(4.2) \quad f_i(\alpha) = P\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} \{ x_j \geq y_\alpha \} \}.
\]

By a similar argument,

\[
P_X\{ \bigcup_{\alpha \in \mathcal{T}(\alpha)} U(x_{\alpha}) \cap E_1 \} = P\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} U(y_{\alpha}) \cup \bigcup_{s \in S(\alpha)} U(y_s) \cap E_1 \}
\]

= \( P_X\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} U(y_{\alpha}) \cap E_1 \} \)

= \( P_X\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} \{ x_j \geq y_\alpha \} \cap \{ x_i < \delta \} \} \)

= \( P_X\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} \{ x_j \geq y_\alpha \} \} P\{ x_i < \delta \} \).

Thus,

\[
(4.3) \quad g_i(\alpha) = P\{ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} \{ x_j \geq y_\alpha \} \}.
\]
From (4.2) and (4.3), we see that \( R_i(\alpha) = 0 \) as claimed.

"Only if" Since, by hypothesis, \( \nu(U_{\alpha}) > 0 \) and since, by Lemma 3.1,
\[
U_{\alpha} = \bigcup_{t \in T(\alpha)} U(y_t),
\]
we see that \( \nu(\bigcup_{t \in T(\alpha)} U(y_t)) > 0 \). Thus, by Lemma 4.1 (ii),
there exists some \( t' \in T(\alpha) \) such that \( \nu(U(y_{t'})) > 0 \). Write
\[
P_{\alpha\alpha} = \{ y \in P_{\alpha} \mid \nu(U(y)) > 0 \} = \{ y_{t'}, t' \in T'(\alpha) \}
\]
\[
P_{\alpha\beta} = \{ y \in P_{\alpha} \mid \nu(U(y)) = 0 \} = \{ y_w, w \in \mathcal{W}(\alpha) \}, \text{ say.}
\]

Then \( P_{\alpha\alpha} \) and \( P_{\alpha\beta} \) form a partition of \( P_{\alpha} \). It is sufficient to show that
if \( R_i(\alpha) = 0 \), then \( y_i = 0 \) for all \( y \in P_{\alpha\alpha} \).

Suppose that there exists a vector \( y \in P_{\alpha\alpha} \) such that \( y_i \neq 0 \). Since
\( \nu(U(y)) > 0 \) for all \( y \in P_{\alpha\alpha} \), it follows from Lemma 4.1(i) that \( y_j \neq 1 \)
for all \( j = 1, 2, \ldots, n \) and so \( 0 < y_i < 1 \). Define the partition
\[
P_{\alpha\alpha} = P_{\alpha\alpha 1} \cup P_{\alpha\alpha 2} \cup P_{\alpha\alpha 3}
\]
where
\[
P_{\alpha\alpha 1} = \{ y \in P_{\alpha} \mid \nu(U(y)) > 0, y_i = 0 \} = \{ y_{\ell}, \ell \in \mathcal{L}(\alpha) \}
\]
\[
P_{\alpha\alpha 2} = \{ y \in P_{\alpha} \mid \nu(U(y)) > 0, 0 < y_i < 1 \} = \{ y_s, s \in \mathcal{S}(\alpha) \}
\]
\[
P_{\alpha\alpha 3} = \{ y \in P_{\alpha} \mid \nu(U(y)) > 0, s < y_i < 1 \} = \{ y_m, m \in \mathcal{M}(\alpha) \}, \text{ say.}
\]

Then, clearly,
\[ P(\gamma(X) \geq \alpha, X_i \geq \delta) = P_X\{U_\alpha \cap D_i\} \]
\[ = P_X\left[ \bigcup_{\alpha \in L(\alpha)} U(\gamma_k) \cup \bigcup_{s \in S(\alpha)} U(\gamma_s) \cup \bigcup_{m \in M(\alpha)} U(\gamma_m) \cap D_i \right] \]
\[ \geq P_X\left[ \bigcup_{\alpha \in L(\alpha)} U(\gamma_k) \cup \bigcup_{s \in S(\alpha)} U(\gamma_s) \cap D_i \right] \]
\[ = P\left[ \bigcup_{\alpha \in L(\alpha)} \bigcap_{j=1}^{n} \{X_j \geq \gamma_{x_j}\} \cap \{X_i \geq \delta\} \cup \bigcup_{s \in S(\alpha)} \bigcap_{j=1}^{n} \{X_j \geq \gamma_{s_j}\} \cap \{X_i \geq \delta\} \right]. \]

Since, by the definitions of \( P_{\alpha 3} \) and \( P_{\alpha 2} \), \( \{X_i \geq \gamma_{x_i}\} \cap \{X_i \geq \delta\} = \{X_i \geq \delta\} \), it follows from the independence of the \( X_i \) 's that
\[ P_X\{U_\alpha \cap D_i\} \geq P\left[ \bigcup_{\alpha \in L(\alpha)} \bigcap_{j \neq i}^{n} \{X_j \geq \gamma_{x_j}\} \cup \bigcup_{s \in S(\alpha)} \bigcap_{j \neq i}^{n} \{X_j \geq \gamma_{s_j}\} \right] P(X_i \geq \delta), \]
and hence
\[ (4.4) \quad p_i(\alpha) \geq P\left[ \bigcup_{\alpha \in L(\alpha)} \bigcap_{j \neq i}^{n} \{X_j \geq \gamma_{x_j}\} \cup \bigcup_{s \in S(\alpha)} \bigcap_{j \neq i}^{n} \{X_j \geq \gamma_{s_j}\} \right]. \]

By a similar argument,
\[ P(\gamma(X) \geq \alpha, X_i \geq \delta) = P_X\{U_\alpha \cap E_i\} \]
\[ = P_X\left[ \bigcup_{\alpha \in L(\alpha)} U(\gamma_k) \cup \bigcup_{s \in S(\alpha)} U(\gamma_s) \cup \bigcup_{m \in M(\alpha)} U(\gamma_m) \cap E_i \right]. \]

Recall that, by the definition of \( P_{\alpha 3} \), \( \delta \leq \gamma_X \) for all \( \gamma_X \in P_{\alpha 3} \), so
\[ \bigcup_{m \in M(\alpha)} U(\gamma_m) \cap E_i = \emptyset, \]
and hence
\[ (4.5) \quad P_X\{U_\alpha \cap E_i\} = P_X\left[ \bigcup_{\alpha \in L(\alpha)} \{U(\gamma_k) \cap E_i\} \cup \bigcup_{s \in S(\alpha)} \{U(\gamma_s) \cap E_i\} \right]. \]
By the definitions of $P_{a2}$ and $E_i$, $U(y_s) \cap E_i$ is of the form 

$[y_{S_1}, 1] \times \cdots \times [y_{S_{i-1}}, 1] \times [y_{S_i}, 0) \times [y_{S_{i+1}}, 1] \times \cdots \times [y_{S_n}, 1]$ for all $s \in S(\alpha)$.

Let 

$E_s = [y_{S_1}, 1] \times \cdots \times [y_{S_{i-1}}, 1] \times [0, \delta) \times [y_{S_{i+1}}, 1] \times \cdots \times [y_{S_n}, 1]$ and 

$E_s' = [y_{S_1}, 1] \times \cdots \times [y_{S_{i-1}}, 1] \times [0, y_{S_i}) \times [y_{S_{i+1}}, 1] \times \cdots \times [y_{S_n}, 1]$ 

for $s \in S(\alpha)$. Then $E_s = E_s' \cup [U(y_s) \cap E_i]$, and $\vee (E_s') > 0$ and $\vee (U(y_s) \cap E_i) > 0$.

It thus follows that 

(4.6) 

$$P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cup \bigcup_{s \in S(\alpha)} E_s)$$

$$= P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cup \bigcup_{s \in S(\alpha)} E_s' \cup \bigcup_{s \in S(\alpha)} U(y_s) \cap E_i)]$$

$$= P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cup \bigcup_{s \in S(\alpha)} U(y_s) \cap E_i)] + P_x(\lambda \bigcup_{s \in S(\alpha)} E_s')$$

$$- P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cup \bigcup_{s \in S(\alpha)} U(y_s) \cap E_i)] \cap \bigcup_{s \in S(\alpha)} E_s')$$

$$= P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cup \bigcup_{s \in S(\alpha)} U(y_s) \cap E_i)] + P_x(\lambda \bigcup_{s \in S(\alpha)} E_s')$$

$$- P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cap \bigcup_{s \in S(\alpha)} E_s') \text{ since } U(y_s) \cap E_i \cap E_s' = \emptyset$$

for all $s \in S(\alpha)$.

Claim: $P_x(\lambda \bigcup_{s \in S(\alpha)} E_s') - P_x(\lambda \{ \bigcup_{i \in L(\alpha)} U(y_{s_i}) \cap E_i \} \cap \bigcup_{s \in S(\alpha)} E_s') > 0.$
Proof of claim: We show, equivalently, that

\[ P(\{ \cup_{x \in L(\alpha)} U(y_n) \cap E_i \cap \cup_{s \in S(\alpha)} E_s^i \}) > 0. \]

Since \( \cup_{x \in L(\alpha)} U(y_n) \cap E_i \subseteq \cup_{x \in L(\alpha)} U(y_n) = U_\alpha \), it follows that

\[ \nu(\{ \cup_{x \in L(\alpha)} U(y_n) \cap E_i \cap \cup_{s \in S(\alpha)} E_s^i \}^c \cap \cup_{s \in S(\alpha)} E_s^i) \geq \nu(\cup_{s \in S(\alpha)} E_s^i) \]

for each \( s \in S(\alpha) \).

Consider the vector \( y'_s = (0_1, y_s) \in \Delta \) for \( s \in S(\alpha) \); clearly \( y'_s \in U_\alpha^c \).

Since \( U_\alpha^c \) is closed, by virtue of the continuity of \( \gamma \), \( U_\alpha^c \) is open, so there exists a vector \( z \in U_\alpha^c \) such that \( z_j > y'_j \) for \( j \neq i \) and \( 0 < z_i < y_{s_i} \).

Define \( E'' = [y'_1, z_1] \times \cdots \times [y'_n, z_n] \); clearly, \( E'' \in U(\alpha) \cap E_s^i \).

Thus

\[ \nu(\cup_{s \in S(\alpha)} E_s^i) > 0, \text{ i.e. } \nu(\{ \cup_{x \in L(\alpha)} U(y_n) \cap E_i \cap \cup_{s \in S(\alpha)} E_s^i \}^c \cap \cup_{s \in S(\alpha)} E_s^i) > 0, \text{ so that,} \]

by Theorem 4.2, \( P(\{ \cup_{x \in L(\alpha)} U(y_n) \cap E_i \cap \cup_{s \in S(\alpha)} E_s^i \}) > 0. \)

This completes the proof of the claim. \( \square \)

It now follows from (4.6) that

\[ P(\{ \cup_{x \in L(\alpha)} U(y_n) \cap E_i \cap \cup_{s \in S(\alpha)} E_s^i \}) > 0. \]
Observe that

\[ P_X[\bigcup_{\alpha \in \mathcal{L}(\alpha)} U(\chi_{\alpha}) \cap E_i \bigcup \bigcup_{s \in \mathcal{S}(\alpha)} U(\chi_{\alpha}) \cap E_s] \]

\[ = P\left[ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} (X_j \geq y_{\alpha j}) \cap (X_i \leq \delta) \right] \cup \bigcup_{s \in \mathcal{S}(\alpha)} \bigcap_{j \neq i} (X_j \geq y_{\alpha j}) \cap (X_i \leq \delta) \]

\[ = P\left[ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} (X_j \geq y_{\alpha j}) \bigcup \bigcup_{s \in \mathcal{S}(\alpha)} \bigcap_{j \neq i} (X_j \geq y_{\alpha j}) \right] P\{X_i \leq \delta\} \]

by independence, and so from (4.5)

\[ g_1(\alpha) < P\left[ \bigcup_{\alpha \in \mathcal{L}(\alpha)} \bigcap_{j \neq i} (X_j \geq y_{\alpha j}) \bigcup \bigcup_{s \in \mathcal{S}(\alpha)} \bigcap_{j \neq i} (X_j \geq y_{\alpha j}) \right]. \]

Thus, by (4.4) and (4.7), \( R_1(\alpha) > 0 \), thereby contradicting the assumption that \( R_1(\alpha) = 0 \).

This completes the proof. \( \square \).

**Corollary 4.5**

Let \( \gamma \) be a continuous CSF such that \( \nu(U_\alpha) > 0 \) for all \( \alpha \in (0,1) \) and suppose that \( X_1, \ldots, X_n \) are independent, absolutely continuous random variables. Then \( R_1(\alpha) > 0 \) for \( \alpha \in (0,1) \) if and only if \( y_i \neq 0 \) for some \( \chi \in P_\alpha \) for which \( \nu(U(y)) > 0 \).
REFERENCES


A continuum structure function is a nondecreasing mapping from the unit hypercube to the unit interval. A definition of the reliability importance, \( R_i(a) \) say, of component \( i \) at level \( a \) \((0 < a < 1)\) is proposed.

Some properties of this function are deduced, in particular conditions under which \( \lim_{a \to 0} R_i(a) = \lim_{a \to 1} R_i(a) = 0 \) and conditions under which \( R_i(a) \) is positive \((0 < a < 1)\).