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Abstract

Stochastic integrals of random functions with respect to a white noise random measure are defined in terms of random series of usual Wiener integrals. Conditions for the existence of such integrals are obtained in terms of the nuclearity of certain operators on $L^2$-spaces. The relation with the Fisk-Stratonovich symmetric integral is also discussed.

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1. Introduction

Before introducing and discussing our results, let us introduce some notation and definitions which are used throughout the paper. \((T,A,m)\) denotes an atomless separable \(\sigma\)-finite measure space and \((\Omega,F,P)\) a probability space.

\(W = \{W(A) : A \subset A_0\}\) is a white noise random measure on \(A_0 = \{A \subset A : m(A) < \infty\}\) with control measure \(m\), i.e. \(W\) is a mean-zero Gaussian stochastic process indexed by sets from \(A_0\) and with covariance function \(EW(A)W(B) = m(AnB)\).

\(z = \{z(t) : t \in T\}\) is a measurable real valued stochastic process such that

\[
\int_T E|z(t)|^2m(dt) < \infty.
\]

Let \(\{\phi_n\}\) be a CONS in (real) \(L^2(T)\) and

\[
a_n = a_n(\omega) = \int_T z(t,\omega)\phi_n(t)m(dt).
\]

Clearly

\[
z(t,\omega) = \sum_n a_n(\omega)\phi_n(t) \quad \text{in} \quad L^2(T \times \Omega).
\]

A stochastic integral \(\int_T z^*dW\) is defined by

\[
(1.1) \quad \int_T z^*dW = \lim_{N \to \infty} \sum_{n=1}^N a_n \int_T \phi_n(t)dW(t),
\]

provided the limit exists in \(L^2(\Omega)\) and does not depend on the choice of a CONS \(\{\phi_n\}\).

This is a very attractive definition of a stochastic integral that does not require any special kind of measurability of \(z\) and the parameter set can be arbitrary.
A stochastic integral of this type has been defined and studied by Balkan [1], Kuo and Rusek [6] and Ogawa [8], independently. Ogawa [8] has proven that $L^1$-convergence in (1.1) with respect to the trigonometric basis $\{\phi_n\}$ implies similar convergence to the same limit with respect to Haar basis. Balkan [1] and Kuo and Rusek [6] studied the case when $\xi(t)$, $t \in T$ is a Wiener $L^2$-functional of $W$, i.e. for every $t \in T$, $\xi(t)$ is $\Gamma^W = \sigma(W(A): A \in A_0)$-measurable. Kuo and Rusek [6] (cf. also [7]) using Hida's white noise analysis studied sufficient conditions for the convergence in (1.1) and proved that under certain assumptions $\int_0^1 \xi \, dW$ can be evaluated utilizing the Fisk-Stratonovich procedure.

Our approach is based on some classical results from the theory of nuclear operators on Hilbert spaces and on the Ito-Wiener expansion of $L^2$-Wiener functionals. In Section 2, we establish characterization of integral operators with a summable tame, which is basic for this paper. In Section 3, we study special cases of integrands. We show that if $\xi$ is a Gaussian process subordinate to $W$, then $\int \xi \, dW$ is a quadratic form in independent standard normal r.v.'s as it was studied by Varberg [10]. A necessary and sufficient condition for the existence of $\int \xi \, dW$ when $\xi$ lies in the $p$-th homogeneous chaos is given in Theorem 3.3. A general sufficient condition for the integrability of $\xi$, is given in Theorem 4.1. In Section 5, we investigate the relationship between $\int_0^1 \xi \, dW$ and the Fisk-Stratonovich integral. Theorem 5.8 provides sufficient condition for the existence and equality of both integrals. This condition, which is given in terms of appropriate Sobolev-space norms, is of the same nature as the one presented in [6], but differs in the value of a coefficient $((p + 1)!$ instead of $p!$ in [6]). Theorem 5.9 gives a quite simple condition which guarantees the evaluation of $\int_0^1 \xi \, dW$ as a limit of corresponding Stiltjes sums.

Throughout this paper the following notations are used:
\[ L^2(T^k) := L^2(T^k, \bigotimes_{j=1}^{k} A_j, \bigotimes_{j=1}^{k} m_j), \]

where \( A_j = A, m_j = m; \)

\[ I_p(g) = \int_T^T \int_T^T g(s_1, \ldots, s_p) dW(s_1) \ldots dW(s_p) \]

is the \( p \)-tuple Itô-Wiener integral of \( g \in L^2(T^p); \) when \((T, m)\) is the unit interval with Lebesgue measure, then \( dW \) is replaced by \( dB \), where \( B(t), t \in [0,1] \) is a standard Brownian motion; \( \int_0^1 \xi(t) dB(t) \) is the usual Itô integral of a nonanticipating process \( \xi(t), t \in [0,1]; \) \( \int_0^1 \xi(t) dB(t) \) denotes the Fisk-Stratonovich integral (cf. [4], p. 101).
2. Integral operators with a summable trace

Let $\mathcal{H}$ be a real separable Hilbert space, and $k: \mathbb{T}^2 \to \mathcal{H}$ be a measurable mapping such that $\int_{\mathbb{T}^2} \|k(s,t)\|^2 \, dm(s) \, dm(t) < \infty$. Define an operator $K: L^2(\mathbb{T}) \to L^2(\mathbb{T}; \mathcal{H})$ by

$$ (K\phi)(t) = \int_{\mathbb{T}} \phi(s) k(s,t) \, dm(s), \quad \phi \in L^2(\mathbb{T}). $$

We say that $K$ has a summable trace if for every CONS $\{\phi_n\} \subset L^2(\mathbb{T})$ the series

$$ \sum_{n} \int_{\mathbb{T}^2} \phi_n(s) k(s,t) \phi_n(t) \, dm(s) \, dm(t) $$

converges in $\mathcal{H}$.

Let $\tilde{k}$ be the symmetrization of $k$, i.e.

$$ \tilde{k}(s,t) = \frac{1}{2} \{k(s,t) + k(t,s)\}, \quad s, t \in \mathbb{T}, $$

and let $\widetilde{K}: L^2(\mathbb{T}) \to L^2(\mathbb{T}; \mathcal{H})$ be the corresponding integral operator with kernel $\tilde{k}$. Note that $K$ has a summable trace if and only if $\tilde{K}$ possesses this property and the limit in (2.2) is the same if $k$ is replaced by $\tilde{k}$.

**Proposition 2.1.** An operator $K$ given by (2.1) has a summable trace if and only if for every $h \in L^2(\mathbb{T})$ the operator $\tilde{K}_h: L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by

$$ (\tilde{K}_h \phi)(t) = \int_{\mathbb{T}} \phi(s) \cdot \tilde{k}(s,t), h>dm(s). $$

is nuclear.
2.2

**Proof.** Let \( \{ \phi_n \} \) be a CONS in \( L^2(T) \) and put \( h_n = \int_{T^2} \phi_n(s)k(s,t)\phi_n(t)dm(s)dm(t) \).

Assume that \( K \) has a summable trace. Since any permutation of \( \{ \phi_n \} \) is also a CONS in \( L^2(T) \), \( \Sigma h_n \) converges unconditionally in \( H \). Hence for every \( h \in H \)

\[
(2.3) \quad \sum_{n} |<\tilde{K}_h \phi_n, \phi_n>| = \sum_{n} |<h_n, h>| < \infty.
\]

Since \( \tilde{K}_h \) is also a selfadjoint operator, \( \tilde{K}_h \) is nuclear (cf. e.g. [2], Theorem 3.4.3).

Conversely, assume \( \tilde{K}_h \) is a nuclear operator for every \( h \in H \). Thus

\[
\sum_{n} |<\tilde{K}_h \phi_n, \phi_n>| < \infty \text{ for every CONS } \{ \phi_n \} \text{ in } L^2(T), \text{ and by (2.3) } \sum h_n \text{ converges}
\]

weakly unconditionally in \( H \). Since \( H \) is weakly complete, \( \sum h_n \) converges strongly in \( H \); cf. e.g. [3], II.5.

**Corollary 2.2.** If \( K \) has a summable trace, then \( h \mapsto \text{Trace}(\tilde{K}_h) \) is a linear functional on \( H \) satisfying the equality

\[
<h_0, h> = \text{Trace}(\tilde{K}_h), \ h \in H,
\]

where \( h_0 \) is the limit of the series (2.2) for some (any) CONS \( \{ \phi_n \} \) in \( L^2(T) \).

Hence the trace of \( K \), denoted by \( \text{tr}_K \) and given by the series

\[
\text{tr}_K := \sum_{n} \int_{T^2} \phi_n(s)k(s,t)\phi_n(t)dm(s)dm(t)
\]

is well-defined, i.e. does not depend on the choice of a CONS \( \{ \phi_n \} \) in \( L^2(T) \).
3. Integration in some special cases

A. Integrals of subordinate Gaussian processes.

Let

\[ \xi(t) = \int_{T} f(s,t) dW(s) = I_{1}(f(\cdot,t)), \quad t \in T, \]

where \( f \) belongs to \( L^{2}(T^{2}) \). Let \( \{\phi_{n}\} \) be a CONS in \( L^{2}(T) \) and consider the orthonormal expansion of \( f \) in \( L^{2}(T^{2}) \):

\[ f(s,t) = \sum_{m,n} f_{mn} \phi_{m}(s) \phi_{n}(t), \]

and the i.i.d. standard normal r.v.'s \( X_{n} = \int_{T} \phi_{n}(s) dW(s) \).

**Proposition 3.1.** Let \( \xi \) be given by (3.1). Then \( \int_{T} \xi^{*} dW \) exists and

\[ \int_{T} \xi^{*} dW = \sum_{m,n} f_{mn} X_{m} X_{n} \quad a.s. \]

provided the operator \( F : L^{2}(T) \to L^{2}(T) \) defined by

\[ (F\phi)(t) = \int_{T} \phi(s)f(s,t) dm(s), \quad \phi \in L^{2}(T), \]

has a summable trace.

Note that in this case \( \int_{T} \xi^{*} dW \) coincides with the double stochastic integral of \( f \) defined by Varberg [10], which is different from the double Itô-Wiener integral \( I_{2}(f) \).

**Proof.** Since \( \sum_{n} f_{nn}^{*} = \text{tr} f \) (cf. Corollary 2.2), the series \( \sum_{n} f_{nn} \) converges unconditionally. Therefore \( \sum_{m,n} f_{mn} X_{m} X_{n} \) converges unconditionally in \( L^{2}(\Omega) \) (cf. [10]), and
3.2

\[ \sum_{m,n} f_{mn} X_n = \lim_{N \to \infty} \sum_{n=1}^{N} \left( \sum_{m=1}^{N} f_{mn} X_n \right) \]

\[ = \lim_{N \to \infty} \sum_{n=1}^{N} \left( \int_{t}^{n} (f_n(t)dm(t))X_n \right) \]

\[ = \int_{t}^{n} (f_n(t)dW(t)). \]

Example 3.2. \( \int_{0}^{1} B(t)dB = B^2(1)/2. \)

Proof. Indeed, \( B(t) = \int_{0}^{1} D(s,t)dB(s), \) where \( D = \{(s,t): 0 \leq s \leq t \leq 1 \}, \)
and \( \sim \phi = 2^{-1}<0,1,\phi>l \) is nuclear as a one-dimensional projection, where \( \sim \phi \) is the symmetrization of \( \phi. \) Also

\[ f_{mn} + f_{nm} = 2<\phi, \phi> = <l, \phi><l, \phi>. \]

Therefore, by Proposition 3.1

\[ \int_{0}^{1} B(t)dB = 2^{-1} \lim_{N \to \infty} \sum_{n=1}^{N} \left( f_{mn} + f_{nm} \right) X_n \]

\[ = 2^{-1} \lim_{N \to \infty} \left( \sum_{n=1}^{N} <l, \phi> X_n \right)^2 = 2^{-1} B^2(1). \]

B. Integrals of multiple Itô-Wiener integrals.

Let

\[ \xi(t) = \int_{T^p} f(s_1, \ldots, s_p, t)dW(s_1)\ldots dW(s_p) = \int_{T^p} f(\cdot, t), \]

\( t \in T, \) where \( f = f(s_1, \ldots, s_p, t) \) belongs to \( L^2(T^{p+1}) \) and is symmetric in \( s_1, \ldots, s_p \) for each fixed \( t. \) We have for every \( t \in T \)
\begin{equation}
\|\xi(t)\|_{L^2(\Omega)}^2 = p!\|f(\cdot, t)\|_{L^2(T^{p+1})}^2
\end{equation}

and

\begin{equation}
\|\xi\|_{L^2(T^*\Omega)}^2 = p!\|f\|_{L^2(T^{p+1})}^2.
\end{equation}

Since for a.e. \((s, t) \in T^2\), \(f(\cdot, s, t) \in L^2(T^{p-1})\), the mapping

\(F : L^2(T) \to L^2(T; L^2(T^{p-1}))\) given by

\begin{equation}
(F\phi)(t) = \int f(s)\phi(\cdot, s, t)dm(s), \quad \phi \in L^2(T),
\end{equation}

is a well-defined linear continuous operator. \(\text{tr} f\) will stand for the trace of \(F\), provided \(F\) has a summable trace (cf. Corollary 2.2). Note that in this case \(\text{tr} f\) is an element of \(L^2(T^{p-1})\).

**Theorem 3.3.** Let \(\xi\) be given by (3.2). Then \(\int_T \xi^*dW\) exists if and only if the operator \(F\) defined by (3.3) has a summable trace. In this case

\begin{equation}
\int_T \xi^*dW = \Pi_{p+1}(f) + \Pi_{p-1}(\text{tr} f).
\end{equation}

**Proof.** We have

\[ a_n = \int_T \xi(t)\phi_n(t)dm(t) \]

\[ = \int_T \left( \int_{T^p} f(s_1, \ldots, s_p, t)dW(s_1) \ldots dW(s_p) \right) \phi_n(t)dm(t) \]

\[ = \Pi_p(g_n), \]

where \(g_n(s_1, \ldots, s_p) = \int_T f(s_1, \ldots, s_p, t)\phi_n(t)dm(t)\), and the interchange of the
multiple Itô-Wiener and usual integration can be easily verified for simple functions and extended to the general case by the usual approximation argument.

By Itô's recurrence formula (cf. [5], Thm. 2.2) we get

\[ a_n \int_{T_n} \phi \, dW = I_p(g_n)I_1(\phi_n) = I_{p+1}(g_n \otimes \phi_n) + pI_{p-1}(h_n), \]

where \( (g_n \otimes \phi_n)(t_1, \ldots, t_{p+1}) = g_n(t_1, \ldots, t_p)\phi_n(t_{p+1}) = \langle f(t_1, \ldots, t_p, \cdot), \phi_n(\cdot) \rangle \phi_n(t_{p+1}) \) and

\[ h_n(s_1, \ldots, s_{p-1}) = \int_s \phi_n(s_1, \ldots, s_{p-1}, s, t)\phi_n(t) \, dm(s) \, dm(t). \]

We observe now that \( \sum_n g_n \otimes \phi_n \) converges to \( f \) in \( L^2(T^{p+1}) \) and consequently \( \sum_n I_{p+1}(g_n \otimes \phi_n) \) converges to \( I_{p+1}(f) \) in \( L^2(\Omega) \). Therefore, in view of (3.4), \( \int \xi \, dW \) exists if and only if \( \sum_n h_n \) converges in \( L^2(T^{p-1}) \) for every \( \text{CONS} \{ \phi_n \} \subset L^2(T) \), which means that \( F \) has a summable trace. Since \( \sum_n h_n = \text{tr} f \), (3.4) completes the theorem.

\[ \square \]

**Example 3.4.**

\[ \frac{1}{n} \int_0 \mathcal{H}_n(B(t), t) \, dB(t) = \frac{1}{n+1} \mathcal{H}_{n+1}(B(1), 1) + \frac{n}{2} \int_0 \mathcal{H}_{n-1}(B(t), t) \, dt, \]

where \( \mathcal{H}_n(x, t) \) is the Hermite polynomial of degree \( n \) defined by

\[ \mathcal{H}_n(x, t) = (-t)^n e^{x^2/2t} \frac{\partial^n}{\partial x^n} e^{-x^2/2t}, \quad t > 0. \]
Proof. Indeed, $H_n(B(t), t) = I_n(f(*, t))$, where

$$f(s_1, \ldots, s_n, t) = 1_{[0, t]}(s_1, \ldots, s_n).$$

Therefore the symmetrization of $f$ in the last two variables is given by

$$\tilde{f}(s_1, \ldots, s_{n-1}, s, t) = \begin{cases} \frac{1}{2} & \text{if } \max\{s_1, \ldots, s_{n-1}\} \leq \max\{s, t\} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\phi_n\}$ be a CONS in $L^2[0,1]$. We have

$$\sum_{n=0}^{11} \int \int \phi_n(s)f(s_1, \ldots, s_{n-1}, s, t)\phi_n(t) dsdt$$

$$= \sum_{n=0}^{11} \int \int \phi_n(s)\tilde{f}(s_1, \ldots, s_{n-1}, s, t)\phi_n(t) dsdt$$

$$= 2^{-1} \sum_{n=0}^{11} \int \int \phi_n(s)(1 - 1[\max\{s_1, \ldots, s_{n-1}\} > s])1[\max\{s_1, \ldots, s_{n-1}\} > t])\phi_n(t) dsdt$$

$$= 2^{-1} \sum_{n=0}^{11} \left< \phi_n, \chi_{[0,1]} + 1 - 1[\max\{s_1, \ldots, s_{n-1}\} > t] \right>^2 \phi_n$$

$$= 2^{-1} (1 - \max\{s_1, \ldots, s_{n-1}\})$$

in $L^2([0,1]^{n-1})$. Hence $F$ has a summable trace and $(trf)(s_1, \ldots, s_{n-1}) = 2^{-1} (1 - \max\{s_1, \ldots, s_{n-1}\})$. By Theorem 3.3 $\int_0^1 H_n(B(t), t) dB(t)$ exists. To evaluate this integral we observe that

$$I_{n+1}(f) = I_{n+1}(1_{[0, s_{n+1}]}(s_1, \ldots, s_n)) = (n+1)^{-1} I_{n+1}(1_{[0,1]}n+1)$$

$$= (n+1)^{-1} H_{n+1}(B(1), 1),$$
and, since a multiple Itô-Wiener integral can be expressed by usual Itô integral (cf. [5], Theorem 5.1),

\[ nI_{n-1}(\text{trf}) = 2^{-1}nI_{n-1}(1 - \max\{s_1, \ldots, s_{n-1}\}) \]

\[ = 2^{-1}n! \int_0^{s_{n-1}} \int_0^{s_2} \cdots \int_0 (1 - s_{n-1}) dB(s_1) \cdots dB(s_{n-2}) dB(s_{n-1}) \]

\[ = 2^{-1}n(n-1) \int_0^1 (1 - s_{n-1})H_{n-2}(s_{n-1}, s_{n-1}) dB(s_{n-1}) \]

\[ = 2^{-1}n \int_0^1 (1 - s) dX(s), \]

where \( X(s) = H_{n-1}(B(s), s) = (n - 1) \int_0^s H_{n-2}(B(u), u) dB(u) \). Integrating by parts (note that sample paths of \( X \) are continuous) we get

\[ \int_0^1 (1 - s) dX(s) = (1 - s)X(s)|_0^1 + \int_0^1 X(s) ds, \]

which yields \( nI_{n-1}(\text{trf}) = 2^{-1}n \int_0^1 X(s) ds \), and completes the example.

**Example 3.5.** \( \int_0^1 B(1 - t) dB(t) \) does not exist.

**Proof.** Indeed, \( B(1 - t) = \int_0^1 f(s,t) dB(s) \), where \( f \) is a symmetric function defined on \([0,1]^2\) by

\[ f(s,t) = \begin{cases} 
1 & \text{if } s + t \leq 1 \\
0 & \text{otherwise.} 
\end{cases} \]

By Theorem 3.3 and Proposition 2.1 it is sufficient to show that \( F = \tilde{F} \) is not a nuclear operator on \( L^2[0,1] \), where
Consider the sequences \( \{\phi_n\} \) and \( \{\psi_n\} \) of orthonormal functions in \( L^2[0,1] \):

\[
\phi_n(t) = \sqrt{2} \cos(2\pi nt), \quad n \geq 1, \ t \in [0,1],
\]

and

\[
\psi_n(t) = \sqrt{2} \sin(2\pi n(1 - t)).
\]

Then

\[
\int_0^1 \int_0^1 \frac{1}{2\pi n} = \infty,
\]

which shows that \( F \) is not a nuclear operator.
4. Integration of general Wiener $L^2$-functionals

Throughout this section we shall assume that

$$\xi \in L^2(T \times \Omega, A \otimes F^W, m \otimes P).$$

According to the well-known Ito-Wiener theorem which says that

$$L^2(\Omega, F^W, P) = \bigoplus_{p=0}^{\infty} K_p,$$

where $K_p$ is the $p$-th homogeneous chaos, we may decompose $\xi(t)$ into an orthogonal series

$$\xi(t) = \sum_{p=0}^{\infty} \xi_p(t),$$

where $\xi_p = \{\xi_p(t): t \in T\} \subset K_p$ and $\xi_0(t) = E\xi(t)$. Since we can always choose $\xi_p$ as measurable processes belonging to $L^2(T \times \Omega)$ we also have

(4.1) $$\xi = \sum_{p=0}^{\infty} \xi_p \text{ in } L^2(T \times \Omega).$$

Moreover, each $\xi_p$, $p \geq 1$ can be represented by a multiple Ito-Wiener integral

(4.2) $$\xi_p(t) = I_p(f_0, t), \quad t \in T$$

where $f_p = f_p(s_1, \ldots, s_p, t) \in L^2(T^{p+1})$ is symmetric in $s_1, \ldots, s_p$ for each fixed $t$. We set $f_0(t) = E\xi(t)$ and as usual $I_0(c) = c$. Further $I_p$ will denote the symmetrization of $f_p$ in the last two variables. For every $p \geq 1$ we define an operator
4.2

\[ F_p : L^2(T) \rightarrow L^2(T; L^2(T^{P-1})) \]

by

\[(F_p \phi)(t) = \int_{T} \phi(s) f_p(\cdot, s, t) d\mu(s), \ \phi \in L^2(T). \]

By Proposition 2.1, \( F_p \) has a summable trace if and only if for every

\[ h \in L^2(T^{P-1}) \]

\[ \tilde{F}_{p,h} : L^2(T) \rightarrow L^2(T) \]

is a nuclear operator, where

\[(\tilde{F}_{p,h}, \phi)(t) = \int_{T} \phi(s) \tilde{f}_p(\cdot, s, t), h(\cdot) dm(s), \ \phi \in L^2(T). \]

We define

\[ (4.3) \quad \| F_p \| = \sup_\| h \| \leq 1, h \in L^2(T^{P-1}), \]

where \( \| A \|_\tau \) denotes the nuclear norm of an operator \( A \) (cf. e.g. [2], p. 111).

Note that (4.3) always makes sense, whether or not \( F_p \) has a summable trace.

Clearly, if \( \| F_p \| < \infty \), then \( F_p \) has a summable trace. The converse is also true

and this simply follows by the Closed Graph Theorem applied to the linear

mapping \( h \rightarrow \tilde{F}_{p,h} \). Finally, \( \text{tr} F_p \) will stand for the trace of \( F_p \), provided

\[ \| F_p \| < \infty. \]

**Theorem 4.1.** Assume that

\[ A^2(\{ f_p \}) := \| f_0 \|_{L^2(T)}^2 + \sum_{p=1}^{\infty} (p+1)! \{ \| f_p \|_{L^2(T^{P+1})}^2 + \| F_p \|^2 \} \]

is finite. Then \( \int_{T} f \, * dW \) exists,
(4.4) \[ \int_{\mathcal{T}} \xi * dW = I_1(f_0) + \sum_{p=1}^{\infty} [I_{p+1}(f_p) + pI_{p-1}(\text{trf}_p)] \]

in \( L^2(\Omega) \) and

\[ \frac{||\int \xi * dW||_2}{L^2(\Omega)} \leq \sqrt{ZA(\{\xi\}_p)}. \]

**Proof.** We have

(4.5) \[ \sum_{n=1}^{N} \int \xi \phi_n \, dm \phi_n \, dW = \sum_{p=0}^{\infty} S_{p,N} \]

where

\[ S_{p,N} = \sum_{n=1}^{N} \int \xi \phi_n \, dm \phi_n \, dW \]

and the series \( \sum_{p=0}^{\infty} S_{p,N} \) converges in \( L^1(\Omega) \) for each \( N \geq 1 \).

By Theorem 3.3 for every \( p \geq 1 \),

(4.6) \[ S_{p,N} \rightarrow I_{p+1}(f_p) + pI_{p-1}(\text{trf}_p) \text{ in } L^2(\Omega) \text{ as } N \rightarrow \infty, \]

and obviously \( S_{0,N} \rightarrow I_1(f_0) \). Using (3.4) we have

(4.7) \[ S_{p,N} = I_{p+1}(f_{p,N}) + pI_{p-1}(k_{p,N}), \]

where

\[ f_{p,N}(t_1, \ldots, t_{p+1}) = \sum_{n=1}^{N} <f_n(t_1, \ldots, t_p, \cdot), \phi_n(\cdot) \phi_n(t_{p+1})> \]

and

\[ k_{p,N}(s_1, \ldots, s_{p-1}) = \sum_{n=1}^{N} \int \phi_n(s_1, \ldots, s_{p-1}, s, t) \phi_n(t) \, dm(s) \, dm(t). \]
Hence \( \| f_{p,N} \| \leq \| f_p \| \) and

\[
\| k_{p,N} \| = \sup \{ \langle k_{p,N}, h \rangle : \| h \| \leq 1, \ h \in L^2(T_{p-1}) \}
\]

\[
\leq \sup \{ \| F_{p,h} \| : \| h \| \leq 1, \ h \in L^2(T_{p-1}) \} = \| F_p \|.
\]

Therefore, for every \( r > q \geq 1 \) and \( N \geq 1 \)

\[
\| \sum_{p=q}^r S_{p,N} \|^2 = \| \sum_{p=q}^r [I_{p+1}(f_{p,N}) + pI_{p-1}(k_{p,N})] \|^2
\]

\[
\leq 2 \| \sum_{p=q}^r I_{p+1}(f_{p,N}) \|^2 + 2 \| \sum_{p=q}^r pI_{p-1}(k_{p,N}) \|^2
\]

\[
\leq 2 \sum_{p=q}^r (p+1)! \| f_{p,N} \|^2 + 2 \sum_{p=q}^r p! \| k_{p,N} \|^2
\]

\[
\leq 2 \sum_{p=q}^r (p+1)! \| f_p \|^2 + \| F_p \|^2,
\]

which shows that \( \| S_p \|_{L^2(\Omega)} \to 0 \) uniformly in \( N \) as \( q,r \to \infty \). Combining this

(4.5) and (4.6) Theorem 4.1 follows.

**Proposition 4.2.** If \( \xi = \sum_{p=0}^q \xi_p \), where \( q < \infty \), then \( A(f_{p}) \to \infty \) is also a

necessary condition for the existence of \( \int f_{p}^* dW \).

**Proof.** Let \( Q_p \) be the orthogonal projection of \( L^2(\Omega,W,P) \) onto \( K_p \). Using

(4.5) and (4.7) we get for each \( 0 \leq p < q \)

\[
Q_p(\int f_{p}^* dW) = \lim_{N \to \infty} \sum_{n=1}^N \int_{\mathbb{T}} \phi_n dm \int_{\mathbb{T}} \phi dW
\]

\[
= \lim_{N \to \infty} \sum_{p=0}^q \sum_{p=0}^q Q_p(S_{p,N})
\]

\[
= \lim_{N \to \infty} \int [I_{p}(f_{p-1,N}) + (p+1)I_{p}(k_{p+1,N})]
\]
\[ = I_p(f_{p-1}) + (1 + p) \lim_{N \to \infty} I_p(k_{p+1},N). \]

Therefore \( \{k_{p+1,N}\}_{N=1}^{\infty} \) converges in \( L^2(T^p) \) for any orthonormal basis \( \{\phi_n\} \subset L^2(T) \).

This implies that \( F_{p+1} \) has a summable trace and \( \|F_{p+1}\| < \infty \).

We do not know whether or not \( A(\{f_p\}) < \infty \) is necessary for the existence of \( \int \xi \ast dB \) in the general case. Nevertheless Theorem 4.1 gives a straightforward way to establish the integrability of \( \xi \). Clearly the basic difficulty is in getting an upper bound for \( \|F_p\| \). We now use certain Sobolev-space type conditions on the \( f_p \)'s to upper bound \( \|F_p\| \).

**Theorem 4.3.** Let \( T = [0,1] \) and \( m \) be Lebesgue measure on \( T \). For \( p \geq 1 \) and \( \alpha > 0 \) we define

\[
U_\alpha^2(f_p) := \|f_p\|_{L^2(T^p)}^2 + \int_{T^2} |u - v|^{-1-2\alpha} \|\tilde{f}_p(\cdot,u) - \tilde{f}_p(\cdot,v)\|_{L^2(T)}^2 \, dudv.
\]

Assume that for some \( \alpha > \frac{1}{2} \),

\[
U_\alpha^2(\{f_p\}) := \|f_0\|_{L^2(T)}^2 + \sum_{p=1}^{\infty} (p + 1)! U_\alpha^2(f_p)
\]

is finite. Then \( \int_0^1 \xi \ast dB \) exists, (4.4) holds and

\[
\frac{1}{0} \int \xi \ast dB \|_{L^2(\Omega)} \leq C U_\alpha(\{f_p\}),
\]

where \( C \) depends only on \( \alpha \).

**Proof.** Since \( \|\tilde{f}_p\| \leq \|f_p\| \) and \( U_\alpha^2(f_p) < \infty \), the function \( [0,1] \times t \rightarrow \tilde{f}_p(\cdot,t) \in L^2([0,1]^p) \) has absolutely convergent Fourier series, i.e.
\[ \tilde{F}_p (s_1, \ldots, s_p, t) = \sum_{n \in \mathbb{Z}} c_{p,n} (s_1, \ldots, s_p) e^{2\pi i nt} \]

in \( L^2([0,1]^p) \), where \( \sum_{n \in \mathbb{Z}} \| c_{p,n} \|_2^2 < \infty \); c.f. [9], proof of Theorem 2.

Moreover
\[ \sum_{n \in \mathbb{Z}} \| c_{p,n} \| \leq C \Gamma_\alpha (f_p), \]

where \( C \) depends only on \( \alpha \). Put \( \chi_n (t) = e^{2\pi i nt} \).

Let \( h \in L^2([0,1]) \) and let \( \{ \phi_n \} \) and \( \{ \psi_n \} \) be two sequences of orthonormal functions in \( L^2[0,1] \). We have
\[
\begin{align*}
\sum \langle \tilde{F}_p, \psi_j, \psi_j \rangle &\leq \sum_{n \in \mathbb{Z}} \sum_{j=0}^{\infty} \int_c (\phi_j, \phi_j) \int_0^1 \chi_n (t) \psi_j (t) dt \int_0^1 \chi_n (s) ds dt \\
&\leq \sum_{n \in \mathbb{Z}} \int_c (\phi_j, \phi_j) \int_0^1 \chi_n (t) \psi_j (t) dt \int_0^1 \chi_n (s) ds dt \\
&\leq \sum_{n \in \mathbb{Z}} \| \psi_n \|_2 \| h \| \leq C \Gamma_\alpha (f_p) \| h \|. 
\end{align*}
\]

Hence \( \| \tilde{F}_p, h \|_1 \leq C \Gamma_\alpha (f_p) \| h \| \) which yields \( \| \tilde{F}_p \| \leq C \Gamma_\alpha (f_p) \). Therefore
\[ A^2 (\{ f_p \}) \leq (C^2 + 1) \Gamma_\alpha^2 (\{ f_p \}) \]

and Theorem 4.1 completes the proof.

A sufficient condition for the integrability of \( \xi \), stronger than that of Theorem 4.3, can be written in terms of the covariance functions of the component processes \( \{ c_p \} \).

**Theorem 4.4.** Let \( T = [0,1] \) and \( m \) be Lebesgue measure on \( T \). If for some \( \alpha > 1/2 \),
\[ N^2_\alpha(\xi) := \|\xi_0\|_{L^2(T)}^2 + \sum_{p=1}^{\infty} p^2 \|\xi_p\|_{L^2(T^2)}^2 + \int \frac{E|\xi_p(u) - \xi_p(v)|^2}{\| u - v \|^{1+2\alpha}} \, du \, dv \]

If finite, then \( \int_0^1 \xi \ast dB \) exists, (4.4) holds and

\[ \int_0^1 \| \xi \ast dB \| \leq C N_\alpha(\xi), \]

where \( C \) depends only on \( \alpha \).

Proof. Since

\[ \|\xi_p\|_{L^2(T^2)}^2 + \int \frac{E|\xi_p(u) - \xi_p(v)|^2}{\| u - v \|^{1+2\alpha}} \]

and

\[ |\langle \tilde{F}_{p, h}^\#, \psi \rangle| \leq 2^{-l} (|\langle F_{p, h}^\#, \psi \rangle| + |\langle F_{p, h}^\#, \phi \rangle|), \]

where \( F_{p, h} \) is defined similarly to \( \tilde{F}_{p, h} \), with \( \tilde{F}_{p, h} \) replaced by \( F_{p, h} \), the inequality \( A(\langle f, h \rangle) \leq \text{Const} \, N_\alpha(f) \) follows by the same arguments as those used in the proof of Theorem 4.3.
5. Evaluation of the integral by the Fisk-Stratonovich procedure

Throughout this section \((T,m)\) will be the unit interval with Lebesgue measure.

Let \(\xi(t), t \in [0,1]\) be a stochastic process. We say that a (generalized) Fisk-Stratonovich integral of \(\xi\) exists if

\[
S_n := \sum_{j=1}^{n} 2^{-1} \{\xi(t_{j-1}) + \xi(t_j)\} [B(t_j) - B(t_{j-1})]
\]

converges in probability as mesh \((n) \to 0\), where \(n\) runs over all finite partitions \(0 = t_0 < t_1 < \ldots < t_n = 1\) \((n \in \mathbb{N})\) of \([0,1]\), and we write

\[
\int_0^1 \xi(t) \, dB(t) = \lim_{\text{mesh}(\tau) \to 0} S_{\tau}
\]

Note that we do not require in this definition any kind of measurability of \(\xi\).

In this section we shall study the relationship between \(\int \xi \, dB\) and \(\int \xi \circ dB\).

Let \(\xi\) be given by (4.1) and (4.2). Put \(D_+ = \{(s,t) : 0 \leq s < t \leq 1\}\) and \(D_- = \{(s,t) : 0 \leq t < s \leq 1\}\). Define \(f^+_p (f^-_p\), respectively) as the restriction of the function

\[
[0,1]^2 : (s,t) \to f_p (s,t) \in L^2([0,1]^{p-1})
\]

to \(D_+ (D_-\), respectively).

Proposition 5.1. Let \(\xi = \sum_{p=q}^\infty f_p\), \(q < \infty\), be a mean-square continuous stochastic process. Assume that for every \(1 \leq p \leq q\) the functions \(f^+_p (f^-_p\) are continuous and possess the extensions (also denoted by \(f^+_p (f^-_p\), respectively) to continuous functions from \(D_+ (D_-\), respectively) into \(L^2([0,1]^{p-1})\). Then

\[
\int_0^1 \xi(t) \, dB(t) \text{ exists and}
\]
\[
\int_0^1 \xi(t) dB(t) = I_1(f_0) + \sum_{p=1}^q \left[ I_{p+1}(f_p) + p I_{p-1}(g_p) \right],
\]

where
\[
g_p(\cdot) = 2^{-1} \int_0^1 [f_p^-(\cdot, s, s) + f_p^+\cdot, s, s)] ds.
\]

**Proof.** Clearly we may assume that \(\xi = \xi_p\), where \(0 \leq p \leq q\). The case \(p = 0\) is obvious. Let \(p \geq 1\) and let \(\pi = \{t_0, \ldots, t_n\}\) be a partition of \([0,1]\). Using Itô's recurrence formula we get

\[
S_\pi = \sum_{j=1}^n 2^{-1}[\xi_p(t_{j-1}) + \xi_p(t_j)](B(t_j) - B(t_{j-1})]
\]

(5.2)

\[
= \sum_{j=1}^n I_p^2[2^{-1}[f_p^-(\cdot, t_{j-1}) + f_p^+\cdot, t_j)] I_1[1[t_{j-1}, t_j]]
\]

\[
= I_{p+1}(f_p, \pi) + p I_{p-1}(g_p, \pi)
\]

where
\[
f_p, \pi(\cdot, t) = \sum_{j=1}^n 2^{-1}[f_p^-(\cdot, t_{j-1}) + f_p^+\cdot, t_j)] I_1[t_{j-1}, t_j]
\]

and
\[
g_p, \pi(\cdot) = 2^{-1} \sum_{j=1}^n \int_{t_j}^{t_{j-1}} [f_p^-(\cdot, s, t_{j-1}) + f_p^+\cdot, s, t_j)] ds.
\]

Since the mapping \([0,1]^2 \times (s, t) \to f_p(\cdot, s, t) \in L^2([0,1])\) is continuous and uniformly bounded on \(D_+ \cup D_-\), the mapping \([0,1] \times f_p(\cdot, t) \in L^2([0,1])\) is continuous. Hence \(f_p, \pi \to f_p\) in \(L^2([0,1])\) as \(\text{mesh (}\pi) \to 0\). By the continuity of \(f_p^+\) and \(f_p^-\) on \(D_+\) and \(D_-\), respectively, \(g_p, \pi \to g_p\) in \(L^2([0,1])\) as \(\text{mesh (}\pi) \to 0\).
5.3

Proposition 5.1. Suppose that \( \xi \) satisfies the assumptions of Proposition 3.1. Then

\[
\int_0^1 \xi(t) \ast dB(t) = \int_0^1 \xi'(t) dB(t).
\]

Proof: We have for a.e. \((s,t) \in [0,1]^2\)

\[
(5.3) \quad \tilde{f}_p(\cdot, s, t) = \frac{1}{2} [f^+_{p}(\cdot, s, t, v, t) + f^-_{p}(\cdot, s, t, v, t)],
\]

and the function on the right-hand side of (5.3) is continuous in \((s,t) \in [0,1]^2\).

By Proposition 4.2 for every \(h \in L^2([0,1)^{p-1})\), \(\tilde{F}_p,h\) is a nuclear operator on \(L^2[0,1]\). Hence for every \(h \in L^2([0,1)^{p-1})\)

\[
\langle trf_p, h \rangle = \text{Trace} \left( \tilde{F}_p,h \right)
\]

\[
= \int_0^1 2^{-1} \langle f^+_{p}(\cdot, s, s), h(\cdot) \rangle ds
\]

\[
= \langle g_p, h \rangle
\]

(cf. e.g. Theorem 3.4.4 [2]). Proposition 5.1 and Theorem 4.1 complete the proof.

Example 5.3. In Example 3.4 we showed that \(\int_0^1 H_n(B(t), t) \ast dB(t)\) exists and we evaluated the integral. Since in this case the assumptions of Proposition 3.1 are satisfied, Proposition 5.2 provides an alternative way of evaluation of

\[
\int_0^1 H_n(B(t), t) \ast dB(t).
\]

Using Proposition 5.2 and Theorem 1.1, Chap. III in [4] we get

\[
\int_0^1 H_n(B(t), t) \ast dB(t) = \int_0^1 H_n(B(t), t) dB(t)
\]

\[
= \int_0^1 H_n(B(t), t) dB(t) + \frac{1}{2} \int_0^1 H_n(B(s), s), B(s). ds.
\]

\[
= \frac{1}{n+1} H_{n+1}(B(1), 1) + \frac{n}{2} \int_0^1 H_n(B(s), s) ds.
\]
Remark 5.4. The assumptions of Proposition 5.1 do not guarantee the existence of $\int f^* dB$. Indeed, it is well-known that there exists a continuous symmetric kernel $k:[0,1]^2 \to \mathbb{R}$ such that the corresponding integral operator is not nuclear (cf. e.g. [2], p. 124). Put $\xi(t) = \int_0^t k(s,t) dB(s)$. Then the assumptions of Proposition 5.1 are satisfied, but by Theorem 3.3 $\int \xi^* dB$ does not exist.

Below are given simple examples of Gaussian processes for which the Fisk-Stratonovich integral exists while the series expansion \((1.1)\) fails to converge.

**Example 5.5.** $\int_0^1 B(1-t) dB(t) = B^2(\frac{1}{2}) + 2 \int_0^1 B(1-t) dB(t)$, but $\int_0^1 B(1-t) dB(t)$ does not exist (cf. Example 3.5).

**Proof.** Indeed, it is easy to check that

$$
\sum_{j=1}^{j_0} 2^{-1}(B(1-t_{j-1}) + B(1-t_j))[B(t_j) - B(t_{j-1})] \to B^2(\frac{1}{2}) + \int B(1-t) dB(t),
$$

and

$$
\sum_{j=j_0+1}^{n} 2^{-1}[B(1-t_{j-1}) + B(1-t_j)][B(t_j) - B(t_{j-1})] \to \int B(1-t) dB(t),
$$

in $L^2(\Omega)$ as $\text{mesh}(\pi) \to 0$, where $\pi = \{t_0, \ldots, t_n\}$ is a partition of $[0,1]$ and $j_0 = \max\{j: t_j < \frac{1}{2}\}$.

**Example 5.6.** Let $\xi(t), t \in [0,1]$ be a non-anticipating stochastic process given by

$$
\xi(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \frac{1}{2} \\
B(t - \frac{1}{2}) & \text{if } \frac{1}{2} < t \leq 1
\end{cases}
$$
Then \( \int_0^1 f(t) dB(t) = \int_0^1 B(t - \frac{1}{2}) dB(t) \), while \( \int_0^1 f dB \) does not exist.

**Proof.** Indeed, it is elementary to check that

\[
\sum_{j=1}^n \left( \xi(t_j^+ - t_j^-) \right) (B(t_j^+ - B(t_j^-)) \rightarrow 0 \text{ in } L^2(\Omega) \text{ as mesh}(\pi) \rightarrow 0,
\]

which implies that the Fisk-Stratonovich integral exists and is equal to the ordinary Itô integral of \( \xi \).

On the other hand \( \xi(t) = \int_0^1 f(s,t) dB(s) \), where \( f(s,t) = 1 \) if \( 0 \leq s \leq \frac{1}{2} \) and \( s + \frac{1}{2} \leq t \leq 1 \) and \( = 0 \) otherwise. Hence \( f \) is given by

\[
\tilde{f}(s,t) = \begin{cases} 
\frac{1}{2} & \text{if } 0 \leq s \leq \frac{1}{2} \text{ and } s + \frac{1}{2} \leq t \leq 1 \\
\frac{1}{2} & \text{if } \frac{1}{2} \leq s \leq 1 \text{ and } 0 \leq t \leq s - \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\]

Consider the sequences \( \{\phi_n\} \) and \( \{\psi_n\} \) of orthonormal vectors in \( L^2[0,1] \) given by

\[
\phi_n(t) = \cos 4\pi nt, \quad 0 \leq t \leq 1
\]

and

\[
\psi_n(t) = \begin{cases} 
\sin 4\pi n(t + \frac{1}{2}) & \text{if } 0 \leq t \leq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\]

Then

\[
\sum_{n} \langle \tilde{f}, \phi_n^* \psi_n \rangle = \int_{0}^{1/2} \int_{0}^{1/2} \tilde{f}(s,t) ds \psi_n(t) dt = \sum_{n} \int_{0}^{1/2} \phi_n^*(t) \int_{0}^{1/2} \tilde{f}(s,t) \psi_n(t) dt dt
\]

\[
= \sum_{n} \frac{-1}{32\pi n} \rightarrow -\infty.
\]
which completes the example.

Intuitively speaking, the existence of the Fisk-Stratonovich integral requires some kind of continuity of the process $\xi$ while conditions for $\int_0^1 \xi dB$ are of a different nature. Hence it is also easy to give an example of a process $\xi$ for which, conversely, $\int \xi dB$ exists but $\int \xi dB$ does not exist.

**Example 5.7.** Let $A$ be a dense Borel subset of $[0,1]$ such that $0 < m(A) < 1$. Put $\psi(t) = 1_A(t)$, $f(s,t) = \psi(s)\psi(t)$ and $\xi(t) = \int_0^1 f(s,t) dB(s) = (\int_0^1 \psi dB)\psi(t)$. Clearly $\tilde{F}_\phi = F\phi = \langle \psi, \phi \rangle$ is nuclear, while $S_{\pi} = 0$ if $\pi = \{t_0, \ldots, t_n\} \subset [0,1]\setminus A$ and $S_{\pi} = (\int_0^1 \psi dB)\tilde{B}(l)$ if $\pi \subset A$, where $S_{\pi}$ is defined by (5.1). Therefore $\int \xi dB$ does not exist.

Proposition 5.2 shows the equality of both integrals $\int \xi dB$ and $\int \xi dB$ under certain additional assumptions. This is an open question if the existence of both integrals suffices for their equality.

Examples 5.5 and 5.6 indicate that the existence of the series expansion (1.1) is a quite strong property of the process $\xi$. Below are given certain Sobolev-space type conditions, similar to those proposed by Kuo and Rusek [6], which imply the existence and equality of both integrals, $\int \xi dB$ and $\int \xi dB$.

Since the proof in [6] seems to contain some gaps and the final condition differs from ours in the value of a coefficient ($p!$ instead of $(p+1)!$), we present a complete proof of this result. Moreover our proof does not use the theory of Sobolev spaces, which makes it more elementary.

In what follows below

$$f^+_p(s_1, \ldots, s_p, t) = f_p(s_1, \ldots, s_{p-1}, s_p \wedge t, s_{p \wedge t})$$

and

$$f^-_p(s_1, \ldots, s_p, t) = f_p(s_1, \ldots, s_{p-1}, s \vee t, s \wedge t).$$
Theorem 5.8. Let $\xi$ be given by (4.1) and (4.2), where $f_0$ is continuous. Assume that for some $\alpha > \frac{1}{2}$,

$$M_\alpha^2(\{f_+\}) = \|f_0\|_{L^2(T)}^2 + \sum_{p=1}^{\infty} (p+1)! [U^2_\alpha(f_-^p) + U^2_\alpha(f_+^p)]$$

is finite, where $U^2_\alpha(\cdot)$ is defined in Theorem 1.3. Then both integrals $\int_0^1 \xi \ast dB$ and $\int_0^1 f_\ast dB$ exist, they are equal a.s. and (4.4) holds. Moreover,

$$\frac{1}{2} \|f_\ast dB\|_{L^2(\Omega)} \leq CM_\alpha(\{f_\ast\}),$$

where $C$ depends only on $\alpha$.

Proof. Since $f_-^p + f_+^p = 2f_0$ and $f_1^p T_p^{-1} x D^-_{-c} + f_1^p T_p^{-1} x D^+_{-c} = f_p$, we get $U^2_\alpha(f_0) \leq \frac{1}{2} [U^2_\alpha(f_+^p) + U^2_\alpha(f_-^p)]$. In view of Theorem 4.3 $\int \xi \ast dB$ exists. Using the same argument as in the proof of Theorem 4.3 we have

$$(5.4) \quad f_\ast^\ast(\cdot, t) = \sum_{n \in \mathbb{Z}} c_{p,n}^\ast(\cdot) \chi_n(t)$$

in $L^2([0,1]^{p+1})$, where $\chi_n(t) = \exp(i2\pi n t)$ and $\sum_{n} \|c_{p,n}^\ast\|_{L^2(T_p)}^2 \leq C U^1_\alpha(f_0)$. A similar expansion we have for $f_\ast^p$, with $c_{p,n}^\ast$ replaced by $c_{p,n}$ in (5.4).

Let $\pi = \{t_0, \ldots, t_k\}$ be a partition of $[0,1]$. We have

$$S_\pi = \sum_{p=0}^{\infty} S_{p,\pi} \in L^2(\Omega),$$

where $S_{p,\pi}$ is defined by (5.1) with $\xi$ replaced by $\xi_p$. Using (5.2) and (5.4) we get
\[ S_{p,\pi} = I_{p+1}(f_{p,\pi}) + pI_{p-1}(g_{p,\pi}), \]

where

\[ f_{p,\pi}(\cdot, t) = \sum_{n \in \mathbb{Z}} c_{p,n}^- (\cdot) \prod_{j=1}^k \chi_{n(t_{j-1})}^-(t_{j-1}) \hat{\chi}_{n(t_j)}^+(t_j) I_{T_{p-1}^+ \times D_+}^{(\cdot, t_j)}(t_j) \]

\[ + \sum_{n \in \mathbb{Z}} c_{p,n}^+ (\cdot) \prod_{j=1}^k \chi_{n(t_{j-1})}^+(t_{j-1}) \hat{\chi}_{n(t_j)}^- \hat{\chi}_{n(t_j)}^-(t_j) I_{T_{p-1}^- \times D_-}^{(\cdot, t_j)}(t_j) \]

\[ = \sum_{n \in \mathbb{Z}} c_{p,n}^- (\cdot) \psi_{n,\pi}^- (\cdot, t) + \sum_{n \in \mathbb{Z}} c_{p,n}^+ (\cdot) \psi_{n,\pi}^+ (\cdot, t), \]

and

\[ g_{p,\pi}(s_1, \ldots, s_{p-1}) = 2^{-1} \sum_{n \in \mathbb{Z}} \int_0^{s_1} \cdots \int_0^{s_{p-1}} \chi_{n,\pi}^-(s) \psi_{n,\pi}^- (s, \cdot) ds \]

\[ + 2^{-1} \sum_{n \in \mathbb{Z}} \int_0^{s_1} \cdots \int_0^{s_{p-1}} \chi_{n,\pi}^+(s) \psi_{n,\pi}^+ (s, \cdot) ds. \]

Here \( \chi_{n,\pi}^- (t_j) = \chi_n(t_{j-1}) \) and \( \chi_{n,\pi}^+ (t_j) = \chi_n(t_j) \) if \( t \in (t_{j-1}, t_j] \), \( j = 1, \ldots, k \).

Since \( |\psi_{n,\pi}^-| \leq 1 \) and \( |\psi_{n,\pi}^+| \leq 1 \) we obtain

\[ \|f_{p,\pi}\|_{L^2(T_{p+1})} \leq \sum_n \|c_{p,n}^-\|_{L^2(T_{p-1})} + \|c_{p,n}^+\|_{L^2(T_{p-1})} \leq C[U_\alpha (f_p^-) + U_\alpha (f_p^+)] \]

and by Schwartz inequality

\[ \|g_{p,\pi}\|_{L^2(T_{p-1})} \leq \sum_{n \in \mathbb{Z}} \|c_{p,n}^-\|_{L^2(T_{p-1})} \]

\[ + \sum_{n \in \mathbb{Z}} \|c_{p,n}^+\|_{L^2(T_{p-1})} \]

\[ \leq \sum_{n \in \mathbb{Z}} \|c_{p,n}^-\|_{L^2(T_{p-1})} + \|c_{p,n}^+\|_{L^2(T_{p-1})} \]

\[ \leq C[U_\alpha (f_p^-) + U_\alpha (f_p^+)]. \]
Therefore

\[ \| \sum_{p=q}^{r} S_{p, \pi} \|^2 \leq 2 \| \sum_{p=q}^{r} I_{p+1}(f_{p, \pi}) \|^2 + 2 \| \sum_{p=q}^{r} I_{p-1}(g_{p, \pi}) \|^2 \]

\[ \leq 2 \sum_{p=q}^{r} (p+1)! \left[ \| f_{p, \pi} \|^2 + \| g_{p, \pi} \|^2 \right] \]

\[ \leq 3C^2 \sum_{p=q}^{r} (p+1)! \left[ U_{\alpha}(f^+_{p}) + U_{\alpha}(f^-_{p}) \right] \to 0 \]

as \( p, q \to \infty \), uniformly in all finite partitions \( \pi \) of \([0,1]\).

To complete the proof it is enough to show that for each \( p \geq 1 \),

\( \sum_{p=1}^{r} I_{p+1}(f_{p}) + pI_{p-1}(\text{trf}_{p}) \) in \( L^2(\Omega) \) as mesh \( (\pi) \to 0 \). To this end we shall show that \( \| f_{p, \pi} - f \| \to 0 \) and \( \| g_{p, \pi} - \text{trf}_{p} \| \to 0 \) as mesh \( (\pi) \to 0 \).

Using (5.4) we have

\[ \| f_{p, \pi} - f \|_{L^2(T^{p+1})} \leq \sum_{n \in \mathbb{Z}} \| c^+_{p, n}(\psi^+_{n, \pi} - 1 T^{p-1} \chi_{n}) \|_{L^2(T^{p+1})} \]

\[ + \sum_{n \in \mathbb{Z}} \| c^-_{p, n}(\psi^-_{n, \pi} - 1 T^{p-1} \chi_{n}) \|_{L^2(T^{p+1})} \to 0 \]

as mesh \( (\pi) \to 0 \) by the Dominated Convergence Theorem.

Since

\[ \text{trf}_{p} = 2^{-\frac{1}{2}} \text{trf}_{p} + 2^{-\frac{1}{2}} \text{trf}_{p}^+ \]

and both \( f^+_{p} \) and \( f^-_{p} \) are symmetric in the last two variables we obtain

\[ \text{trf}_{p}^+ = \sum_{n \in \mathbb{Z}} \int_{T^{p-1}} \chi_{n}(s) f^-_{p}(\cdot, s, t) \chi_{n}(t) ds dt \]

\[ = \sum_{n \in \mathbb{Z}} \int c^-_{p, n}(\cdot, s) \chi_{n}(s) ds, \]

\[ = \sum_{n \in \mathbb{Z}} \int c^-_{p, n}(\cdot, s) \chi_{n}(s) ds, \]

\[ = \sum_{n \in \mathbb{Z}} \int c^-_{p, n}(\cdot, s) \chi_{n}(s) ds, \]
where \( \overline{a} \) denotes the complex conjugate to \( a \). A similar expression we obtain for \( \operatorname{trf}_p^+ \). Finally

\[
\| g_{p,n} - \operatorname{trf}_p \|_{L^2(T^{p-1})} \leq 2^{-l} \sum_{n \in \mathbb{Z}} \left\| \int c_{p,n}^-(\cdot,s)(\chi_{n,n}^-(\cdot,s) - \chi_n(\cdot,s))ds \right\|_{L^2(T^{p-1})} + 2^{-l} \sum_{n \in \mathbb{Z}} \left\| \int c_{p,n}^+(\cdot,s)(\chi_{n,n}^+(\cdot,s) - \chi_n(\cdot,s))ds \right\|_{L^2(T^{p-1})}
\]

\[
\leq 2^{-l} \sum_{n \in \mathbb{Z}} \left\| c_{p,n}^- \right\|_{L^2(T^p)} \left\| \chi_{n,n}^- - \chi_n \right\|_{L^2(T)} + 2^{-l} \sum_{n \in \mathbb{Z}} \left\| c_{p,n}^+ \right\|_{L^2(T^p)} \left\| \chi_{n,n}^+ - \chi_n \right\|_{L^2(T)} 
\]

as \( \operatorname{mesh}(n) \to 0 \) by the Dominated Convergence Theorem. The proof of Theorem 5.8 is complete.

It occurs that a simple condition \( N_\alpha(\xi) < \infty \), for some \( \alpha > \frac{1}{2} \), given in Theorem 4.4 implies not only the existence of \( \int \xi dB \) but also the integrability of \( \xi \) in the Stieltjes sense.

**Theorem 5.9.** Assume that for some \( \alpha > \frac{1}{2} \), \( N_\alpha(\xi) < \infty \), where \( N_\alpha(\xi) \) is defined in Theorem 4.4. Then for every partition \( \pi = \{t_0, \ldots, t_k\} \) of \([0,1]\) and any choice \( \{t_j \} \subset \{t_{j-1}, t_j\} \), \( j = 1, \ldots, k \),

\[
S_\pi^* = \sum_{j=1}^{k} \xi(t_j^*)(B(t_j) - B(t_{j-1})]
\]

converges to \( \int_0^1 \xi dB \) as \( \operatorname{mesh}(\pi) \to 0 \).

**Proof.** By (4.8), \( t \to f_p(\cdot,t) \) has absolutely convergent Fourier series, i.e. \( f_p \) can be presented in the form similar to (5.4). Starting from this representation and following essentially all the steps in the proof of Theorem 5.8 we complete the proof of Theorem 5.9.
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