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ON THE EFFECT OF TWO-STAGE SAMPLING
ON THE F STATISTIC

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ABSTRACT

We investigate the effect of intracluster correlation in two-stage samples on the ordinary F procedures in linear models. A measure is proposed as a diagnostic and basis for correction to the F statistic. A decomposition of this measure is given in terms of the contributions of the individual regressors and their cross products. For known intracluster correlation the proposed correction to F performs very well in the numerical study. For unknown intracluster correlation a simple alternative to the generalized least squares procedure is proposed and is shown to perform favorably in the simulation study.

AMS (MOS) Subject Classifications: 62J05, 62D05

Key Words: Two-stage sampling; intracluster correlation; F statistic; generalized least squares; design effect; misspecification effect.

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Users of survey data often ignore the effect of survey design on analysis. This may be due to the unavailability of information on design such as cluster labels. Another reason is that standard packages do not take into account the design effect. We study the effect of intracluster correlation in two-stage sampling on the validity of statistical procedures based on the $F$ statistic. A measure is proposed as a diagnostic and basis for correction to the $F$ statistic. The proposed correction and related modifications perform well in the simulation study. We also explain the design effect in terms of the individual variables and their cross products.
ON THE EFFECT OF TWO-STAGE SAMPLING ON THE F STATISTIC

C. F. J. Wu, D. Holt and D. Holmes

1. Introduction

The assumption of independent and identically distributed observations which underlies many statistical procedures is called into question when analyzing complex survey data. The population structure, and particularly the existence of clusters in two-stage samples which usually exhibit positive intracluster correlation, invalidates the independence assumption. The impact of this in regression analysis has been investigated in the standard sample survey theory framework by Kish and Frankel (1974) and in the linear model framework by Campbell (1979) and Scott and Holt (1982). The overall picture is that while ordinary least squares (OLS) procedures are unbiased, but not fully efficient, for estimation of the regression coefficients, serious difficulties can arise in using the OLS estimators for second order terms. Variances of the OLS estimators for the regression coefficients can be larger, sometimes much larger, than the usual OLS variance expression would indicate and estimators for the variances of coefficient estimators do not take this into account. This leads to underestimation of variances with consequences for confidence intervals.

This paper is concerned with following this impact through to the F statistic because of its central importance to hypothesis tests and confidence ellipsoids. Our

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first aim is to investigate the effect of intracluster correlation on the F statistic. We then seek modifications to the F statistic which will restore its usual properties without needing the full set of information and numerical complexity of the alternative generalized least squares (GLS) procedure. We also seek diagnostic statistics which will identify when the ordinary F statistic is likely to be affected and explore the various factors which contribute to this effect. Finally we compare our alternative procedures with GLS.

Sections 2 and 3 contain the basic framework and theoretical development and lay the groundwork for modifications to the F statistic. Section 4 considers examples of one and two covariates as special cases of the general theory. Section 5 presents numerical results for the case of two independent variables, which is the simplest allowing many of the factors to be explored. A further modification to the F statistic is proposed in Section 5 and comparisons made with the iterative GLS procedure when the intracluster correlation coefficient is unknown. The proposed modifications perform much better than the OLS procedures. They perform almost as well as the GLS procedures for large values of the intracluster correlation and better than the GLS for small values. A summary of the numerical and simulation results and relevant remarks are given in Section 6.

2. F Statistic Under a Regression Model for Two-Stage Samples

Following Campbell (1974) and Scott and Holt (1982), we utilize a regression model with an error structure which allows for intracluster correlation of the residual errors:

\[ y = \mathbf{x} \mathbf{\beta} + \epsilon, \quad (2.1) \]

where there are \( n \) observations from a two-stage sample with \( c \) clusters drawn at the first stage of sampling and \( m_x \) elements drawn from the \( i \)th sampled cluster at the second stage, \( n = \sum_{i=1}^{c} m_x \). Assume \( \epsilon \) is normal with mean zero and variance-covariance matrix \( \sigma^2 \mathbf{V} \). The sample observations are written in the natural order
with the first \( m_1 \) elements from the first cluster and so on and \( V \) is assumed to have a block diagonal form \( \sum_1^c V_k \) with

\[
V_k = \begin{bmatrix}
1 & \rho & \ldots & \rho \\
\rho & 1 & \ldots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \ldots & \rho & 1 
\end{bmatrix} \quad (2.2)
\]

If no account is taken of the variance structure, the standard OLS procedures are

\[
\hat{\beta} = (X^T X)^{-1} X^T y \\
\text{var}(\hat{\beta}) = \hat{\sigma}^2 (X^T X)^{-1}
\]

where \( \hat{\sigma}^2 = y^T (I - X(X^T X)^{-1} X^T) y / (n - k) \) and there are \( k \) explanatory variables.

The \( F \) statistic for hypothesis testing and confidence ellipsoids is

\[
F(\hat{\beta}) = \frac{\hat{\beta}^T X \hat{\beta}^2 / k}{1 y - \hat{\beta}^2 / (n - k)} \quad (2.3)
\]

If the cluster labels are known, the natural approach is to use GLS or iterative GLS (when \( \rho \) is unknown and must be estimated.) We return to this alternative in Section 5. However, the standard OLS procedures and the \( F \) statistic are often used either because the cluster labels are unavailable (particularly when the survey data is used for secondary analysis) or because users of the survey data ignore the effects of \( \rho \) on their analysis. Thus, to test \( \beta = \beta_0 \), the hypothesis is rejected at the \( \alpha \) significance level if

\[
F(\hat{\beta}_0) > F_{\alpha}(k, n - k) \quad (2.4)
\]

and the associated \( (1 - \alpha) \) confidence ellipsoid is

\[
\{ \beta : F(\beta) < F_{\alpha}(k, n - k) \} \quad (2.5)
\]

where \( F_{\alpha}(k, n - k) \) is the upper \( \alpha \) point of the \( F \) distribution with \( k \) and \( n - k \) degrees of freedom.
Under the model (2.1)-(2.2) the F statistic does not in general have an F distribution and the F procedure is invalid. The F test will not have true significance level equal to the nominal α value and the coverage property of the confidence ellipsoid will be similarly distorted. By writing \( \delta = \mathbf{V}^{-1/2} \mathbf{c} \), F can be written as

\[
F = \frac{\delta^T \mathbf{V}^{1/2} \mathbf{P} \mathbf{V}^{1/2} \delta / k}{\delta^T \mathbf{V}^{1/2} (\mathbf{I} - \mathbf{P}) \mathbf{V}^{1/2} \delta / (n - k)},
\]

(2.6)

where \( \delta = (\delta_1, \ldots, \delta_n) \) are independent \( \mathcal{N}(0,1) \) and \( \mathbf{P} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \) is the projection matrix onto the column space of \( \mathbf{X} \). Apart from the two scalar factors \( k \) and \( n - k \), the numerator and denominator in (2.6) are each separately weighted sums of independent chi-square random variables with the added complication that they are correlated. Thus, \( \delta^T \mathbf{V}^{1/2} \mathbf{P} \mathbf{V}^{1/2} \delta \) is distributed as \( \sum_{i=1}^{k} \lambda_i \xi_i \), where the \( \{\xi_i\} \) are independent \( \chi^2_1 \) and the \( \{\lambda_i\} \) are the eigenvalues of \( \mathbf{P} \mathbf{V} \). Similarly for the denominator the weights are the eigenvalues of \( (\mathbf{I} - \mathbf{P}) \mathbf{V} \).

The actual coverage probability of the ellipsoid (2.5) can be shown to be

\[
\text{Prob}\{\delta^T \mathbf{V}^{1/2} \left[ \mathbf{P} - k(n - k)^{-1} \mathbf{I}\right] \mathbf{V}^{1/2} \delta < 0\} \quad \text{(see (A.5))},
\]

which is not tractable.

We note that survey data is usually large and the denominator of \( F, \hat{\sigma}^2 \), has mean \( \hat{\sigma}^2 = n - \text{tr}(\mathbf{P} \mathbf{V})/(n - k) \), where \( \text{tr} \) is the trace of a matrix (see Scott and Holt, 1982). Since \( n \) is large and \( \text{tr}(\mathbf{P} \mathbf{V}) \) is of the order of \( k \), \( \hat{\sigma}^2 \) is nearly unbiased for \( \sigma^2 \). This suggests that the correlation between the numerator and denominator of \( F \) may be weak and its effect on the validity of \( F \) is small. This is borne out by the numerical results in Section 5.

A simple and revealing way of studying the effect of intracluster correlation on \( F \) is to approximate its numerator and \( \mathbf{P} \) denominator by constant multiples of \( \chi^2_k \) and \( \chi^2_{n-k} \). By matching the first moments the constants are \( \text{tr}(\mathbf{P} \mathbf{V})/k \) and \( [n - \text{tr}(\mathbf{P} \mathbf{V})]/(n - k) \) respectively. Because of the almost unbiasedness of the
denominator for large $n$, we focus on $\text{tr}(PV)/k$. If it is substantially different from one, the distribution of the $F$ statistic is not adequately approximated by the $F$ distribution. For example, if $n$ is large and $\text{tr}(PV)/k < F_{\alpha_1}(k,n-k)/F_{\alpha_2}(k,n-k)$, $\alpha_2 > \alpha_1$, the ordinary $F$ test with nominal level $\alpha_1$ has actual level at least $\alpha_2$. One may therefore use $\text{tr}(PV)/k$ as a measure of the effect of the intracluster correlation on the ordinary $F$ procedure.

A better approximation to the true distribution of $F$ can be obtained by approximating $\mathbf{X}^T \mathbf{X} - \mathbf{X}^T \mathbf{V} \mathbf{X}$ and $\mathbf{I} - \mathbf{X}^T \mathbf{X}$ in $F$ by $c_1 \chi^2_1$ and $c_2 \chi^2_2$, where $c_1$, $\mu_1$, $c_2$, $\mu_2$ are determined by matching the first two moments (Satterthwaite, 1946). However, $c_1$ and $\mu_1$ depend on $\text{tr}(PV)^2$ (similarly $c_2$ and $\mu_2$ depend on $\text{tr}(V - PV)^2$), which is not as readily available as $\text{tr}(PV)$. Moreover, the additional gain in accuracy by this more refined approximation is small. Therefore it is not further pursued.

If $\rho = 0$ then $V = I$ and $\text{tr}(PV)/k = 1$. In general the true covariance matrix of the OLS estimators for $\hat{\beta}$ is given by $C = \sigma^2 (X'X)^{-1} D$, where $D = (X'VX)(X'X)^{-1}$ has been termed the misspecification effect (Scott and Holt, 1982). If $X'X$ or $D$ were diagonal, $k^{-1} \text{tr}(PV) = k^{-1} \text{tr}(D) = k^{-1} \sum \{\text{var}(\hat{\beta}_j)/\text{var}(\hat{\beta}_j|\rho = 0)\}$ would represent the average inflation (due to nonzero $\rho$) in variance for the OLS estimators. More generally $k^{-1} \text{tr}(PV)$ often captures the main components of the variance inflation and may be termed the 'approximate misspecification effect'.

The term $k^{-1} \text{tr}(PV)$ suggests a simple adjustment to $F$, whose properties are discussed in the next section.

3. A Modified $F$ Statistic

The foregoing discussion suggests the following simple modification to the $F$ statistic
\[ F' = \frac{\|\hat{y} - \hat{x}_B\|^2 / \text{tr}(PV)}{\|y - x_B\|^2 / (n - \text{tr}(PV))} = F \frac{k[n - \text{tr}(PV)]}{\text{tr}(PV)(n - k)}. \]  

We call (2.4) or (2.5) with \( F \) replaced by \( F' \) a modified \( F \) procedure. Here we assume \( \rho \) is known. Unknown \( \rho \) will be considered in Section 5. It will be shown later that the modified \( F \) procedure (3.1) is almost exact in most situations in the simulation study.

For testing a subhypothesis or setting confidence regions for some linear combinations of the \( B_j \)'s, let the parameters of interest be \( A_B = (a_B^T, \ldots, a_B^T)^T \), where \( A \) is a \( q \times k \) matrix of rank \( q < k \). The ordinary \( F \) procedure (Draper and Smith, 1981) for \( A_B \) is based on the \( F_A \) statistic and the critical value \( F_a(q, n - k) \).

\[
F_A = \frac{(A_B - \hat{A}_B)^T (X_A^T X_A)^{-1} (A_B - \hat{A}_B) / q}{\|y - x_B\|^2 / (n - k)}, \quad X_A = X(x^T x)^{-1} x^T. \tag{3.2}
\]

Note that the denominator of \( F_A \) is the same as that of \( F \). By approximating the numerator of \( F_A \) by a constant multiple of \( X^2 \) with the same first moment, a simple adjustment to \( F_A \) is given by

\[
F'_A = \frac{(A_B - \hat{A}_B)^T (X_A^T X_A)^{-1} (A_B - \hat{A}_B) / \text{tr}(P_A V)}{\|y - x_B\|^2 / (n - \text{tr}(PV))} = F_a \frac{q[n - \text{tr}(PV)]}{\text{tr}(P_A V)(n - k)}, \tag{3.3}
\]

where \( P_A = X_A (X_A^T X_A)^{-1} X_A^T \) is the projection matrix onto the column space of \( X_A \), which is of dimension \( q \). The simulation study in Section 5 shows that the modified \( F \) procedure (3.3) is almost exact in most situations considered there.

Since for large \( n \) the denominator of \( F'_A \) is nearly unbiased for \( \sigma^2 \), the difference between \( F'_A \) and \( F_A \) is primarily due to the difference between \( \text{tr}(P_A V) \) and \( q \). One may use \( \text{tr}(P_A V)/q \) as a measure of the effect of intracluster
correlation on the \( F_A \) procedure. A value larger than \( F_{q_1}(q, n - k)/F_{q_2}(q, n - k) \), \( a_2 > a_1 \), indicates that the ordinary \( F_A \) test with nominal level \( a_1 \) has actual level at least \( a_2 \).

The special case \( q = 1 \) deserves further attention. By writing \( A \) as a \( 1 \times k \) vector \( a^T \) and \( X_A \) as a \( n \times 1 \) vector \( v \), \( P_A = vv^T/\|v\|^2 \) and

\[
\text{tr}(P_A V) = \frac{v^T v}{\|v\|^2} = \frac{a^T (X^T X)^{-1} X^T VX (X^T X)^{-1} a}{a^T (X^T X)^{-1} a}
\]

(3.4)

is equal to

\[
\text{var}(a^T \beta)/\text{var}(a^T \beta | \rho = 0).
\]

Scott and Holt (1982) call this the misspecification effect (meff) for estimating \( a^T \beta \). Since \( q = 1 \), the numerator of \( F_A \) has a \( \chi^2 \) distribution. This explains our later empirical finding that the modified \( F \) procedure (3.3) works extremely well in the one-parameter case.

The applicability of the proposed modified \( F \) procedures depends very much on the accessibility of the values \( \text{tr}(PV) \) and \( \text{tr}(P_A V) \). Let us consider a special situation where the \( k \) column vectors of \( X \), denoted by \( x_1, \ldots, x_k \), are orthogonal to each other. Then the projection matrix

\[
P = \sum_{j=1}^{k} \frac{x_j x_j^T / \|x_j\|^2}{1}
\]

and

\[
\text{tr}(PV) = \sum_{j=1}^{k} \frac{x_j^T v x_j / x_j^T x_j}{1} = \sum_{j=1}^{k} \frac{\text{var}(\beta_j) / \text{var}(\beta_j | \rho = 0)}{1}
\]

is the sum of misspecification effects for estimating the \( k \) orthogonal parameters \( \beta_1, \ldots, \beta_k \). Similarly for testing or estimating \( q \) parameters out of \( \beta_1, \ldots, \beta_k \), \( q < k \), \( \text{tr}(P_A V) \) is the sum of the corresponding \( q \) meffs. The proposed procedures
require only \( k \) meffs, which may be provided by the sampler, no matter how large \( n \) or \( c \) is. Note that the orthogonality condition on \( X \) is satisfied by balanced ANOVA and simple linear regression. The latter will be studied in more detail in Section 4.1. A simple model not satisfying the orthogonality condition will be considered in Section 4.2.

4. Examples

Here we allow the intracluster correlation coefficients \( \rho_k \) in cluster \( k \) to be possibly unequal.

4.1. Simple linear regression

The \( i \)th observation in the \( k \)th cluster can be expressed as

\[
y_{ki} = \alpha + \beta(x_{ki} - \bar{x}_k) + \epsilon_{ki},
\]

where \( \bar{x}_k \) is the mean of \( x_{ki} \) in the sample and \( \epsilon_{ki} \) satisfies the conditions (2.1)-(2.2). The correction factor \( \text{tr}(PV)/2 \) in \( F' \), (3.1), can be computed as indicated above since the vector \( 1 = (1, \ldots, 1)^T \) and \( \bar{x} = (x_{ki} - \bar{x}_k), k, i \) are orthogonal. That is,

\[
\frac{1}{2} \text{tr}(PV) = \frac{1}{2} \left( \frac{1^T V 1}{1^T 1} + \frac{\bar{x}^T V \bar{x}}{\bar{x}^T \bar{x}} \right) = \frac{1}{2} (D_\alpha + D_\beta)
\]

(4.2)

is an average of the overall meffs \( D_\alpha \) and \( D_\beta \) for estimating \( \alpha \) and \( \beta \). It can be shown that

\[
D_\alpha = \frac{1}{n} \sum_{k=1}^c \left( m_k + m_k(m_k - 1)\rho_k \right)
\]

\[
= \frac{c}{n} \sum_{k=1}^c (1 + (m_k - 1)\rho_k)
\]

(4.3)

is a weighted average of the meffs.
for estimating $a$ in cluster $\ell$, and by writing $T_{x,\ell} = \frac{1}{L} \sum_{i=1}^{L} (x_{i\ell} - \bar{x}_{\ast \ast})^2$ and

$$T_x = \frac{1}{L} \sum_{\ell=1}^{L} T_{x,\ell},$$

$$D_{\alpha,\ell} = 1 + (m_{\ell} - 1) \rho_{\ell} \rho_{x},$$

for estimating $\beta$ in cluster $\ell$, where

$$\rho_{\ell,x} = \frac{1}{m_{\ell} - 1} \frac{m_{\ell}(\bar{x}_{\ell \ast} - \bar{x}_{\ast \ast})^2}{T_{x,\ell} - \rho_{x} \rho_{\ell} \bar{x}_{\ell \ast} - \bar{x}_{\ast \ast})^2}$$

is a weighted average of the meff

$$D_{\beta,\ell} = 1 + (m_{\ell} - 1) \rho_{\ell} \rho_{x},$$

for estimating $\beta$ in cluster $\ell$, where

$$\rho_{\ell,x} = \frac{1}{m_{\ell} - 1} \frac{m_{\ell}(\bar{x}_{\ell \ast} - \bar{x}_{\ast \ast})^2}{T_{x,\ell}}$$

can be regarded as a sample analog to the intracluster correlation of $x$ in cluster $\ell$. The derivations of (4.3)-(4.4) become straightforward once the component $V_k$ of $V$ is rewritten as $(1 - \rho_{\ell})I_k + \rho_{\ell}J_k$, where $I_k$ is the identity matrix and $J_k$ the matrix of one's, both of order $m_{\ell}$. In the special case $m_{\ell} = m$ and $\rho_{\ell} = \rho$, $D_{\alpha} = 1 + (m - 1) \rho$ and $D_{\beta} = 1 + (m - 1) \rho \rho_{x}$, where

$$\rho_{x} = \frac{1}{m - 1} \frac{m_{\ell}(\bar{x}_{\ell \ast} - \bar{x}_{\ast \ast})^2}{T_x - \rho_x (\bar{x}_{\ell \ast} - \bar{x}_{\ast \ast})^2}$$

can be regarded as a sample analog to the overall intracluster correlation of $x$.

Since this is a relatively easy problem, we are able to do a more refined analysis. As shown in Section 2, the numerator of $F$, apart from $\sigma^2$, is distributed as $\lambda_1^2 X_1^2 + \lambda_2^2 X_1^2$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $PV$ and can
be determined by

\[ \lambda_1 + \lambda_2 = \text{tr}(PH) = D_a + D_b \]

\[ \lambda_1^2 + \lambda_2^2 = \text{tr}(PH)^2 = D_a^2 + D_b^2 + 2(\lambda_1^T V_X) / (nT_x) \tag{4.5} \]

For any fixed \( \lambda_1 + \lambda_2 \), which is independent of \( \lambda_1^T V_X \), it follows from (4.5) that \( \lambda_1 \) and \( \lambda_2 \) would be wider apart if \( \lambda_1^T V_X \) were not zero. One implication is that the approximation to \( \lambda_1^2 + \lambda_2^2 \) by \( (\lambda_1 + \lambda_2)^2 \) is less accurate for nonzero \( \lambda_1^T V_X \) and fixed \( \lambda_1 + \lambda_2 \), and therefore the effect of intracluster correlation on \( F \) is more pronounced. Note that

\[ \lambda_1^T V_X = \Sigma L_k (1 + (m_k - 1)\rho_k)(x_k^* - x^*) = 0 \]

if \((m_k - 1)\rho_k\) is constant. We conjecture that, for fixed \( D_a + D_b \) the meff on \( F \) is smaller if \( m_k \) and \( \rho_k \) are both constant.

Since \( \lambda \) and \( \chi \) are orthogonal, it is easy to see that the correction factor \( \text{tr}(P_A V) \) in \( F_A \) for testing \( \alpha \) and for testing \( \beta \) is respectively \( D_a \) and \( D_b \). Consider now the problem of testing a more general parameter \( c_1^\alpha + c_2^\beta \), which can be handled by the formula (3.4). The vector \( \nu \) in (3.4) is \( n^{-1}(1 + T_x^2 c_2^\chi) \) and

\[ \text{tr}(PH) = \left( \frac{c_1^2}{n} D_a + \frac{c_2^2}{T_x} D_b + \frac{2c_1 c_2}{nT_x} \lambda_1^T V_X \right) / (\frac{c_1^2}{n} + \frac{c_2^2}{T_x}) , \]

which is not a weighted average of \( D_a \) and \( D_b \) (unless \( c_1, c_2 \) or \( \lambda_1^T V_X = 0 \)) as one might expect.

4.2. Regression with two covariate variables

This is the simplest example in which the column vectors of \( X \) are not orthogonal. The response value \( y_{k_1} \) is related to the two covariates \( x_{k_1} \) and \( z_{k_1} \) by

\[ y_{k_1} = \alpha + \beta(x_{k_1} - \bar{x}) + \gamma(z_{k_1} - \bar{z}) + \epsilon_{k_1} \tag{4.6} \]

where \( \epsilon_{k_1} \) satisfies (2.1)-(2.2). It is proved in Appendix A that
\[ \text{tr}(\text{PV}) = D_a + \frac{1}{1 - r^2} (D_\beta + D_\gamma) - \frac{2r^2}{1 - r^2} D_{\beta, \gamma}, \]  
(4.7)

where \( D_a \) and \( D_\beta \) are the meffs for estimating \( \alpha \) and \( \beta \), given by (4.3) and (4.4), \( D_\gamma \) is the meff for estimating \( \gamma \) and is analogous to \( D_\beta \) with \( x_{ki} \) in (4.4) replaced by \( z_{ki} \),

\[ r = \frac{T}{\sum_i x_i z_i} \]  
is the correlation coefficient between \( x \) and \( z \) = \( (z_{ki} - \bar{z}_{..})(x_{ki} - \bar{x}_{..}) \), and

\[ D_{\beta, \gamma} = \frac{\sum_i x_i z_i}{T} = \frac{\sum_i T_{xz,i}}{T} (1 + (m - 1) \rho_{xz,i}) \]  
(4.8)

is a weighted average of \( D_{\beta, \gamma,i} = 1 + (m_i - 1) \rho_{xz,i} \) with \( T_{xz,i} = \sum_x T_{xz,i} \),

\[ T_{xz,i} = \sum_j (x_{ki} - \bar{x}_{..})(z_{ki} - \bar{z}_{..}), \]  

and

\[ \rho_{xz,i} = \frac{1}{m_i - 1} \left( \frac{m_i(x_{ki} - \bar{x}_{..})(z_{ki} - \bar{z}_{..})}{T_{xz,i}} - 1 \right). \]

Unlike \( D_\beta \) and \( D_\gamma \), the weight in (4.8) may be negative. For \( m_i = m \), \( \rho_{xz,i} = \rho \),

\[ D_{\beta, \gamma} = 1 + (m - 1) \rho \]  
where

\[ \rho_{xz} = \frac{1}{m - 1} \left( \frac{\sum_i m_i(x_{ki} - \bar{x}_{..})(z_{ki} - \bar{z}_{..})}{T_{xz}} - 1 \right) \]

can be regarded as a sample analog to the intracluster correlation between the \( x_{ki} \)'s and the \( z_{ki} \)'s in the population.

When \( r \) is small the third term of (4.7) is relatively small and \( \text{tr}(\text{PV}) \) is approximately the sum of the three meffs as in the orthogonal case already discussed. For example, when \( r = 0.2, 1 - r^2 = 0.96 \) and \( 2r^2 = 0.08 \) so that the contribution
of the third term is negligible since $D_{\beta,\gamma}$ is at most of the same order as $D_{\beta}$ and $D_{\gamma}$.

It should be pointed out that, although $D_\alpha$ and $D_\beta$ in the two-variable regression model are formally the same as in the simple linear regression model, they are actually different since the intracluster correlation coefficient $\rho_k$ varies with model. Typically, if the additional covariate variable is effective in explaining the variation due to the clustering variable, the $\rho_k$ in the augmented model is smaller.

For testing $\alpha$, the correction factor $\text{tr}(P_A V)$ is $D_\alpha$, and for testing $\beta$ and $\gamma$, the correction factor $\text{tr}(P_A V)$ is

$$\frac{1}{1 - r^2} \left( D_\beta + D_\gamma - 2r^2 D_{\beta,\gamma} \right), \quad (4.9)$$

the second and third terms of (4.7). This is easy to see because the projection matrix $P_A$ for the two problems are respectively the first term and the sum of the second and third terms of $P$ in (A.1). For testing $\gamma$ only, the correction factor $\text{tr}(P_A V)$ is computed as follows. To use the general formula (3.4), the vector $v = X(X^T X)^{-1}(0,0,1)^T$ in (3.4) turns out to be proportional to

$$Z - \frac{X^T \bar{z}}{\|X\|^2} X,$$

which is denoted by $\bar{y}$ in (A.1). Then

$$\text{tr}(P_A V) = \frac{\bar{y}^T \bar{y}}{\|\bar{y}\|^2},$$

which is the third term of (A.2) and from formulae (A.3)-(A.4),

$$\text{tr}(P_A V) = \frac{1}{1 - r^2} \left[ D_\gamma - 2r^2 D_{\beta,\gamma} + r^2 D_\beta \right]. \quad (4.10)$$

When $r$ is close to zero, $\text{tr}(PV)$ is approximately equal to $D_\gamma$ but is in general not equal to $D_\gamma$. 

-12-
5. Empirical Investigations

5.1. Introduction

Numerical results are presented for the case of two independent variables

\[ E(y) = \alpha + \beta x + \gamma z, \]

the simplest model which allows us to explore the impact of various factors on the \( F \) statistic. Values of \( x_{li} \) and \( z_{li} \) for the \( i^{th} \) unit in the \( \ell^{th} \) cluster have been generated from the bivariate normal distribution with additional random effects components to allow for intracluster correlation on both \( x \) and \( z \),

\[
x_{li} = \mu_x + \alpha_x l + \varepsilon_{xli}
\]

\[
z_{li} = \mu_z + \alpha_z l + \varepsilon_{zli}
\]

where \( \alpha_x l \sim N(0, \sigma_x^2), \alpha_z l \sim N(0, \sigma_z^2), \varepsilon_{xli} \sim N(0, \sigma_x^2), \varepsilon_{zli} \sim N(0, \sigma_z^2), \)

\( \rho_x = \sigma_{\alpha_x}^2 / (\sigma_{\alpha_x}^2 + \sigma_{\varepsilon_x}^2) \) and \( \rho_z = \sigma_{\alpha_z}^2 / (\sigma_{\alpha_z}^2 + \sigma_{\varepsilon_z}^2) \). The two cluster effects \( \alpha_x \) and \( \alpha_z \) are correlated with coefficient \( \rho_{xz} \) for the same cluster and similarly the two individual terms \( \varepsilon_x \) and \( \varepsilon_z \) for the same unit are correlated. Otherwise random terms are uncorrelated. For given values of the various parameters, values for \( x \) and \( z \) were generated for \( c = 10 \) clusters and \( m = 10 \) observations per cluster. In the linear model framework inference is conditional on the values of \( x \) and \( z \) and data sets were retained for use in the numerical investigations only if the specific data set generated exhibited the required structure (i.e., achieved estimates for \( \rho_x, \rho_z \), etc. were close to the desired values).

For the initial results describing the actual significance levels of the \( F \) test when \( \rho \) is known (Table 1), the results were obtained for given values of \( x \) and \( z \) without simulation by using the approximation described in Appendix B. Subsequent results on the performance of \( F' \) when \( \rho \) is unknown, on a further modification to \( F \) and on the GLS procedure for comparative purposes were obtained by computer simulation. Conditional on the values of \( x \) and \( z \), values of \( y_{li} \) were generated with the required intracluster correlation structure using random effects terms.
where \( \text{var}(a_k) = \sigma_a^2 \), \( \text{var}(\varepsilon_{kl}) = \epsilon^2 \) and 

\[
\rho = \frac{\sigma_a^2}{\sigma_a^2 + \epsilon^2}.
\]

For each simulated set of \( y \) values, each of the test procedures was carried out. For each set of \( x \) values the simulation was repeated 10,000 times to obtain estimates of the actual significance level of each test procedure. Thus, actual significance levels presented for each procedure are accurate to \( \pm 0.5\% \).

5.2. **Known \( \rho \)**

Table 1 contains the actual significance level of the nominal 5% \( F \) test for various values of \( \rho, \rho_x, \rho_z \) and \( \text{corr}(x, z) \). In this table \( \rho_{xz} = 0 \). The main points to note are as follows.

1. \( \text{tr}(P_{AV})/q \) is a good indicator of the level of distortion of the \( F \) test by intracluster correlation.
2. When \( \rho_z, \rho_x = 0 \), the \( F \) test is unaffected as we would expect.
3. Similarly when \( \rho = 0 \), there is no effect.
4. For testing \( \gamma = 0 \) the strongest effect comes, as we would expect, from \( \rho_z \neq 0 \) with \( \rho \neq 0 \).
5. Even when \( \rho_z = 0 \), there is an effect from \( \rho_x \neq 0, \rho \neq 0 \) when \( \text{corr}(x, z) \) is large. The correlation between \( x \) and \( z \) allows the intracluster correlation for the \( x \) variable to have an impact on the test for \( \gamma = 0 \) (the coefficient of the \( z \) variable). This effect is not as large as the direct effect of \( \rho_z \neq 0 \).

This point can be explained by formula (4.10).

6. The effects for testing \( \beta = \gamma = 0 \) tend to be larger than on the test for \( \gamma = 0 \). This is easily justified by comparing (4.9) and (4.10).
7. Although not shown in the table, the actual significance level for $F'$ is always 5% when testing $\gamma = 0$ since the numerator of the $F$ statistic is a simple multiple of only one $X_1^2$ and the modification is exact, restoring the properties of $F$ entirely. For testing $\beta = \gamma = 0$ the approximation used in $F'$ is not perfect but the actual significance level achieved was in the range 5% to 7% in all cases.

### TABLE 1: Actual significance level of the nominal 5% $F$ test and the value of $\text{tr}(P_AV)/q$ (in brackets) for $c = 10$ clusters and $m = 10$ observations per cluster when $\rho_{xz} = 0$.

Testing $\gamma = 0$

<table>
<thead>
<tr>
<th>$\rho_z$</th>
<th>$\rho_x$</th>
<th>corr(x,z)</th>
<th>0</th>
<th>.05</th>
<th>.1</th>
<th>.2</th>
<th>.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.1</td>
<td>5 (1.0)</td>
<td>5 (.99)</td>
<td>5 (.98)</td>
<td>5 (.96)</td>
<td>5 (.92)</td>
<td></td>
</tr>
<tr>
<td>.4</td>
<td>5 (1.0)</td>
<td>5 (.99)</td>
<td>5 (.97)</td>
<td>5 (.94)</td>
<td>4 (.88)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.7</td>
<td>5 (1.0)</td>
<td>5 (1.01)</td>
<td>5 (1.03)</td>
<td>6 (1.06)</td>
<td>7 (1.11)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| .1      | 5 (1.0) | 5 (.97)    | 5 (.95) | 4 (.89) | 3 (.79) |       |       |
| .4      | 5 (1.0) | 5 (1.01)   | 5 (1.02) | 6 (1.04) | 7 (1.08) |       |       |
| .5      | 5 (1.0) | 7 (1.13)   | 8 (1.25) | 11 (1.50) | 18 (2.00) |       |       |

| 0       | 5 (1.0) | 7 (1.21)   | 10 (1.41) | 15 (1.82) | 24 (2.65) |       |       |
| .5      | 5 (1.0) | 8 (1.24)   | 11 (1.47) | 16 (1.94) | 26 (2.88) |       |       |
| .6      | 5 (1.0) | 9 (1.32)   | 13 (1.64) | 20 (2.28) | 31 (3.56) |       |       |

| 0       | 5 (1.0) | 8 (1.21)   | 10 (1.43) | 15 (1.85) | 25 (2.71) |       |       |
| .4      | 5 (1.0) | 8 (1.23)   | 11 (1.45) | 16 (1.91) | 26 (2.81) |       |       |
| .8      | 5 (1.0) | 9 (1.37)   | 14 (1.74) | 22 (2.48) | 35 (3.97) |       |       |
TABLE 1 - cont'd.

Testing $\beta = \gamma = 0$

<table>
<thead>
<tr>
<th>$\rho_z$</th>
<th>$\rho_x$</th>
<th>corr($x,z$)</th>
<th>0</th>
<th>.05</th>
<th>.1</th>
<th>.2</th>
<th>.4</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>.1</td>
<td>5 (1.0)</td>
<td>5 (.99)</td>
<td>5 (.99)</td>
<td>5 (.97)</td>
<td>5 (.95)</td>
<td></td>
</tr>
<tr>
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<td>5 (.99)</td>
<td>5 (.98)</td>
<td>5 (.97)</td>
<td>4 (.93)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>.7</td>
<td>5 (1.0)</td>
<td>5 (1.01)</td>
<td>5 (1.01)</td>
<td>6 (1.03)</td>
<td>6 (1.05)</td>
<td></td>
</tr>
<tr>
<td>.1</td>
<td>5 (1.0)</td>
<td>6 (1.07)</td>
<td>8 (1.15)</td>
<td>11 (1.30)</td>
<td>17 (1.59)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.3</td>
<td>5 (1.0)</td>
<td>7 (1.09)</td>
<td>8 (1.18)</td>
<td>12 (1.37)</td>
<td>19 (1.74)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.5</td>
<td>5 (1.0)</td>
<td>8 (1.15)</td>
<td>10 (1.30)</td>
<td>16 (1.60)</td>
<td>26 (2.19)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>.5</td>
<td>5 (1.0)</td>
<td>7 (1.09)</td>
<td>8 (1.19)</td>
<td>12 (1.37)</td>
<td>20 (1.74)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>.6</td>
<td>5 (1.0)</td>
<td>7 (1.01)</td>
<td>9 (1.22)</td>
<td>13 (1.43)</td>
<td>22 (1.86)</td>
<td></td>
</tr>
<tr>
<td>.5</td>
<td>5 (1.0)</td>
<td>8 (1.15)</td>
<td>10 (1.30)</td>
<td>16 (1.60)</td>
<td>27 (2.20)</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>5 (1.0)</td>
<td>8 (1.19)</td>
<td>12 (1.38)</td>
<td>19 (1.77)</td>
<td>33 (2.53)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.4</td>
<td>5 (1.0)</td>
<td>8 (1.19)</td>
<td>12 (1.39)</td>
<td>19 (1.77)</td>
<td>33 (2.54)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.8</td>
<td>5 (1.0)</td>
<td>10 (1.27)</td>
<td>15 (1.54)</td>
<td>25 (2.08)</td>
<td>41 (3.17)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In practice it is unlikely that $\rho_{xz} = 0$ since for many situations one would expect that the cluster effects for $x$ and $z$ are derived from a common source or influence. In this case the cluster effects for $x$ and $z$ would be positively correlated. Figure 1 presents graphically an extension of the pattern of results in Table 1 for testing $\gamma = 0$ to include $\rho_{xz} \neq 0$.

We note the following points.

1. When corr($x,z$) = 0, the value of $\rho_{xz}$ makes no difference. This is because the only term involving $\rho_{xz}$ in (4.10) is zero if corr($x,z$) = 0.

2. In general, the effect on the F test is accentuated if there is a strong difference between $\rho_{xz}$ and corr($x,z$). High values of $|\text{corr}(x,z)|$ show particularly strong effects when associated with $\rho_{xz} = 0$. This is because the second term of (4.10) is negative for $\rho_{xz} > 0$ and will have a larger effect in
Figure 1: Actual significance level of the nominal 5% F test for $c = m = 10$ for testing $\alpha = 0$, for varying $\rho_{xz}$. 
reducing $\text{tr}(P_A V)$ when $|\text{corr}(x, z)|$ is large. When $\rho_{xz} = 0$, the second term in (4.10) is zero and there is no reduction to $\text{tr}(P_A V)$.

3. The affects described here, of the differential effect of $\rho_{xz}$ have a smaller impact than the direct effects of $\rho_z \neq 0$ and $\rho \neq 0$. This is obvious from (4.10) since $D_{\beta, Y}$, which involves $\rho_{xz}$, has coefficient $r^2 = \text{corr}^2(x, z) (< 1)$ and $D_Y$, which involves $\rho_z$ and $\rho$, has coefficient 1.

5.3. Unknown $\rho$

In general $\rho$ is unknown and must be estimated in order to modify the $F$ statistic or alternatively to use GLS. As a first approximation, which requires no iteration, we may use OLS to estimate the regression coefficients and then obtain an estimate of $\rho$ from the residuals

$$(n - k)\hat{\sigma}^2 = y'(I - P)y.$$ Let $P_B = \frac{1}{m} J'$, where $J' = (1, \ldots, 1)$ of length $m$, $P_B = \bigoplus_{l=1}^{c} P_{B_l}$ and $P_W = I - P_B$. Here $P_B$ and $P_W$ are symmetric, idempotent projection matrices and are orthogonal to each other. Now

$$(n - k - c + 1)\hat{\sigma}_c^2 = y'(I - P)P_Wy$$

and we may use $\hat{\rho} = 1 - \frac{\hat{\sigma}_c^2}{\hat{\sigma}^2}$. Typical simulation results for testing $Y = 0$ are presented in Table 2. The adjustment to $F$ using $\hat{\rho}$ is better than using the unadjusted $F$ statistic but not as good as when $\rho$ is known.

Part of the problem with using $\hat{\rho}$ is that it is biased not simply because it is a ratio estimator but also because $\hat{\sigma}_c^2$ and $\hat{\sigma}^2$ are biased.

Holt and Scott (1981) show that $\hat{\sigma}^2$ is biased since

$$(n - k)E(\hat{\sigma}^2)/\sigma^2 = \text{tr}[(I - P)V]$$

$$= n - k - \rho(m \text{tr}(P_P) = k)$$

$$= n - k.$$ (5.1)

In practice the extra term is small. For example, with $m = c = 10$, $n = 100$, $k = 3$, $\rho = 0.1$, the downward bias in $\hat{\sigma}^2$ is about 2% of $\sigma^2$. Similarly,
\[(n-c-k+1)E(\sigma^2_\varepsilon)/\sigma^2 = \text{tr}[(I - P)P_W(I - P)V]\]
\[= n - c - k + \text{tr}(PP_B) + \frac{m\rho}{1 - \rho} [\text{tr}(PP_B) - \text{tr}(PP_BPP_B)]. \quad (5.2)\]

TABLE 2: Actual significance levels for various tests, nominal 5% level; testing \(\gamma = 0\).

<table>
<thead>
<tr>
<th>(\rho_x)</th>
<th>(\rho_z)</th>
<th>(\rho_{xz})</th>
<th>corr(x,z)</th>
<th>Test</th>
<th>.01</th>
<th>.05</th>
<th>.1</th>
<th>.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>.2</td>
<td>.0</td>
<td>.69</td>
<td>F(OLS)</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F'(known (\rho))</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F'(estimated (\rho))</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>.5</td>
<td>.2</td>
<td>.2</td>
<td>.0</td>
<td>F(OLS)</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F'(known (\rho))</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F'(estimated (\rho))</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>.1</td>
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<td>F(OLS)</td>
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<td>F'(known (\rho))</td>
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<td>.5</td>
<td>.2</td>
<td>.64</td>
<td>F(OLS)</td>
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<td>9</td>
<td>14</td>
<td>29</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>F'(known (\rho))</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F'(estimated (\rho))</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

Now \(\text{tr}(PP_B) < k\) and \(\text{tr}(PP_B) - \text{tr}(PP_BPP_B) > 0\). Once again for \(n \gg m, c, k\) and small values of \(\rho\), the bias in \(\hat{\sigma}^2_\varepsilon\) is small. With the values of \(m, c, n, k\) and \(\rho\) given above, the upward bias is 2% of \(\sigma^2_\varepsilon\).

These small biases (in opposite directions) have a large impact on the bias of \(\hat{\rho}\) since for the example given

\[
\text{E}(\hat{\rho}) = \text{E}(1 - \frac{\hat{\sigma}^2_\varepsilon}{\sigma^2}) = 1 - \frac{(1.02)\sigma^2_\varepsilon}{0.98 \sigma^2} = 1.04\rho - 0.04.
\]

Thus, when \(\rho = 0.1\), the bias is almost 0.04 (i.e. about one-half the value). This suggests that small biases in estimating \(\sigma^2_\varepsilon\) and \(\sigma^2\) will have a large effect on...
Figure 2: Actual significance levels for various tests, nominal 5% level, with $\rho_X = .1$, $\rho_Z = .5$, and $m = c = 10$; testing $\alpha = 0$. 
the bias of $\hat{\beta}$. Also, if $\hat{\sigma}^2$ and $\hat{\sigma}^2_{\epsilon}$ were unbiased, then even small coefficients of variation in the estimators would result in large fluctuations in $\hat{\beta}$.

Equations (5.1) and (5.2) suggest a further adjustment to the $F$ statistic by adjusting both $\hat{\sigma}^2$ and $\hat{\sigma}^2_{\epsilon}$ to yield approximately unbiased estimates of $\sigma^2$ and $\sigma^2_{\epsilon}$ in a two-stage process.

1. Use the OLS residuals to obtain a first estimate $\hat{\rho}$ as described above.
2. Replace $\rho$ in (5.1) and (5.2) by $\hat{\rho}$ to obtain approximately unbiased estimates of $\sigma^2$ and $\sigma^2_{\epsilon}$ and hence a new estimate $\hat{\rho}$ of $\rho$.
3. Use this estimate, $\hat{\rho}$, in adjusting the $F$ statistic as described for $F'$. We call this statistic $F''$.

This procedure is a two-stage process (although it could be iterated) and is approaching the complexity of the iterative GLS procedure. Computer simulations were carried out to estimate the actual significance level of this procedure under various conditions and also to compare with the iterative GLS procedure. Figure 2 shows the actual significance levels for $F''$ and GLS for various situations. When the true value of $\rho$ is very small (0.01), the GLS procedure has convergence problems and the significance levels reported are a slight underestimate since they are based only on those cases where convergence was achieved.

The main points to note are as follows:

1. Both the $F''$ procedure and GLS remove the substantial impact on the ordinary $F$ statistic when $\rho$ is large, although at the cost of numerical complexity and a slight increase in the significance level when $\rho$ is very small (0.01).
2. The $F''$ procedure is superior to the GLS procedure for small (0.01) and moderate (0.1) values of $\rho$.
3. The achieved significance levels are above 5% but the remaining distortion is small. The worst situations are when $\text{corr}(x, z)$ is strong.
4. Only one choice of values of $\rho_x$ and $\rho_z$ is reported in Figure 2 but other values confirm the same pattern.
5.4. Unequal intracluster correlations in different clusters

The theory in Section 4 allows for the possibility that \( \rho \) may vary across clusters and we conjectured that variation in \( \rho \) would result in an increased distortion to the \( F \) statistic. To explore this question we have carried out further simulations which are from a model even more general than that used in Section 4. The previous simulation procedure was modified so that the random effect generated for clusters was multiplied by a constant \( \omega \) for one-half of the clusters and left as before for the other half. Thus for half of the clusters

\[
\text{var}(y|x) = \sigma_a^2 + \sigma_c^2, \quad \rho_k = \sigma_a^2/(\sigma_a^2 + \sigma_c^2) = \rho_1, \quad \text{say;}
\]

for the other half of the clusters

\[
\text{var}(y|x) = \omega^2 \sigma_a^2 + \sigma_c^2, \quad \rho_k = \omega^2 \sigma_a^2/(\omega^2 \sigma_a^2 + \sigma_c^2) = \rho_2, \quad \text{say.}
\]

Thus the simulations allow for unequal \( \rho_k \) and also unequal variances of the error terms from the model.

Table 3 contains the actual significance levels of the various test procedures for the case when \( \rho_x = .48, \rho_z = .49 \) and \( \rho_xz = .42 \), and for various values of \( \rho_1 \) and \( \rho_2 \). The table also contains corresponding results for the case of a common value \( \tilde{\rho} \) for all clusters which was chosen to yield the same overall intracluster correlation.

The main points to note are as follows:

1. The \( F \) test is slightly more distorted for unequal \( \rho_k \), although the main effect comes from \( \rho \neq 0 \) rather than the variation in \( \rho_k \).

2. Both the modified procedure \( F^m \) and the GLS procedure continue to perform reasonably well, although there is slightly more distortion than with constant \( \rho \).

The GLS procedure seems slightly less affected than \( F^m \).
TABLE 3: Comparison of actual significance levels for various tests, nominal 5% level, for equal and unequal $\rho_k$; $\rho_x = .48$, $\rho_z = .49$, $\rho_xz = .42$, and $m = c = 10$; testing $\gamma = 0$.

<table>
<thead>
<tr>
<th></th>
<th>$\omega$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\tilde{\rho}$</th>
<th>$F$</th>
<th>$F''$</th>
<th>GLS</th>
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<td>8.1</td>
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<td>Equal</td>
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6. Conclusion

It is clear that the ordinary F test may be seriously distorted in two-stage sampling when there is positive intracluster correlation. Our theoretical results show the importance of $\text{tr}(P_AV)$ as a diagnostic and basis for correction to the F statistic. For cases where there is low correlation between the regressor variables, $\text{tr}(P_AV)$ is seen to be approximated by the sum of the meffs for the variables involved in any subset of regressors. A general ANOVA-type decomposition for $\text{tr}(P_AV)$ is given in terms of the contributions of the individual regressors and their cross products. The numerical results given show the possible levels of distortion to the
significance level of the nominal 5% test for the case when there are 10 observations per cluster. Larger cluster sample sizes will lead to greater effects and vice versa.

When \( \rho \) is known (or when the sample size and number of clusters is so large that \( \rho \) is very accurately estimated) a simple modification to the \( F \) statistic seems to work well and to provide a test procedure of approximately the correct significance level.

When \( \rho \) is unknown and must be estimated, we note the large effect on \( \hat{\rho} \) which comes from relatively small variations in \( \hat{\sigma}_a^2 \) and \( \hat{\sigma}_\epsilon^2 \). This, perhaps, explains the common practice in the sample survey literature of pooling estimates of deff (and in a similar way the correlated components of response variance) to achieve some stability. In this case the usual alternative is to use GLS but for small values of \( \rho \) (less than 0.1) we have suggested an alternative which is numerically simpler and seems to work better in practice. In the survey context, experience suggests that \( 0 < \rho < 0.1 \) is a likely range of possible values and so the \( F^* \) procedure suggested here is a realistic alternative to GLS. In our view, larger values of \( \rho \) often suggest an inadequately specified model so that the first step should be to introduce additional explanatory variables which account for some of the between-cluster variation rather than simply accepting such a high value of \( \rho \) and modifying the \( F \) statistic.

The final set of numerical results allow for a situation with unequal within-cluster variances and correlations, which is more general than the population model used in the theory. The limited numerical results presented suggest that these sources of variation between clusters, increase the distortion of the usual \( F \) procedure. Both the GLS and the alternative modification to \( F \) continue to work reasonably well.
REFERENCES


Appendix A. Proof of (4.7) and (4.8).

Decompose \( z \) into two orthogonal components

\[
\frac{T}{\|x\|^2} x + (z - \frac{T}{\|x\|^2} x)
\]

and denote its second component by \( y \). Then the projection matrix \( P \) can be expressed simply as

\[
\frac{1}{n} \sum_j x_j x_j^T + \frac{y^T y}{\|y\|^2}, \quad (A.1)
\]

since \( j, x \) and \( y \) are mutually orthogonal. From (A.1),

\[
\text{tr}(PV) = \frac{1}{n} \sum_j x_j x_j^T + \frac{y^T y}{\|y\|^2}, \quad (A.2)
\]

whose first and second terms are \( D_a \) and \( D_\beta \) respectively. To compute the third term, note that

\[
y^T y = x^T x + \frac{(x^T x)^2}{\|x\|^4} x^T x - 2 \frac{x^T x}{\|x\|^2} x^T x
\]

\[
= \|x\|^2 D_y + \frac{(x^T x)^2}{\|x\|^2} D_y - 2 \frac{(x^T x)^2}{\|x\|^2} D_y \quad (A.3)
\]

and

\[
\|y\|^2 = \|x\|^2 - \frac{(x^T x)^2}{\|x\|^2} = \|x\|^2 (1 - r^2) \quad (A.4)
\]

Formula (4.7) follows from (A.2)-(A.4).
To prove (4.8), note that

\[ T_{V_Z} = \sum \left[ (1 - \rho) T_{xZ}, \ell + m^2 \rho \frac{(\bar{x}_{\ell} - \bar{x}_s)(\bar{z}_{\ell} - \bar{z}_s)}{T_{xz}, \ell} \right] \]

\[ = \sum T_{xZ}, \ell \left( 1 + m^2 \rho \frac{m(\bar{x}_{\ell} - \bar{x}_s)(\bar{z}_{\ell} - \bar{z}_s)}{T_{xz}, \ell} \right) \]

\[ = \sum T_{xZ}, \ell \left( 1 + (m - 1) \rho \rho_{xz}, \ell \right). \]
Appendix B. Approximation to the true significance level of F.

From (2.6), the true significance level of $F$

$$\text{Prob}(F > F_a(k, n-k)) = \text{Prob}\{\delta^T \sqrt{\frac{1}{2}} [P-d(I-P)] \sqrt{\frac{1}{2}} \delta > 0\}, \quad (A.5)$$

where $\delta \sim N(0,I)$ and $d = k(n-k)^{-1} F_a(k, n-k)$. Let

$$\lambda_1 > \cdots > \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_s > \lambda_{s+1} > \cdots > \lambda_n$$

be the $n$ eigenvalues of $[P - d(I - P)]V$. Then (A.5) equals

$$\text{Prob}\left\{\sum_{i=1}^{r} \lambda_i \xi_i / \sum_{i=s+1}^{n} |\lambda_i| \xi_i > 1\right\}, \quad (A.6)$$

where the $\xi_i$'s are independent $\chi^2$ random variables. Use the approximation

$$\sum_{i=1}^{r} \lambda_i \xi_i = a \chi^2_\mu, \quad \sum_{i=s+1}^{n} |\lambda_i| \xi_i = b \chi^2_v \quad (A.7)$$

where

$$a = \sum_{i=1}^{r} \frac{\lambda_i^2}{\lambda_i}, \quad b = \sum_{i=s+1}^{n} |\lambda_i|, \quad \mu = \frac{1}{r} \sum_{i=1}^{r} \lambda_i^2, \quad v = \frac{1}{s+1} \sum_{i=s+1}^{n} \lambda_i^2$$

are obtained by matching the first two moments. The approximation in (A.7) is known to be very accurate. Now (A.6) can be approximated by

$$\text{Prob}\{a \chi^2_\mu / b \chi^2_v > 1\} = \text{Prob}\{F(\mu, v) > \frac{b \chi^2_v}{a \mu}\},$$

which can be evaluated from the $F$ distribution. The problem of non-integral degrees of freedom $\mu$ and $v$ is handled by interpolation.
We investigate the effect of intracluster correlation in two-stage samples on the ordinary F procedures in linear models. A measure is proposed as a diagnostic and basis for correction to the F statistic. A decomposition of this measure is given in terms of the contributions of the individual regressors and their cross products. For known intracluster correlation the proposed correction to F performs very well in the numerical study. For unknown intracluster correlation a simple alternative to the generalized least squares procedure is proposed and is shown to perform favorably in the simulation study.