AN EXPLICIT SCHEME FOR THE PREDICTION OF OCEAN ACOUSTIC PROPAGATION IN TH. (U) YALE UNIV NEW HAVEN CT DEPT OF COMPUTER SCIENCE T F CHAN ET AL. JUL 95
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AN EXPLICIT SCHEME FOR THE PREDICTION OF OCEAN ACOUSTIC PROPAGATION IN THREE DIMENSIONS

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SUMMARY

Because of excessive computation time, solving the parabolic equation in higher dimensions by means of implicit finite difference schemes seems to be impractical even if the scheme is unconditionally stable. To economize the computation time and computer storage, a stable explicit finite difference scheme is introduced for the solution of the parabolic equation of the Schrödinger type. This explicit scheme involves five spatial points and is conditionally stable by introducing an additional dissipative term. The complete theory with respect to the stability is proved. An application to a three-dimensional ocean acoustic propagation problem is included to demonstrate its validity.

INTRODUCTION

Many physical problems result in the real application of parabolic equations. A familiar representative parabolic equation is the heat equation with real coefficients. A number of applications (other than heat conduction) arise in the area of quantum mechanics, plasma physics, optics, seismology, ocean acoustics, etc. [1], and result in a form of parabolic equation with complex coefficients. A familiar representative parabolic equation with complex coefficients is the Schrödinger equation. For discussion, the theory of a new stable explicit finite difference scheme as well as a real application are chosen to deal with the Schrödinger equation with dimensions in the form

\begin{equation}
  u_{r} = \sum_{l} \alpha_{l} u_{l}^n.
\end{equation}

For a more general expression, we can include the low order terms to give

\begin{equation}
  u_{r} = \sum_{l} (\beta_{l} u_{l+1}^n + \eta_{l} u_{l-1}^n) + \varphi_{l}.
\end{equation}

As an application, a one-way ocean acoustic sound propagation in three dimensions is represented by

\begin{equation}
  u_{r} = \frac{1}{2} k_{0} \ln \left[ \frac{\gamma_{0} (r, \omega, z)}{\eta_{0}} \right] u
\end{equation}

where \( k_{0} \) is a reference wavenumber and \( \gamma(r, \omega, z) \) is the three-dimensional index of refraction, which is defined as a ratio of a reference sound speed to a three-dimensional sound speed. Eq. (3) is in three-dimensional cylindrical coordinates [2].

A solution exists [3] for Eq. (3) that uses an unconditionally stable implicit finite difference scheme, which discretizes Eq. (3) by means of central finite differences for both \( r \) and \( z \) derivatives. Then the Crank-Nicolson scheme is applied to formulate a large system of sparse matrices. This system was solved by a Yale University [3] preconditioning sparse matrix technique. Results, produced by the Crank-Nicolson scheme [2] are reasonably accurate. However, due to the step-by-step
iteration to solve the system. Excessive computer time was required. This motivated us to develop a more economical, stable explicit finite difference scheme. In the sections to follow, the main discussion is on the introduction of a conditionally stable explicit finite difference thoroughly examining its consistency, stability, and convergence. A theorem to describe the stability of this new scheme is developed and proved. Following the theoretical section, we use a three-dimensional acoustic wave equation arising from the application of underwater wave propagations as a test case to examine the validity of the theory. We examine the accuracy and speed of the theory by comparing it with the solution produced by the Crank-Nicolson scheme. As a physical illustration of the three-dimensional problem, a plot is included to describe intensity effects of the three-dimensional ocean wave propagation.

**A STABLE EXPLICIT SCHEME FOR HIGH DIMENSIONS**

Chan, Shen, and Lee [4] discussed the solution to a model Schrödinger equation, i.e.,

\[ u_t = i u_{zzz} \quad (4) \]

by the finite difference scheme

\[ \frac{u^{n+1}_j - u^n_j}{\Delta t} = \left( \alpha + i \beta \right) \left( \frac{u^n_{j+1} + u^n_{j-1}}{2} - \frac{u^n_{j+2} + u^n_{j-2}}{2} \right) \frac{1}{\Delta x^2} \]

(5)

which is UNSTABLE where \( \alpha \) = \( \alpha \), \( \beta = \beta \).

As a consequence, a number of stable explicit schemes were introduced [4] to solve the parabolic equation of the Schrödinger type. In this paper, a scheme is selected for application and replaces scheme (5) by introducing a dissipative term, which is added to scheme (5) to give

\[ \frac{u^{n+1}_j - u^n_j}{\Delta t} = \left( \alpha + i \beta \right) \left( \frac{u^n_{j+1} + u^n_{j-1}}{2} - \frac{u^n_{j+2} + u^n_{j-2}}{2} \right) \frac{1}{\Delta x^2} \]

(6)

where \( \alpha \) and \( \beta \) are determined to be \( \alpha = -1/4 \), \( \beta = 1/4 \) for least restrictive stability condition. As a generalization of the scheme (6) to the Schrödinger equation of high order [1], consider the multi-dimensional Schrödinger equation

\[ u_{rr} = \sum_{i=1}^m b_i u_{i i} \]

(7)

where the \( b_i \)'s are assumed to have the same sign. Without loss of generality, we assume \( D > 0 \) for \( i = 1, 2, \ldots, m \). We consider the natural extension of scheme (6). The stability requires that \( \| u \| < 1 \). After some simplification, the stability condition can be written as

\[ \left( \sum_{i=1}^m b_i \right) \left( \| u \| \right)^2 < 1 \]

and is obtained when \( \alpha = -1/4 \) and \( \beta = 1/4 \).

The proof appears in its entirety in reference 1 and is outlined below.

**PROOF:** For economy in writing, define

\[ f_j = x_j^2 \quad \eta_j = 4 \sin^2 \frac{b_j}{2}, \quad f_j = b_j \eta_j \]

(8)

The amplification factor \( R \) can be determined to be

\[ R = 1 - \sum_{j=1}^m b_j \eta_j \left( \| u \| \right)^2 \]

(9)

The least restrictive stability constraint is

\[ k \leq \frac{1}{2} \sum_{j=1}^m b_j \eta_j \]

and is obtained when \( \alpha = -1/4 \) and \( \beta = 1/4 \).

The proof appears in its entirety in reference 1 and is outlined below.
we consider two cases: \( a \geq 1/4 \) and \( a < 1/4 \).

**CASE I: \( a \geq 1/4 \)**

\[ g(u_1, u_2, \ldots, u_m) = -a/\left\{ \frac{m}{\sum_{k=1}^{m} f_k} \right\} \]

\[ \text{where} \]

\[ S(w) = -2a/\left\{ \frac{2}{\left[ \sum_{k=1}^{m} f_k \right]^{1/2}} - \frac{2a/\sum_{k=1}^{m} f_k}{\left[ \sum_{k=1}^{m} f_k \right]^{1/2}} \right\} \]

and \( w^r = \sup \{ w : w = \sum_{k=1}^{m} f_k^{1/2} \} \).

From Eqs. (10) and (11), it can be verified that for \( a \geq 1/4 \), the stability condition is

\[ k \leq \min \left\{ -a/\left( \frac{m}{\sum_{k=1}^{m} f_k} \right), -2a/\sum_{k=1}^{m} f_k \right\} \]

**CASE II: \( a < 1/4 \)**

Clearly \( U_1 \) is empty and \( U = U_2 \). It is seen that

\[ G \leq \min \left\{ -2a/\sum_{k=1}^{m} f_k, -a/\left\{ \frac{m}{\sum_{k=1}^{m} f_k} \right\} \right\} \]

The general stability condition is therefore (12).

Clearly, we must have \( a < 0 \). To choose \( a \) and \( b \) such that the stability condition is the least restrictive, we must take \( a = 1/4 \) so that

\[ k \leq \min \left\{ \frac{1}{2b \sum_{k=1}^{m} f_k}, -\frac{2a}{\sum_{k=1}^{m} f_k} \right\} \]

To maximize the right-hand side above, we take \( a = -1/4 \), which gives

\[ k \leq \frac{1}{2 \sum_{k=1}^{m} f_k} \]

establishing the result.

**APPLICATION**

In the three-dimensional ocean, a class of sound wave propagation problems can be represented by a parabolic equation of the Schrödinger type [4]. For prescribed environmental conditions, an application of a three-dimensional problem in sector can be shown as in Figure 1 for its sector region of propagation, where \( r_0 \leq r \leq r_0^*, \) \( 0 \leq \omega \leq 5^\circ, \) and \( 0 \leq z \leq 1600. \) In actual simulation the sector is taken to be \(-20^\circ \leq \omega \leq 20^\circ.\)

An exact solution \( u(r, \omega, z) \) has been obtained [2] and takes the form

\[ u(r, \omega, z) = \sin(\Omega z) e^{ \frac{-m^2}{2\alpha} \frac{1}{r^2} } \]

which satisfies the three-dimensional parabolic equation (3).

The computation speed among explicit finite schemes [1] and an implicit finite scheme [2] using two-dimensional as well as three-dimensional examples has been examined by Chan et al. [5]. The same three-dimensional problem with known exact solution was used by Chan et al. [5]. In particular, the computation speed between each explicit scheme, as described in [1], against the implicit finite difference scheme, as described in [2]. Their findings show a more favorable computation speed for the explicit scheme than the implicit scheme. We extend their study to some three-dimensional effects using the explicit scheme, expressed by Eq. (6).

In the application, the \( \Omega \) is assigned to be \( \pi/100 \) and the modal index \( m \) is taken to be \( 3. \) The source is placed at 50m below the surface and propagates the sound in a regular three-dimensional cylindrical region. The propagation is required to reach the maximum range at 50m, where we can see three-dimensional effects. We limit the propagation to a sector of \( 40^\circ \) (i.e., from \(-20^\circ \) to \( +20^\circ \)) and centered at the origin \((0,0,0).\) For simplicity, the three-dimensional sound speed \( c(r, \omega, z) \) is taken as a constant and the medium is assumed homogeneous. Initial boundary values are generated from the exact solution.

Since our numerical results produce field intensity information at all receiver depths, we can output contour plots for each angle \( \omega \). Figure 2 presents a contour plot of energy flow at \( \omega = 0^\circ.\)
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REFERENCES


