PRODUCT STOCHASTIC MEASURES, MULTIPLE STOCHASTIC INTEGRALS
AND THEIR EXTENSIONS TO NUCLEAR SPACE VALUED PROCESSES

by

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TECHNICAL REPORT 107
June 1985
**Title:** Product Stochastic Measures, Multiple Stochastic Integrals and Their Extensions to Nuclear Space Valued Processes

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A dissertation submitted to the faculty of
the University of North Carolina at Chapel
Hill in partial fulfillment of the require-
ments for the degree of Doctor of Philosophy
in the Department of Statistics

Chapel Hill
1985

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ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to Professor Gopinath Kallianpur for the guidance and encouragement that he has given me as my advisor on this thesis. It is a pleasure to acknowledge him not only for this but for the many hours of stimulating discussion, his constructive criticism and his unlimited suggestions.

In addition I would like to thank Professors C. Baker, S. Cambanis, M. R. Leadbetter and W. L. Smith for their illuminating classes of Probability and Stochastic Processes and for being part of my advisory committee.

I am grateful to the Consejo Nacional de Ciencia y Tecnologia of Mexico for providing financial support during the first three years and to the Statistics Department for its assistance during the last two.

Thanks also go to Ms Ruth Bahr for her epic and careful typing of the manuscript.

Finally, a word of thankfulness to my family for its immense support and to my very close friends in Chapel Hill for the constant and sometimes not so easy discussions that were always a source of second thoughts.
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CHAPTER I

INTRODUCTION

Beginning with the works by Wiener (1938) and Itô (1951), the notion of multiple stochastic integrals (m.s.i.) has been a useful tool in applied and theoretical areas of Probability and Statistics, and several efforts have been made to build a general theory.

The purpose of the first part of this thesis (Chapters 2 and 3) is to define the symmetric tensor product measure (s.t.p.m.) and use this concept systematically to construct multiple stochastic integrals. The latter will be obtained as integrals w.r.t. appropriate product stochastic measures (p.s.m.). The concept of symmetric tensor product measures is central to the construction of product stochastic measures.

In Chapter II of this work we develop the theory of tensor and symmetric tensor products of orthogonally scattered measures. It turns out that the different powers (in the symmetric tensor product sense) of the s.t.p.m.'s are orthogonal and take values in a common, appropriately defined (exponential) Hilbert space. Then we construct multiple integrals with respect to the s.t.p. measure using the theory of integration w.r.t. vector valued measures as developed, for example, in the book by Dunford and Schwartz (1958).

In Chapter III we apply the results of the previous chapter to study symmetric tensor product stochastic measures and multiple stochastic integrals of dependent, non-identically distributed $L^2$-valued independently scattered measures. We identify the appropriate exponential Hilbert space
which is the common range for the s.t.p. stochastic measures and m.s.i. of different orders. We investigate the Gaussian and Poisson situations separately since a more general treatment is possible for these cases. We show that the symmetric tensor product measure approach includes the usual multiple Wiener and Poisson integrals defined by Itô (1951) and Ogura (1972) respectively, as well as the multiple Wiener integrals with dependent integrators of Fox and Taqqu (1984). This establishes a clear relationship between the theory of multiple stochastic integrals and the theory of vector valued measures. We conclude Chapter III with a comparison of our results with previous attempts to define product stochastic measures including the one by Engel (1982) on the $L^2$-theory of products of independently scattered measures.

The second part of this work (Chapters 3 and 4) deals with stochastic processes taking values in infinite dimensional spaces. Realistic models for the investigation of many important problems in Physics, Statistical Mechanics, Geophysics and certain areas of Biology, lead to stochastic processes of this kind. A convenient choice of infinite dimensional spaces is the class of nuclear spaces (more precisely, duals of nuclear spaces).

Nuclear space valued stochastic processes have been considered in the works of K. Itô (1978a, 1978b, 1983) and the papers, among others, of Dawson and Salehi (1980), Shiga and Shimizu (1980), Mitoma (1981a, 1981b) and Miyahara (1981). In most of the above papers, the nuclear space considered is $S(\mathbb{R}^d)$, whose dual $S(\mathbb{R}^d)'$ is the space of tempered distributions. However, in several practical problems, e.g. those occurring in neurophysiology, it is not possible to fix in advance the space in which the stochastic processes take their values (see Kallianpur and Wolpert (1984)).

We begin Chapter IV by defining the nuclear space $\Phi$ we are going to
consider in the remainder of this work. Then we define \( \Phi' \)-valued Wiener processes with continuous positive definite bilinear form \( Q \) on \( \Phi \times \Phi \), and study some related concepts that are used later in this work. We conclude Chapter IV by presenting stochastic integrals with respect to a \( \Phi' \)-valued Wiener process.

The main object of the second part of this work is to develop techniques for the study of nonlinear functionals of \( W_t \). In this direction we construct in Chapter V multiple Wiener integrals (m.W.i.) with respect to a \( \Phi' \)-valued Wiener process \( W_t \). This construction leads to real valued, finite dimensional multiple stochastic integrals w.r.t. dependent non-identically distributed independently scattered measures of the kind considered in the first part of this work. In addition we obtain the Wiener decomposition of the space of \( \Phi' \)-valued nonlinear functionals of \( W_t \) as an inductive limit of appropriate Hilbert spaces. Multiple stochastic integral expansions and stochastic integral representations for nonlinear functionals of \( W_t \) are also obtained in Chapter V. It turns out that the stochastic integrals constructed in Chapter IV are the ones useful in representing nonlinear functionals and \( \Phi' \)-valued square integrable martingales.
CHAPTER II

TENSOR PRODUCT AND MULTIPLE INTEGRALS OF
ORTHOGONALLY SCATTERED MEASURES

We begin this chapter by presenting results on tensor products of orthogonally scattered measures. We include the infinite tensor product case (Theorem 2.1.4) which seems to be considered for the first time and appears as a natural generalization of Theorem 2.1.3. In Section 2.2 we obtain the symmetric tensor product measure of different orthogonally scattered measures which are mutually orthogonal over disjoint sets, and present some of its properties. Finally, in Section 2.3 we apply the theory of integration with respect to vector valued measures to define multiple integrals for the symmetric tensor product measure.

2.1 Tensor and infinite tensor product of orthogonally scattered measures.

The theory of orthogonally scattered measures with values in a Hilbert space has been presented by Masani (1968) among others. We begin this section by presenting a definition and a theorem which are given in the above named work.

Definition 2.1.1. Let $H$ be a real separable Hilbert space with inner product $<\cdot,\cdot>_H$ and norm $||\cdot||_H$, and let $(T,A,\mu)$ be a measure space where $\mu$ is a non-negative, countably additive, $\sigma$-finite measure. Let $A_\mu$ denote the ring

$$A_\mu = \{ A \in A : \mu(A) < \infty \}. $$
The $H$-valued set function $X$ is said to be an **orthogonally scattered measure** (o.s.m.) on $(T,A)$ with values in $H$ and control measure $\mu$ if the next two conditions hold:

i) For each sequence $\{A_n\}_{n \geq 1}$ of disjoint elements in $A_\mu$ such that
$$
\bigcup_{n=1}^{\infty} A_n \in A_\mu, \quad \sum_{n=1}^{m} X(A_n) \text{ converges in } H \text{ to } X\left(\bigcup_{n=1}^{\infty} A_n\right) \text{ when } m \to \infty.
$$

ii) For $A, B \in A_\mu$

$$
\langle X(A), X(B) \rangle_H = \mu(AnB).
$$

It follows from (2.1.1) that

$$
\mu(A) = \|X(A)\|^2_H \quad A \in A_\mu
$$

and that if $A, B \in A_\mu, AnB = \emptyset$ then $X(A)$ and $X(B)$ are orthogonal elements in $H$.

The linear space of $H$

$$
H_X = \text{sp}\{X(A): A \in A_\mu\}
$$

is called the **subspace** of $X$. Condition (2.1.2) enables us to define a well-known isometry between $H_X$ and $L^2(T,A,\mu)$ as follows:

For an $A_\mu$-simple function $f(t) = \sum_{i=1}^{r} c_i 1_{A_i}(t), A_i \in A_\mu, c_i \in \mathbb{R}$ define

$$
I_X(f) = \int_T f(t)dX(t) = \sum_{i=1}^{r} c_i X(A_i)
$$

and for $f \in L^2(T,A,\mu)$ define

$$
I_X(f) = \int_T f(t)dX(t) = \lim_{n \to \infty} \int_T f_n(t)dX(t)
$$

where $\{f_n\}_{n \geq 1}$ is a sequence of $A_\mu$-simple functions converging to $f$ in
The following result is known as the isomorphism theorem.

**Theorem 2.1.1** (Masani (1968)). Let $X$ be as in Definition 2.1.1. Then

$$I_X : f \rightarrow \int f(t)dX(t)$$

is an isometry on $L^2(T,A,\mu)$ onto $H_X \in H$ such that for $A \in A_\mu$, $X(A) = I_X(1_A)$. Then every o.s.m. $X$ carries with it two Hilbert spaces $H_X$ and $L^2(T,A,\mu)$ which are isomorphic under the isometric integral $I_X$.

Similar to the classical theory of real valued product measures (Halmos (1950)), it is possible to develop a theory of tensor products of orthogonally scattered measures. In this direction we give a proof of the next theorem which is established in Chevet (1981). Before doing this we first introduce some notation: For two Hilbert spaces $H_1$ and $H_2$, $H_1 \otimes H_2$ denotes their Hilbert space tensor product (Reed and Simon (1980)) with inner product $<.,.>_H$, and $h_1 \otimes h_2$ denotes the tensor product of the elements $h_1 \in H_1$, $h_2 \in H_2$. Given two real valued measures $\mu$ and $\nu$, $\mu \otimes \nu$ denotes their product measure.

**Theorem 2.1.2** (Chevet (1981)). For each $i=1,\ldots,n$, let $(T_i, A_i, \mu_i)$ be a $\sigma$-finite measure space, $H_i$ a real separable Hilbert space and $X_i$ an o.s.m. on $(T_i, A_i)$ with values in $H_i$ and control measure $\mu_i$. Then there exists a unique orthogonally scattered measure $\otimes X_i$ on $(T_1 \times \ldots \times T_n, A_1 \times \ldots \times A_n)$ with values in $H_1 \otimes \ldots \otimes H_n$ and control measure $\mu_1 \otimes \ldots \otimes \mu_n$ such that

$$\otimes X_i (A_1 \times \ldots \times A_n) = X_1(A_1) \otimes \ldots \otimes X_n(A_n)$$

for $A_i \in A_i = \{A \in A_i : \mu_i(A) < \infty\}$. The o.s.m. $\otimes X_i$ is called the tensor product o.s.m. of $X_1,\ldots,X_n$.

**Proof** It is enough to show it for $n = 2$. Let
\[ R = \{A_i \times A_2 : A_i \in A_i, \ i=1,2\}. \]

For \( A_i \in A_i, \ i=1,2 \) define the \( H_1 \otimes H_2 \)-valued set function
\[
2 \quad \otimes X_i (A_i \times A_2) = X_1(A_i) \otimes X_2(A_2).
\]

Next let \( C(2) = F_0(R) \) be the field generated by \( R \), i.e. the collection of all disjoint unions of elements in \( R \). For \( C \in C(2) \) with \( \mu_1 \otimes \mu_2(C) < \infty \), i.e.
\[
C = \bigcup_{j=1}^{r} (A_j \times B_j)
\]

where \( \mu_1(A_j) < \infty, \mu_2(B_j) < \infty \) and \( A_j \times B_j, j=1,\ldots,r \) are disjoint elements in \( R \), define
\[
2 \quad \otimes X_1(C) \overset{\text{def}}{=} \sum_{j=1}^{r} \otimes X_1(A_j \times B_j) = \sum_{j=1}^{r} X_1(A_j) \otimes X_2(B_j).
\]

Thus
\[
\begin{align*}
\| \otimes X_1(C) \|_{H_1 \otimes H_2}^2 & = \sum_{j=1}^{r} \sum_{k=1}^{r} \langle X_1(A_j), X_1(A_k) \rangle_{H_1} \langle X_2(B_j), X_2(B_k) \rangle_{H_2}. \\
\end{align*}
\]

But \( (A_j \times B_j) \cap (A_k \times B_k) = \emptyset, j \neq k \), then
\[
\langle X_1(A_j), X_1(A_k) \rangle_{H_1} \langle X_2(B_j), X_2(B_k) \rangle_{H_2} = 0 \quad j \neq k
\]

and therefore
\[
\begin{align*}
\| \otimes X_1(C) \|_{H_1 \otimes H_2}^2 & = \sum_{j=1}^{r} \mu_1(A_j) \mu_2(B_j) = \mu_1 \otimes \mu_2(C) < \infty. \\
\end{align*}
\]

Since \( \mu_1 \otimes \mu_2 \) is a \( \sigma \)-finite measure on \( A_1 \times A_2 \), the result follows from the Hahn extension theorem for o.s.m. given in Masani (1968).

Q.E.D.

For convenience of later reference we present the following theorem which is a special case of Theorem 2.1.2. We shall use it frequently on the remainder of the chapter. In this result we require the o.s.m. \( X_1 \)'s
to take values in the same Hilbert space $H$ and the control measures $\mu_i$ to be finite.

Theorem 2.1.3 Let $X_i$ for $i=1,\ldots,n$ be orthogonally scattered measures on $(T_i,A_i)$ with values in a common Hilbert space $H$ and corresponding finite control measures $\mu_i$ for $i=1,\ldots,n$. Then there exists a unique $n$th tensor product orthogonally scattered measure $\otimes_{i=1}^n X_i$ on $(T^n,A^n)$ with values in $H^n = \otimes_{i=1}^n H$ and finite control measure $\mu = \otimes_{i=1}^n \mu_i$ such that for $A_1 \in A_i$ for $i=1,\ldots,n$

\begin{equation}
\otimes_{i=1}^n X_i(A_1 \times \ldots \times A_n) = X_1(A_{11}) \otimes \ldots \otimes X_n(A_{nn})
\end{equation}

and

\begin{equation}
\left\| \otimes_{i=1}^n X_i(A) \right\|_2^2 = \otimes_{i=1}^n \mu_i(A) \quad A \in A^n
\end{equation}

where $T^n = T_1 \times \ldots \times T_n$ and $A^n = A_1 \times \ldots \times A_n$ is the n-fold product $\sigma$-field.

We now consider the extension of Theorem 2.1.3 to infinitely many dimensions. Suppose that $(T_i,A_i,\mu_i)$ for $i \geq 1$ is a sequence of probability spaces and that for each $i=1,2,\ldots$ $X_i$ is an o.s.m. on $(T_i,A_i)$ with values in a common Hilbert space $H$ and control measure $\mu_i$. An example of this situation is given at the end of Section 4.1.2 in Chapter 4 of this work.

We use the following notation (as close as possible to Halmos (1950)):

\begin{align*}
T^\infty &= \otimes_{i=1}^\infty T_i, \\
A^\infty &= \otimes_{i=1}^\infty A_i, \\
T(n) &= \otimes_{i=1}^{n+1} T_i, \\
T^n &= \otimes_{i=1}^n T_i, \\
A^n &= \otimes_{i=1}^n A_i
\end{align*}

and

\begin{equation}
\mu^\infty = \otimes_{i=1}^\infty \mu_i.
\end{equation}

For notation, definitions and basic results concerning infinite tensor products of Hilbert spaces, see Appendix A. Let $u = (u_i)_{i=1}^\infty$ where $u_i = X_i(T_i)$ be the sequence of unit vectors in $H$ from which we construct the infinite
tensor product Hilbert space $\bigotimes_{i=1}^{\infty} H$ whose inner product is denoted by $\langle \cdot, \cdot \rangle_\infty$. For each $n \geq 1$, the $n^{th}$ tensor product $\bigotimes_{i=1}^{n} X_i$ of Theorem 2.1.3 takes values in $\bigotimes_{i=1}^{\infty} H$. This space may be seen as a subspace of $\bigotimes_{i=1}^{\infty} H$ through the injection

$$h_1 \otimes \cdots \otimes h_n \rightarrow h_1 \otimes \cdots \otimes h_n \otimes X_i(T_i)$$

$h_i \in H$ for $i=1, \ldots, n$.

Let $\mathcal{C}$ be the field of cylindrical sets in $\mathcal{A}^\infty$. For $A \in \mathcal{C}$, $A = B \times T^{(n)}$ for some $n \geq 1$, define the $\bigotimes_{i=1}^{\infty} H$-valued set function $X^\infty$ as

$$X^\infty(A) = \bigotimes_{i=1}^{n} X_i(B) \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i).$$

Lemma 2.1.1 The $\bigotimes_{i=1}^{\infty} H$-valued set function $X^\infty$ is well defined and finitely additive on $\mathcal{C}$. Moreover,

$$\|X^\infty(A)\|_\infty^2 = \mu_{\infty}(A) \quad A \in \mathcal{C}.$$

Proof Suppose $A \in \mathcal{C}$, $A = B \times T^{(n)}$, $B \in \mathcal{A}^n$ and $A = C \times T^{(m)}$, $C \in \mathcal{A}^m$, $m < n$.

Then $B = C \times T^{m+1} \times \cdots \times T^n$ and

$${\bigotimes}_{i=1}^{n} X_i(B) = \bigotimes_{i=1}^{m} X_i(C) \otimes X_{m+1}(T_{m+1}) \otimes \cdots \otimes X_n(T_n).$$

Then

$${\bigotimes}_{i=1}^{n} X_i(B) \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i) = \bigotimes_{i=1}^{m} X_i(C) \otimes \bigotimes_{i=m+1}^{\infty} X_i(T_i),$$

and hence $X^\infty$ is unambiguously defined for $A \in \mathcal{C}$.

Next, if $A_1, A_2 \in \mathcal{C}$, $A_1 \cap A_2 = \phi$, then for some $n \geq 1$, $A_1 = B_1 \times T^{(n)}$, $A_2 = B_2 \times T^{(n)}$ and $B_1 \cap B_2 = \phi$. Hence

$$X^\infty(A_1 \cup A_2) = \bigotimes_{i=1}^{n} X_i(B_1 \cup B_2) \otimes \bigotimes_{i=n+1}^{\infty} X_i(T_i)$$
\[ n \mathbf{X}(B_1) \otimes \mathbf{X}(T_1) \otimes \mathbf{X}(B_2) \otimes \mathbf{X}(T_1) = \mathbf{X}(A_1) + \mathbf{X}(A_2) \]

(by Theorem 2.1.3)

\[ n \mathbf{X}(B_1) \otimes \mathbf{X}(T_1) \otimes \mathbf{X}(B_2) \otimes \mathbf{X}(T_1) = \mathbf{X}(A_1) + \mathbf{X}(A_2) \]

i.e., \( \mathbf{X} \) is additive on \( C \).

Finally, if \( A \in C, A = B \times T^{(n)}, B \in A^n \) some \( n \geq 1 \)

\[ || \mathbf{X}(A) ||^2_\infty = || \mathbf{X}(B) ||^2_H \Rightarrow \mu_1 \cdots \mu_n(B) = \mu_\infty(A). \]

Q.E.D.

In fact we now prove that \( \mathbf{X} \) has a \( \sigma \)-additive extension to \( A^\infty \). The following result appears to be new and due to the finiteness of \( \mu_\infty \) its proof does not need the Hahn extension theorem for o.s.m. of Masani (1968).

**Theorem 2.1.4** There exists a unique \( \otimes H \)-valued orthogonally scattered measure \( \mathbf{X} \) on \( (T^\infty, A^\infty) \) with control measure \( \mu_\infty \) (a probability measure), such that for every \( A \in C \) \( A = B \times T^{(n)}, B \in A^n \)

\[ (2.1.8) \]

\[ \mathbf{X}(A) = \mathbf{X}(B) \otimes \mathbf{X}(T_1) \]

and for every \( A \in A^\infty \)

\[ (2.1.9) \]

\[ || \mathbf{X}(A) ||^2_\infty = \mu_\infty(A). \]

**Proof**

**Step 1:** \( \mathbf{X} \) is \( \sigma \)-additive on \( C \).

Let \( A_i \in C \ i=1,2,\ldots A_i \neq \emptyset \) (null set). We need to show that

\[ || \mathbf{X}(A_i) ||^2_\infty \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty. \]

But this follows since from Lemma 2.1.1

\[ || \mathbf{X}(A_i) ||^2_\infty = \mu_\infty(A_i) \quad \text{and} \quad \mu_\infty(A_i) \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty \]

for \( \mu_\infty \) is a probability measure on \( A^\infty \).

**Step 2:** Extension to \( A^\infty \).

Let \( A \in A^\infty = \sigma(C) \), then there exists a sequence of sets \( \{A_n\}_{n \geq 1} \) in \( C \)
such that $\mu^\infty(A\Delta A_n) + 0$ as $n \to \infty$ and $\{X^\infty(A_n)\}_{n \geq 1}$ is a Cauchy sequence in
$\mathcal{H}$:

$$\left\| X^\infty(A_n) - X^\infty(A_m) \right\|_\infty^2 = \mu^\infty(A_n \Delta A_m) + 0 \quad n, m \to \infty.$$ 

Then define $X^\infty(A) = \lim_\infty \cdot \left\| X^\infty(A_n) \right\|$ which has the required properties.

Q.E.D.

**Corollary 2.1.1**

Let $E_i \in A_i \quad i \geq 1$, $A = \times_{i=1}^\infty E_i$. Then

$$X^\infty(A) = \otimes_{i=1}^\infty X_i(E_i).$$

**Proof**

It is known (Halmos (1950)) that $A \in A^\infty$ and

$$\mu^\infty(A) = \prod_{i=1}^\infty \mu_i(E_i).$$

This last expression implies that

$$\sum_{i=1}^\infty \left| (\mu_i(E_i))^{1/2} - 1 \right| < \infty$$

and

$$\sum_{i=1}^\infty \left| \mu_i(E_i) - 1 \right| < \infty.$$ 

Then conditions (1) and (2) of Proposition A.1 in Appendix A are satisfied
and hence $\otimes_{i=1}^\infty X_i(E_i)$ is a decomposable vector. The result follows by taking

$$A_n = E_1 \times \cdots \times E_n \times T(n) \quad n \geq 1$$

in the final part of the proof of the last theorem.

Q.E.D.

**Corollary 2.1.2**

$$X^\infty(T^\infty) = \otimes_{i=1}^\infty X_i(T_i)$$

and

$$\left\| X^\infty(T^\infty) \right\|_\infty^2 = \mu^\infty(T^\infty) = 1.$$
2.2 The symmetric tensor product measure

In this section we obtain results regarding symmetric tensor products of different orthogonally scattered measures with values in the same Hilbert space. We have restricted ourselves to the case where each o.s.m. is bounded (finite control measure) and defined on a σ-field. The reason for these requirements is that we are primarily interested in using the well established theory of vector valued measures in order to construct product stochastic measures and their corresponding integrals, and in this theory these requirements are needed. This limitation may not be very restrictive since one could always study the limit behavior of the product measure or the integral when the control measure goes to infinite, as indeed we do in Chapter 5. Moreover, bounded orthogonally scattered measures have recently become of interest (see Niemi (1984)).

Assumption 2.2.1 Throughout this section, unless otherwise stated, we will make the following assumptions: Let \((T,A)\) be an arbitrary measurable space, \(H\) a separable real Hilbert space with inner product \(<\cdot,\cdot>_H\) and norm \(\| \cdot \|_H\) and \(n\) a fixed but arbitrary natural number. Let \(X_i, i=1,...,n\) be o.s.m.'s on \(A\), taking values in \(H\) and with corresponding finite control measures \(\mu_i, i=1,...,n\). Assume there exist real valued set functions \(\mu_{ij}\) such that for \(A,B \in A\)

\[
(2.2.1) \quad \mu_{ij}(A \cap B) = <X_i(A),X_j(B)>_H \quad i,j=1,...,n.
\]

Applying Cauchy-Schwartz inequality it follows that for each \(i,j=1,...,n\), \(\mu_{ij}\) is a signed measure of bounded variation, where \(\mu_{ii} = \mu_i, i=1,...,n\). At times we shall use the facts that if \(\mu_o\) is a σ-finite non-negative measure on \((T,A)\) such that \(\mu_i << \mu_o, i=1,...,n\), then \(\mu_{ij} << \mu_o, i,j=1,...,n\).
and if

\[(2.2.2) \quad r_{ij}(t) = \frac{du_{ij}}{du_0}(t)\]

then \(R(t) = (r_{ij}(t))\) is an \(n \times n\) non-negative definite matrix a.e. \(du_0\).

**Symmetric tensor products** In order to set up notation we now present some facts about symmetric tensor products of Hilbert spaces (see Guichardet (1972)). As in Section 2.1 let \(H^n\) denote the \(n\)-fold tensor product of \(H\). For \(h_1 \otimes \ldots \otimes h_n \in H^n\) define

\[(2.2.3) \quad \sigma(h_1 \otimes \ldots \otimes h_n) = \frac{1}{n!} \sum_{\Pi} (h_{\Pi_1} \otimes \ldots \otimes h_{\Pi_n})\]

where \(\Pi = (\Pi_1, \ldots, \Pi_n)\) is a permutation of \((1,2,\ldots,n)\). The \(n\)-fold symmetric tensor product Hilbert space \(H^n\) is the closed subspace of \(H^n\) generated by elements of the form

\[
\sum_{k=1}^{m} c_k \sigma^n(h_{i_k} \otimes \ldots \otimes h_{i_{m-k}})
\]

where \(c_k \in \mathbb{R}, h_{i_k} \in H, k=1,\ldots,m, i=1,\ldots,n\).

The operator \(\sigma^n\) can be extended to an orthogonal projection operator on \(H^n\) whose range is \(H^n\). We write

\[
h_1 \otimes \ldots \otimes h_n = \sigma^n(h_1 \otimes \ldots \otimes h_n)
\]

where

\[(2.2.4) \quad h_1 \otimes \ldots \otimes h_n = \sigma^n(h_1 \otimes \ldots \otimes h_n) \quad h_i \in H, i=1,\ldots,n\]

and notice that for \(g_i \in H, i=1,\ldots,n\)

\[(2.2.5) \quad \langle \otimes h_i, \otimes g_i \rangle_{H^n} = \frac{1}{n!} \sum_{\Pi} \langle h_{\Pi_1}, g_{\Pi_1} \rangle_{H} \ldots \langle h_{\Pi_n}, g_{\Pi_n} \rangle_{H}\]

Thus in particular, if \(h^n = \otimes h, h \in H\), then
For \( h_i \in H \) \( i=1,\ldots,n \), \( h_1 \otimes \cdots \otimes h_n \) can be represented as a linear combination of elements of the form \( h^\otimes_n \) \( h \in H \):

\[
(2.2.7) \quad h_1 \otimes \cdots \otimes h_n = \sum_{P_L} \frac{1}{n!} \sum_{j=0}^{n-1} (-1)^j \sum_{N \in P_L} \sum_{N^c} \left( (1 \cdot c(1) h_1 + \cdots + 1 \cdot c(n) h_n \right)^\otimes_n
\]

where \( P_L \) is the set of subsets of \( \{1,\ldots,n\} \) with \( l \) elements.

**Symmetric tensor product measure** The first result of this section gives the symmetric tensor product measure of \( X_1,\ldots,X_n \), which is a Hilbert space \( (H^\otimes_n\text{-valued}) \) measure. Although it can be proved for any o.s.m.'s not necessarily bounded, we assume them as in the beginning of this section.

**Theorem 2.2.1** Let \( X_1,\ldots,X_n \) be orthogonally scattered measures as in Assumption 2.2.1. Then there exists a unique \( H^\otimes_n\)-valued measure \( \otimes X_i \) on \( (T^n,A^n) \) such that for \( A_i \in A \) \( i=1,\ldots,n \)

\[
(2.2.8) \quad \otimes X_i(A_1 \times \cdots \times A_n) = X_1(A_1) \otimes \cdots \otimes X_n(A_n).
\]

**Proof** Let \( \otimes X_i \) be the \( H^\otimes_n\)-valued o.s.m. on \( (T^n,A^n) \) given by Theorem 2.1.3. For \( A \in A^n \) define

\[
(2.2.9) \quad \otimes X_i(A) = \sigma^\otimes_n (\otimes X_i(A))
\]

where \( \sigma^\otimes_n \) is the projection operator on \( H^\otimes_n \) with range \( H^\otimes_n \) defined in (2.2.3). Then for \( A \in A^n \)
\[
(2.2.10) \quad \left\| \otimes X_i(A) \right\|_{\mathcal{H}^n} \leq \sum_{i=1}^{n} \left\| \otimes X_i(A) \right\|_{\mathcal{H}^n}
\]
and \( \otimes X_i \) is a finitely additive \( \mathcal{H}^n \)-valued measure on \( A^n \). Next since
\[
\otimes X_i \text{ is } \sigma\text{-additive on } A^n \quad \text{ thus continuous by above at the empty set - it }
\]
follows from (2.2.10) that \( \otimes X_i \) is continuous by above at the empty set
\[
\text{ and then } \sigma\text{-additive on } A^n.
\]
Finally, by Theorem 2.1.3, \( \otimes X_i \) is the unique \( \sigma \)-s.m. on \( (T^n, A^n) \) with
values in \( \mathcal{H}^n \) such that
\[
\otimes X_i(A_1 \times \ldots \times A_n) = X_i(A_1) \otimes \ldots \otimes X_n(A_n)
\]
for \( A_i \in A, \ i = 1, \ldots, n \), from which (2.2.8) follows.

Q.E.D.

**Corollary 2.2.1** Under the assumptions of the last theorem, for \( A \in A^n \)
\[
(2.2.11) \quad \left\| \otimes X_i(A) \right\|_{\mathcal{H}^n}^2 \leq \sum_{i=1}^{n} \mu_i(A).
\]
The proof follows using (2.2.10) above and (2.1.7) in Theorem 2.1.3.

The above corollary gives an upper bound for the norm of \( \otimes X_i(A) \),
\( A \in A^n \). We shall obtain an exact expression for this norm (Corollary 2.2.2)
which uses the signed measures \( \mu_{ij} \) defined in (2.2.1). We first present
a more general result.

**Lemma 2.2.1** For \( A, B \in A^n \)
\[
(2.2.12) \quad < \otimes X_i(A), \otimes X_i(B) >_{\mathcal{H}^n} = \frac{1}{n!} \sum_{\Pi} \mu_1 \Pi_1 \otimes \ldots \otimes \mu_n \Pi_n \left( \Pi(A^B) \right) = \frac{1}{n!} \sum_{\Pi} \mu_1 \Pi_1 \otimes \ldots \otimes \mu_n \Pi_n \left( \Pi(A^B) \right)
\]
where for \( A \in A^n \) and \( \Pi = (\Pi_1, \ldots, \Pi_n) \) a permutation of \( (1, \ldots, n) \) \( \Pi_n \) is
defined by

\[(2.2.13) \quad A^n = \{(t_1, \ldots, t_n) \in T^n: (t_{\pi_1}, \ldots, t_{\pi_n}) \in A\}.\]

**Proof.**

**Step 1.** Assume that \(A, B \in A^n\), \(A = A_1 \times \ldots \times A_n\), \(B = B_1 \times \ldots \times B_n\)

Where \(A_i, B_i \in A\) \(i=1, \ldots, n\). Then from (2.2.1), (2.2.5) and Theorem 2.2.1

\[
< \otimes X_i(A), \otimes X_i(B)>_{\mathcal{H}\mathcal{O}_n} = \left< X_1(A_1) \otimes \ldots \otimes X_n(A_n), X_1(B_1) \otimes \ldots \otimes X_n(B_n) \right>_{\mathcal{H}\mathcal{O}_n}
\]

\[
= \frac{1}{n!} \sum_{i=1}^{n} <X_1(A_i), X_1(B_i)>_{\mathcal{H}\mathcal{O}_n} \ldots <X_n(A_i), X_n(B_i)>_{\mathcal{H}\mathcal{O}_n}
\]

\[
= \frac{1}{n!} \sum_{i=1}^{n} \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \mu_1^{m_1} \otimes \ldots \otimes \mu_n^{m_n} ((A_1^1 \times \ldots \times A_n^1) \cap (B_1^1 \times \ldots \times B_n^1))
\]

\[
= \frac{1}{n!} \sum_{i=1}^{n} \mu_1^{m_1} \otimes \ldots \otimes \mu_n^{m_n} (A \cap B).
\]

**Step 2.** Assume \(A, B \in A^n\) are of the form

\[
A = \bigcup_{j=1}^{m_1} (A_j^1 \times \ldots \times A_{j'}^1), \quad B = \bigcup_{k=1}^{m_2} (B_k^1 \times \ldots \times B_{k'}^1)
\]

Where \(A_j^1 \times \ldots \times A_{j'}^1, B_k^1 \times \ldots \times B_{k'}^1\) \(j=1, \ldots, m_1, k=1, \ldots, m_2\) are as in Step 1 and

\[
(A_j^1 \times \ldots \times A_{j'}^1) \cap (A_{j'}^1 \times \ldots \times A_j^1) = \emptyset \quad \text{if} \quad j \neq j', \quad \text{and} \quad (B_k^1 \times \ldots \times B_{k'}^1) \cap (B_{k'}^1 \times \ldots \times B_k^1) = \emptyset \quad \text{if} \quad k \neq k'.
\]

Then since \(\otimes X_i\) is additive on \(A^n\), from Step 1 we have that

\[
< \otimes X_i(A), \otimes X_i(B)>_{\mathcal{H}\mathcal{O}_n} = \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \sum_{i=1}^{n} \mu_1^{m_1} \otimes \ldots \otimes \mu_n^{m_n} ((A_j^1 \times \ldots \times A_{j'}^1) \cap (B_k^1 \times \ldots \times B_{k'}^1))
\]

\[
= \frac{1}{n!} \sum_{i=1}^{n} \mu_1^{m_1} \otimes \ldots \otimes \mu_n^{m_n} (A \cap B).
\]
Step 3 Assume \( A, B \in A^n \). Then there exist sequences \( \{A_m\}_{m \geq 1}, \{B_m\}_{m \geq 1} \) of sets in \( A^n \) as in Step 2 such that
\[
\forall i \leq n, \quad \mu_i(A, AA) \to 0, \quad \mu_i(B, AB) \to 0 \\
i=1 \quad m=1 \quad m=1 \quad m \to \infty
\]
and
\[
\forall i \leq n, \quad \langle \diamond X_i(A), \diamond X_i(B) \rangle_{H^n} = \lim \langle \diamond X_i(A_m), \diamond X_i(B_m) \rangle_{H^n} \\
i=1 \quad i=1 \quad \forall n \quad m=1 \quad i=1 \quad m \to \infty
\]
Then it is enough to show that
\[
\sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B)_{\Pi} \to \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B)_{\Pi} \\
\]
Using the fact that for all \( \Pi \) and \( A, B \in A^n \)
\[
|\mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B)_{\Pi}| \leq \{ \diamond \mu_i(A) \}_{i=1}^n \{ \diamond \mu_i(B) \}_{i=1}^n
\]
we have that
\[
= \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B)_{\Pi} - \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A_m \cap B_{m\Pi_n}) \\
- \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A_{m\Pi_n} \cap (A \cap B)_{\Pi}) \\
= \frac{1}{n!} \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B \cap A_m^C) \\
+ \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B \cap (B_m^C)^C) \\
- \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A_m \cap B \cap A_m^C) \\
- \sum_{\Pi} \mu_{1\Pi_1} \ldots \mu_{n\Pi_n} (A \cap B \cap (B_m^C)^C)
\[
\frac{1}{n!} \sum_{\Pi} \mu_1^{\Pi_1} \ldots \mu_n^{\Pi_n} (A \cap B \cap \cdots \cap C) \\
+ \frac{1}{n!} \sum_{\Pi} \mu_1^{\Pi_1} \ldots \mu_n^{\Pi_n} (A_1 \cap B_1 \cap \cdots \cap B_m) \\
+ \frac{1}{n!} \sum_{\Pi} \mu_1^{\Pi_1} \ldots \mu_n^{\Pi_n} (A_1 \cap B_1 \cap \cdots \cap B_m) \\
+ \frac{1}{n!} \sum_{\Pi} \mu_1^{\Pi_1} \ldots \mu_n^{\Pi_n} (A_1 \cap B_1 \cap \cdots \cap B_m) \\
\leq \left\{ \sum_{i=1}^{n} \mu_i(A \setminus A_m) \right\}^2 \left\{ \sum_{i=1}^{n} \mu_i(B) \right\}^2 + \left\{ \sum_{i=1}^{n} \mu_i(A) \right\}^2 \left\{ \sum_{i=1}^{n} \mu_i(B \setminus B_m) \right\}^2 \\
+ \left\{ \sum_{i=1}^{n} \mu_i(A_m \setminus A) \right\}^2 \left\{ \sum_{i=1}^{n} \mu_i(B_m) \right\}^2 + \left\{ \sum_{i=1}^{n} \mu_i(A_m) \right\}^2 \left\{ \sum_{i=1}^{n} \mu_i(B_m \setminus B) \right\}^2
\]

which goes to zero as \( m \to \infty \), since \( \mu_i(A \Delta A_m) \to 0 \) and \( \mu_i(B \Delta B_m) \to 0 \).

Corollary 2.2.2 If \( A \in \mathcal{A}^n \), then

\[
(2.2.14) \quad \left\| \sum_{i=1}^{n} \mu_i(A) \right\|_{L^2}^2 = \frac{1}{n!} \sum_{\Pi} \mu_1^{\Pi_1} \ldots \mu_n^{\Pi_n} (A \cap A) \\
\]

We now consider special cases for which the previous results simplify.

Symmetric sets A set \( A \in \mathcal{A}^n \) is called a symmetric set if \( A^\Pi = A \) for all permutation \( \Pi \) of \( \{1, \ldots, n\} \), where \( A^\Pi \) is defined in (2.2.13). The \( \sigma \)-field of symmetric sets of \( \mathcal{A}^n \) (Dellacherie and Meyer (1978)) is denoted by \( \mathcal{A}^\mathcal{A}^n \). Since for \( A \in \mathcal{A}^\mathcal{A}^n \) \( A \cap A = A \) for all \( \Pi \), we obtain the following result which is a generalization to the case of several orthogonally scattered measures of corollary to Theorem 1 on Chevet (1981).

Corollary 2.2.3 The vector measure \( \sum_{i=1}^{n} X_i \) is an \( \mathcal{H}^n \)-valued orthogonally scattered measure on \( (T^n, \mathcal{A}^\mathcal{A}^n) \) with control measure \( \mathcal{H}^n \) given by
Antisymmetric sets  Let $\Pi$ denote the identity permutation of $(1, 2, \ldots, n)$. A set $A \in A^n$ is called an antisymmetric set if $A \cap A^\Pi = \emptyset$ for all permutation $\Pi$ of $(1, \ldots, n)$ distinct from $\Pi$. For this kind of set Corollary 2.2.2 simplifies as follows:

Corollary 2.2.4  If $A \in A^n$ is an antisymmetric set, then

$$
|| \bigoplus_{i=1}^n X_i(A) ||^2_{H^{\otimes n}} = \frac{1}{n!} \bigoplus_{i=1}^n \mu_i(A).
$$

The next result is an application of the last corollary to the case when $T$ is an interval of the real line and the measures $\mu_1, \ldots, \mu_n$ are non-atomic. This will be a useful result in the next section.

Corollary 2.2.5  Let $T \subseteq \mathbb{R}$ be an interval of the real line, $A = \mathcal{B}(T)$ and suppose that $\mu_1, \ldots, \mu_n$ are finite non-atomic measures on $(T, A)$. For each permutation $\Pi = (\Pi_1, \ldots, \Pi_n)$ of $(1, \ldots, n)$ let

$$
T^n_\Pi = \{(t_1, \ldots, t_n) \in T^n : t_{\Pi_1} < \ldots < t_{\Pi_n}\}.
$$

Then for each $\Pi$ $T^n_\Pi$ is an antisymmetric set of $A^n$ and for each $A \in A^n$

$$
\bigoplus_{i=1}^n \mu_i(A) = n! \sum_{\Pi} || \bigoplus_{i=1}^n X_i(A \cap T^n_\Pi) ||^2_{H^{\otimes n}}.
$$

The $\sigma$-field

$$
A^n_\Pi = A \cap T^n_\Pi
$$

is called the $\sigma$-field of antisymmetric subsets of $T^n_\Pi$ corresponding to $\Pi$. 
Proof First note that if \( \Pi \) and \( \Pi^* \) are two distinct permutations of \((1, \ldots, n)\) then \( T_\Pi^n \cap T_{\Pi^*}^n = \emptyset \). Let \( S^n = \cup T_\Pi^n \) where the union is taken over all permutations \( \Pi \) of \((1, \ldots, n)\). Then

\[
(S^n)^c = \{(t_1, \ldots, t_n) \in T^n: t_i = t_j \text{ for some } i \neq j\}
\]

and since the measures \( \mu_1, \ldots, \mu_n \) are non-atomic

\[
\sum_{i=1}^{n} \mu_i((S^n)^c) = 0.
\]

Hence

\[
\sum_{i=1}^{n} \mu_i(A) = \sum_{i=1}^{n} \mu_i(\bigcup \{T_\Pi^n: \Pi \in \Pi\}) \quad A \in \mathcal{A}^n.
\]

But for each permutation \( \Pi \), \( T_\Pi^n \) is an antisymmetric set; then using Corollary 2.2.4

\[
\left\| \sum_{i=1}^{n} x_i(\bigcup \{T_\Pi^n: \Pi \in \Pi\}) \right\|_H^2 = \frac{1}{n!} \sum_{i=1}^{n} \mu_i(\bigcup \{T_\Pi^n: \Pi \in \Pi\}) \quad A \in \mathcal{A}^n
\]

and hence

\[
\sum_{i=1}^{n} \mu_i(A) = n! \sum_{i=1}^{n} \left\| \sum_{i=1}^{n} x_i(\bigcup \{T_\Pi^n: \Pi \in \Pi\}) \right\|_H^2 \quad A \in \mathcal{A}^n.
\]

Q.E.D.

We now take into consideration a concept that plays an important role in the theory of integration with respect to vector valued measures.

Semivariation of \( \sum_{i=1}^{n} x_i \). We first review the definition of semivariation of a bounded vector valued measure and related concepts. We follow Kussmaul (1977).

Let \( M \) be a bounded vector valued measure on a field \( \mathcal{A}_0 \) of a set \( S \) with values in a Banach space \((E, \| \cdot \|)\). The semivariation of \( M \) is the extended nonnegative function \( sv(M; \cdot) \) whose value on a set \( A \in \mathcal{A}_0 \) is given by

\[
sv(M; A) = \sup \left\| \sum_{j} \alpha_j M(A_j) \right\|
\]
where the supremum is taken over all finite partitions \((A_j)\) of \(A_0\) into disjoint sets \(A_j \in A_0\), and all finite collections \((\alpha_j)\) of scalars \(\alpha_j\) satisfying \(|\alpha_j| \leq 1\). The set function \(sv(M;\cdot)\) is extended to the family of all subsets \(B\) of \(S\) by defining

\[
sv(M;B) = \inf\{sv(M;A) : B \subseteq A, A \in A_0\}.
\]

A subset \(B \subseteq S\) is called an \(M\)-null set if \(sv(M;B) = 0\). The exceptional sets for \(M\)-almost everywhere convergence are these \(M\)-null sets. For two scalar valued functions \(f\) and \(g\) on \(S\) we define

\[
d(f,g) = \inf\{\alpha + sv(M;|f-g| > \alpha) : \alpha \geq 0\}.
\]

Convergence with respect to the topology generated by the pseudometric \(d\) is called convergence in \(M\)-measure.

The following result (Diestel and Uhl (1977) page 14) is known in the theory of vector valued measures as the Bartle-Dunford-Schwartz Theorem. It will play a key role in Section 2.3. We write it here adapted to the \(H^n\)-valued bounded measure \(\otimes X_i\) of Theorem 2.2.1.

**Lemma 2.2.2** There exists a finite nonnegative measure \(\nu\) on \(A^n\) such that

\[
a) \quad \nu(A) \leq \inf_{i=1}^{n} sv(\otimes X_i;A) \\
and \\
b) \quad \lim_{\nu(A) \to 0} \inf_{i=1}^{n} sv(\otimes X_i;A) = 0.
\]

We are not able to compute exact expressions for the semivariation of \(\otimes X_i\) nor for the measure \(\nu\) of Lemma 2.2.2. However, we will find very useful upper and lower bounds for \(sv(\otimes X_i;A)\).

**Lemma 2.2.3** Let \(X_1, \ldots, X_n\) be o.s.m.'s and \(\nu_1, \ldots, \nu_n\) be measures as in
Assumption 2.2.1. Then

\[ (2.2.19) \quad sv( \otimes X_i;A) \leq \{ \mu_1, \ldots, \mu_n(A) \}^{\frac{1}{2}}. \]

Proof. Let \((\alpha_j)_{j=1}^m \) be real numbers such that \(|\alpha_j| \leq 1\) and let \(A_1, \ldots, A_m\) be disjoint elements in \(A^n\) such that \(\bigcup_{j=1}^m A_j = A\) for \(m \geq 1\). Then from (2.2.9) and Theorem 2.1.3

\[
\left\| \sum_{j=1}^m \alpha_j \otimes X_i(A_j) \right\|_{H^m}^2 = \left\| \alpha \left( \sum_{j=1}^m \alpha_j \otimes X_i(A_j) \right) \right\|_{H^m}^2 \leq \left\| \sum_{j=1}^m \alpha_j \otimes X_i(A_j) \right\|_{H^m}^2 = \sum_{j=1}^m \sum_{k=1}^m \alpha_j \alpha_k \otimes X_i(A_j), \otimes X_i(A_k) \right\|_{H^m} \leq \sum_{j=1}^m \alpha_j^2 \otimes \mu_n(A_j) \leq \sum_{j=1}^m \mu_j \otimes \ldots \otimes \mu_n(A) = \mu_1 \otimes \ldots \otimes \mu_n(A)
\]

and hence (2.2.19) follows. Q.E.D.

To obtain a lower bound for the semivariation of \( \otimes X_i \) we have to assume an additional condition on the measurable space \((T,A)\) and on the control measures \(\mu_1, \ldots, \mu_n\).

Lemma 2.2.4 Let \(T \subseteq \mathbb{R}\) be an interval of the real line, \(A = B(T)\) and \(X_1, \ldots, X_n\) be o.s.m.'s as in Assumption 2.2.1 with finite non-atomic control measures \(\mu_1, \ldots, \mu_n\). Then for \(A \in A^n\)

\[ (2.2.20) \quad \frac{1}{n!} \sum_{i=1}^n \{ \otimes \mu_i(A) \}^{\frac{1}{2}} \leq sv( \otimes X_i;A) \leq \{ \otimes \mu_i(A) \}^{\frac{1}{2}}. \]

Proof. By Corollary 2.2.5, if \(A \in A^n\)

\[
\otimes (A) = n! \sum_{\Pi} \left\| \otimes X_i(A^n_{\Pi}) \right\|_{H^m}^2
\]

where for each permutation \(\Pi\) of \((1, \ldots, n) \) \(T^n_{\Pi}\) is defined in (2.2.17).
Next from Proposition 11 of Diestel and Uhl (1977), giving a lower bound for the semivariation of a vector valued measure,

\[ \sup_{i=1}^{n} \left\| \bigotimes_{i=1}^{n} X_i(B) \bigotimes_{i=1}^{n} A \right\|_{H^n} \leq \text{sv}(\bigotimes_{i=1}^{n} X_i;A). \]

Then

\[ n! \sum_{i=1}^{n} \left\| \bigotimes_{i=1}^{n} X_i(A_{nT_i}) \right\|_{H^n}^2 \leq n! \sum_{i=1}^{n} \{\text{sv}(\bigotimes_{i=1}^{n} X_i;A)\}^2 \]

and therefore

\[ \frac{1}{n!} \left\{ \bigotimes_{i=1}^{n} \mu_i(A) \right\} \leq \text{sv}(\bigotimes_{i=1}^{n} X_i;A). \]

The upper bound in (2.2.20) follows from the last lemma. Q.E.D.

The next two results are consequences of the above lemma. They characterize convergence in \( \bigotimes_{i=1}^{n} X_i \)-measure in terms of the control measures \( \mu_1, \ldots, \mu_n \).

**Corollary 2.2.6** Under the assumptions of Lemma 2.2.4, for a sequence \( \{A_m\}_{m \geq 1} \) in \( A^n \), \( \text{sv}(\bigotimes_{i=1}^{n} X_i;A_m) \to 0 \) if and only if \( \bigotimes_{i=1}^{n} \mu_i(A_m) \to 0 \).

**Corollary 2.2.7** Under the assumptions of Lemma 2.2.4, a sequence of real \( A^n \)-measurable functions \( \{f_m\}_{m \geq 1} \) converges in \( \bigotimes_{i=1}^{n} X_i \)-measure to real valued function \( f \) on \( T^n \) if and only if \( f_m \) converges to \( f \) in \( \bigotimes_{i=1}^{n} \mu_i \)-measure.

**Orthogonality** We now study a special property of symmetric tensor product measures. We assume that for each \( n \geq 1 \) we have \( X_1, \ldots, X_n \) orthogonally scattered measures as in Assumption 2.2.1, all taking values in the same Hilbert space \( H \).

The Hilbert Exponential space of \( H \) (Guichardet (1972)) is the Hilbert space
(2.2.21) \[ \text{EXP}(H) = \sum_{n \geq 0} H^\otimes n \quad (H^0 \equiv \mathbb{R}) \]

that is, the set of all sequences \( x = (x_n) \), \( x_n \in H^\otimes n \), \( n \geq 0 \)

such that

\[ \sum_{n=0}^{\infty} \| x_n \|_H^2 < \infty, \text{ with inner product} \]

(2.2.22) \[ \langle x, y \rangle_e = \sum_{n=0}^{\infty} \langle x_n, y_n \rangle_H^\otimes n. \]

Since for each \( n \geq 0 \) \( H^\otimes n \) may be seen as a subspace of \( \text{EXP}(H) \)

\((0, \ldots, x_n, \ldots) \), then for each \( n \geq 1 \)

\[ n \cdot x_i : A^n \rightarrow \text{EXP}(H) \]

is a vector valued measure with values in \( \text{EXP}(H) \). Therefore symmetric tensor product measures of different orders may be realized as taking values in the same Hilbert space, viz, \( \text{EXP}(H) \) and we have the following result.

**Lemma 2.2.5** If \( n_1 \neq n_2 \), then for all \( A_1 \in A^n_1 \) and \( A_2 \in A^n_2 \), \( n_1 \cdot x_i (A_1) \) is orthogonal to \( n_2 \cdot x_i (A_2) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_e \) in \( \text{EXP}(H) \).

The proof follows since \( n_1 \cdot x_i \) and \( n_2 \cdot x_i \) are \( H^\otimes_{n_1} \) and \( H^\otimes_{n_2} \) valued respectively and \( H^\otimes_{n_1} \) is orthogonal to \( H^\otimes_{n_2} \) in \( \text{EXP}(H) \) if \( n_1 \neq n_2 \).

The \( n \)th symmetric tensor product measure of an o.s.m. with itself To conclude this section we turn to the special case of symmetric tensor product measures of an orthogonally scattered measure with itself. We show how previous results are simplified and other new results can be obtained. We take \( X \) to be an o.s.m. defined on an arbitrary measurable space \((T, \mathcal{A})\), with values in a real separable Hilbert space \( H \) and finite control measure \( \mu \) (which is not assumed to be nonatomic). We may take \( H \) to be \( H^X \), the linear subspace
generated by $X$ as in (2.1.3). For $n \geq 1$ let $\chi^\otimes n$ be the $n^{th}$ symmetric tensor product measure on $A^n$ with values in $\text{EXP}(H_\chi)$ (or $H_\chi^\otimes n$) given by Theorem 2.2.1, and let $\mu^\otimes n$ denote the $n$-fold product measure of $\mu$ with itself. The main properties of $\chi^\otimes n$ are summarized in the next result in which some possible simplifications of earlier results are shown.

**Proposition 2.2.1**

a) For $A_1, \ldots, A_n \in A$, $n \geq 1$.

$$\chi^\otimes n(A_1 \times \ldots \times A_n) = \chi_1(A_1) \otimes \ldots \otimes \chi_n(A_n).$$

b) If $A \in A^n$ and $B \in A^m$.

$$\langle \chi^\otimes n(A), \chi^\otimes m(B) \rangle_H = \delta_{nm} \langle \chi^\otimes n(A), \chi^\otimes m(B) \rangle_{H_\chi^\otimes n}$$

$$= \delta_{nm} \frac{1}{n!} \sum \mu^\otimes n(AnB) = \delta_{nm} \frac{1}{n!} \sum \mu^\otimes n(A^\otimes nB).$$

c) $||\chi^\otimes n(A)||^2_{H_\chi^\otimes n} = \frac{1}{n!} \sum \mu^\otimes n(AnA^\otimes n) \leq \mu^\otimes n(A)$, $A \in A^n$, $n \geq 1$.

d) The vector measure $\chi^\otimes n$ is an $H_\chi^\otimes n$-valued orthogonally scattered measure on $(T^n, A^\otimes n)$ with control measure $\mu^\otimes n$.

e) If $A \in A^n$ is an antisymmetric set

$$||\chi^\otimes n(A)||^2_{H_\chi^\otimes n} = \frac{1}{n!} \mu^\otimes n(A).$$

The proof of (a) follows from Theorem 2.2.1. Lemmas 2.2.1 and 2.2.5 imply (b) and Corollaries 2.2.2, 2.2.3 and 2.2.4 imply (c), (d) and (e) respectively.

In particular, Lemma 2.2.4 applies to $\chi^\otimes n$. However, in that lemma we have assumed that $T$ is an interval of the real line and the control measure is nonatomic. In the case of $\chi^\otimes n$ it is possible to improve Lemma 2.2.4 and
compute an exact expression for the semivariation of $X^n$, without assuming that $T \subseteq R$ or $\mu$ is nonatomic.

Lemma 2.2.6  Let $X$ be an o.s.m. on an arbitrary measurable space $(T, \mathcal{A})$ with values in $H_X$ and finite control measure $\mu$ (not necessarily nonatomic). Then for $A \in \mathcal{A}^n$

$$sv(X^n; A) = \left\{ \frac{1}{n!} \sum_{\Pi} \mu^\oplus_n(A \cap A') \right\}^{\frac{1}{2}} = \| X^n(A) \|_{H_X}.$$  

Proof  From (c) in the last proposition and the definition of $sv(X^n; A)$

$$\frac{1}{n!} \sum_{\Pi} \mu^\oplus_n(A \cap A') = \| X^n(A) \|_{H_X}^2 \leq sv(X^n; A).$$  

On the other hand if $|a_i| \leq 1$ for $i = 1, \ldots, m$ are real numbers and $A_1, \ldots, A_m$ are disjoint sets in $\mathcal{A}^n$, $U A_i = A$, then for each permutation $\Pi$ of $(1, \ldots, n)$

$A_1 \supseteq \ldots \supseteq A_m$ are disjoint sets in $\mathcal{A}^n$ and $A = U_i A_i$. Thus, using (b) in the last proposition and the fact that $\mu^\oplus_n$ is a positive measure on $\mathcal{A}^n$ we obtain

$$\| \sum_{i=1}^m \sum_{j=1}^m a_i a_j \|_{H_X}^2 = \sum_{i=1}^m \sum_{j=1}^m \sum_{A_i \cap A_j} \mu^\oplus_n(A_i \cap A_j) \leq \frac{1}{n!} \sum_{i=1}^m \sum_{j=1}^m \sum_{A_i \cap A_j} \mu^\oplus_n(A_i \cap A_j)$$

Therefore

$$sv(X^n; A) ^2 \leq \frac{1}{n!} \sum_{\Pi} \mu^\oplus_n(A \cap A').$$

and the lemma follows by using (2.2.24).

Q.E.D

As one may expect Corollaries 2.2.6 and 2.2.7 can be improved for $X^n$. 

Corollary 2.2.8 Under the hypotheses of Lemma 2.2.6 for \( A \in \mathcal{A}^n \)

\[(2.2.25) \quad \left\{ \frac{1}{n!} \mu \otimes^n(A) \right\}^{\frac{1}{2}} \leq \text{sv}(X \otimes^n; A) \leq \left\{ \mu \otimes^n(A) \right\}^{\frac{1}{2}} \]

and for a sequence \((A_m)_{m \geq 1}\) in \( \mathcal{A}^n \) \( \text{sv}(X \otimes^n; A_m) \to 0 \) if and only if \( \mu \otimes^n(A_m) \to 0 \).

**Proof** Since \( \mu \otimes^n \) is a positive measure on \( \mathcal{A}^n \), then for each \( A \in \mathcal{A}^n \)

\[ \mu \otimes^n(A) \leq \sum_{\Pi} \mu \otimes^n(A \cap A^\Pi) \leq n! \mu \otimes^n(A) \]

and hence \((2.2.25)\) follows from \((2.2.23)\). The second part of the corollary follows from \((2.2.25)\).

Q.E.D.

Corollary 2.2.9 Under the hypothesis of Lemma 2.2.6 a sequence of real valued \( \mathcal{A}^n \)-measurable functions \( \{f_m\}_{m \geq 1} \) on \( T^n \) converges to a real valued function \( f \) on \( T^n \) in \( X \otimes^n \)-measure if and only if \( f_m \) converges to \( f \) in \( \mu \otimes^n \)-measure.

The proof follows by the above corollary and the definition of convergence in \( M \)-measure for a vector valued measure \( M \), given before Lemma 2.2.2.

2.3 Integrals with respect to the symmetric tensor product measure

We now apply the theory of integration with respect to vector valued measures (Dunford and Schwartz (1958)) to define a multiple integral

\[ \int_{T^n} \ldots \int_{T^n} f(t_1, \ldots, t_n) dX_1(t_1) \ldots dX_n(t_n) = \int_{T^n} f(t) d(t^\otimes^n X_1(t)) \]

where \( f \) is a real valued function and \( \otimes X_1 \) is the \( H \otimes^1 \)-valued measure of \( i=1 \)

Theorem 2.2.1. We assume, unless otherwise stated, that \( H, X_1, \ldots, X_n, \mu_{ij} \)

and \( (T, A) \) are as in Assumption 2.2.1 of the beginning of Section 2.2.
We begin by presenting a definition and a proposition from the theory of integration with respect to bounded vector valued measures. They are given, for example, in the book by Kussmaul (1977) (Definition 10.3 and Proposition 10.4). We shall write them here using the notation for the vector valued measure \( \bigotimes X_i \).

**Definition 2.3.1** Let \( f(t) \) be an \( A^n \)-measurable simple function on \( T^n \), that is

\[
f(t) = \sum_{j=1}^{k} a_j 1_{A_j}(t) \quad t = (t_1, \ldots, t_n)
\]

where \( a_j \in \mathbb{R} \) \( j=1, \ldots, k \) and \( A_1, \ldots, A_k \) are disjoint elements in \( A^n \). The integral of \( f \) with respect to \( \bigotimes X_i \), denoted by \( \int_{T^n} f(t) \, d \bigotimes X_i(t) \), is the element of \( H_0^{\otimes n} \) given by

\[
\int_{T^n} f(t) \, d \bigotimes X_i(t) = \sum_{j=1}^{k} a_j \bigotimes X_i(A_j).
\]

A real valued function \( f \) on \( T^n \) is said to be \( \bigotimes X_i \)-integrable if there exists a sequence \( \{f_m\}_{m \geq 1} \) of \( A^n \)-measurable simple functions on \( T^n \) such that

\[
f_m \text{ converges to } f \text{ in } \bigotimes X_i \text{-measure and}
\]

\[
\lim_{m \to \infty} \int_{T^n} 1_{A^m}(t) \, d \bigotimes X_i(t) = 0
\]

uniformly in \( m = 1, 2, \ldots \), i.e.: for each \( \epsilon > 0 \) there exists \( \delta > 0 \) (independent of \( m \)) such that for every set \( A \) for which \( \text{sv}(\bigotimes X_i; A) < \delta \) we have

\[
\left| \int_{T^n} 1_{A^m}(t) \, d \bigotimes X_i(t) \right| < \epsilon \quad \text{for } m = 1, 2, \ldots.
\]

We denote by \( L_1(\bigotimes X_i) \) the class of all \( \bigotimes X_i \)-integrable functions.

**Proposition 2.3.1** Let \( f(t) \in T^n \) be a \( \bigotimes X_i \)-integrable function and \( \{f_m\}_{m \geq 1} \) be a sequence of \( A^n \)-simple functions satisfying (2.3.3) and (2.3.4).
Then for every \( A \in \mathcal{A}^n \) \( (1_{A}(f))(t) \) is \( \oplus X_i \)-integrable and the sequence

\[
\{ \int_{T^n} (1_{A_m}(f))(t) d \oplus X_i(t) \}_{m \geq 1}
\]

converges to an element in \( \mathcal{H}^n \) uniformly in \( A \in \mathcal{A}^n \). The element

\[
(2.3.5) \quad \int_{T^n} (1_{A}(f))(t) d \oplus X_i(t) = \lim_{m \to \infty} \int_{T^n} (1_{A_m}(f))(t) d \oplus X_i(t)
\]

is called the integral of \( f \) with respect to \( \oplus X_i \) over the set \( A \).

Sometimes we will use the following notation

\[
I_n(f; X_1, \ldots, X_n) = \int_{T^n} f(t) d X_i(t)
\]

and

\[
\int_{A} f(t) d X_i(t) = \int_{T^n} (1_{A}(f))(t) d \oplus X_i(t)
\]

We now obtain a sufficient condition for the \( \oplus X_i \)-integrability of a function \( f \) in terms of the control measures \( \mu_1, \ldots, \mu_n \).

**Theorem 2.3.1** If \( f \in L^2(T^n, \mathcal{A}^n, \oplus \mu_i) \) then \( f \) is \( \oplus X_i \)-integrable.

**Proof** Since \( f \in L^2(T^n, \mathcal{A}^n, \oplus \mu_i) \), there exists a sequence \( \{f_m\}_{m \geq 1} \) of \( \mathcal{A}^n \)-measurable simple functions such that \( |f_m| \leq |f| \) a.e. \( \oplus \mu_i \) for \( m \geq 1 \) and \( f_m \) converges to \( f \) in \( L^2(T^n, \mathcal{A}^n, \oplus \mu_i) \). Then \( f_m \) converges to \( f \) in \( \oplus \mu_i \)-measure \( m \geq 1 \) and by Lemma 2.2.3 \( f_m \) converges to \( f \) in \( \oplus X_i \)-measure. Thus condition (2.3.3) in Definition 2.3.1 is satisfied.

Next, for \( A \in \mathcal{A}^n \) \( 1_{A}(t)(f_m(t) - f_k(t)) \) is a simple function for all \( m, k \geq 1 \), i.e.

\[
1_{A}(t)(f_m(t) - f_k(t)) = \sum_{j=1}^{g} \alpha_j 1_{A_j}(t)
\]

for some \( \alpha_j \in \mathbb{R} \) and \( A_1, \ldots, A_g \) disjoint elements in \( \mathcal{A}^n \). Then by definition
of $\mathbf{X}^n$ (Theorem 2.2.1)
\[
\left\| \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j) \right\|_{H^0_{\mathcal{A}^n}}^2 = \left\| \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j) \right\|_{H^0_{\mathcal{A}^n}}^2 = \sum_{j=1}^{\ell} \left\| \alpha_j \otimes X_i(A_j) \right\|_{H^0_{\mathcal{A}^n}}^2
\]
and using Theorem 2.1.3
\[
\left\| \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j) \right\|_{H^0_{\mathcal{A}^n}}^2 = \left\| \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j) \right\|_{H^0_{\mathcal{A}^n}}^2 = \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j)
\]
which goes to zero as $m,k \to \infty$ for $A \in \mathcal{A}^n$ because $\{f_m\}_{m\geq 1}$ is a Cauchy sequence in $L^2(T^n,\mathcal{A}^n,\mathcal{P}^n_i)$. Therefore for each $A \in \mathcal{A}^n$, $\lim_{m \to \infty} \int_{T^n} f_m(t) d \otimes X_i(t)$ exists.

Next, since for each $A \in \mathcal{A}^n$ and $m \geq 1$ $1_{A_m}$ is a simple function, i.e.
there exist $\alpha_j, j=1,\ldots,\ell$ and disjoint elements $A_1,\ldots,A_\ell$ of $\mathcal{A}^n$ such that
\[
1_{A_m}(t) = \sum_{j=1}^{\ell} \alpha_j \mathbf{1}_{A_j}(t),
\]
then from (2.3.2) and the definition of $sv(\otimes X_i;\mathcal{A})$
\[
\left\| \int_{T^n} f_m(t) d \otimes X_i(t) \right\|_{H^0_{\mathcal{A}^n}} = \left\| \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j) \right\|_{H^0_{\mathcal{A}^n}} = \sum_{j=1}^{\ell} \alpha_j \otimes X_i(A_j) \leq sv(\otimes X_i;\mathcal{A})
\]
where
\[
\left\| f_m \right\|_{\infty} = \sup_{t \in T^n} \left\| f_m(t) \right\| = \max(|a_1^m|,\ldots,|a_\ell^m|).
\]
Hence we have that for each $m \geq 1$, $\int_{T^n} f_m(t) d \otimes X_i(t)$ is $sv(\otimes X_i, \cdot)$-continuous. Then by the Vitali-Hahn-Saks Theorem (Dunford and Schwartz (1958)) and Lemma 2.2.2 we have that
uniformly in \( m=1,2,\ldots \). Then condition (2.3.4) in Definition 2.3.1 is satisfied and hence \( f \) is \( \chi_i \)-integrable.

Q.E.D.

Under additional conditions on \((T,A)\) and the control measures \( \mu_1,\ldots,\mu_n \) we are able to give a converse of Theorem 2.3.1.

**Theorem 2.3.2** Let \( T \subseteq \mathbb{R} \) be an interval of the real line, \( A = B(T) \) and suppose that \( \mu_1,\ldots,\mu_n \) are finite non-atomic measures on \((T,A)\). Then a real valued function \( f \) on \( T^n \) is \( \chi_i \)-integrable if and only if \( f \in L^2(T^n,\mathcal{A}^n,\otimes \mu_i) \).

**Proof** Sufficiency follows from Theorem 2.3.1. So assume that \( f \) is \( \chi_i \)-integrable, i.e. there exists a sequence \( \{f_m\}_{m \geq 1} \) of \( \mathcal{A}^n \)-measurable simple functions that satisfies conditions (2.3.3) and (2.3.4) of Definition 2.3.1.

Next, since for each \( k,m \) \( f_m-f_k \) is a simple function, \( f_m(t)-f_k(t) = \sum_{j=1}^\ell \alpha_j \chi_{A_j}(t) \) say where \( \alpha_j \in \mathbb{R} \) \( j=1,\ldots,\ell \) and \( A_1,\ldots,A_\ell \) are disjoint elements in \( \mathcal{A} \), then for each \( A \in \mathcal{A}^n \) using Lemma 2.2.1 we have that

\[
\left\| \int_{T^n} f_{m-k}(t) d \otimes \chi_i(t) \right\|^2_{H_{\diamond \mu_i}} =
\end{align}

\[
= \frac{1}{n!} \sum_{j_1=1}^\ell \sum_{j_2=1}^\ell \sum_{j_1=1}^\ell \alpha_j \alpha_k \sum_{j_1}^\ell \mu_{\Pi} \otimes \ldots \otimes \mu_{n,\Pi}(A_{j_1} \cap (A_{j_2} \cap A_{j_3}) \cap \ldots)
\]
Next, using the notation of Corollary 2.2.5, for each permutation $\pi$, $T^n_\pi$ defined in (2.2.17) is an antisymmetric set, i.e. $T^n_\pi \cap (T^n_\pi)^* = \emptyset$ for each permutation $\pi^* \neq \text{identity permutation}$. Then for each $\pi$ the above expression (2.3.6) simplifies as

$$\left\| \frac{1}{n!} \sum_{\pi} 1_{A_n \pi}(t)(f_m - f_k)(f_m - f_k)_{\pi}(t) \right\|_{H^0}^2 = \frac{1}{n!} \int_{T^n} (f_m - f_k)(f_m - f_k)^2 \mu_1 \otimes \ldots \otimes \mu_n \, dt.$$ 

But from Proposition 2.3.1, if $m, k \to \infty$

$$\left\| \frac{1}{n!} \sum_{\pi} 1_{A_n \pi}(t)(f_m - f_k)(f_m - f_k)_{\pi}(t) \right\|_{H^0}^2 \to 0$$

converges to zero uniformly in $A \in A^n$. Then if $S^n = \cup_{\pi} T^n_\pi$

$$\frac{1}{n!} \sum_{\pi} 1_{A_n \pi}(t)(f_m - f_k)(f_m - f_k)_{\pi}(t) \, dt \otimes \mu_1(t) \to 0 \quad \text{as} \quad m, k \to \infty$$

and since the measures $\mu_1, \ldots, \mu_n$ are non-atomic

$$\mu_1((S^n)^c) = 0$$

which implies that

$$\frac{1}{n!} \sum_{\pi} 1_{A_n \pi}(t)(f_m - f_k)(f_m - f_k)_{\pi}(t) \, dt \otimes \mu_1(t) \to 0 \quad \text{as} \quad m, k \to \infty.$$ 

Thus $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^2(T^n, A^n, \mu_i)$ and since by Corollary 2.2.7 $f \in L^2(T^n, A^n, \mu_i)$, then $f$ belongs to $L^2(T^n, A^n, \mu_i)$.
has been constructed using the theory of integration w.r.t. vector valued measures, then this integral inherits the properties from that theory. For the sake of completeness we present here some of these properties. They are Propositions 2.3.2-2.3.5 whose proofs are given in Kussmaul (1977). We write them here using our notation even though they hold for every bounded vector valued measure.

On the other hand we are able to prove another kind of properties for \( \int_{\mathbb{T}^n} f(t) \odot X_i(t) \) which use the special structure of symmetric tensor product of \( \odot X_i \) and Assumption 2.2.1. They do not necessarily hold for every vector valued measure. They are presented in Theorem 2.3.3, Lemmas 2.3.1-2.3.2 and Corollaries 2.3.1-2.3.3.

**Proposition 2.3.2** Let \( f \) be a \( \otimes X_i \)-integrable function on \( \mathbb{T}^n \), i.e. \( f \in L_1(\otimes X_i) \). Then

\[
\int_{A} \sum_{i=1}^{n} a f(t) \odot X_i(t) = \sum_{i=1}^{n} a f(t) \odot X_i(t)
\]

a) If \( g \in L_1(\otimes X_i) \) and \( a, b \in \mathbb{R} \), for each \( A \in A^n \) we have

\[
\int_{A} \sum_{i=1}^{n} a f(t) \odot X_i(t) = a \int_{A} f(t) \odot X_i(t) + b \int_{A} g(t) \odot X_i(t)
\]

b) \( \int_{A} \sum_{i=1}^{n} f(t) \odot X_i(t) \) is a countable additive bounded measure on \( (\mathbb{T}^n, A^n) \)

with values in \( H^n \) such that

\[
\text{sv}(\sum_{i=1}^{n} f(t) \odot X_i(t); A) \rightarrow 0 \quad \text{as} \quad \text{sv}(\otimes X_i; A) \rightarrow 0.
\]

**Proof** See Kussmaul (1977) Proposition 10.4.

**Proposition 2.3.3** Let \( S \) be the vector space of real \( A^n \)-measurable simple functions. Define for \( f \in S \)

\[
\|f\| = \text{sv}(\sum_{i=1}^{n} f(t) \odot X_i(t); A^n)
\]

**Definition 2.3.7**
Then

a) $\|f\|$ is a norm on $S$.

b) Let $\nu$ be the nonnegative measure on $A^n$ given by Lemma 2.2.2. Then every $\|\cdot\|_1$-Cauchy sequence $\{f_n\}_{n \geq 1}$ of elements in $S$ converges in $\omega$-measure (and hence in $\nu$-measure) to an $A^n$-measurable function $f$ on $T^n$.

c) The space $L_1(\omega_i, \ldots, \omega_n)$ of $\omega_i$-integrable functions can be identified with the completion of $S$ with respect to the norm $\|\cdot\|$ given by (2.3.7).

d) The linear operator $I_n(\omega_i; \ldots, \omega_n) : L_1(\omega_i, \ldots, \omega_n) \rightarrow H^n$ defined by

$$I_n(f; \omega_i, \ldots, \omega_n) = \int_{T^n} f(t) d \omega_i(t)$$

is continuous with norm $\|I_n\|_1 \leq 1$.


Proposition 2.3.4 a) Let $f \in L_1(\omega_i, \ldots, \omega_n)$. Then the element

$$\int_{T^n} f(t) d \omega_i(t) \in H^n$$

is uniquely determined by

$$\int_{T^n} f(t) d \omega_i(t), F > H^n = \int_{T^n} f(t) d < \omega_i(f), F > (t)$$

for all $F \in H^n$, where $< \omega_i(f), F > (\cdot)$ is the signed measure on $A^n$ given by

$$< \omega_i(f), F > (A) = \int_{T^n} f(t) d < \omega_i(f), F > (t)$$

for all $F \in H^n$, where $< \omega_i(f), F > (\cdot)$ is the signed measure on $A^n$ given by

b) An $A^n$-measurable function on $T^n$ is $\omega_i$-integrable if and only if

$$\int_{T^n} f(t) d < \omega_i(F), F > (t)$$

for every $F \in H^n$ is $< \omega_i(F), F >$-integrable and the family

$$\{\int_{T^n} f(t) d < \omega_i(F), F > (t) : \|F\|_1 H^n \leq 1, F \in H^n\}$$
is weakly sequentially compact.


Proposition 2.3.5 (Lebesgue Dominated Convergence Theorem).

Let \( \{f_m\}_{m=1}^{\infty} \) be a sequence of \( \oplus X_i \)-integrable functions which converges to a function \( f \) in \( \oplus X_i \)-measure and \( |f_m| \leq g \) \( m \geq 1 \) where \( g \) is a \( \oplus X_i \)-integrable function. Then \( f \) is \( \oplus X_i \)-integrable, \( f_m \) converges to \( f \) in \( \Pi_{i=1}^{\infty} L_1(\oplus X_i) \)-norm of Proposition 2.3.3 and

\[
\int_{\mathbb{R}^n} f(t) d\oplus X_i(t) = \lim_{m \to \infty} \int_{\mathbb{R}^n} f_m(t) d\oplus X_i(t).
\]


In all the above properties (Propositions 2.3.2-2.3.5) we have only used the fact that \( \oplus X_i \) is a bounded vector valued measure. Now we shall use the special structure of symmetric tensor product of \( \oplus X_i \) and the hypotheses in Assumption 2.2.1 to show additional properties of the integral

\[
\int_{\mathbb{R}^n} f(t) d\oplus X_i(t).
\]

Our first result gives an expression for the inner product of two integrals and consequently for their norm. We first introduce some new notation: Let \( \mu_0 \) be a \( \sigma \)-finite non-negative measure on \((\mathbb{T},\mathcal{A})\) such that \( \mu_{ij} \ll \mu_0 \) \( i,j=1,...,n \) (\( \mu_0 = \sum_{i=1}^{n} \mu_i \) for example) and \( R(s) = (r_{ij}(s)) \) be the non-negative definite matrix a.e. \( d\mu_0 \) given by (2.2.2), that is

\[
\begin{align*}
    r_{ij}(s) &= \frac{d\mu_{ij}}{d\mu_0}(s) & \text{a.e. } d\mu_0.
\end{align*}
\]

For each \( t = (t_1,...,t_n) \in \mathbb{R}^n \), let

\[
(2.3.8) \quad \hat{R}^{\otimes n}(t) = R(t_1)\otimes...\otimes R(t_n)
\]
where \( \hat{\otimes} \) denotes the Kronecker product for matrices, i.e. if \( A = (a_{ij}) \) is an \((m \times n)\) matrix and \( B = (b_{kj})\) is a \(p \times q\) matrix, then \( A \hat{\otimes} B\) is the \((mp \times nq)\) matrix

\[
A \hat{\otimes} B = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix}
\]

Then \( R^\otimes_n(\xi)\) defined in (2.3.8) is an \( (n^n \times n^n)\) non-negative definite matrix a.e. \( d\mu_0^n\).

Let \( (e_i^\otimes)_\xi \) be the canonical basis in \( \mathbb{R}^n \). For each permutation \( \Pi = (\Pi_1, \ldots, \Pi_n) \) of \( (1, \ldots, n) \) let

\[
e^\Pi_{\otimes \xi} = e^\Pi_1 \otimes \cdots \otimes e^\Pi_n
\]

and for a given real valued function \( f \) on \( T^n \) define the \( (\mathbb{R}^n)^\otimes \xi \)-valued functions on \( T^n \)

\[
f^\Pi_{\otimes \xi}(t) = f(t^{\Pi_1}) e^\Pi_{\otimes \xi}
\]

and

\[
f_{\otimes \xi}(t) = \frac{1}{n!} \sum_{\Pi} f^\Pi_{\otimes \xi}(t).
\]

We shall denote by \( f_{\otimes \xi}(t)' \) the transpose of \( f_{\otimes \xi}(t) \).

Using the above notation we now establish the next result.

**Theorem 2.3.3**

a) If \( f, g \in L^2(T^n, A^n, \bigotimes_{i=1}^n u_i) \)

\[
<\int_T^n f(t) d \otimes X_i(t), \int_T^n g(t) d \otimes X_i(t) >_{\otimes \xi} = \int_T^n f_{\otimes \xi}(t)' R^\otimes_n(t) g_{\otimes \xi}(t) d\mu_0^n(t).
\]
b) If \( f \in L^2(T^n, A^n, \mu_i) \)

\[
\| \int T_i f(t) d \otimes X_i(t) \|^2 = \int T_i f \otimes X_i f d \mu_i(t)
\]

\[
\leq \int T_i |f(t)|^2 d \otimes \mu_i(t).
\]

**Proof** First we will show that for \( f \in L^2(T^n, A^n, \mu_i) \)

(2.3.13) \[ \int T_i f \otimes X_i f d \mu_i(t) < \infty. \]

Since \( \int T_i f \otimes X_i f d \mu_i(t) \) is a semi-inner product in the space of \( A^n \)-measurable \( (\mathbb{R}^n)^n \)-valued functions on \( T^n \), it follows by the triangle inequality that

(2.3.14) \[ 0 \leq \int T_i f \otimes X_i f d \mu_i(t) \]

\[ = \int T_i \left( \frac{1}{n!} \sum_{\Pi} f(t_{\Pi}) e_{\otimes}^{\Pi} \right) \otimes X_i \left( \frac{1}{n!} \sum_{\Pi} f(t_{\Pi}) e_{\otimes}^{\Pi} \right) d \mu_i(t) \]

\[ \leq \left\{ \frac{1}{n!} \sum_{\Pi} \left( \int T_i f(t_{\Pi}) e_{\otimes}^{\Pi} \right) \otimes X_i \left( f(t_{\Pi}) e_{\otimes}^{\Pi} \right) d \mu_i(t) \right\}^2. \]

Next using (2.3.8), (2.3.10), the transformation theorem and the fact that

\[ r_{ii}(s) = \frac{d \mu_i}{d \mu_0}(s) \text{ a.e. } d \mu_0, \]

we have that for each permutation \( \Pi = (\Pi_1, \ldots, \Pi_n) \)

\[ \int T_i \left( \frac{1}{n!} \sum_{\Pi} f(t_{\Pi}) e_{\otimes}^{\Pi} \right) \otimes X_i \left( f(t_{\Pi}) e_{\otimes}^{\Pi} \right) d \mu_i(t) \]

\[ = \int T_i f^2(t_{\Pi}) r_{\Pi_1} r_{\Pi_2} \ldots r_{\Pi_n} d \mu_i(t) \]

\[ = \int T_i f^2(s) r_{i1} r_{i2} \ldots r_{in} d \mu_i(s) = \int T_i f^2(s) \frac{d \otimes \mu_i}{d \mu_0}(s) d \mu_0(s) \]

\[ = \int T_i |f(s)|^2 d \otimes \mu_i(s). \]
Then from (2.3.14) we obtain that if \( f \in L^2(\mathbb{T}^n, \mathbb{A}_n, \cdot \mu_i) \)

\[
(2.3.15) \quad \int_{\mathbb{T}^n} f \otimes \mu_i(t) \, d\mu_i(t) < \infty
\]

which proves (2.3.13) and the inequality in (b).

Thus applying Cauchy-Schwarz inequality we have that if \( f, g \in L^2(\mathbb{T}^n, \mathbb{A}_n, \cdot \mu_i) \), then

\[
\int_{\mathbb{T}^n} f \otimes \mu_i(t) \, d\mu_i(t) < \infty.
\]

Now we shall show that (2.3.12) holds if \( f \) and \( g \) are \( \mathbb{A}_n \)-measurable real simple functions, that is

\[
f(t) = \sum_{k=1}^{m} a_k 1_{A_k}(t)
\]

\[
g(t) = \sum_{k=1}^{m} b_k 1_{A_k}(t)
\]

where \( a_k, b_k \in \mathbb{R} \) \( k=1, \ldots, m \) and \( A_1, \ldots, A_m \) are disjoint sets in \( \mathbb{A}_n \). Then using the definition of the integral \( \int_{\mathbb{T}^n} f \otimes X_i(t) \) for simple functions and Lemma 2.2.1

\[
< \int_{\mathbb{T}^n} f(t) d \otimes X_i(t), \int_{\mathbb{T}^n} g(t) d \otimes X_i(t) >_{\mathbb{H} \otimes \mu_i}
\]

\[
= \sum_{j=1}^{m} \sum_{k=1}^{m} a_j b_k < \otimes \mu_i(A_j), \otimes \mu_i(A_k) >_{\mathbb{H} \otimes \mu_i}
\]

\[
= \frac{1}{n!} \sum_{j=1}^{m} \sum_{k=1}^{m} a_j b_k \int_{\mathbb{T}^n} \mu_1 \otimes \cdots \mu_n (A_j \cap A_k)
\]

\[
= \frac{1}{n!} \sum_{j=1}^{m} \int_{\mathbb{T}^n} f(t) g(t) d\mu(t) \int_{\mathbb{T}^n} \otimes \mu_n (t)
\]

\[
= \frac{1}{n!} \sum_{j=1}^{m} \int_{\mathbb{T}^n} f(t) g(t) r_{n \cap 1}(t) \cdots r_{n \cap 1}(t) \, d\mu(t)
\]
\[ \frac{1}{n!} \sum_{\Pi} \int_{T^n} f_{\otimes n}(t)^{R^{\otimes n}(t)} g_{\otimes n}(t) \, du_0^n(t) \]

\[ = \int_{T^n} f_{\otimes n}(t)^{R^{\otimes n}(t)} g_{\otimes n}(t) \, du_0^n(t) \]

where \( f_{\otimes n}(t) = f_{\otimes n}(t) \) for \( \Pi \) the identity permutation.

Next, for each permutation \( \Pi \), applying the transformation theorem we obtain

\[ \int_{T^n} f_{\otimes n}(t)^{R^{\otimes n}(t)} g_{\otimes n}(t) \, du_0^n(t) \]

\[ = \frac{1}{n!} \sum_{\Pi} \int_{T^n} f_{\Pi n}(t) g_{\Pi n^*}(t) r_{\Pi n}(t) \, du_0^n(t) \]

\[ = \frac{1}{n!} \sum_{\Pi} \int_{T^n} f(s) g_{\Pi n^*}(s) r_{\Pi n^*}(s) \, du_0^n(s) \]

\[ = \int_{T^n} f_{\otimes n}(t)^{R^{\otimes n}(t)} g_{\otimes n}(t) \, du_0^n(t). \]

Then (2.3.12) holds for \( A^n \)-simple functions using (2.3.11).

Now suppose that \( f, g \in L^2(T^n, \mathbb{A}^n, \sigma_{u_i}) \). Then there exist sequences \( \{f_m\}_{m \geq 1}, \{g_m\}_{m \geq 1} \) of \( A^n \)-measurable simple functions such that \( f_m \rightarrow f \) and \( g_m \rightarrow g \) in \( L^2(T^n, \mathbb{A}^n, \sigma_{u_i}) \) and by Theorem 2.3.1

\[ \int_{T^n} f_m(t) d \otimes X_i(t) \rightarrow \int_{T^n} f(t) d \otimes X_i(t) \]

and

\[ \int_{T^n} g_m(t) d \otimes X_i(t) \rightarrow \int_{T^n} g(t) d \otimes X_i(t). \]

Then it is enough to show that

\[ \int_{T^n} (f_m \otimes n(t))^{R^{\otimes n}(t)} (g_m \otimes n(t)) \, du_0^n(t) \]

\[ \rightarrow \int_{T^n} f_{\otimes n}(t)^{R^{\otimes n}(t)} g_{\otimes n}(t) \, du_0^n(t). \]
Denote \( L^2(\bigotimes_{i=1}^n, \mu_i) = L^2(T^n, A^n, \bigotimes_{i=1}^n) \). Then using (2.3.15)

\[
\left| \int_{T^n} ((f_m) \bigotimes_{i=1}^n(t))' R^\bigotimes_{i=1}^n(t) (g_m) \bigotimes_{i=1}^n(t) d\mu_0^n(t) \right|
\]

\[
- \int_{T^n} f_\bigotimes_{i=1}^n(t)' R^\bigotimes_{i=1}^n(t) g_\bigotimes_{i=1}^n(t) d\mu_0^n(t)
\]

\[
\leq \int_{T^n} ((f_m) \bigotimes_{i=1}^n(t) - f_\bigotimes_{i=1}^n(t))' R^\bigotimes_{i=1}^n(t) ((g_m) \bigotimes_{i=1}^n(t) - g_\bigotimes_{i=1}^n(t)) d\mu_0^n(t)
\]

\[
+ \int_{T^n} ((f_m) \bigotimes_{i=1}^n(t) - f(t))' R^\bigotimes_{i=1}^n(t) g_\bigotimes_{i=1}^n(t) d\mu_0^n(t)
\]

\[
+ \int_{T^n} f_\bigotimes_{i=1}^n(t)' R^\bigotimes_{i=1}^n(t) ((g_m) \bigotimes_{i=1}^n(t) - g_\bigotimes_{i=1}^n(t)) d\mu_0^n(t)
\]

\[
\leq \| f_m - f \| \sum_{i=1}^n \| g_m - g \| \| \bigotimes_{i=1}^n \| \sum_{i=1}^n \| g \| \sum_{i=1}^n \| \bigotimes_{i=1}^n \| + \| f_m - f \| \sum_{i=1}^n \| g_m - g \| \sum_{i=1}^n \| \bigotimes_{i=1}^n \|
\]

\[
+ \| f \| \sum_{i=1}^n \| g - g_m \| \sum_{i=1}^n \| \bigotimes_{i=1}^n \| \to 0 \text{ as } m \to \infty
\]

because \( f_m \to f \) and \( g_m \to g \) in \( L^2(\bigotimes_{i=1}^n) \).

(b) follows from (a) taking \( f = g \). Q.E.D.

Corollary 2.3.1 If \( A \) is an antisymmetric set in \( A^n \) then for all \( f \in L^2(T^n, A^n, \bigotimes_{i=1}^n) \)

\[
\| \int_{A} f(t) d \bigotimes_{i=1}^n x_i(t) \|^2_{H^\bigotimes_{n}} = \frac{1}{n!} \int_{A} |f(t)|^2 d \bigotimes_{i=1}^n \mu_i(t).
\]

Proof Since \( A \cap A^\Pi = \phi \) for all \( \Pi \) distinct from the identity permutation \( \Pi \), then using (b) in the last theorem

\[
\| \int_{A} f(t) d \bigotimes_{i=1}^n x_i(t) \|^2_{H^\bigotimes_{n}} = \int_{A} f_\bigotimes_{i=1}^n(t)' R^\bigotimes_{i=1}^n(t) f_\bigotimes_{i=1}^n(t) d\mu_0^n(t)
\]
\[ f_{ij} = \mu_i, j = 1, \ldots, n \text{ and } f \in L^2(T^n, A^n, \mu^n), \text{ then} \]
\[ \| \int_{T^n} f(t) d \otimes X_i(t) \|_{H^2}^2 = \int_{T^n} |\tilde{f}(t)|^2 d\mu^o(t) \]
where \[ \tilde{f}(t) = \frac{1}{n!} \sum_{\pi} f_{\pi}(t). \]

**Proof** Taking \( \mu_0 = \mu \), then \( r_{ij}(t) = 1 \) all \( i, j = 1, \ldots, n \) and from Theorem 2.3.3 (b)
\[ \| \int_{T^n} f(t) d \otimes X_i(t) \|_{H^2}^2 = \int_{T^n} f_{\otimes n}(t) R_{\otimes n}(t) f_{\otimes n}(t) d\mu^o(t) \]
\[ = \frac{1}{n!} \sum_{\pi} \int_{T^n} f(t) f_{\pi}(t) d\mu^o(t) = \int_{T^n} |\tilde{f}(t)|^2 d\mu^o(t). \]

**Corollary 2.3.3** If \( \mu_{ij} = \mu \) \( i, j = 1, \ldots, n \) and \( f \) is a symmetric function in \( L^2(T^n, A^n, \mu^n) \) then
\[ \| \int_{T^n} f(t) d \otimes X_i(t) \|_{H^2}^2 = \int_{T^n} |f(t)|^2 d\mu^o(t). \]

The proof follows from the last corollary since \( \tilde{f} = f \).

The next lemma may be seen as a Fubini's type theorem for multiple stochastic integrals. We write
\[ I_n(f; X_1, \ldots, X_n) = \int_{T^n} f(t) d \otimes X_i(t). \]

**Lemma 2.3.1** Let \( f(t_1, \ldots, t_n) = f_1(t_1) \cdots f_n(t_n) \) where \( f_i \in L^2(T, A, \mu_i) \)
\[ i = 1, \ldots, n. \text{ Then } f \text{ is } \otimes X_i - \text{integrable and} \]
\[ I_n(f; X_1, \ldots, X_n) = I_{\otimes X_1} f_1 \otimes \cdots \otimes I_{\otimes X_n} f_n. \]
where for each $i = 1, \ldots, n$, $I_{X_i}$ is the isometry between $L^2(T, A, \mu_i)$ and $H_{X_i}$ given by Theorem 2.1.1.

**Proof** By Fubini's Theorem $f(t) = f_1(t_1) \ldots f_n(t_n)$ belongs to $L^2(T^n, A^n, \otimes \mu_i)_{i=1}^n$ and hence by Theorem 2.3.1 $f$ is $\otimes X_i$-integrable.

First assume that each $f_i$ is a simple function on $(T, A)$, i.e.

$$f_i(s) = \sum_{j=1}^{k_i} a_{ij} \chi_{A_{ij}}(s)$$

where $a_{ij} \in \mathbb{R}$, $j = 1, \ldots, k_i$, and $A_{i1}, \ldots, A_{ik_i}$ are disjoint sets in $A$, $i = 1, \ldots, n$.

Then

$$f(t) = \sum_{j_1=1}^{k_1} \ldots \sum_{j_n=1}^{k_n} a_{jj_1} \ldots a_{jfn} \chi_{A_{j1j_1} \ldots A_{jfnj_n}}(t)$$

and by Definition 2.3.1 and Theorem 2.2.1

$$I_n(f; X_1, \ldots, X_n) = \sum_{j_1=1}^{k_1} \ldots \sum_{j_n=1}^{k_n} a_{jj_1} \ldots a_{jfn} \chi_{X_1(A_{j1j_1} \ldots A_{jfnj_n})} = \sum_{j_1=1}^{k_1} \ldots \sum_{j_n=1}^{k_n} a_{jj_1} \chi_{X_1(A_{j1j_1})} \cdots \chi_{X_n(A_{jfnj_n})} = I_{X_1}(f_1) \cdots I_{X_n}(f_n).$$

Next since each $f_i \in L^2(T, A, \mu_i)$, there exist sequences $(f^m_i)_{m \geq 1}$ of simple functions on $(T, A)$ such that $f^m_i \to f$ in $L^2(T, A, \mu_i)$ for each $i = 1, \ldots, n$. Define

$$f^m(t) = f^m_1(t_1) \ldots f^m_n(t_n)$$

then $f^m \to f$ in $L^2(T^n, A^n, \otimes \mu_i)$. Next since $I_n(\otimes X_1, \ldots, X_n)$ is a bounded
linear operator, to prove the lemma it is enough to show that

$$I_{X_1}(f^m) \cdots I_{X_n}(f^m) + I_{X_{1}\cdots X_m}(f).$$

Let $\sigma^m_n$ be the projection operator defined in (2.2.3), then

$$|| I_{X_1}(f^m) \cdots I_{X_n}(f^m) - I_{X_{1}\cdots X_m}(f)||_H^2$$

$$= || \sigma^m_n(I_{X_1}(f^m) \cdots I_{X_n}(f^m) - I_{X_{1}\cdots X_m}(f)||_H^2$$

$$\leq || I_{X_1}(f^m) \cdots I_{X_n}(f^m) - I_{X_{1}\cdots X_m}(f)||_H^2 + || I_{X_{1}\cdots X_m}(f)||_H^2$$

$$- 2 \langle I_{X_1}(f^m), I_{X_{1}\cdots X_m}(f) \rangle_H \leq \langle I_{X_{1}}(f^m), I_{X_{1}\cdots X_m}(f) \rangle_H$$

which goes to zero as $m \to \infty$ since by Theorem 2.1.1

$$|| I_{X_i}(f^m)||_H^2 = || f^m ||_{L^2(\mu_i)}^2 \to || f_i ||_{L^2(\mu_i)}^2 = || I_{X_i}(f_i)||_H^2$$

for each $i=1,\ldots,n$, where $L^2(\mu_i) = L^2(T,\Lambda,\mu_i).$

The next result gives the orthogonality of multiple integrals of different order. We use the notation of Lemma 2.2.5.

**Lemma 2.3.2 (Orthogonality)** If $n_1 \neq n_2$ and $f \in L^1(\otimes_{i=1}^{n_1}X_i)$, $g \in L^1(\otimes_{i=1}^{n_2}X_i)$

then $I_{n_1}(f; X_1,\ldots,X_{n_1})$ is orthogonal to $I_{n_2}(g; X_1,\ldots,X_{n_2})$ with respect to the inner product $\langle \cdot,\cdot \rangle_e$ in $\exp(H)$.
Proof From Definition 2.3.1 \( I_{n_1}(f; X_1, ..., X_{n_1}) \) takes values in \( H_{n_1} \) and \( I_{n_2}(f; X_1, ..., X_{n_2}) \) in \( H_{n_2} \). Then the lemma follows since for \( n_1 \neq n_2 \), \( H_{n_1} \) and \( H_{n_2} \) are orthogonal in \( \text{EXP}(H) \). Q.E.D.

The case of only one o.s.m. \( X \) To conclude this section we consider the special case of only one measure \( X \), i.e. \( X \) is an orthogonally scattered measure on a measurable space \( (T,A) \) with values in \( H_\chi \) and control measure \( \mu \). We use the hypotheses and notation of Proposition 2.2.1 and Lemma 2.2.6. For \( f \) an \( X^\infty \)-integrable function we will write

\[
I_{n\infty}(f) = \int f(t) dX^n(t) = I_n(f;X).
\]

Additional properties to those given above are now presented for the integral \( I_{n\infty}(f) \). They are similar to those for the multiple Wiener integral with respect to a Gaussian random measure presented in Itô (1951).

For a real valued function \( f \) on \( T^n \) we denote by \( \tilde{f} \) the symmetrization of \( f \) defined as

\[
\tilde{f}(t) = \frac{1}{n!} \sum_{\pi} f(t_{\pi}).
\]

The exponential space \( \text{EXP}(H_\chi) \) is defined in (2.2.21) with inner product \( \langle \cdot, \cdot \rangle_\chi \) given by (2.2.22) with corresponding norm \( \| \cdot \|_\chi \).

Proposition 2.3.6 Let \( f, g \in L^2(\mu_\infty) = L^2(T^n, A^n, \mu_\infty) \). Then

a) \( I_{n\infty}(\tilde{f}) = I_{n\infty}(f) \).

b) \( \langle I_{n\infty}(f), I_{n\infty}(g) \rangle_\chi \in L^2(\mu_\infty) = \langle \tilde{f}, \tilde{g} \rangle_\chi \in L^2(\mu_\infty) \).

d) \( \| I_{n\infty}(f) \|_\chi^2 = \| I_{n\infty}(f) \|_\chi^2 \leq \| f \|_\chi^2 \leq \| f \|_\chi^2 \)
Proof. a) Let $f$ be an elementary function of the form

$$f(t) = \frac{1}{n!} \sum_{\Pi} \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} A_{i_1} \times \cdots \times A_{i_n}(t)$$

where $a_{i_1, \ldots, i_n} \in \mathbb{R}$ and $A_{i_1}, \ldots, A_{i_n}$ are disjoint sets in $A$. Then

$$\tilde{f}(t) = \frac{1}{n!} \sum_{\Pi} \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} A_{i_1} \times \cdots \times A_{i_n}(t)$$

and from Definition 2.3.1

$$I_n(f) = \frac{1}{n!} \sum_{\Pi} \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X(A_{i_1}) \cdots X(A_{i_n})$$

and

$$I_n(f) = \frac{1}{n!} \sum_{\Pi} \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} X(A_{i_1}) \cdots X(A_{i_n})$$

But for all $\Pi = (\Pi(1), \ldots, \Pi(n))$ permutation of $(1, \ldots, n)$

$$X(A_{i_1}) \cdots X(A_{i_n}) = X(A_{\Pi(1)} \cdots X(A_{\Pi(n)})$$

Then $I_n(f) = I_n(\tilde{f})$ if $f$ is an elementary function as in (2.3.17).

If $f \in L^2(T^n, A^n, \mu^\otimes)$ a limit argument applies, since elementary functions form a dense linear manifold in $L^2(T^n, A^n, \mu^\otimes)$.

b) It follows from Theorem 2.3.3 as in Corollary 2.3.2 that

$$\langle I_n(f), I_n(g) \rangle_{H_X^\otimes} = \frac{1}{n!} \sum_{\Pi} \int_{\Pi} f(t) g_{\Pi}(t) d\mu^\otimes(t)$$

$$= \int_{\Pi} f(t) g(t) d\mu^\otimes(t) = \langle f, g \rangle_{L^2(\mu^\otimes)}$$

and the second equality of (b) follows from (a). The proof of (c) and (d) follows from (b) and Lemma 2.3.2.

Q.E.D.
We denote by $L^2(\mu^m)$ the subspace of $L^2(T^n, A^n, \mu^m)$ consisting of all symmetric functions \( f(t_\Pi) = f(t) \) for all $\Pi$.

**Proposition 2.3.7** (Orthogonal expansions). Let $\psi \in \text{EXP}(H_\chi)$. Then

\[
\psi = \sum_{n=0}^{\infty} I_{n\Theta}(\tilde{f}_n) \quad (|| \cdot ||_e - \text{convergence})
\]

where $\tilde{f}_n \in L^2(\mu^m)$, $n \geq 1$. Moreover

\[
||\psi||_e^2 = \sum_{n=0}^{\infty} || f_n ||^2_{L^2(\mu^m)} = \sum_{n=0}^{\infty} || I_{n\Theta}(\tilde{f}_n) ||^2_{H^2_X} < \infty.
\]

i.e. the system of multiple integrals \( \{ I_{n\Theta}(\tilde{f}_n) : \tilde{f}_n \in L^2(\mu^m) \} \) is complete in $\text{EXP}(H_\chi)$.

**Proof** For $h \in H_\chi$ let

\[
\exp \Theta(h) = (1, h, \frac{1}{\sqrt{2!}} h^2, \frac{1}{\sqrt{3!}} h^3, \ldots).
\]

It is known (Guichardet (1972)) that \( \{ \exp \Theta(h) : h \in H_\chi \} \) generates the space $\text{EXP}(H_\chi)$ and therefore if $\psi \in \text{EXP}(H_\chi)$

\[
(2.3.18) \quad \psi = \sum_{n=0}^{\infty} \psi_n \quad \psi_n \in H_X^m \quad n \geq 1 \quad \psi_o = \text{constant}.
\]

Next, let $h_i \in H_\chi$ \( i=1, \ldots, n \). Then by Theorem 2.1.1 $h_i = I_X(g_i) \quad g_i \in L^2(T, A, \mu)$ \( i=1, \ldots, n \) and by Lemma 2.3.1 and Proposition 2.3.6 (b)

\[
h_1 \otimes \ldots \otimes h_n = I_X(g_1 \otimes \ldots \otimes g_n) = I_{n\Theta}(g_1 \ldots g_n) = I_{n\Theta}(\tilde{f}_n)
\]

where $f_n = g_1 \ldots g_n$. But from Proposition 2.3.6 (b) there is an isometry between $L^2(\mu^m)$ and a closed subspace $R_n$ of $H_X^m$ where $R_n = \{ I_{n\Theta}(\tilde{f}) : f \in L^2(\mu^m) \}$.

Then since elements of the form $h_1 \otimes \ldots \otimes h_n$ generate $H_X^m$, it follows by continuity of $I_{n\Theta}$ that if $\psi_n \in H_X^m$, then $\psi_n = I_{n\Theta}(f_n)$ where $f_n \in L^2(\mu^m)$. Then the proposition follows from (2.3.18).
CHAPTER III

PRODUCT STOCHASTIC MEASURES AND MULTIPLE STOCHASTIC INTEGRALS OF $L^2$-INDEPENDENTLY SCATTERED MEASURES

Let $(T,A)$ be a measurable space and $(\Omega,F,P)$ be a complete probability space. The real valued set function $X$ on $(T,A)$ is said to be an independently scattered measure (i.s.m.) on $(T,A)$ if for each sequence of pairwise disjoint sets $\{A_k\}_{k \geq 1}$ in $A$, $\{X(A_k)\}_{k \geq 1}$ is a sequence of independent random variables on $(\Omega,F,P)$ and

$$X(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} X(A_k) \quad \text{a.s.}$$

We say that $X$ is an $L^2(\Omega)$-valued i.s.m if $X(A)$ belongs to $L^2(\Omega,F,P)$ for each $A \in A$ and the above series converges in $L^2(\Omega)$. Zero mean $(E(X(A)) = 0$ as $A \in A$) $L^2(\Omega)$-valued independently scattered measures are special cases of orthogonally scattered measures.

In this chapter we apply the results obtained in the last one to study $L^2$-valued product stochastic measures and multiple stochastic integrals of non-identically distributed $L^2$-independently scattered measures. We include the Gaussian (Section 3.1), Poisson (Section 3.2) and general $L^2$-independent increments process (Section 3.3) cases. Although the last situation includes the first two, we gain generality in the measurable space $(T,A)$ by studying them separately, apart from the fact that results presented in the first two sections are later used in Section 3.3. In each case we present the identification of the exponential space and the symmetric tensor products of the common Hilbert space where the indepen-
dently scattered measures take values. We conclude the chapter with Section 3.4 where we make comparisons with recent works in the literature including the one by Engel (1982) on the $L^2$-theory of products of different stochastic measures.

3.1 Gaussian random measures

In this section we consider non-identically distributed Gaussian random measures and their symmetric tensor products. We use the well-known identification of the exponential space of a Gaussian process (Neveu (1968), Kallianpur (1970)) and our results of Section 2.2 to obtain an $L^2$-valued product stochastic measure. Further, applying the theory of Section 2.3, we construct multiple integrals, obtaining as special cases the multiple Wiener integral of Itô (1951) and the multiple stochastic integral with dependent integrators of Fox and Taqqu (1984).

Let $(\Omega, F, P)$ be a complete probability space, $T \subseteq \mathbb{R}^d$ ($d \geq 1$) a measurable subset, $A = B(T)$ its Borel subsets and $A_c$ its relatively compact subsets. Let $W(A) = (W_1(A), \ldots, W_n(A))$ $A \in A_c$ be a zero mean $n$-dimensional Gaussian random field on $(\Omega, F, P)$, such that $W(A)$ and $W(B)$ are independent if $A \cap B = \emptyset$, $A, B \in A_c$. Then for each $i=1, \ldots, n$ $W_i$ is a Gaussian random measure and therefore an $L^2(\Omega, F, P)$-valued orthogonally scattered measure, where $F^W = \sigma(W(A); A \in A_c)$. Further, $W_i$ and $W_j$ are independent (and hence orthogonal) over disjoint sets for $i, j=1, \ldots, n$. Define

(3.1.1) \[ \mu_i(A) = E[W_i(A)^2] \quad A \in A_c \quad i=1, \ldots, n \]

(3.1.2) \[ \mu_{ij}(A \cap B) = E(W_i(A)W_j(B)) \quad A, B \in A_c \quad i, j=1, \ldots, n \]

and let

(3.1.3) \[ H = \overline{sp \{a'W(A): a \in \mathbb{R}^n, A \in A_c \}} \]
be the Gaussian space, closed subspace of $L^2(\Omega,F^W,P)$ generated by $W$.

Then each $W_i$ is an $H$-valued orthogonally scattered measure on $(T,A)$ with control measure $\mu_i$, which we assume finite for $i=1,\ldots,n$. (If $W$ is a Wiener process this finiteness assumption means $T$ has to be a bounded set, since $\mu_i$ is Lebesgue measure.)

It is known (Proposition 7.3 of Neveu (1968)) that

(3.1.4) \[ \exp(H) = L^2(\Omega,F^W,P) \]

where for all $h \in H$

(3.1.5) \[ \psi(\exp \circ(h)) = \exp(h - \frac{1}{2} E(h^2)) \]

\[ \exp \circ(h) = \sum_{n \geq 0} \left( \frac{1}{n!} \right)^2 h^\otimes n \in \exp(H) \]

and that $\{\psi(\exp \circ(h)) : h \in H\}$ generates $L^2(\Omega,F^W,P)$.

In our first result of this chapter we obtain an $L^2$-valued product stochastic measure of the Gaussian random measures $W_1,\ldots,W_n$.

Proposition 3.1.1 Let $W_i$ $i=1,\ldots,n$ be Gaussian random measures on $(T,A)$ satisfying the above conditions. Then there exists a unique $L^2(\Omega,F^W,P)$-valued measure $\circ W_i$ on $(T^n,A^n)$ such that for $A_i \in A$ $i=1,\ldots,n$

(3.1.6) \[ \circ W_i(A_1 \times \ldots \times A_n) = W_i(A_1) \circ \ldots \circ W_i(A_n) \]

and for $A \in A^n$

\[ E(\circ W_i(A)) = 0 \]

\[ \text{VAR}(\circ W_i(A)) = \frac{1}{n!} \sum_{\Pi} \mu_{\Pi_1} \circ \ldots \circ \mu_{\Pi_n} (A \cap A^\Pi) \]

Proof Existence and uniqueness of the $H^\otimes n$-valued measure $\circ W_i$ follow \[ i=1 \]
from Theorem 2.2.1. By (3.1.4) \( n \sum_{i=1}^{n} \cdot W_i \) is seen as an \( L^2(\Omega, F^W, P) \)-valued measure. Since for each \( n \geq 1, n \sum_{i=1}^{n} \cdot W_i \) is \( H^k \)-valued and \( H^k \) is orthogonal to \( H^0 = \mathbb{R} \), then \( E( n \sum_{i=1}^{n} \cdot W_i(A)) = 0 \) for \( A \in A^k \). The expression for the variance follows from (2.2.14) in Corollary 2.2.2.

Q.E.D.

The \( H^k \)-valued measure \( n \sum_{i=1}^{n} \cdot W_i \) is an example of the symmetric tensor product measure constructed in Section 2.2. Then all results of that section may be applied to this \( L^2(\Omega) \)-valued product stochastic measure \( n \sum_{i=1}^{n} \cdot W_i \).

In order to compute the symmetric tensor product measure \( n \sum_{i=1}^{n} \cdot W_i \) for some sets \( A \in A^k \) we shall use the identification of the Exponential space of a general Gaussian space studied by Neveu (1968) and Kallianpur (1970). The next well known result is Proposition 7.5 in the work of the first named author. We present it here for the sake of completeness and later reference.

**Proposition 3.1.2** (Neveu (1968)). If \( H \) is a Gaussian space and \( h_1, \ldots, h_k \) are orthogonal elements in \( H \), then

\[
\begin{equation}
(3.1.7)
\frac{n^k}{k!} h_n(h_1; \sigma^2) = (-\sigma^2)^n \frac{d^n}{dx^n} e^{-x^2/(2\sigma^2)} \sigma^2 > 0, m \geq 0.
\end{equation}
\]

where \( \sigma_j^2 = E(h_j^2), n = \sum_{j=1}^{k} n_j \) and \( h_m(x; \sigma^2) \) are Hermite polynomials defined by

\[
\begin{equation}
(3.1.7)
\frac{n^k}{k!} h_n(h_1; \sigma^2) = (-\sigma^2)^n \frac{d^n}{dx^n} e^{-x^2/(2\sigma^2)} \sigma^2 > 0, m \geq 0.
\end{equation}
\]

Using the above proposition and the notation in (2.2.7) we obtain the following.

**Proposition 3.1.3** Let \( H \) be a Gaussian space and \( h_i \in H, i=1, \ldots, n \). Then

\[
(3.1.8)
\frac{n^k}{k!} h_n(h_1; \sigma^2) = (-\sigma^2)^n \frac{d^n}{dx^n} e^{-x^2/(2\sigma^2)} \sigma^2 > 0, m \geq 0.
\]
\[(n!)^{3/2} \sum_{\ell=0}^{n-1} (-1)^\ell \sum_{N \in \mathbb{N}^\ell} h_N \left( \sum_{i=1}^n c_i h_i \right)^2 \right) \cdot \sum_{i=1}^n c_i (i) h_i^2 \).

The proof follows from (2.2.7) and (3.1.7).

The first few expressions given by (3.1.8) are

\[(3.1.9) \quad h_1 \otimes h_2 = (2)^h \{ h_1 h_2 - E(h_1 h_2) \} \]

\[(3.1.10) \quad h_1 \otimes h_2 \otimes h_3 = (3!)^{1/2} \{ h_1 h_2 h_3 - h_1 E(h_2 h_3) - h_2 E(h_1 h_3) - h_3 E(h_1 h_2) \} \]

\[(3.1.11) \quad h_1 \otimes h_2 \otimes h_3 \otimes h_4 = (4!)^{1/2} \{ h_1 h_2 h_3 h_4 - h_1 h_2 E(h_3 h_4) - h_1 h_3 E(h_2 h_4) - h_1 h_4 E(h_2 h_3) - h_2 h_3 E(h_1 h_4) - h_2 h_4 E(h_1 h_3) - h_3 h_4 E(h_1 h_2) - h_4 E(h_1 h_2 h_3) \} \]

where \(h_1, \ldots, h_4 \in \mathbb{H}\). They are called Multivariate Hermite polynomials (Fox and Taqqu (1984)).

Using the last two propositions we are now able to compute the symmetric tensor product measure for some sets in \(A^n\).

**Corollary 3.1.1** Let \(W_i, i=1, \ldots, n\) be Gaussian random measures as in Proposition 3.1.1. Then if \(A_1, \ldots, A_n\) are disjoint sets in \(A\)

\[(3.1.12) \quad \bigotimes_{i=1}^n W_i(A_1 \times \ldots \times A_n) = (n!)^{-\frac{1}{2}} W_1(A_1) \ldots W_n(A_n). \]

**Proof** By (3.1.6) in Proposition 3.1.1

\[(3.1.12) \quad \bigotimes_{i=1}^n W_i(A_1 \times \ldots \times A_n) = W_1(A_1) \otimes \ldots \otimes W_n(A_n). \]

Since \(A_1, \ldots, A_n\) are disjoint sets in \(A\), then by (3.1.2) \(W_1(A_1), \ldots, W_n(A_n)\) are orthogonal elements in \(H\) where the latter is defined in (3.1.3). Then (3.1.12) follows using Proposition 3.1.2 with \(k=n, n_i=1, h_i=W_i(A_i)\)
Corollary 3.1.2 Let $W_1, ..., W_n$ be Gaussian random measures as in Proposition 3.1.1. Then if $A_i \in A, i = 1, ..., n$

\begin{equation}
(3.1.13) \quad \otimes_{i=1}^{n} W_i (A_1 \times ... \times A_n)
\end{equation}

\[
(n!)^{3/2} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{N \in \mathcal{P}_n} \left( \sum_{i=1}^{n} c_i \right)^2 \left( \sum_{i=1}^{n} c_i \right)^2 \left( \sum_{i=1}^{n} c_i \right)^2
\]

The proof follows by (3.1.6) in Proposition 3.1.1 and Proposition 3.1.3.

In the next corollary we consider the special case in which $W_1 = ... = W_n$ and $\mu_{ij} = \mu_{ji}, i, j = 1, ..., n$. We use the notation of Proposition 2.2.1.

Corollary 3.1.3 If $A \in A$

\[
W_{(A)} = [W(A)]_{\otimes n} = (n!)^{1/2} h_n(W(A); \mu(A)).
\]

The proof follows from Proposition 3.1.2 since $\mu(A) = E(W(A))^2$ and

\[
h_n = (n!)^{-1/2} h_n(h, E(h^2)).
\]

Multiple integrals Let $W_1, ..., W_n$ and $\otimes W_i$ be as in Proposition 3.1.1. Using the notation of Section 2.3 we have that if $f \in L_1(\otimes W_i)$, i.e. $f$ is $\otimes W_i$-integrable (see Definition 2.3.1), then

\begin{equation}
(3.1.14) \quad I_n(f; W_1, ..., W_n) = \int f(t) d \otimes W_i(t)
\end{equation}

is an element of $H_{\otimes n}$ (and by (3.1.4) an element of $L^2(\Omega, F^n, P)$) which satisfies all properties of the integral w.r.t. the symmetric tensor product measure constructed in Section 2.3. Moreover, we have the next result, in which we use the notation of Theorem 2.3.3.
Lemma 3.1.1  Let $W_1, \ldots, W_n$ and $\otimes W_i$ be as in Proposition 3.1.1. Then

\[ I_n \left( f; W_1, \ldots, W_n \right) = 0. \]

a) If $f \in L^1(\otimes W_i)$, $E(I_n(f; W_1, \ldots, W_n)) = 0$.

b) If $f \in L^2(T^n, A^n, \otimes \mu_1)$ and $g \in L^2(T^n, A^n, \otimes \mu_1)$,

\[ E(I_n(f; W_1, \ldots, W_n)I_m(g; W_1, \ldots, W_m)) = \delta_{nm} \int f_{\otimes n}^\infty(t) R_{\otimes n}(t) g_{\otimes n}(t) d\mu_0^n(t) \]

where $\mu_0 = \sum_{i=1}^n \mu_i$, $R_{\otimes n}(t)$ is defined in (2.3.8) for $\mu_{ij}$ $i, j=1, \ldots, n$ as in (3.1.1) and (3.1.2), and $f_{\otimes n}(t)$ is given by (2.3.11).

Proof a) Since $I_n(f; W_1, \ldots, W_n)$ is $H^{\otimes n}$-valued and $H^{\otimes n}$ is orthogonal to $H^{\otimes 0} = \mathbb{R}$, then $E(I_n(f; W_1, \ldots, W_n)) = 0$.

b) It follows from Theorem 2.3.3 and (3.1.4).

Q.E.D.

Two special cases of the multiple stochastic integral $I_n(f; W_1, \ldots, W_n)$

\[ \int f(t) d \otimes W_i(t) \]

are now considered. First assume the situation studied by Fox and Taqqu (1984): Let $\mu_{ij}(A) = s_{ij} \mu(A)$ $A \in A$, $i, j=1, \ldots, n$, where $\mu$ is a ($\sigma$-finite)-measure on $(T, A)$ and $S = (s_{ij})$ in an $n \times n$ non-negative definite matrix, i.e. $W_1, \ldots, W_n$ are Gaussian random measures such that for $i, j=1, \ldots, n$

\[ E(W_i(A)W_j(B)) = s_{ij} \mu(AnB) \quad A, B \in A. \]

Fox and Taqqu following Itô (1951), define the multiple stochastic integral with dependent integrators $J_n(f; W_1, \ldots, W_n)$ in the following manner: Let $f$ be a special elementary function on $T^n$, i.e.

\[ f(t) = \sum_{i=1}^n a_i \prod_{i=1}^n A_i(t) \]

(3.1.16)
where $A_1, \ldots, A_p$ is a collection of disjoint sets in $A$ and $a_{i_1 \ldots i_n} \in \mathbb{R}$ are zero unless $i_1, \ldots, i_n$ are all distinct. Denote by $S_n$ the class of all special elementary functions. If $\mu$ satisfies the continuity property (i.e. $\forall \varepsilon > 0$ and $A \in A$, $\mu(A) < \varepsilon$, there exist some disjoint $B_j \in A$, $\mu(B_j) < \varepsilon$ for $j = 1, \ldots, m$ and $A = \bigcup B_j$) then $S_n$ is a dense linear manifold in $L^2(T^n, \mathbb{A}^n, \mu^n)$ (Itô (1951)). For $f \in S_n$ define

$$J_n(f; W_1, \ldots, W_n) = \sum_{i_1 \ldots i_n=1}^p a_{i_1 \ldots i_n} W_1(A_{i_1}) \ldots W_n(A_{i_n}).$$

Then $J_n$ can be extended to a bounded linear operator from $L^2(T^n, \mathbb{A}^n, \mu^n)$ to $L^2(\Omega, F, \mathbb{P})$ (Fox and Taqqu (1984)).

On the other hand, using Definition 2.3.1 and Corollary 3.1.1 we have that for $f \in S_n$

$$I_n(f; W_1, \ldots, W_n) = \sum_{i_1 \ldots i_n=1}^p a_{i_1 \ldots i_n} W_1(A_{i_1}) \ldots W_n(A_{i_n}) = (n!)^{-\frac{1}{2}} \sum_{i_1 \ldots i_n=1}^p a_{i_1 \ldots i_n} W_1(A_{i_1}) \ldots W_n(A_{i_n}).$$

Therefore from (3.1.17) and (3.1.18)

$$I_n(f; W_1, \ldots, W_n) = (n!)^{-\frac{1}{2}} J_n(f; W_1, \ldots, W_n) \quad \text{if} \ f \in S_n.$$

**Proposition 3.1.4** Let $\mu$ be a finite measure on $(T, A)$ satisfying the continuity property. Then if $f \in L^2(T^n, \mathbb{A}^n, \mu^n)$

$$I_n(f; W_1, \ldots, W_n) = (n!)^{-\frac{1}{2}} J_n(f; W_1, \ldots, W_n).$$

The proof follows since $I_n$ and $(n!)^{-\frac{1}{2}} J_n$ are $L^2(\Omega)$-valued continuous bounded operators on $L^2(T^n, \mathbb{A}^n, \mu^n)$ which agree (see (3.1.19)) on the dense linear manifold $S_n$. 

Hence the multiple stochastic integral with dependent integrators of Fox and Taqqu (1984) is a special case of our integral $I_n(f; W_1, \ldots, W_n)$ of (3.1.14). Moreover, we do not need to assume the continuity property on $\mu$ to construct this integral, although we require $\mu$ to be finite (see beginning of Section 2.1).

Next assume the situation considered by Itô (1951): $W = W_1, \ldots, W_n$ is a Gaussian random measure on $(T, A)$ and $\mu$ satisfies the continuity property. Itô's multiple Wiener integral $J_n(f; W)$ is constructed as in the Fox and Taqqu case above and it is a bounded linear operator from $L^2(T^n, A^n, \mu^\otimes n)$ to $L^2(\Omega, F^W, P)$. As in Proposition 3.1.4 we have that for $f \in L^2(T^n, A^n, \mu^\otimes n)$ $I_n(\phi; W) = (n!)^{-1/2} J_n(f, W)$ where $I_{n\phi}$ is as in Proposition 2.3.6.

The next two propositions were obtained by Itô (1951). They relate multiple Wiener integrals with Hermite polynomials. We prove them here using the symmetric tensor product set up. We remark that since in the construction of $I_{n\phi}(f; W)$ in Section 2.3 it was not required that $\mu$ satisfies the continuity property (as it is required in Itô's case) this will not be assumed in the next result.

**Proposition 3.1.5** Let $\phi_1(t), \ldots, \phi_m(t)$ be an orthogonal system of real valued functions in $L^2(T, A, \mu)$ and $h_k$ be the Hermite polynomial of degree $k$ given in Proposition 3.1.2. Define

$$f(t) = \phi_1(t_1) \ldots \phi_1(t_{p_1}) \phi_2(t_{p_1+1}) \ldots \phi_2(t_{p_1+p_2}) \ldots \phi_m(t_{p_1+\ldots+p_{m-1}})$$

Then if $n = p_1 + \ldots + p_m$

$$I_{n\phi}(f) = (n!)^{-1/2} \prod_{i=1}^m h_{p_i}(I_1(\phi_i); E(I_1(\phi_i))^2)$$
where \( I_1(g) = \int g(s)dW(s) \) is the isometric integral w.r.t. \( W \) given in Theorem 2.1.1.

**Proof** From Lemma 2.3.1 \( f \) is \( W^n \)-integrable and

\[
I_n(f;W) = I_1(\phi_1) \otimes I_1(\phi_2) \otimes \cdots \otimes I_1(\phi_m).
\]

Next, since for \( i \neq j \) \( E(I_1(\phi_i)I_1(\phi_j)) = \int_T \phi_i(s)\phi_j(s)d\mu(s) = 0 \) then by Proposition 3.1.2

\[
I_1(\phi_1) \otimes I_1(\phi_2) \otimes \cdots \otimes I_1(\phi_m) = (n!)^{-1/2} \prod_{j=1}^m h_1(I_1(\phi_j) ; E(I_1(\phi_j))^2).
\]

Q.E.D.

**Proposition 3.1.6** Let \( T = [0,1] \) and \( \mu \) be non-atomic. Then if

\[
T^n_1 = \{ (t_1, \ldots, t_n) \in T^n : 0 \leq t_1 < \cdots < t_n \leq 1 \}
\]

\[
I_n(1_{T^n_1};W) = \frac{1}{(n!)^{3/2}} h_n(W(T);\mu(T)).
\]

That is formally

(3.1.20) \[
\int_{0 \leq t_1 < \cdots < t_n \leq 1} \cdots \int \cdots dW(t_1) \cdots dW(t_n) = \frac{1}{(n!)^{3/2}} h_n(W(T);\mu(T)).
\]

**Proof** By Definition 2.3.1 \( I_n(1_{T^n_1};W) = W^n(T^n_1) \). Then the result follows by Corollary 3.1.3, since \( W^n \) is finitely additive, \( \mu \) is non-atomic and Proposition 2.2.1 (c).

Q.E.D.

A different proof of (3.1.20) above is given in Theorem 6.5 of Engel (1982).

3.2 Poisson random measure

Let \((\Omega,F,\mathbb{P})\) be a complete probability space and \((T,A)\) be any measurable space such that all singleton sets are measurable, i.e. \( \{t\} \in A \) \( \forall t \in T \).
Let $M(T)$ be the set of all $\sigma$-finite measures on $(T,A)$. We assume that $N$ is a Poisson random measure on $(T,A)$ with intensity $\mu \in M(T)$, i.e. $N(A)_{A \in A}$ is an integer-valued random measure such that the following two conditions hold:

(i) for each $A \in A$, $\mu(A) < \infty$, $N(A)$ is a Poisson random variable with mean $\mu(A)$,

(ii) if $A_1, \ldots, A_k$ are disjoint sets in $A$ then $N(A_1), \ldots, N(A_k)$ are independent random variables on $(\Omega, F, P)$.

The signed random measure $q(A) = N(A) - \mu(A)_{A \in A}$, $\mu(A) < \infty$ is called a centered Poisson random measure. Since for $A, B \in A$ with $\mu(A) < \infty$, $\mu(B) < \infty$, $E(q(A)q(B)) = \mu(AnB)$, then $q$ is an orthogonally scattered measure on $(T,A)$ with control measure $\mu$. We assume $\mu$ is a non-atomic measure.

In this section we consider symmetric tensor product measures of the centered Poisson random measure $q$ with itself for all $n \geq 1$. That is, using the notation of Proposition 2.2.1 we construct $q^\otimes n$ for $n \geq 1$ and multiple integrals w.r.t. $q^\otimes n$.

Let $F^q = \sigma(N(A): A \in A, \mu(A) < \infty)$, $I_q(f)$ be the isometric integral of $f$ w.r.t. $q$ (Theorem 2.1.1) for $f \in L^2(T,A,\mu)$, and $H_q \subseteq L^2(\Omega, F^q, P)$ generated by $q$ (see (2.1.3)), i.e.

$$H_q = \{I_q(f): f \in L^2(T,A,\mu)\}.$$

The exponential space $\text{EXP}(H^q)$ associated with a Poisson random measure has been studied by Neveu (1968) and Surgailis (1984) for the cases when the control measure $\mu$ is finite and $\sigma$-finite respectively. We do not follow the identification of $\text{EXP}(H^q)$ given by the second named author since he uses multiple Poisson integrals to obtain this identification and we want to proceed in the opposite way: first identify $\text{EXP}(H^q)$, then obtain...
the symmetric tensor product stochastic measure \( q^m \) and finally construct multiple integrals w.r.t. \( q^m \), as we did in the last section for the Gaussian random measure. Although for the purpose of this section the identification of \( \text{EXP}(H_q) \) given by Neveu (1968) (\( \mu \) finite) is sufficient, in the next theorem we extend Neveu's result to the case when \( \mu \) is \( \sigma \)-finite. This result will be used in Section 3.3 where we study the general \( L^2 \)-independent increments processes case.

**Theorem 3.2.1**  Let \((\Omega,F,P)\) be a probability space in which there is defined a centered Poisson random measure \( q \) on a measurable space \((T,A)\) with \( \sigma \)-finite non-atomic control measure \( \mu \). Then

\[
\text{EXP}(H_q) = L^2(\Omega,F^q,P)
\]

where for \( f \in L^2(T,A,\mu) \)

\[
(3.2.1) \quad \phi(\exp \Theta(I_q(f))) = \left\{ \frac{1}{\prod \prod} (1+f(z^{(i)}))e^{i} \right\}
\]

where

(i) \( T_i \) are disjoint sets in \( A \), \( 0 < \mu(T_i) < \infty \) and \( U_{i=1}^\infty T_i = T \),

(ii) for each \( i=1,2,... \) and \( j=1,2,... \), \( z_j^{(i)} \) is a \( T_i \)-valued random element with distribution given by the measure \( \mu(T_i)^{-1}\mu(\cdot) \), and for each \( i=1,2,... \) \( N(T_i) \) follows a Poisson distribution with parameter \( \mu(T_i) \),

(iii) \( z_j^{(i)}, N(T_i) \) are mutually independent.

Moreover,

\[
E(\phi(\exp \Theta(I_q(f))))^2 = \exp(E(I_q(f))^2) = \exp(\int_T f^2 d\mu) < \infty.
\]

In order to prove the theorem we first prove the following technical result.
Lemma 3.2.1 \ Let $\mu$ and $T_i, N(T_i)$, $Z_j^{(i)}, j=1,2, \ldots, i=1,2, \ldots$ be as in (i)-(iii) in the above theorem. If $g \in L^1(T_i, \Lambda, \tau, \mu)$ for some $i \geq 1$ then

$$E \left[ \prod_{j=1}^{N(T_i)} g(Z_j^{(i)}) \right] = \int_{T_i} (g-1) d\mu.$$

Proof \ Since $N(T_i)$ follows the Poisson distribution with parameter $\mu(T_i) < \infty$ and for each $j=1,2, \ldots$ $Z_j^{(i)}$ has distribution $\mu(T_i)^j \mu(\cdot)$, then using the independence of $N(T_i), Z_1^{(i)}, Z_2^{(i)}, \ldots$

$$E \left[ \prod_{j=1}^{N(T_i)} g(Z_j^{(i)}) \right] = \sum_{n=0}^{\infty} \frac{(-\mu(T_i))^n}{n!} \int_{T_i^n} \prod_{j=1}^{n} g(t_j) \prod_{j=1}^{n} d\mu(t_j)$$

$$= e^{-\mu(T_i)} \int_{T_i} g d\mu = e^{-\mu(T_i)} \int_{T_i} (g-1) d\mu.$$

Q.E.D.

Proof of Theorem 3.2.1 \ For any Hilbert space $K \{\exp \ θ(k) : k \in K\}$ generates $\text{EXP}(K)$ (Guichardet (1972)), where

$$\exp \ θ(k) = (1,k, (2!)^{-\frac{1}{2}}k θ^2, \ldots)$$

and

$$< \exp \ θ(k_1), \exp \ θ(k_2) >_{\text{EXP}(K)} = e^{<k_1,k_2>_{\text{K}}} \quad k_1, k_2 \in \text{K}.$$ 

Then since $L^2(T, \Lambda, \mu) \cong H_q$, in order to prove the theorem we have to show the following three conditions:

a) for each $f \in L^2(T, \Lambda, \mu) \quad \phi(\exp \ θ(I_q f)) \in L^2(\Omega, f^Q, p)$.

b) for $f_1, f_2 \in L^2(T, \Lambda, \mu) \quad E(\phi(\exp \ θ(I_q f_1)) \phi(\exp \ θ(I_q f_2)))$

$$= < \exp \ θ(I_q f_1), \exp \ θ(I_q f_2) >_{\text{EXP}(H_q)} = e^{\int_{T} f_1 f_2 d\mu}.$$

c) \{\phi(\exp \ θ(I_q f)): f \in L^2(T, \Lambda, \mu)\} generates $L^2(\Omega, f^Q, p)$. 

Since $\mu$ is a $\sigma$-finite measure on $(T,A)$, there exists a sequence of sets $\{T_i\}_{i\geq 1}$ in $A$ such that $0 < \mu(T_i) < \infty$ and $\bigcup_{i=1}^{\infty} T_i = T$. The existence of the random elements $Z_j^{(i)} j=1,2,\ldots, i=1,2,\ldots$ satisfying (ii) and (iii) follows from the construction of a Poisson random measure $N$ with control measure $\mu$ (Theorem 8.1 in Ikeda and Watanabe (1981)).

Let $f \in L^2(T,A,\mu)$, then for each $i \geq 1$ $f$ belongs to $L^2(T_i,\mathcal{A}T_i,\mu)$ and $L^1(T_i,\mathcal{A}T_i,\mu)$. Then by taking $g = (1+f)$ in Lemma 3.2.1 we obtain

$$E \left[ \prod_{j=1}^{N(T_i)} (1+f(Z_j^{(i)})) e^{-\int_{T_i} f \, d\mu} \right] = 1 \quad i=1,2,\ldots$$

Then using (iii) $G_i = \prod_{j=1}^{N(T_i)} (1+f(Z_j^{(i)})) e^{-\int_{T_i} f \, d\mu}$ is a sequence of independent random variables with $E(G_i) = 1 \quad i \geq 1$ and therefore

$$D_n = \prod_{i=1}^{n} G_i$$

is a martingale. Next, using Lemma 3.2.1 with $g = (1+f)^2$ and the independence of $Z_j^{(i)}, N(T_i)$ $j=1,2,\ldots, i=1,2,\ldots$

$$E D_n^2 = \prod_{i=1}^{n} E \left[ \prod_{j=1}^{N(T_i)} (1+f(Z_j^{(i)})) e^{-\int_{T_i} f \, d\mu} \right] = \prod_{i=1}^{n} e^{\int_{T_i} f^2 \, d\mu}$$

$$= \prod_{i=1}^{n} e^{\int_{T_i} f^2 \, d\mu} \leq \exp\left( \int_{T_i} f^2 \, d\mu \right) \leq \exp\left( \int_{\bigcup_{i=1}^{n} T_i} f^2 \, d\mu \right) < \infty$$

i.e.

$$E D_n^2 \leq \exp\left( \int_{T} f^2 \, d\mu \right) < \infty \quad \text{all } n \geq 1.$$
\[
E \left( \prod_{i=1}^{\infty} \prod_{j=1}^{N(T_i)} (1+f(Z_{j}^{(i)})) e^{-\int_{T_i} f d\mu} \right) = E D_n = 1
\]
and
\[
E \left( \prod_{i=1}^{\infty} \prod_{j=1}^{N(T_i)} (1+f(Z_{j}^{(i)})) e^{-\int_{T_i} f d\mu} \right)^2 = \lim_{n \to \infty} E D_n^2
\]

\[
= \lim_{n \to \infty} \exp \left( \int_{T_i} f^2 d\mu \right) = \exp \left( \int_{T} f^2 d\mu \right) < \infty
\]

which shows (a).

Let \( f_1, f_2 \in L^2(T,A,\mu) \), then applying Lemma 3.3.1 to \( g = (1+f_1)(1+f_2) \)

one shows in a similar way as above that

\[
E(\exp \Theta(I(q(f_1))) \exp \Theta(I(q(f_2)))) = \lim_{n \to \infty} \prod_{i=1}^{N(T_i)} (1+f_1)(1+f_2)(Z_{j}^{(i)}) e^{\int_{T_i} f_1^2 f_2 d\mu}
\]

\[
= \lim_{n \to \infty} \prod_{i=1}^{N(T_i)} e^{\int_{T_i} f_1^2 f_2 d\mu} = e
\]

proving (b).

Finally, to prove (c) let \( G \in L^2(\Omega,F^q,P) \) and suppose that

\[
E(\exp \Theta(I_q(f)))G = 0 \quad \text{for all } f \in L^2(T,A,\mu).
\]

We want to show that \( G = 0 \) a.e. \( dP_{F^q} \), where

\[
F^q = \sigma(I_q(f) : f \in L^2(T,A,\mu)).
\]

Using (3.2.1) we have that for all \( f \in L^2(T,A,\mu) \)

\[
E \left( \prod_{i=1}^{\infty} \prod_{j=1}^{N(T_i)} (1+f(Z_{j}^{(i)})) e^{-\int_{T_i} f d\mu} \right) G = 0.
\]

Next let \( i \geq 1 \) be fixed and for \( g \in L^2(T_1,\mathbb{A} \cap T_1,\mu) \) define \( f : T \to \mathbb{R} \) by

\[
f(t) = g(t) \quad t \in T_i \quad \text{and zero if } t \notin T_i.
\]

Then \( f \in L^2(T,A,\mu) \) and
\[
\mathbb{E} \left( \prod_{j=1}^{N(T_i)} (1 + g(Z_j)) e^{-\int_{T_i} g d\mu} \right) = 0 \quad \text{all } g \in L^2(T_i A \cap T_i, \mu).
\]

Hence by Proposition 7.13 in Neveu (1968) (Lemma 3.3.2 below)
\[
\mathbb{E}(G | F_i^n) = 0 \quad \text{a.s. where } F_i^n = \sigma(I_q(g): g \in L^2(T_i A \cap T_i, \mu)), \text{ and } F_i^n \subset F_i^n
\]
all \( i \geq 1 \).

Hence for all \( n \geq 1 \) \( \mathbb{E}(G | \bigvee_{i=1}^n F_i^n) = 0 \) a.s. since \( F_1^n, \ldots, F_n^n \) are independent \( \sigma \)-fields.

Next let \( F = \bigvee_{i=1}^n F_i^n \) and \( F = \bigvee_{i=1}^\infty F_i^n \). Then since \( \mathbb{E}(G^2) < \infty \) by the martingale convergence theorem \( G = 0 \) a.s. \( dP \). Thus it remains to show that \( F^n \subset F^\infty \).

Let \( f \in L^2(T, A, \mu) \), then
\[
f(t) = \sum_{i=1}^\infty f(t) 1_{T_i}(t)
\]
and
\[
I_q(f) = \sum_{i=1}^\infty I_q(f 1_{T_i}) \quad \text{a.s.}
\]
Thus \( I_q(f) \) is \( F^\infty \)-measurable all \( f \in L^2(T, A, \mu) \) since for each \( i \geq 1 \) \( I_q(f 1_{T_i}) \) is \( F_i^n \)-measurable. That is, \( F^n \subset F^\infty \) and \( G = 0 \) a.e. \( dP \).

Q.E.D.

We are now going to apply our results of Chapter 2 to the construction of the product stochastic measure \( q^\otimes n \). Therefore in the remainder of this section we will assume that the control measure \( \mu \) is finite and the following version (Proposition 7.13 in Neveu (1968)) of the last theorem will be enough for our purpose.

**Lemma 3.2.2** (Neveu (1968)). Let \( q \) be a centered Poisson random measure as in Theorem 3.2.1 with finite control measure \( \mu \). Then
\[
\text{EXP}(H_q) \cong L^2(\Omega, F^q, P)
\]
where for \( f \in L^2(T,A,\mu) \)

\[
\eta(\exp \circ (I_q(f))) = \frac{N(T)}{\prod_{j=1}^{N(T)} (1+f(Z_j))} e^{-\int_T f d\mu}
\]

(3.2.2)

where \( \{Z_j\}_{j=1} \) is a sequence of independent random elements, independent of \( N(T) \), each \( Z_j \) taking values in \( T \) and having distribution \( \mu(T)^{-1}\mu(\cdot) \).

Using the identification of \( \text{EXP}(H_q) \) given by the above lemma, in our next result we obtain an \( L^2 \)-valued product stochastic measure of the Poisson random measure \( q \).

**Proposition 3.2.1** Let \( q \) be a centered Poisson random measure on \((T,A)\) with finite non-atomic control measure \( \mu \). Then for each \( n \geq 1 \) there exists a unique \( L^2(\Omega,F^q,P) \)-valued measure \( q^{en} \) on \((T^n,A^n)\) such that if \( A_1, \ldots, A_n \) belong to \( A \)

\[
q^{en}(A_1 \times \ldots \times A_n) = q(A_1) \otimes \ldots \otimes q(A_n)
\]

and for \( A \in A^n \)

\[
E(q^{en}(A)) = 0
\]

\[
\text{VAR}(q^{en}(A)) = \frac{1}{n!} \sum_{\Pi} \mu^{\cdot \eta}(A \eta A^n).\]

Moreover, Proposition 2.2.1 (b)-(d), Lemma 2.2.6 and Corollaries 2.2.8-2.2.9 hold for \( q^{en} \).

**Proof** For each \( n \geq 1 \), existence and uniqueness of the \( H_q^{en} \)-valued measure \( q^{en} \) follow from Proposition 2.2.1 (a). It is seen as an \( L^2(\Omega,F^q,P) \)-valued element by Lemma (3.2.2). The expressions for the mean and the variance follow similar to Proposition 3.1.1.

Q.E.D.

We now compute special cases of (3.2.3). If \( A \in A \) then \( q(A) = I_q(1,A) \) and from (3.2.2) we have that
\[ q(A)^{\otimes n} = (n!)^{-\frac{1}{2}} \sum_{j=0}^{n} \exp \Theta(zq(A)) \]

\[ = (n!)^{-\frac{1}{2}} \sum_{j=0}^{n} \exp \theta \left( (1+z)^{A} (z_A) e^{-z\mu(A)} \right) \]

But \((1+z)^x e^{-z\lambda} z > -1\) is the generating function of the Poisson-Charlier polynomials (Chihara (1978)) with parameter \(\lambda > 0\), denoted by \(c_n(x;\lambda)\), i.e.

\[
e^{-\lambda z} (1+z)^x = \sum_{n=0}^{\infty} \frac{z^n}{n!} c_n(x;\lambda) \quad \lambda > 0.
\]

Then for \(A \in \mathcal{A}\)

\[ q^{\otimes n}(A) = (n!)^{-\frac{1}{2}} c_n(N(A);\mu(A)). \]

We now obtain a similar result to Proposition 3.1.2 for symmetric tensor products of Poisson random measures.

**Proposition 3.2.2** Let \(q\) be a centered Poisson random measure as in Proposition 3.2.1 and \(A_1, \ldots, A_k\) disjoint sets in \(A\). Then

\[ q(A_1)^{\otimes n_1} \cdots q(A_k)^{\otimes n_k} = (n!)^{-\frac{1}{2}} \prod_{j=1}^{k} c_{n_j}(N(A_j);\mu(A_j)) \]

where \(n = \sum_{j=1}^{k} n_j\) and \(c_m(x;\lambda)\) are Poisson-Charlier polynomials with parameter \(\lambda\) defined in (3.2.4).

**Proof** Since \(A_1, \ldots, A_k\) are disjoint sets in \(A\), then \(q(A_1), \ldots, q(A_n)\) are mutually orthogonal in \(H \subset L^2(\Omega, F^q, \mu)\) and therefore the family \(\{q(A_1)^{\otimes n_1} \cdots q(A_k)^{\otimes n_k}\} \quad n_1 \geq 0, \ldots, n_k \geq 0\) is orthogonal in \(\text{EXP}(H^q)\). Next for \(z_1, \ldots, z_k \in \mathbb{R}\)

\[ \exp \Theta(z_1 q(A_1) + \cdots + z_k q(A_k)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z_1 q(A_1) + \cdots + z_k q(A_k))^n \]
On the other hand from (3.2.2)

\[ n(\exp z_1q(A_1)+\ldots+z_kq(A_k)) \]

\[ = \prod_{i=1}^{k} (1+z_1A_1(z_i)+\ldots+z_kA_k(z_i)) e^{-z_iu(A_i)} \]

Then the assertion of the proposition follows from the last two expressions since

\[ q(A_1)\ldots q(A_k) = (n!)^{-k} \left( \frac{n_1^{n_1} \ldots n_k^{n_k}}{n_1! \ldots n_k!} \right) \left( \prod_{i=1}^{k} (1+z_1z_i) \right) e^{z_iu(A_i)} \]

Expression (3.2.6) is related to the multivariate Poisson-Charlier polynomials (Ogura (1972), Chihara (1978)) defined by

\[ K_n((x_1)n_1,\ldots,(x_k)n_k; (\lambda_1)n_1,\ldots,(\lambda_k)n_k) = \]

\[ \left( \frac{\partial^n}{\partial z_1 \ldots \partial z_k} \right)_{z=0}^{k} (1+z_1z_i)^{-\lambda_i} e^{-\lambda_i} \]

where \( n = \sum_{i=1}^{k} n_i \), \( (x_i)_{n_i} = (x_1,\ldots,x_i) \) and \( \lambda_i > 0 \) \( i=1,\ldots,k \).
The first few expressions are

\[ K_0(x; \lambda) = 1, \quad K_1(x; \lambda) = c_1(x; \lambda) = x - \lambda \]

\[ K_2(x_1, x_2; \lambda_1, \lambda_2) = x_1(x_2 - \delta_{12}) - x_1 \lambda_1 - x_2 \lambda_2 + \lambda_1 \lambda_2 \]

\[ K_3(x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3) = x_1(x_2 - \delta_{12})(x_3 - \delta_{13} - \delta_{23}) \]

\[ - [x_1(x_2 - \delta_{12})\lambda_1 \lambda_2 + x_2(x_3 - \delta_{23})\lambda_2 \lambda_3 + x_3(x_1 - \delta_{13})\lambda_1 \lambda_3] \]

\[ + x_1 \lambda_1 + x_2 \lambda_2 + x_3 \lambda_3 - \lambda_1 \lambda_2 \lambda_3 \]

where \( \delta_{ij} = 1 \) if \( i=j \), and 0 otherwise.

**Corollary 3.2.1** If \( A_1, \ldots, A_n \) are disjoint sets in \( \mathcal{A} \),

\[ q(A_1) \otimes \cdots \otimes q(A_n) = (n!)^{-\frac{1}{2}} q(A_1) \cdots q(A_n). \]

**Proof** Taking \( k=n, n_i=1 \) \( i=1, \ldots, n \) in Proposition 3.2.2 since \( q(A_1), \ldots, q(A_n) \) are orthogonal

\[ q(A_1) \otimes \cdots \otimes q(A_n) = (n!)^{-\frac{1}{2}} \prod_{i=1}^{n} c_1(N(A_i), \mu(A_i)). \]

Then the corollary follows since \( c_1(N(A); \mu(A)) = N(A) - \mu(A) = q(A) \) for all \( A \in \mathcal{A} \).

Q.E.D.

**Multiple Poisson integrals** Using the notation of Section 2.3., if \( f \in L_1(q^\otimes n) \), \( n \geq 1 \) then

\[ I_n(f; q) = \int_{T^n} f(t) dq^\otimes n(t) \]

is an element of \( H^\otimes_n \) (and hence of \( L^2(\Omega, F^3, P) \) by (3.2.1)). This integral satisfies all properties of the integral w.r.t. the symmetric tensor product measure of Section 2.3, and in particular Propositions 2.3.6 and 2.3.7.
now written in the following manner.

**Lemma 3.2.3**  

(a) If \( f \in L^1(\Omega^n, I_m) \), \( \mathbb{E}(I_m(f; q)) = 0. \)

(b) If \( f \in L^2(T^n, A^n, \mu) \) and \( g \in L^2(T^m, A^m, \mu) \)

\[
\mathbb{E}(I_m(f; q) I_m(g; q)) = \delta_{nm} <f, g>_{L^2(T^n, A^n, \mu)}. 
\]

(c) \( \{ I_m(f; q) : f \in L^2(T^n, A^n, \mu) \} \) constitutes a complete orthogonal system in \( L^2(\Omega, F^q, P) \), where \( I_0(\cdot) = 1. \)

**Proof**  
(a) follows from Proposition 3.2.1, (b) from Proposition 2.3.6 and (c) by (3.2.1) and Proposition 2.3.7.

Q.E.D.

Multiple Poisson Wiener integrals were introduced by Ito (1956) and studied later by Ogura (1972) and Surgailis (1984). H. Ogura uses multivariate Poisson-Charlier polynomials to define a multiple stochastic integral with respect to \( q \) (when \( T \subseteq \mathbb{R} \)), which we denote by \( J_n(f) \). If \( f \) is an elementary function defined in (2.3.17)

\[
J_n(f) = \sum_{i_1 \ldots i_n=1}^{p} a_{i_1} \ldots i_n K_n(N(A_{i_1}), \ldots, N(A_{i_n}); \mu(A_{i_1}), \ldots, \mu(A_{i_n}))
\]

where \( K_n \) is multivariate Poisson-Charlier polynomial. Then \( J_n \) extends to a bounded linear operator from \( L^2(T^n, A^n, \mu) \) to \( L^2(\Omega, F^q, P) \) (Ogura (1972)).

On the other hand, using Proposition 3.2.2, if \( f \) is an elementary function

\[
I_n(f; q) = \sum_{i_1 \ldots i_n=1}^{p} a_{i_1} \ldots i_n q(A_{i_1}) \ldots q(A_{i_n})
\]

\[
= (n!)^{\frac{1}{2}} \sum_{i_1 \ldots i_n=1}^{p} a_{i_1} \ldots i_n K_n(N(A_{i_1}), \ldots, N(A_{i_n}); \mu(A_{i_1}), \ldots, \mu(A_{i_n})).
\]

Thus \( I_n(f; q) \) and \( (n!)^{\frac{1}{2}} J_n(f) \) agree on a dense linear manifold of...
$L^2(T^n,A^n,\mu^n)$ and therefore the following result is obtained.

**Proposition 3.2.3** If $f \in L^2(T^n,A^n,\mu^n)$

$$I_{n\phi}(f;q) = (n!)^{-1/2} J_n(f).$$

**Proposition 3.2.4** Let $T = [0,1]$ and $\mu$ be non-atomic. Then if

$$T_1^n = \{(t_1,\ldots,t_n) \in T^n: 0 \leq t_1 < \ldots < t_n \leq 1\}

$$

$$I_{n\phi}(T_1^n;q) = (n!)^{-3/2} c_n(N(T); \mu(T)).$$

That is, formally

$$(3.2.7) \int_{0 \leq t_1 < \ldots < t_n \leq 1} dq(t_1) \ldots dq(t_n) = (n!)^{-3/2} c_n(N(T); \mu(T))$$

where $c_n$ is the Poisson-Charlier polynomial of degree $n$ defined in (3.2.4).

**Proof** By definition $I_{n\phi}(T_1^n;q) = q^\Theta(T_1^n)$. The rest of the proof follows similar to Proposition 3.1.6 using Corollary 3.2.1.

The expression (3.2.7) above is Theorem 6.9 in Engel (1982), who gives a different proof.

### 3.3 Nonidentically distributed $L^2$-independently scattered measures

In this section we study $L^2$-valued product stochastic measures and multiple stochastic integrals of nonidentically distributed $L^2$-independently scattered measures that are mutually independent over disjoint sets. The identification of the exponential space of $H$, the common Hilbert space where the i.s.m.'s take values, is obtained using results given in the last two sections. The parameter set $T$ considered in this section is an interval of the real line. This is an important assumption throughout the section since we use martingale theory to identify symmetric tensor products of $H$. 
We conclude this section by giving characterizations in terms of multiple stochastic integrals of Poisson and Gaussian processes with independent increments.

Our first result of this section is a general one, in the sense that it identifies the exponential space of any Hilbert space \( H \) which is a direct sum of an arbitrary Gaussian space \( H_W \) and an arbitrary Poisson space \( H_q \), where \( H_w \) and \( H_q \) are stochastically independent.

**Theorem 3.3.1** Let \((\Omega, F, P)\) be a complete probability space and \( q \) be a centered Poisson random measure on a measurable space \((E, E)\) defined on \((\Omega, F, P)\), with \( \sigma \)-finite non-atomic control measure \( \nu \) and generating the Poisson space

\[
H_q = \{ I_q(f): f \in L^2(E, \mathcal{E}, \mu) \}
\]

where \( I_q \) is the isometric integral of \( f \) w.r.t. \( q \). Let \( H_W \) be a Gaussian space on \((\Omega, F, P)\) stochastically independent of the system of random variables \( H_q \). Define the \( \sigma \)-fields \( F_W = \sigma(H_W), F^q = \sigma(H_q) \) and the Hilbert space

\[
H = H_W \otimes H_q
\]

Then

\[
\text{EXP}(H) = \sum_{n \geq 0} \otimes_{H} \otimes_{\nu} \gamma \in L^2_{\mathbb{R}}(\Omega, F_{W} \vee F^{q}, P)
\]

where for \( h \in H, h = h_W + h_q, h_W \in H_W, h_q \in H_q \)

\[
\gamma: \text{EXP}(H) \rightarrow L^2_{\mathbb{R}}(\Omega, F_{W} \vee F^{q}, P)
\]

is defined by

\[
\gamma(\exp \otimes(h)) = \psi(\exp \otimes(h_W)) \phi(\exp \otimes(h_q))
\]

and \( \psi, \phi \) are the isometrics given in (3.1.4) and Theorem 3.2.1 respectively.
Proof. We first prove that for all $h \in H$, $\gamma(\exp \Theta(h))$ is an element of $L^2(\Omega, F^W \vee F^q, P)$ and that

\[(3.3.3)\]

$$E(\gamma(\exp \Theta(h)))^2 = \exp(Eh^2).$$

By Theorem 3.2.1 for all $h_q \in H_q$

$$E(\phi(\exp \Theta(h_q)))^2 = \exp(Eh_q^2) < \infty$$

and by Proposition 7.3 in Neveu (1968) for all $h_W \in H_W$

$$E(\psi(\exp \Theta(h_W)))^2 = \exp(Eh_W^2) < \infty.$$

Then if $h = h_w + h_q$, $\gamma(\exp \Theta(h)) = \psi(\exp \Theta(h_W)) \phi(\exp \Theta(h_q))$ belongs to $L^2_{IR}(\Omega, F^W \vee F^q, P)$ since $h_W$ and $h_q$ are independent. Moreover, from the above expressions we have that

$$E(\gamma(\exp \Theta(h)))^2 = \exp(Eh_W^2 + Eh_q^2) = \exp(Eh^2).$$

Next we shall prove that $\{\gamma(\exp \Theta(h)): h \in H\}$ generates $L^2_{IR}(\Omega, F^W \vee F^q, P)$, which will imply (3.3.1) since for any Hilbert space $K \{\exp \Theta(k): k \in K\}$ generates the Hilbert space $\text{EXP}(K)$ (Guichardet (1972)).

Let $Z \in L^2_{IR}(\Omega, F^W \vee F^q, P)$ and suppose that

$$E(Z(\gamma(\exp \Theta(h)))) = 0 \text{ for each } h \in H.$$ 

Then for all $h_W \in H_W$ and $h_q \in H_q$

$$E(Z(\psi(\exp \Theta(h_W)) \phi(\exp \Theta(h_q)))) = 0.$$

But from Proposition 7.3 in Neveu (1968) and Theorem 3.2.1 in this thesis $\{\psi(\exp \Theta(h_W)): h_W \in H_W\}$ and $\{\phi(\exp \Theta(h_q)): h_q \in H_q\}$ generate $L^2_{IR}(\Omega, F^W, P)$ and $L^2_{IR}(\Omega, F^q, P)$ respectively. Then for all $A_1 \epsilon F^W$ and $A_2 \epsilon F^q$
\[ \int_{A_1 \cap A_2} Z \, dP = 0. \]

But it is known (Dellacherie and Meyer (1978)) that if two \( \sigma \)-fields \( F_1 \) and \( F_2 \) are independent, then \( F_1 \cup F_2 \) is generated by the field \( C_0 \) of all finite disjoint unions of sets \( A_1 \cap A_2 \), \( A_1 \in F_1 \), \( A_2 \in F_2 \).

Thus since \( Z \) is \( P \)-integrable

\[ C = \{ A \in F : \int Z \, dP = 0 \} \]

is a monotone class, and by the monotone class theorem

\[ \int Z \, dP = 0 \quad \forall A \in F^W \vee F^Q \]

since \( C_0 \subseteq C \). That is, \( Z = 0 \) a.e. \( dP \), \( F^W \vee F^Q \).

Then

\[ \{ \gamma(\exp \Theta(h) : h \in H) \} \]

generates the space \( L^2(\Omega, F^W \vee F^Q, P) \).

Q.E.D.

**Assumption 3.3.1** Throughout the remainder of this section we will make the following assumptions and notations: Let \( (\Omega, F, P) \) be a complete probability space, \( T = [0, T_0] \), \( T_0 > 0 \), \( A = B(T) \), \( F_0^0 = \mathbb{R}^n - \{ 0 \} \) and \( B^0_n = B(\mathbb{R}^0_n) \) for \( n \geq 1 \). Suppose that

\[ Y_t = (X_1(t), \ldots, X_n(t)) \quad t \in T, \quad Y_0 = 0 \]

is an \( n \)-dimensional stochastically continuous, right continuous, zero mean, \( L^2 \)-stochastic process with independent increments defined on \( (\Omega, F, P) \), with characteristic function given by

\[ \Phi_y(a) = \exp\{-\frac{1}{2} a' \sigma(t) a + \int_{\mathbb{R}^n} (e^{ia'x} - 1 - ia'x) \nu(t, dx)\} \quad a \in \mathbb{R}^n \]
and Levy-Itô representation (Gikhman-Skorokhod (1969))

\[(3.3.6) \quad Y_t = W_t + \int_0^t \int_{\mathbb{R}^n} xq(ds,dx) \quad \text{all } t \in T\]

where

\[(3.3.7) \quad W_t \text{ is an } n\text{-dimensional Gaussian process with independent increments, } W_0 = 0 \text{ and positive definite diffusion matrix } \sigma(t);\]

\[(3.3.8) \quad q(B,\Gamma) = N(B,\Gamma) - \nu(B,\Gamma) \quad \Gamma \in B^O_n, \quad B \in \mathcal{A}\]

is a centered Poisson random measure on \((\mathbb{T} \times \mathbb{R}^n, \mathcal{A} \times B^O_n)\) with \(\sigma\)-finite control measure \(\nu\), and independent of \(\{W_t\} \quad t \in T;\)

\[(3.3.9) \quad N(B,\Gamma) = \sum_s 1_{B \times \Gamma}(s,\Delta Y_s) \quad \Delta Y_s = Y_s - Y_{s-},\]

\[(3.3.10) \quad \nu(B,\Gamma) = E(N(B,\Gamma)),\]

\[\int_0^t \int_{\mathbb{R}^n} |x|^2 \wedge 1 \nu(ds,dx) < \infty \quad t \in T\]

and \(\nu_t(\Gamma) = \nu([0,t],\Gamma)\) is increasing and continuous.

From (3.3.5) we have that for \(i,j=1,\ldots,n\) and \(t \in T\)

\[(3.3.11) \quad \mu_{ij}(t) = E_{\mathbb{T}^i}(t)X_j(t) = \sigma_{ij}(t) + \lambda_{ij}(t) < \infty\]

where

\[\lambda_{ij}(t) = \int_0^t x_i x_j \nu(t,dx) \quad X = (x_1,\ldots,x_n).\]

Define

\[(3.3.12) \quad \mu_0(t) = \sum_{i=1}^n \sigma_{ii}(t) + \sum_{i=1}^n \lambda_{ii}(t)\]

and denote by \(\mu_{ij}, \sigma_{ij}, \lambda_{ij}\) and \(\mu_0\) the corresponding finite (signed) measures on \((\mathbb{T},\mathcal{A})\) generated by \(\mu_{ij}(t), \sigma_{ij}(t), \lambda_{ij}(t)\) and \(\mu_0(t)\) respectively.
Then for each i,j=1,...,n \( \lambda_{ij} \ll \mu_0 \), \( \sigma_{ij} \ll \mu_0 \), \( \mu_{ij} \ll \mu_0 \) and

\[
(3.3.13) \quad R_1(t) = \left( \frac{d\sigma_{ij}}{du_0}(t) \right)_{i,j=1}^{n} \text{ a.e. } du_0(t)
\]

\[
(3.3.14) \quad R_2(t) = \left( \frac{d\lambda_{ij}}{du_0}(t) \right)_{i,j=1}^{n} \text{ a.e. } du_0(t)
\]

are non-negative definite matrices a.e. \( du_0(t) \) and so is the matrix

\[
R(t) = (r_{ij}(t))_{i,j=1}^{n}, \text{ where for } i,j=1,...,n
\]

\[
(3.3.15) \quad r_{ij}(t) = \frac{du_{ij}}{du_0}(t) = \frac{d\sigma_{ij}}{du_0}(t) + \frac{d\lambda_{ij}}{du_0}(t) \text{ a.e. } du_0(t).
\]

Finally if \( A(t) \) is an \( n \times n \) non-negative definite matrix a.e. \( du_0(t) \), we denote by \( L^2_A(\mu_0) \) the linear space of functions

\[
(3.3.16) \quad L^2_A = \{ f: T \to \mathbb{R}^n : \mathbb{E} \int_T f(t)^T A(t) f(t) du_0(t) < \infty \}.
\]

In the next two results we use Theorem 3.3.1 to identify some functionals of the process \( Y_t \) as elements of an appropriate Hilbert space \( H_Y \) and its exponential space \( \text{EXP}(H_Y) \). Having this and using the framework of Chapter II, we will be able to study symmetric tensor product measures and multiple stochastic integrals of the independently scattered measures \( X_i \)'s where \( X_i([0,t]) = X_i(t) \) \( i=1,...,n \) and the latter are given in (3.3.4).

**Proposition 3.3.1** Let \( Y_t, t \in T \) be a stochastic process as in Assumption 3.3.1. Let \( H_W \) be the Gaussian space generated by \( W_t \), i.e.

\[
H_W = \text{sp}\{ a^T W_t ; a \in \mathbb{R}^n, t \in T \}
\]

and \( H_q \) the "Poisson" space generated by \( q \), i.e.

\[
(3.3.17) \quad H_q = \{ I_q(g) : g \in L^2(T \times \mathbb{R}^n, A \times B_n^0, \nu) \}.
\]
Define

\[ H_Y = H_W \oplus H_q. \]

Let \( R(t) \) be as in (3.3.15). Then if \( f \in L^2(\mu_0) \), \( f(t) = (f_1(t), \ldots, f_n(t)) \), the random variable

\[ (3.3.18) \quad \int_T f^* d\gamma = \sum_{i=1}^{n} I_{W_i} (f_i) + I_{q} (f'x) \]

is an element of \( H_Y \), where \((f'x)(t,x) = f(t)'x \quad t \in T, \quad x \in \mathbb{R}_n\), and \( I_{X}(\phi) \) is the isometric integral (Theorem 2.1.1) of \( \phi \) w.r.t. the orthogonally scattered measure \( X \). Moreover,

\[ f(t)'x \in L^2(T \times \mathbb{R}_n^0, A \times B_n^0, \nu) \cap L^1(T \times \mathbb{R}_n^0, A \times B_n^0, \nu). \]

**Proof** To prove that \( \int_T f^* d\gamma \in H_Y \) it is enough to show that \( f_i \in L^2(T, A, \sigma_{i1}) \)

\( i=1, \ldots, n \) and \( f'x \in L^2(T \times \mathbb{R}_n^0, A \times B_n^0, \nu) \), since \( I_{W_i}(f_i) \in H_W \quad i=1, \ldots, n \) and \( I_{q}(f'x) \in H_q \).

Since \( f \in L^2(\mu_0) \) then from (3.3.15) \( f \in L^2_R(\mu_0) \cap L^2_R(\mu_0) \) where \( R_1 \) and \( R_2 \) are given in (3.3.13) and (3.3.14) respectively. Then

\[ \sum_{i=1}^{n} \int_T f_i^2(t) d\sigma_{i1}(t) = \sum_{i=1}^{n} \int_T f_i(t) \frac{d\sigma_{i1}}{d\mu_0}(t)f_i(t)\,d\mu_0 < \infty \]

i.e.

\[ f_i \in L^2(T, A, \sigma_{i1}) \quad i=1, \ldots, n. \]

On the other hand, using (3.3.12)

\[ (3.3.19) \quad \int_T \int_{\mathbb{R}_n^0} (f(t)'x)^2 \,d\nu(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_T \int_{\mathbb{R}_n^0} f_i(t)f_j(t)x_j \,d\nu(t,x) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_T f_i(t)f_j(t) \,d\lambda_{ij}(t) = \int_T (f(t)'R_2(t)f(t))\,d\mu_0(t) < \infty. \]

Finally, to prove that \( f(t)'x \in L^1(T \times \mathbb{R}_n^0, A \times B_n^0, \nu) \) it is enough to show that
\( f_i(t)x_i \in L^1(T \times \mathbb{R}^n, A \otimes B^o_n, \nu) \) \( i=1,...,n \) since \( f(t)'x = \sum_{i=1}^{n} f_i(t)x_i \). By (3.3.19) for each \( i=1,...,n \)

\[
\int_T f_i^2(t) d\lambda_i(t) < \infty.
\]

Then since each \( \lambda_i \) is a finite measure on \((T,A)\)

\[
\int_T |f_i(t)| d\lambda_i(t) < \infty
\]

and therefore using (3.3.12)

\[
\int_T \int_{\mathbb{R}^n} f_i(t)x_i d\nu(t,x) = \int_T f_i(t) d\lambda_i(t) < \infty.
\]

Q.E.D.

For the process \( Y_t \), Theorem 3.3.1 is written in the following manner.

**Proposition 3.3.2** Under the assumptions of Proposition 3.3.1

(3.3.20) \[ \text{EXP}(H^\gamma) = L^2(\Omega, F^Y, P) \]

where \( \gamma \) is given by (3.3.2) in Theorem 3.3.1 and

\[ F^Y = \sigma(Y_t: t \in T) \cup \{\text{P-null sets of } \Omega\}. \]

**Proof** From the construction of the Levy-Itô decomposition (3.3.6) of \( Y_t \) (Gikhman and Skorokhod (1969)) \( W_t \) and \( q \) are \( F^Y \) measurables. On the other hand, by (3.3.6) and Proposition 3.3.1, \( Y_t \) is \( F^W \cup F^Q \)-measurable all \( t \in T \), where \( F^W = \sigma(H^W) \) and \( F^Q = \sigma(H^Q) \). Then the result follows using Theorem 3.3.1 and the fact that \( H^W \) and \( H^Q \) are independent.

Q.E.D.

For purposes of later reference, the main properties of the integral

\[ \int_T f \cdot dY \]

defined in Proposition 3.3.1 are summarized in the next result.

**Lemma 3.3.1** Under the assumptions of Proposition 3.3.1, for \( f \in L^2_K(\nu) \) define
\[ F_t = \int f \cdot dY = \int_{[0,t]} \mathbf{1}_{[0,t]}(s) f(s) \cdot dY_s \quad t \in T \]

where right hand side is defined in (3.3.18). Then \( (F_t)_{t \in T} \) is a zero mean \( L^2 \)-stochastic process with independent increments and \( (F_t, F_Y^t) \) is an \( L^2 \)-right continuous martingale such that

a) \( F_t \in H_Y \quad t \in T \).

b) \( \langle F \rangle_t = \int_0^t f(s)' R(s) f(s) d\mu_o(s) < \infty \quad \forall \ t \in T \)

where \( \langle F \rangle \) denotes the predictable quadratic variation of \( F \).

c) If \( g \in L^2_R(\mu_o) \)

\[ \langle F, G \rangle_t = \int_0^t f(s)' R(s) g(s) d\mu_o(s) = \mathbb{E}(F_t G_t) \quad t \in T. \]

d) The characteristic function of \( F_t \) is given by

\[ \Phi_F(a) = \exp \{ -\frac{a}{2} \langle F \rangle_t + \frac{1}{2} \int_0^t e^{iaf'(s)X - 1 - iaf'(s)X} d\nu(s, X) \} \]

where

\[ \langle F^c \rangle_t = \int_0^t f(s)' R^c(s) f(s) d\mu_o(s) \quad t \in T. \]

**Proof** Since the process \( Y_t \) is right continuous, then \( (F^Y_t)_{t \in T} \) is a right continuous filtration. From Proposition 3.3.2 \( F^Y_t \subset F^Y = \sigma(H_W) \cup \sigma(H^Y) \) all \( t \in T \) and by (3.3.18)

\[ F_t = \sum_{i=1}^n I_{\mathbb{W}_i}(1_{[0,t]}^t f_i^t) + \int_0^t f'(s) X 1_{[0,t]}(s) dq(s, X) \in H_Y. \]

It is known (Galthouck (1976)) that if \( f_i \in L^2(T, A, \sigma_{\mathbb{W}_i}) \quad i=1, \ldots, n \) and

\[ \phi^c_t = \sum_{i=1}^n I_{\mathbb{W}_i}(1_{[0,t]} f_i^t) \quad f(t) = (f_1(t), \ldots, f_n(t)) \]

then \( (F^c_t, F^Y_t)_{t \in T} \) is an \( L^2 \)-continuous martingale with \( \langle F^c \rangle_t \) given by (3.3.22).

Also it is known that if
Jo F (s)^' x 1 [0, t] (s) dq(s, x) t \in T.

(3.3.25) \quad \langle F^d \rangle_t = \int_0^t f(s)^' R_2(s) f(s) d\mu_0(s) t \in T.

Thus \( (F_t, F_t') \) is an \( L^2 \)-right continuous martingale and since \( F^C_t \in H_W \) and \( F^d_t \in H_q \), then \( E(F_t') = 0 t \in T \) and

\[
\langle F \rangle_t = E(F^2_t) = E(F^C_t)^2 + E(F^d_t)^2 = \langle F^C \rangle_t + \langle F^d \rangle_t
\]

\[
= \int_0^t f(s)^' R(s) f(s) d\mu_0(s) < \infty t \in T.
\]

The proof of (c) follows from (b) and the polarization identity

\[
\langle F, G \rangle_t = \frac{1}{4} \{ \langle F + G, F + G \rangle_t - \langle F - G, F - G \rangle_t \} t \in T.
\]

Finally (d) follows since \( F_t = F^C_t + F^d_t \) \( F^C_t \in H_W, F^d_t \in H_q, H_W \) and \( H_q \) are independent and using (3.3.5).

Q.E.D.

Now we shall apply Propositions 3.3.1 and 3.3.2 and our results in Chapter 2 to construct an \( L^2(\Omega) \)-valued product stochastic measure of \( X_1, \ldots, X_n \).

The next theorem gives an \( L^2(\Omega) \)-valued product stochastic measure of non-identically distributed independently scattered measures. It is the main result of this section.

**Theorem 3.3.2** Let \( \{X_1, \ldots, X_n\} \) be a system of \( n \geq 1 \) \( L^2 \)-independently...
scattered measures on \((T, \mathcal{A})\) such that 
\[ Y_t = (X_1(t), \ldots, X_n(t)) \quad t \in T, \]

is an \(n\)-dimensional stochastic process with independent increments as in 
Assumption 3.3.1, where \(X_i(t) = X_i([0,t])\). Then there exists a unique 
\(L^2(\Omega, \mathcal{F}^Y, \mathbb{P})\)-valued measure \(\bigotimes_{i=1}^{n} X_i\) on \((T^n, \mathcal{A}^n)\) such that for \(A_i \in \mathcal{A}\) \(i=1, \ldots, n\)

\[
\bigotimes_{i=1}^{n} X_i(A_1 \times \ldots \times A_n) = X_1(A_1) \otimes \ldots \otimes X_n(A_n)
\]

and for \(A \in \mathcal{A}^n\)

\[
\bigotimes_{i=1}^{n} X_i(A) \in H_Y
\]

\[
E(\bigotimes_{i=1}^{n} X_i(A)) = 0
\]

and

\[
\text{VAR}(\bigotimes_{i=1}^{n} X_i(A)) = \frac{1}{n!} \sum_{\Pi} \mu_1^{\Pi_1} \otimes \ldots \otimes \mu_n^{\Pi_n}(A \Pi^n)
\]

where

\[
\mu_{ij}(C \cap B) = \text{EX}_i(C)X_j(B) \quad C, B \in \mathcal{A} \quad i, j = 1, \ldots, n.
\]

**Proof** Let \(\{e_i\}_{i=1}^{n}\) be the canonical basis in \(\mathbb{R}^n\) and define

\[
f^i_A(t) = 1_A(t)e_i \quad t \in T \quad A \in \mathcal{A} \quad i=1, \ldots, n.
\]

Then for each \(i=1, \ldots, n\) \(f^i_A \in L^2_R(\mu_o)\) since

\[
\int_T f^i_A(t)^*R(t)f^i_A(t)\text{d}\mu_o(t) = \int_A f^i_A(t)\text{d}\mu_o(t) = \int_A \text{d}\mu_i(t)
\]

\[
= \mu_i(A) < \infty.
\]

Next, by Proposition 3.3.1 for each \(A \in \mathcal{A}\)

\[
X_i(A) = \int_T f^i_A \text{d}Y \in H_Y
\]

where \(H_Y = H_W \otimes H_q\). Thus each independently scattered measure \(X_i\) is an 
orthogonally scattered measure on \((T, \mathcal{A})\) with values in the common Hilbert 
spaces \(H_Y\) and control measure \(\mu_i\). Therefore existence and uniqueness of 
the \(H_Y^{\otimes n}\)-valued measure \(\bigotimes_{i=1}^{n} X_i\) follow by Theorem 2.2.1. On the other hand
by Proposition 3.3.2 we can see $\otimes \mathbf{X}_i$ as an $L^2(\Omega, F^\mathbf{Y}_i, \mathcal{P})$-valued measure.

Finally, the equalities (3.3.27) and (3.3.28) follow in the same way as for $\otimes W_i$ in Proposition 3.1.1.

Q.E.D

In order to compute the symmetric tensor product measure $\otimes \mathbf{X}_i(A)$ for $A \in \mathcal{A}^n$, we have to find concrete expressions for general symmetric tensor products of elements in $H_\mathbf{Y}$ where $\mathbf{Y}$ is an $n$-dimensional stochastic process with independent increments as in (3.3.4). This last problem was studied by Kailath and Segall (1976) for the case of a one-dimensional stochastic process with stationary and independent increments. Although the main ideas behind the next two results originated from the above named work, we were not able to find them in the literature in the generality that they are presented and proven here.

The next result is a generalization of Propositions 3.1.2 and 3.2.2.

We remark that for this theorem to hold it is required that $T$ is an interval of the real line.

**Theorem 3.3.3** Let $(\mathbf{Y}_t)$ $t \in T = [0, T_0]$ and $H_\mathbf{Y}$ be as in Proposition 3.3.1. For each $i=1, \ldots, n$ let $f^i \in L^2(\mathcal{L}_0)$ and

\begin{align*}
(3.3.30) & \quad F_i(t) = \int_{[0,t]} f^i \cdot d\mathbf{Y}, \quad F_i = \int_T f^i \cdot d\mathbf{Y} \\
(3.3.31) & \quad F_1 \otimes \ldots \otimes F_n = (n!)^{-\frac{1}{2}} P_T^n(F_1, \ldots, F_n)
\end{align*}

where the $f^i$'s are not necessarily all different. Then

\begin{align*}
(3.3.32) & \quad P_T^0(\cdot) = 1 \quad t \in T
\end{align*}

where $P_T^n(F_1, \ldots, F_n)$ are multivariate functionals of the process $\mathbf{Y}$ (Kailath and Segall (1976), Meyer (1976)) defined by
The proof of this theorem is based on the following lemma.

**Lemma 3.3.2** Let $Y_t$, $t \in T$ and $H_Y$ be as in Theorem 3.3.3, $f \in L^2(\mu_0)$, $F(t) = \int_0^t f^* dY$ and $F = F_{T_0} = \int_T f^* dY$. Then

$$F \otimes n = (n!)^{1/2} p^n_{T_0}(F)$$

where $p^n(F)$ $n \geq 1$ are univariate functionals of the process $Y$ (Kailath and Segall (1976), Meyer (1976)) defined by

\begin{align*}
\text{(3.3.35)} & \quad p^0_t(\cdot) = 1 \quad \forall t \in T \\
\text{(3.3.36)} & \quad p^1_t(F) = F(t) \\
\text{(3.3.37)} & \quad p^n_t(F) = \int_{[0,t]} p^{n-1}_s(F) dF(s).
\end{align*}

**Proof** We first show that the functionals (3.3.35)-(3.3.37) belong to $L^2(\Omega,F,Y,P)$ and that they are square integrable $F^Y_t$-martingales: each $p^n_t(F)$ is $F^Y_t$ adapted all $n \geq 1$ $t \in T$ and by Lemma 3.3.1 ($p^1_t(F), F^Y_t$) is an $L^2$-martingale. Next using Lemma 3.3.1 (b) we have that

$$E\left( \int_{[0,t]} [p^{n-1}_s(F)]^2 d\mathcal{F}_s \right)$$

$$= \int_{[0,t]} E(p^{n-1}_s(F))^2 f(s)^T R(s) f(s) d\mu_0(s)$$

$$\leq E(p^{n-1}_{T_0}(F))^2 \int_{[0,t]} f(s)^T R(s) f(s) d\mu_0(s).$$

Hence, using induction, if $E(p^{n-1}_{T_0}(F))^2 < \infty$, then
which shows that \( P^n_t(F) = \int_{[0,t]} p^{n-1}(s) dF(s) \) is a square integrable \( F^Y_t \)-martingale.

Next from Lemma 3.3.1 and Theorem 3.3.1, if \( \xi \in L^2(\Omega_0) \)

\[
F^c_T = F^c_T + \int_0^T \int_0^T \mu dW \eta
\]

and

\[
\gamma(\exp \Theta(F)) = \exp \left( F^c_T - \frac{1}{2} \int_0^T \frac{1}{2} \left( 1 + \frac{d}{ds} (s) \Delta Y_s \right) e^{-s} \right) \prod_{s \leq T} \left( 1 + \frac{d}{ds} (s) \right) e^{-s} = \exp(F)
\]

(3.3.39)

since \( \xi(s) \in L^2(T \times \mathbb{R}_+^d, \sigma(A \times B^0_n, \nu) \cap L^1(T \times \mathbb{R}_+^d, \sigma(A \times B^0_n, \nu) \) by Proposition 3.3.1.

But under this last condition

\[
\begin{align*}
F^d_T &= \int_0^T \xi(s) dN(s, x) = \int_0^T \int_0^T \xi(s) xdN(s, x) - \int_0^T \int_0^T \xi(s) xdY(s, x).
\end{align*}
\]

Then using (3.3.9)

\[
F^d_T = \sum_{s \leq T} \xi(s) \Delta Y_s - \int_0^T \int_0^T \xi(s) xdY(s, x)
\]

and therefore from the last expression and (3.3.8)

\[
\gamma(\exp \Theta(F)) = \exp \left( F^c_T + \int_0^T \int_0^T \mu dW \eta \right) \prod_{s \leq T} \left( 1 + \frac{d}{ds} (s) \Delta Y_s \right) e^{-s} = \exp(F)
\]

where \( \exp(F) \) is the exponential semimartingale of \( F \) (Doleans-Dade (1970)).

Then the lemma follows using Proposition 3.3.2, since from Section 3 in

Doleans-Dade (1970), for any \( \beta \in \mathbb{C} \)

(3.3.39) \( \exp(\beta F)_t = \sum_{n=0}^{\infty} \beta^n p^t_n(F) \) a.s. \( \forall t \in T \)

and on the other hand

\[
\exp(\Theta F) = \sum_{n=0}^{\infty} \frac{(n!)^{-1}}{\sqrt{n}} \beta^n F \omega_n.
\]

Q.E.D.
Proof of Theorem 3.3.3  Using the notation in (2.2.7) and by Lemma 3.3.2

\[ F_1 \otimes \cdots \otimes F_n = (n!)^{-\frac{1}{2}} \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{M \in P_\ell \, T} P^n \left( \sum_{i=1}^{n} M^c(i)F_i \right). \]

Next, using induction on \( n \) one shows that for all \( t \)

\[ p^n_t(F_1, \ldots, F_n) = \frac{1}{n!} \sum_{\ell=0}^{n-1} (-1)^{\ell} \sum_{M \in P_\ell \, T} P^n \left( \sum_{i=1}^{n} M^c(i)F_i \right) \]

from which the theorem follows. Q.E.D.

We now obtain an extension of Proposition 3.1.2.

Lemma 3.3.3  Let \( Y \subset T \) be as in Theorem 3.3.3. Let \( f^1, \ldots, f^k \)
be elements in \( L_R(\mu_0) \) with disjoint support,

\[ F_i(t) = \int_{[0,t]} f^i \cdot dY \quad \text{and} \quad F_i = F_i(T_0) = \int_{T_0} f^i \cdot dY. \]

Then

\[ \otimes_{n^1}^k \otimes_{n^k}^n F_1 \otimes \cdots \otimes F_k = (n!)^{-\frac{1}{2}} \prod_{i=1}^{k} \left( P_{n_i}^i (F_i) (n_i!)^{\frac{1}{2}} \right) \]

\[ \text{where } P_i \text{ are defined in (3.3.35)-(3.3.37) and } n = \sum_{i=1}^{k} n_i. \]

Proof  Since \( f^1, \ldots, f^k \) have disjoint support, then by Lemma 3.3.1 (c),
\( F_1, \ldots, F_k \) are mutually orthogonal elements in \( H_Y \) and therefore the family

\[ \{ F_1 \otimes \cdots \otimes F_k: n^1 \geq 0, \ldots, n^k \geq 0 \} \]

is orthogonal in \( \exp(H_Y) \cong L^2(\Omega, F^Y, \mu) \).

Next, for \( \beta_1, \ldots, \beta_k \in \mathbb{R} \)

\[ \exp \otimes (\sum_{i=1}^{k} \beta_i F_i) = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \left( \sum_{i=1}^{k} \beta_i F_i \right) \otimes_{n^1}^n \otimes_{n^k}^k \]

\[ = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \sum_{n_1 + \ldots + n_k = n} \prod_{i=1}^{k} \frac{\beta_{n_i}^{n_i}}{n_i!} F_1 \otimes \cdots \otimes F_k \]
(n!)

\[ = \sum_{n_1 \ldots n_k} \frac{(n_1 + \ldots + n_k)!}{n_1! \ldots n_k!} n_1 \ldots n_k \beta_1 \ldots \beta_k F_1 \ldots F_k. \]

On the other hand since for \( i \neq j \) \( \tilde{f}^i \) and \( \tilde{f}^j \) have disjoint support, then if \( [ \cdot, \cdot ]_t \) denotes the optional quadratic variation, using Lemma 3.3.1 (b) we obtain

\[ [\beta_i F_i, \beta_j F_j]_t = \beta_i \beta_j <F_i^C, F_j^C>_t + \beta_i \beta_j \sum_{s \leq t} \Delta F_i(s) \Delta F_j(s) \]

\[ = \beta_i \beta_j \sum_{s \leq t} f^i(s) \Delta Y(s) f^j(s) \Delta Y(s) = 0. \]

Then

\[ \text{Exp}(\beta_i F_i + \ldots + \beta_k F_k)_t = \prod_{i=1}^{k} \text{Exp}(\beta_i F_i)_t \quad \text{a.s. } \forall \, t \in T. \] (3.3.42)

Hence the assertion of the lemma follows from (3.3.39), since

\[ \text{Exp}(\beta_i F_i + \ldots + \beta_k F_k)_t = \prod_{i=1}^{k} \text{Exp}(\beta_i F_i)_t \quad \text{a.s. } \forall \, t \in T. \]

With the above lemma we are able to compute the symmetric tensor product measure \( \otimes X_i(A) \), for special sets \( A \in A^n \). The following result will be used in Section 3.4.

**Corollary 3.3.1** Let \( \{X_1, \ldots, X_n\} \) be a system of independently scattered measures as in Theorem 3.3.2. Let \( A_1, \ldots, A_k \) be disjoint sets in \( A \) for \( k > 0 \). Then if \( i_1, \ldots, i_k \in \{1, \ldots, n\} \)

\[ \text{Exp}(\beta_i F_i + \ldots + \beta_k F_k)_t = \prod_{i=1}^{k} \text{Exp}(\beta_i F_i)_t \quad \text{a.s. } \forall \, t \in T. \]

The proof follows by Lemma 3.3.3 since \( F^1_{T_0} (F_i) = F_i (T_0) \) and by taking
\( f^i(t) = 1_{A_i}(t)e_i \), where \( \{e_i\}_{i=1}^n \) is the canonical basis in \( \mathbb{R}^n \).

The main properties of the functionals \( P^n(Z_1, \ldots, Z_n) \) and \( P^n(Z) \) defined in (3.3.32)-(3.3.34) and (3.3.35)-(3.3.37) respectively, are given in Kailath and Segall (1976) and Meyer (1976), including recursive expressions to compute them. In particular, the univariate functionals \( P^n(Z) \) are Hermite polynomials in the case when \( Z \) is a Gaussian martingale.

The first few expressions for \( P^n(Z_1, \ldots, Z_n) \) are:

1. \( \mathbb{P}_t^0(\cdot) = 1 \)
2. \( \mathbb{P}_t^1(Z_i) = Z_i(t) \quad \forall \in T \)

\[ \mathbb{P}_t^2(Z_i, Z_j) = \frac{1}{2}[Z_i(t)Z_j(t) - [Z_i, Z_j]_t] \]

\[ \mathbb{P}_t^3(Z_i, Z_j, Z_k) = \frac{1}{6}[Z_i(t)Z_j(t)Z_k(t)] \\
- Z_i(t)[Z_j, Z_k]_t - Z_j(t)[Z_i, Z_k]_t \\
- Z_k(t)[Z_i, Z_j]_t + 2 \sum_{s \leq t} \Delta Z_i(s)\Delta Z_j(s)\Delta Z_k(s) \]

where \( Z_1, \ldots, Z_n \) are semimartingales not necessarily all distinct.

**Multiple stochastic integrals**

As in the Gaussian and Poisson cases (Sections 3.1 and 3.2 respectively), integrals with respect to \( \bigoplus_{i=1}^n X_i \) can be constructed using the theory of Section 2.3. Under Assumption 3.1.1 \( T \) is an interval of the real line and the measures \( \mu_i \)'s are non-atomic. Then by Theorem 2.3.2 a function \( f \) is \( \bigoplus_{i=1}^n X_i \)-integrable if and only if \( f \in L^2(T^n, A^n, \bigoplus_{i=1}^n \mu_i) \), i.e. \( L_1(\bigoplus_{i=1}^n X_i) = L^2(T^n, A^n, \bigoplus_{i=1}^n \mu_i) \).

Thus from Proposition 3.3.2 we have that if \( f \in L^2(T^n, A^n, \bigoplus_{i=1}^n \mu_i) \)

\[ I_n(f; X_1, \ldots, X_n) = \int_{T^n} f(t) d \bigoplus_{i=1}^n X_i(t) \]
is an element of $L^2(\Omega, \mathcal{F}^Y, \mathbb{P})$ with all the properties of the integral of Section 2.3. In the next result we summarize some of these properties.

We use the notation of Theorem 2.3.3.

**Proposition 3.3.3** Let $X_1, \ldots, X_n$ and $\otimes X_i$ be as in Theorem 3.3.2. Then a function $f$ is $\otimes X_i$-integrable if and only if $f \in L^2(\mathbb{T}^n, \mathcal{A}^n, \otimes \mu_i)$ in which case

a) $I_n(f; X_1, \ldots, X_n) \in L^2(\Omega, \mathcal{F}^Y, \mathbb{P})$.

b) $E(I_n(f; X_1, \ldots, X_n)) = 0$.

c) If $g \in L^2(\mathbb{T}^m, \mathcal{A}^m, \otimes \mu_i)$, $m \neq n$

$$E(I_n(f; X_1, \ldots, X_n)I_m(g; X_1, \ldots, X_m)) = \delta_{nm} \int_{\mathbb{T}^n} f_{\otimes n}(t) R^{\otimes n}(t) g_{\otimes m}(t) d\mu^{n}_o(t).$$

The proof follows analogous to the proof of Lemma 3.1.1.

**One dimensional case** We now consider the case when $X = X_1 = \ldots = X_n$ i.e. $Y_t = X_t$ is a one dimensional stochastic process with independent increments as in Assumption 3.1.1 with $n = 1$. Then $X$ is an $H_X$-valued orthogonally scattered measure and by (3.3.11) it has control measure

$$\mu(A) = \sigma_1(A) + \int \int_{\mathbb{T}} x^2 d\nu(t, x) \quad A \in \mathcal{A}.$$  \hspace{1cm} (3.3.46)

Then in this case $\mu_o = \mu$ and $r(t) = 1$ all $t \in T$. If we identify functions which are equal a.e. $d\mu_o(t)$, the space of functions $L^2_{\mu_o}$ can be taken to be the Hilbert space $L^2(\mu_o)$, that in this situation is equal to $L^2(T, \mathcal{A}, \mu)$. Then is this case

$$I_X(f) = \int_{T} f \cdot dY$$
where the last integral is defined in (3.3.18) for \( n=1 \).

Following the notation of Propositions 2.2.1 and 2.3.6, by Proposition 3.3.2 we obtain that for \( n \geq 1 \) the symmetric tensor product stochastic measure \( X^\otimes n \) on \( (T^n, \mathcal{A}^n) \) and the multiple stochastic integral \( I_{n\otimes}(f;X) \), \( f \in L^2(T^n, \mathcal{A}^n, \mu^\otimes n) \), are \( L^2(\Omega, F^X, \mathbb{P}) \)-valued elements. They satisfy the properties of Proposition 2.2.1 and Proposition 2.3.5 respectively.

The next two results are extensions of Theorem 3.1 in Ito (1951) (Propositions 3.1.5 and 3.1.6 in our work) to the case of a general \( L^2 \)-independent increments process.

**Proposition 3.3.4** Let \( X \) be a one dimensional stochastic process with independent increments as above. Let \( P_k^k(\cdot) \) be the functionals defined in \((3.3.35)-(3.3.37)\). Assume that \( f_1(t), ..., f_m(t) \) is a system of real valued functions in \( L^2(T, \mathcal{A}, \mu) \) with disjoint support. Define

\[
f(t) = f_1(t_1) ... f_1(t_{k_1}) f_2(t_{k_1+1}) ... f_2(t_{k_1+k_2}) ... f_m(t_{k_1+...+k_{m-1}}) ... f_m(t_{k_1+...+k_m}).
\]

Then if \( n = k_1 + ... + k_m \)

\[
I_{n\otimes}(f;X) = (n!)^{-\frac{1}{2}} \prod_{i=1}^{m} P_{T_i}^{k_i}(F_i)(k_i)
\]

where \( F_i = \int_{T} f_i dX = I_X(f_i) \).

**Proof** By Lemma 2.3.1 \( f \) is \( X^\otimes \)-integrable and

\[
I_{n\otimes}(f;X) = \mathbb{E}(f_1^{\otimes k_1}) ... \mathbb{E}(f_m^{\otimes k_m}).
\]

Next using Lemma 3.3.1 (c)

\[
E(F_i F_j) = \int_{T} f_i(s) f_j(s) d\mu(s) = 0
\]
since \( f_i, f_j \) have disjoint support. Then by Lemma 3.3.3

\[
I_X(f_1) \otimes \cdots \otimes I_X(f_m) = (n!)^{1/2} \prod_{j=1}^{m} p_{T_1}(F_j)(k_i!).
\]

Q.E.D.

**Corollary 3.3.2** Let \( T = [0,1] \). Then if

\[
T^n_1 = \{(t_1, \ldots, t_n) : 0 \leq t_1 < \ldots < t_n \leq 1\}
\]

\[
I_{n^n}(1^n; X) = (n!)^{-1/2} p_{T_1}(X)
\]

That is, formally

\[
\int \cdots \int_{0 \leq t_1 < \ldots < t_n \leq 1} dX(t_1) \cdots dX(t_n) = (n!)^{-1/2} p_{T_1}(X).
\]

**Proof** By Definition 2.3.1, \( I_{n^n}(1^n; X) = X^{n^n}(T^n_1) \). Then the result follows from the last proposition since \( X^{n^n} \) is finitely additive, \( \mu \) is continuous and Proposition 2.2.1 (c).

Q.E.D.

Finally, in this section we discuss the completeness of the multiple stochastic integrals \( I_n(\cdot; X) \) \( n \geq 0 \) in \( L^2(\Omega, F^X, P) \), obtaining a characterization of the Gaussian and certain Poisson random measures on \((T, \mathcal{A})\).

**Proposition 3.3.5** Let \( X \) be an \( L^2 \)-stochastic process with independent increments as in Proposition 3.3.4. Then the system

\[
\{I_n(f_n; X) : f_n \in L^2(T^n, \mathcal{A}^n, \mu^{n^n}), n \geq 1\}
\]
generates \( L^2(\Omega, F^X, P) \) if and only if

a) \( \nu(t, A) = 0 \quad \forall A \in \mathcal{B}^0, t \in T \) (Gaussian situation) or

b) \( \sigma_1(t) = 0 \) and \( \nu(t, \cdot) \) is concentrated in one point \( x_o \neq 0 \).

**Proof** Assume (a) holds. Then \( \mu(A) = \sigma_1(A) \quad \forall A \in A, \quad X_t = W_t \quad t \in T, \quad I_X(f) = I_W(f) \quad f \in L^2(T, A, \mu) \) and \( H_X = H_W \). Thus from Proposition 2.3.7 the system (3.3.47) generates the space \( \text{exp}(H_W) \) and by (3.1.4) \( \text{exp}(H_W) \) is identified with \( L^2(\Omega, F^W, P) \). Then the system (3.3.47) generates \( L^2(\Omega, F^W, P) \).

Now assume (b) holds, then \( \mu(A) = x_o^2 \nu(A; x_o) \quad \forall A \in A \) for \( x_o \neq 0 \), \( q(A) = q(A; x_o) \) is a Poisson random measure on \( (T, A) \), \( X_t = q([0, t]) \), \( I_X = I_q(f) \quad f \in L^2([1, A, \mu]) \) and \( H_X = H_q \). Then as above, from Proposition 2.3.7 the system (3.3.47) generates the space \( \text{exp}(H_q) \) and by Lemma 3.2.2, \( \text{exp}(H_q) \) is identified with \( L^2(\Omega, F^q, P) \). Then the system (3.3.47) generates \( L^2(\Omega, F^q, P) \).

Next assume that the system (3.3.47) generates the space \( L^2(\Omega, F^X, P) \) which is identified with \( \text{exp}(H) \) by Proposition 3.3.2, where \( H = H_W \oplus H_q \). Then by Proposition 2.3.7 the system (3.3.47) generates \( \text{exp}(H_X) \). Therefore, \( \text{exp}(H_X) = \text{exp}(H) \) which implies \( H_X = H \). Then for each \( g_1 \in L^2(T, A, \sigma_1) \) and \( g_2 \in L^2(T \times \mathbb{R}^0, A \times \mathcal{B}^0, \nu) \) there exists \( f \in L^2(T, A, \mu) \) such that

\[
I_W(g_1) + I_q(g_2) = I_X(f).
\]

But from (3.3.18) in Proposition 3.3.1

\[
I_X(f) = I_W(f) + I_q(f_x)
\]

Therefore

\[
I_W(g_1) - I_W(f) = 0 \quad \text{a.e.}
\]

and

\[
I_q(g_2) - I_q(f_x) = 0 \quad \text{a.e.}
\]

which implies
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\[ g_1 = f \quad \text{a.e. } d\sigma_1 \]

and

\[ g_2(t,x) = f(t)x \quad \text{a.e. } dv. \]

Suppose there exists \( t_1 \in T \) and \( A \in \mathcal{B}^0 \) such that

\[ 0 < \nu([0,t_1] \times A) < \infty. \]

Then \( 0 < \nu(t,A) < \infty \) for \( t > t_1 \) since \( \nu(t,A) = \nu([0,t] \times A) \) is a non-decreasing function. Let \( g_1(t) = \alpha 1_{[0,t_1]}(t) \) for an arbitrary \( \alpha \neq 0 \) and \( g_2(t,x) = 1_{[0,t_1]}(t,x) \). Then \( g_1 \in L^2(T,A,\sigma_1) \) and \( g_2 \in L^2(T \times \mathbb{R}^0, A \times \mathcal{B}^0, \nu) \) and by the above argument there exists \( f \in L^2(T,A,\mu) \) such that

\[ f(t) = \alpha 1_{[0,t_1]}(t) \quad \text{a.e. } d\sigma_1 \]

\[ f(t) = 1_{[0,t_1]}(t,x) \quad \text{a.e. } dv. \]

Let \( t \in [0,t_1] \) and \( x_1, x_2 \in A \). Then \((t,x_1) \in [0,t_1] \times A \) and \((t,x_2) \in [0,t_1] \times A \) and by (3.3.52) \( f(t)x_1 = f(t)x_2 = 1 \) i.e. \( f(t) \neq 0 \) and \( f(t) = 1/x_1 = 1/x_2 \)

which implies \( x_1 = x_2 \) and therefore \( \nu(t,A) \) is concentrated in one point \( x_0 \), say, and \( f(t) = 1/x_0 \ \ t \in [0,t_1] \). Then by (3.3.51) if \( x_0 \neq A^{-1} \), \( \sigma_1(t) = 0 \ \ t \in [0,t_1] \) and since \( \alpha \) is arbitrary, then \( \sigma_1(t) = 0 \ \ \text{all } t \in [0,t_1] \). Then since \( \nu(t,A) \) is non-decreasing the above argument holds for all \( t > 0 \), i.e. \( \nu(t,A) \) is concentrated in one point and \( \sigma_1(t) = 0 \ \ \text{all } t > 0 \). Hence condition (b) holds.

On the other hand, if there do not exist \( t_1 \in T \) and \( A \in \mathcal{B}^0 \) such that (3.3.50) is satisfied, then \( \nu(t,A) = 0 \ \ \text{all } t \in T \) and \( A \in \mathcal{B}^0 \) since \( \nu \) is a \( \sigma \)-finite measure on \((T \times \mathbb{R}^0, A \times \mathcal{B}^0)\). Then condition (a) holds.

Q.E.D.

3.4 Comparisons with earlier results

Engel (1982) and Rosinski and Szulga (1982) have recently studied
L²-valued product stochastic measures. In all these works the usual product of independently scattered measures has always been considered and in none of them has the symmetric tensor product been used. In this section we point out the main differences between the two approaches and the advantages of the symmetric tensor product measure.

Engel and Kakutani (Engel (1982)) have considered the construction of an L²(Ω)-valued product stochastic measure from a system of n≥1 stochastic processes \( \{X_1(t), \ldots, X_n(t)\} \), \( t \in T = [0,T_0] \), on a probability space \((Ω,F,P)\), satisfying the following four conditions (R1-R4 in Engel's notation (1982)):

**(3.4.1)** The function \( m_k(t) = EX_k(t) \) is a continuous function of bounded variation on \( T \), for each \( k=1,\ldots,n \).

**(3.4.2)** The function \( \mu_k(t) = E(X_k(t) - m_k(t))^2 \) is a continuous monotonely increasing function on \( T \), for each \( k=1,\ldots,n \).

**(3.4.3)** If \( \{I_1,\ldots,I_q\} \) is any set of disjoint intervals contained in \( T \) and \( \{i_1,\ldots,i_q\} \) is any set of integers where \( 1 \leq k \leq n \), \( k=1,\ldots,q \), then \( \{X_{i_1}(I_1),\ldots,X_{i_q}(I_q)\} \) forms an independent system of random variables, where for \( I = (s,t] \), \( X_i(I) = X_i(t) - X_i(s) \).

**(3.4.4)** If \( I \subset T \) is any interval and \( 1 \leq j_1 < \ldots < j_k \leq n \) is any sequence of integers between 1 and \( n \), then

\[
E\left|X_{j_1}(I)X_{j_2}(I)\ldots X_{j_k}(I)\right|^2 < \infty
\]

and

\[
E(|X_{j_1}(I)\ldots X_{j_k}(I)|)² \to 0 \quad \text{as} \quad |I| \to 0.
\]

Let \( A^n_0 \) (\( F^n_0 \) in Engel's notation) be the field of elementary subsets of \( T^n \) of the form
(3.4.5) \[ B = \bigcup_{i_1 \ldots i_n=1}^{q} c_{i_1} \ldots c_{i_n} I_{i_1} \times \ldots \times I_{i_n} \]

where \( \{I_1, \ldots, I_q\} \) is a partition of \( T \) into disjoint intervals (depending on \( B \)) for which \( I_k < I_{k+1} \) (i.e. if \( s_1 \in I_k \) and \( s_2 \in I_{k+1} \) then \( s_1 < s_2 \)) \( k=1, \ldots, q-1 \) and \( c_{i_1} \ldots c_{i_n} \) is either zero, indicating that \( I_{i_1} \times \ldots \times I_{i_n} \) is not included in the union, or is one indicating that it is included. Engel (1982) defines the finitely additive \( L^2(\Omega) \)-valued product measure \( Y^{(n)} \) on \( A^n_0 \) as

\[ Y^{(n)}(B) = \sum_{i_1 \ldots i_n=1}^{q} c_{i_1} \ldots c_{i_n} X_1(I_{i_1}) \ldots X_n(I_{i_n}) \quad B \in A^n_0 \]

and proves the following theorem which extends the measure \( Y^n \) to \( A^n = \sigma(A^n_0) \).

**Theorem 3.4.1** (Theorem 4.5 Engel (1982)). Let \( \{X_1(t), \ldots, X_n(t)\}, t \in T = [0, T_0] \) be a system of \( n \geq 1 \) stochastic processes satisfying the regularity conditions (3.4.1)-(3.4.4). Then the \( L^2(\Omega) \)-valued measure \( Y^{(n)} \) defined by (3.4.6) on \( A^n_0 \) can be extended to a countably \( L^2(\Omega) \)-valued measure (also denoted by \( Y^{(n)} \)) on the Borel \( \sigma \)-field \( A^n = \sigma (A^n_0) \).

The idea of the proof of this theorem is to partition the set \( T^n \) into disjoint pieces on which an appropriate countably \( L^2(\Omega) \)-valued measure can be defined and then show that the sum of all these measures is the required measure. This procedure uses a complicated double induction and involves prior knowledge of what the measure \( Y^{(n)} \) should look like, even when the mean function of each process is assumed to be zero. On the other hand the construction of the symmetric tensor product measure \( \otimes X_i \) follows the more natural ideas from the theory of product real valued measures. Moreover, the assumption that \( T \) is a subset of the real line is essential in the construction of Engel's product measure \( Y^{(n)} \) and therefore more general.
parameter sets $T$, as those considered in Sections 3.1 and 3.2 of this chapter, cannot be contemplated in Engel's framework.

Assumptions (3.4.2) and (3.4.3) imply that

$$Y_t = (X_1(t),...,X_n(t)) \quad t \in T$$

is an $n$-dimensional $L^2$-stochastic process with independent increments. Then if we assume $E X_i(t) = 0$ $\forall t \in T$ $i=1,...,n$, we are able to construct the $L^2(\Omega,F^Y,P)$-valued product stochastic measure $\otimes X_i$ as in Section 3.3. This "zero mean" condition is satisfied in many interesting cases and allows us to use Hilbert space techniques in the construction of $\otimes X_i$. On the other hand, although the "zero mean" condition simplifies the proof of some of Engel's results (e.g. Theorem 4.1 Engel (1982)) it does not make easier the proof of his Theorem 4.5 using his method. In the zero mean case Engel's problem (the construction of an $L^2(\Omega)$-valued product stochastic measure) is more easily solved using Section 3.3, although the product stochastic measure obtained is not the same, as we shall see later. Multiple Wiener integrals have been applied to obtain expansions and stochastic integral representations of $L^2$-functionals of $Y_t$ (Ito (1951), Kallianpur (1980)). Since the $\sigma$-fields generated by the processes $(X_1(t),...,X_n(t))$ and $(X_1(t)-m_1(t),...,X_k(t)-m_k(t))$ are the same, from the point of view of this application the zero mean assumption is unimportant. Moreover, this assumption enables us to use the techniques of Chapter II to construct integrals with respect to $\otimes X_i$ and identify the class of $\otimes X_i$-integrals that in this case is equal to $L^2(T^n,A^n,\otimes \mu_i)$. Engel (1982) does not consider stochastic integration w.r.t. his product stochastic measure $Y^{(n)}$.

Assumption (3.4.4) is used in Engel's work to assure that $Y^{(n)}$, as defined in (3.4.5), is an $L^2(\Omega)$-valued measure, i.e. Hilbert space valued.
In our framework, this additional moment condition is not necessary to obtain a (Hilbert space) $H_i$-valued measure $\otimes X_i, \forall i = 1, \ldots, n$, where $H$ is given in Proposition 3.3.1. Furthermore, for $n_1 \neq n_2$ the measures $\otimes^1 X_1$ and $\otimes^{n_2} X_1$ take values in the same Hilbert space $\text{EXP}(H)$ and they are orthogonal w.r.t. the inner product in $\text{EXP}(H)$. Moreover, the identification of symmetric tensor products of $H$ provides more insight into the structure of $\{X_1(t), \ldots, X_n(t)\}$. Such identification is known for important cases (Neveu (1968), Kallianpur (1970)), and we have obtained in Sections 3.2 and 3.3 identifications for other important situations.

As well as differences in the assumptions and in the techniques used, there are also important differences between the resulting product stochastic measures $Y^{(n)}$ and $\otimes X_i, \forall i = 1, \ldots, n$. We now study some of these differences.

If $E \in A^n_0$ is as in (3.4.5), then

\begin{equation}
Y^{(n)}(E) = \sum_{1 \leq i_1 < \ldots < i_n} \prod_{i=1}^n c_{i_1} \ldots c_{i_n} X_{i_1}(I_{i_1}) \ldots X(I_{i_n})\tag{3.4.7}
\end{equation}

and

\begin{equation}
\otimes X_i(E) = \sum_{1 \leq i_1 < \ldots < i_n} \prod_{i=1}^n c_{i_1} \ldots c_{i_n} X_{i_1}(I_{i_1}) \otimes \ldots \otimes X_n(I_{i_n})\tag{3.4.8}
\end{equation}

Thus if $j_1 < \ldots < j_n$, using Corollary 3.3.1 we obtain

\begin{equation}
\otimes X_i(I_{j_1} \times \ldots \times I_{j_n}) = X_{i_1}(I_{j_1}) \otimes \ldots \otimes X_n(I_{j_n}) = (n!)^{\frac{1}{2}} Y^{(n)}(I_{j_1} \times \ldots \times I_{j_n})\tag{3.4.9}
\end{equation}

which suggests that $(n!)^{\frac{1}{2}} Y^{(n)}$ and $\otimes X_i$ agree on antisymmetric sets (see Corollary 2.2.4), as it is shown by the following result.

**Proposition 3.4.1** For each permutation $\Pi = (\Pi_1, \ldots, \Pi_n)$ of $(1, \ldots, n)$ let
\[ T^n_\Pi = \{(t_1, \ldots, t_n) \in T^n : t_{\Pi 1} < \ldots < t_{\Pi n}\} \]

and

\[ A^n_\Pi = A^n \cap T^n_\Pi. \]

Then

\[ \Phi X_i(A) = (n!)^{q_i} Y^n(A) \quad A \in A^n_\Pi. \]  

**Proof** \( A^n_\Pi \) is the \( \sigma \)-field generated by the field of all elementary subsets of \( T^n_\Pi \) of the form

\[ B = \bigcup_{1 \leq i_1 < \ldots < i_n \leq q} c_{i_1} \ldots c_{i_n} I_{i_1} \times \ldots \times I_{i_n} \]

where \( I_1 < \ldots < I_q \) is a partition of intervals of \( T \) and \( c_{i_1} \ldots c_{i_n} \)'s are as in (3.4.5). By (3.4.9) and the additivity property of the measures \( Y^n(A) \) and

\[ \Phi X_i \]

\[ \Phi X_i(B) = (n!)^{q_i} Y^n(B). \]

Therefore from Theorem 3.3.2, since \( B \) is an antisymmetric set

\[ (n!) \mu(\Phi Y^n(B))^2 = \sum_{i=1}^n \mu_i \Phi X_i(B). \]

Then an approximation argument shows (3.4.10) for all \( B \in A^n_\Pi \).

Q.E.D.

If \( A \) is not an antisymmetric set then (3.4.10) does not hold. Consider for example the case \( n=2 \), then from (3.4.7) and (3.4.8) if \( B \in A^2_\circ \) is given by (3.4.5)

\[ Y^2(B) = \sum_{i_1 < i_2} c_{i_1} c_{i_2} X_1(I_{i_1}) X_2(I_{i_2}) \]

and using Theorem 3.3.3 and (3.3.44)
\[ X_i(B) = \sum_{i_1, i_2=1}^q c_{i_1 i_2} X_1(I_{i_1}) \otimes X_2(I_{i_2}) \]
\[ = (2!)^{-\frac{1}{2}} \sum_{i_1, i_2=1}^q c_{i_1 i_2} \{ X_1(I_{i_1}) X_2(I_{i_2}) - [X_1, X_2](I_{i_1} \cap I_{i_2}) \} \]
\[ = (2!)^{-\frac{1}{2}} \sum_{i_1, i_2=1}^q c_{i_1 i_2} \{ X_1(I_{i_1}) X_2(I_{i_2}) - \sum_{i_1=1}^q c_{i_1 i_1} [X_1, X_2](I_{i_1}) \} \]
\[ = (2!)^{-\frac{1}{2}} \{ Y(2)(B) - \int_T 1_B(s, s) d[X_1, X_2](s) \} \]
i.e.
\[ Y(2)(B) = (2)^{-\frac{1}{2}} \sum_{i=1}^2 X_{1i}(B) + \int_T 1_B(s, s) d[X_1, X_2](s) \quad B \in A^n \]

Then it follows from the above expression that even in the "zero mean" case we have that
\[ E(Y(2)(B)) = \int_T 1_B(s, s) du_{12}(s) = \mu_{12}(B) \]
since \[ E(\sum_{i=1}^2 X_{1i}(B)) = 0 \] by Theorem 3.3.2 and
\[ \mu_{12}(B) = E(X_1(B) X_2(B)) = E([X_1, X_2](B)). \]

Therefore Engel's product stochastic measure \( Y(n) \) is an uncentered measure which gives rise to an uncentered stochastic integral, while \( \sum_{i=1}^n X_{1i} \) is centered.

Let \( T = [0,1], A = B(T) \) and \( X \) be a single zero mean \( L^2 \)-independently scattered measure on \((T, A)\) with control measure \( \nu \). Rosinski and Szulga (1982) have considered the random product measure of \( X \) with itself in such a way that
\[ X(2)(A_1 \times A_2) = X(A_1) X(A_2) \quad A_1, A_2 \in A \]
can be extended to an \( L^1(\Omega) \)-valued vector measure on \((T \times T, A \times A)\). Under the additional assumption of \( X(A) \in L^4(\Omega) \) all \( A \in A \), they have shown that \( X^{(2)} \)
can be extended to an $L^2(\Omega)$-valued vector measure on $(T \times T, A \times A)$. This last case corresponds to the Engel's situation $n=2$ and $X = X_1 = X_2$. They do not go beyond the case $n=2$. One of the complications that will appear is that for $n=3$, if $X(A) \in L^2(\Omega)$, $A \in A$; then

$$X^{(3)}(A_1 \times A_2 \times A_3) = X(A_1)X(A_2)X(A_3) \quad A_i \in A \quad i=1,2,3$$

is not necessarily an element of $L^1(\Omega)$. This means more moment conditions about $X$ are required and that product random measures of different orders take values in distinct spaces. Rosinski and Szulga (1982) use the theory of integration with respect to vector valued measures to construct integrals with respect to $X^{(2)}$, characterizing the class of $X^{(2)}$-integrable functions. For purposes of comparison some results of Rosinski and Szulga (1982) are summarized in the next two propositions.

**Proposition 3.4.2** (Rosinski and Szulga (1982)). Let $(T,A,\mu)$ be as above and $X$ be a zero mean $L^2$-independently scattered measure on $(T,A)$ with control measure $\mu$. For $A_1,A_2 \in A$ define $X^{(2)}(A_1 \times A_2) = X(A_1)X(A_2)$. Then

a) $X^{(2)}$, as a vector measure in $L^1(\Omega)$, has a countably additive extension to $(T^2,A^2)$.

b) For a real valued measurable function $f$ on $T^2$ define

$$N(f) = \int_{T^2} |f(t,t)| \mu(t) + \int_{T^2} \left( \int_{T^2} |f(s,t)|^2 \mu(s) \mu(t) \right)^{1/2} \mu(t)$$

where $\Delta = \{(s,t) \in T^2 : s=t\}$. If $N(f) < \infty$ then $f$ is $X^{(2)}$-integrable with $L^1(\Omega)$-valued integral denoted by

$$\int_{T^2} f(s,t)dX^{(2)}(s,t).$$

c) If $N(f) < \infty$ then
where $V_1(t) = [X,X]_t$ is the optional quadratic variation process of $X(t) = X([0,t])$ and $V_2(A) = X^{(2)}(A \setminus A)$, $A \in \mathcal{A}^2$.

Moreover, the first integral on the RHS of (3.4.13) belongs to $L^1(\Omega)$ while the second belongs to $L^2(\Omega)$.

**d)** Let $X$ be a Gaussian random measure. Then $f$ is $X^{(2)}$-integrable if and only if $N(f) < \infty$.

---

**Proposition 3.4.3** (Rosinski and Szulga (1982)). Let $X$ be an independently scattered measure as in Proposition 3.4.2 and assume that $E(X(A))^4 < \infty$ for all $A \in \mathcal{A}$. For $A_1, A_2 \in \mathcal{A}$ define $X^{(2)}(A_1 \times A_2) = X(A_1)X(A_2)$. Then

a) $X^{(2)}$, as a vector measure in $L^2(\Omega)$, has a countably additive extension to $(T^2, \mathcal{A}^2)$.

b) A real valued function on $T^2$ is $X^{(2)}$-integrable if and only if the next three conditions are satisfied.

(i) $\int_T |f(t,t)| d\mu(t) < \infty$

(ii) $\int_{T^2 \setminus A} |f(t,s)|^2 d\mu(t) d\mu(s) < \infty$

(iii) $\int_T |f(t,t)|^2 |G|(dt) < \infty$

where $|G|$ is the variation of the signed measure

$G(A) = E(X(A))^4 - 3E(X(A))^2$ $A \in \mathcal{A}$.

Denote by

$\int_{T^2} f(s,t) dX^{(2)}(s,t)$

the $L^2(\Omega)$-valued integral of $f$ w.r.t. $X^{(2)}$. 

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c) If \( f \) is \( X^{(2)} \)-integrable
\[
E\left(\int_T f \, dX^{(2)}\right)^2 = \left(\int_T f(t,t) \, du(t)\right)^2 + \int_T \int_T f^2(s,t) \, du(s) \, du(t)
+ \int_T \int_T f(s,t) f(t,s) \, du(s) \, du(t) + \int_T |f(t,t)|^2 G(dt).
\]

d) If \( X \) is Gaussian then \( G = 0 \) and the random integrals with respect to \( X^{(2)} \) in the sense of \( L_1 \) and \( L_2 \) coincide.

On the other hand, following the notation of Proposition 2.2.1, let \( X^{\otimes 2} \) be the \( L^2(\Omega, F^X, P) \)-valued symmetric tensor product stochastic measure on \( (T^2, A^2) \) constructed at the end of Section 3.3 \((n=2)\) under the only moment assumption of \( X \) being \( L^2 \)-valued. From Theorem 3.3.3 and (3.3.44) if \( A_1, A_2 \in A \)
\[
X^{\otimes 2}(A_1 \times A_2) = X(A_1) \otimes X(A_2) = c(X(A_1)X(A_2) - [X,X](A_1 \cap A_2))
\]
where \( c = (2)^{-1} \), and therefore by (3.4.12)
\[
(3.4.14) \quad X^{\otimes 2}(A_1 \times A_2) = c(X^{(2)}(A_1 \times A_2) - [X,X](A_1 \cap A_2)).
\]

Hence since \( E(X^{\otimes 2}(A_1 \times A_2)) = 0 \)
\[
E(X^{(2)}(A_1 \times A_2)) = E([X,X](A_1 \cap A_2)) = \mu(A_1 \cap A_2)
\]
i.e. \( X^{(2)} \) is not necessarily a centered product random measure and gives rise to an uncentered integral as it is shown in (3.4.13) of Proposition 3.4.2. Moreover, from (3.4.14) and (3.4.13) we obtain that
\[
c \, V_2(A) = X^{\otimes 2}(A) \quad A \in A^2.
\]
If \( f \in L^2(T^2, A^2, \mu^{\otimes 2}) \) then \( f \) is \( X^{\otimes 2} \)-integrable and the converse holds. If \( \mu \) is non-atomic. The integral with respect to \( X^{\otimes 2} \) is centered (Proposi-
tion 3.3.3 taking $n=2$ and $X = X_1 = X_2$.

Then the advantages of the symmetric tensor product measure approach are that it does not need additional higher moment conditions to construct an $L^2(\Omega)$-valued random product measure and it gives rise to centered multiple stochastic integrals. Moreover, this approach can be used (as it was done in Sections 3.1, 3.2 and 3.3) to construct product stochastic measures of order $n \geq 1$, all taking values in the same space $L^2(\Omega, F^X, P)$, without needing extra higher moment conditions.

**Concluding remarks** We have seen in this chapter that the symmetric tensor product approach is an appropriate tool to obtain product stochastic measures and to construct multiple stochastic integrals w.r.t. them. A clear relationship between the theory of multiple stochastic integrals and the theory of vector valued measures has been established. Moreover, Theorem 2.1.4 suggests that this approach could be used to construct infinite product stochastic measures, which we have not done since we have not been successful in defining the concept of infinite symmetric tensor product. This last notion was not found in the literature.
CHAPTER IV

NUCLEAR SPACE VALUED WIENER PROCESS
AND STOCHASTIC INTEGRALS

In this chapter we bring together several notions and results about nuclear space valued Wiener processes that will be used in the next chapter. We begin by presenting the Countably Hilbert Nuclear Space $\Phi$ that we are going to consider in the remaining part of this work (Assumption 4.1.1). Then we define a $\Phi'$-valued Wiener process $(W_t)_{t \geq 0}$ with a continuous positive definite bilinear form $Q$ on $\Phi \times \Phi$ and study some of its properties such as the corresponding Rigged Hilbert Space associated with it; the Wiener integral and its associated Gaussian space; and an infinite system of independently scattered measures that are non-identically distributed and mutually independent on disjoint sets. At the end of Section 4.1 we present some examples of $\Phi'$-valued Wiener processes that show how our framework includes many cases already considered in the literature. In Section 4.2 we discuss real valued and $\Phi'$-valued stochastic integrals with respect to $W_t$ in a manner that they can be used in Chapter V in representing nonlinear functionals of $W_t$.

4.1 Nuclear space valued Wiener process

4.1.1 The Countably Hilbert Nuclear Space $\Phi$ and its $n^{th}$ tensor product $\Phi^\otimes_n$

Suppose $E$ is a real linear space whose topology is determined by a countable family of Hilbertian semi-norms $|| \cdot ||_n$ ($<, >_n$) $n \geq 0$. For each $n$
let \( E_n \) be the Hilbert space completion of \( E \) with respect to \( \| \cdot \|_n \). For \( n < m \) suppose we have \( \| \phi \|_n \leq \| \phi \|_m \) \( \phi \in E \). Then

\[
E \subset E_m \subset E_n \subset E'
\]

\( E' \) being the topological dual space of \( E \). Furthermore, let \( E = \bigcap_{n \geq 0} E_n \) and suppose that for every \( n \) there is an \( m \geq n \) such that the injection of \( E_m \) into \( E_n \) is a Hilbert-Schmidt map. Then \( E \) is said to be a Countably Hilbert Nuclear Space (CHNS).

Among important properties of a CHNS \( E \) we have that (Gelfand and Vilenkin (1964)) \( E \) is a complete metrizable locally convex space (Frechet space) which is separable and every bounded closed set in \( E \) is compact. A useful result that we will use several times in this work is the following lemma. Although it can be proved for any linear topological space of the second category (see Xia (1972) page 386), we shall establish and prove it in the case when \( E \) is a CHNS. A result of this type, involving a Baire category argument, was first used in the study of \( E' \)-valued stochastic processes in Mitoma (1981a, 1981b).

**Lemma 4.1.1** Let \( E \) be a CHNS and let \( V(\phi) \) be a non-negative, lower semi-continuous functional on \( E \) (i.e. \( \phi_n + \phi \) in \( E \) implies that \( V(\phi) \leq \liminf V(\phi_n) \)), satisfying the following conditions:

a) For any \( \phi, \psi \in E \) \( V(\phi + \psi) \leq V(\phi) + V(\psi) \).

b) For any \( \phi \in E \) and \( a \in \mathbb{R} \) \( V(a\phi) = |a|V(\phi) \).

c) \( V(\phi) < \infty \) for any \( \phi \in E \).

Then \( V(\phi) \) is continuous on \( E \) and there exist a positive real number \( \theta \) and a positive integer \( r \) such that

\[
V(\phi) \leq \theta \| \phi \|_r \quad \forall \phi \in E.
\]
Proof Let

\[ D_n = \{ \phi \in E : V(\phi) \leq n \} . \]

Since \( V \) is a lower semicontinuous function on \( E \) then for each \( n \geq 1 \) \( D_n \) is a closed set of \( E \) (see Reed and Simon (1980)). Condition (c) implies that

\[ E = \bigcup_{n=1}^{\infty} D_n . \]

Then by the Baire category theorem, since \( E \) is a complete metric space, it is never the union of a countable number of nowhere dense sets. Therefore there exists \( n_0 \) such that \( D_{n_0} \) is not a nowhere dense set, i.e. there exist \( \phi_0 \in E, \delta_1 > 0 \) and a positive integer \( r \) such that

\[ U_{\phi_0} = \{ \phi \in E : \| \phi - \phi_0 \|_r < \delta_1 \} \subseteq D_{n_0} . \]

Then for any \( \phi \in E \) \( \phi \neq 0 \) if \( \delta < \delta_1 \)

\[ \delta \| \phi \|_r + \phi_0 \in U_{\phi_0} \quad \text{and} \quad \phi - \delta \| \phi \|_r \in U_{\phi_0} \]

and hence they belong to \( D_{n_0} \), i.e.

\[ V(\delta \| \phi \|_r + \phi_0) \leq n_0 \quad \text{and} \quad V(\phi_0 - \delta \| \phi \|_r) \leq n_0 . \]

But using (b) with \( a = -1 \)

\[ V(\delta \| \phi \|_r - \phi_0) = V(\phi_0 - \delta \| \phi \|_r) \leq n_0 . \]

Then by (a)

\[ V(2\delta \| \phi \|_r) \leq V(\delta \| \phi \|_r + \phi_0) + V(\delta \| \phi \|_r - \phi_0) \leq 2n_0 \]

and hence using (b), if \( \theta = n_0 / \delta \)

\[ V(\phi) \leq \theta \| \phi \|_r \quad \forall \phi \in E . \]

Then the continuity of \( V \) follows since using (a) we obtain
\[ |V(\phi) - V(\psi)| \leq V(\phi - \psi) \leq \theta \| \phi - \psi \|_x. \]

Q.E.D.

As examples of CHNS we have \( S(\mathbb{R}^d) \) the Schwartz space of all rapidly decreasing functions on \( \mathbb{R}^d \), \( d \geq 1 \) and \( S(\mathbb{Z}^d) \) the space of all rapidly decreasing sequences on \( \mathbb{Z}^d \), the \( d \)-dimensional lattice space. Stochastic processes taking values in duals of these spaces have been considered in the recent works of K. Itô (1978a, 1978b, 1983), Dawson and Salehi (1980) and Shiga and Shimizu (1980) among others. However, in several practical problems, like those occurring in neurophysiology, it is not possible to fix in advance the space in which the stochastic processes take their values (see Kallianpur and Wolpert (1984)). The next example is taken from the work of the last named authors (see also Daletskii (1967)).

**Example 4.1.1 (Kallianpur and Wolpert (1984)).** Suppose a strongly continuous semigroup \( (T_t)_{t \geq 0} \) given on a Hilbert space \( H_0 \) (that can be taken as \( H_0 = L^2(X, \mu) \) for some \( \sigma \)-finite measure space \( (X, \Sigma, \Gamma) \)). The semigroup \( (T_t)_{t \geq 0} \) usually describes the evolutionary phenomenon being studied, such as the behavior of the voltage potential of a neuron (Kallianpur and Wolpert (1984)). Suppose that the strongly continuous and self adjoint semigroup \( (T_t)_{t \geq 0} \) satisfies the following two conditions:

\[(4.1.1) \quad \text{The resolvent } R_\alpha = \int_0^\infty e^{-\alpha t}T_t \text{ dt is compact for each } \alpha > 0.\]

\[(4.1.2) \quad \text{For some } r_1 > 0 (R_\alpha)^{r_1} \text{ is a Hilbert-Schmidt operator.}\]

By the Hille-Yosida theorem \( (T_t) \) has a negative definite infinitesimal generator \( -L \). Then by Corollaries 4.4.1 and 4.4.2 in Balakrishnan (1981), \( H_0 \) admits a complete orthonormal set \( \{ \phi_j \}_{j \geq 1} \) of eigenvectors of \( L \) with
eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ satisfying

\begin{equation}
\sum_{j=1}^{\infty} (\alpha + \lambda_j)^{-2r} \lambda_1^{-2r} < \infty \quad (r_1 > 0).
\end{equation}

Set

\begin{equation}
\theta_1 = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r},
\end{equation}

Denote by $<\cdot, \cdot>_0$ the inner product in $H_0$ and let

\begin{equation}
\Phi = \{ \phi \in H_0 : \sum_{j=1}^{\infty} <\phi, \phi_j>_0^2 < (1 + \lambda_j)^{2r} < \infty \text{ for all } r \in \mathbb{R} \}.
\end{equation}

For each $r \in \mathbb{R}$ define an inner product $<\cdot, \cdot>_{r}$ and norm $\| \cdot \|_r$ on $\Phi$ by

\begin{equation}
<\phi, \psi>_r = \sum_{j=1}^{\infty} <\phi, \phi_j>_0 <\psi, \phi_j>_0 (1 + \lambda_j)^{2r},
\end{equation}

\begin{equation}
\| \phi \|_r^2 = <\phi, \phi>_r
\end{equation}

and let $H_r$ be the Hilbert space completion of $\Phi$ in the inner product $<\cdot, \cdot>_r$. Then $\Phi$ with the Frechet topology determined by the family $\{ \| \cdot \|_r \}_{r \in \mathbb{R}}$ of Hilbertian norms is a Countably Hilbert Nuclear Space. Let $\Phi' = \bigcup_r H_r$ with the inductive limit topology. Then $\Phi'$ is identified with the dual space (in the weak topology) to $\Phi$. The following properties hold (see Kallianpur and Wolpert (1984)):

\begin{equation}
H_r \text{ and } H_r' \text{ are in duality under the pairing}
\end{equation}

\begin{equation}
<\xi, [\phi]>_r = \sum_{j=1}^{\infty} <\xi, \phi_j>_r, \quad \xi \in H_r', \quad \phi \in H_r.
\end{equation}

\begin{equation}
\phi \in H_s \subset H_r \subset \Phi' \text{ if } r < s \text{ and the injection of } H_s \text{ into } H_r
\end{equation}

is a Hilbert-Schmidt map if $s > r + r_1$.

\begin{equation}
\text{Finite linear combinations of } \{\phi_j\} \text{ are dense in } \Phi \text{ and in every } H_r; \text{ moreover, } \{\phi_j\}_{j \geq 1} \text{ is an orthogonal system in}
\end{equation}
each $H_r$, and then $\{(1+\lambda_j)^{-r}f_j\}_{j \geq 1}$ is a CONS for $H_r$.

**Assumption 4.1.1** From now on, unless explicitly stated otherwise, we will assume that $\Phi$ is the CHNS of (4.1.5) and that $L$, $\{\lambda_j\}_{j \geq 1}$, $\{\phi_j\}_{j \geq 1}$, $r_1, \theta_1$, $H_r$, $-\infty < r < \infty$, and $\Phi'$ are as in the last example.

We will see in Examples 4.1.2 and 4.1.3 how the CHN spaces $S(\mathbb{R}^d)$ and $S(\ell^d)$ may be obtained within this framework.

**The tensor product nuclear space $\Phi^\otimes_n$** Under the notation and the hypotheses of Example 4.1.1, for each $r \in \mathbb{R}$ let $H^\otimes_r$ be the n-fold tensor product Hilbert space of $H_r$ with inner product $\langle \cdot, \cdot \rangle_r^\otimes$ and norm $\| \cdot \|_r^\otimes$.

Define the linear space

$$\Phi^\otimes = \{ \psi \in H^\otimes_0 : \psi \in H^\otimes_r \text{ all } r \in \mathbb{R} \}$$

with the topology determined by the family of norms $\{\| \cdot \|_r^\otimes\}_{r \in \mathbb{R}}$.

**Proposition 4.1.1** For each $n \geq 1$ $(\Phi^\otimes, \| \cdot \|_r^\otimes r \in \mathbb{R})$ is a Countably Hilbert Nuclear Space. It is called the n-fold tensor product nuclear space of $\Phi$. If $r < s$ then

$$\Phi^\otimes \subset H^\otimes_s \subset H^\otimes_r \subset (\Phi^\otimes)'$$

where $(\Phi^\otimes)'$ is the inductive limit of $H^\otimes_r$ $r \in \mathbb{R}$, identified with the dual to $\Phi^\otimes$ in the weak topology.

**Proof** Since for $r < s$ $H^\otimes_s \subset H^\otimes_r$ and $\| \cdot \|_r^\otimes \leq \| \cdot \|_s^\otimes$, then for each $n \geq 1$

$$H^\otimes_s \subset H^\otimes_r \quad \text{and} \quad \| \cdot \|_r^\otimes \leq \| \cdot \|_s^\otimes.$$ 

Then

(4.1.11) $\Phi^\otimes = \bigcap_{k=0}^{\infty} H^\otimes_k$
and therefore \( \Phi^\otimes_n \) is a complete metric space.

Thus we only have to prove that if \( s > r + r_1 \), the injection from \( H^\otimes_n_s \) into \( H^\otimes_r \) is a Hilbert-Schmidt map. Let \( \{ e_j = (1+\lambda_j)^{-s/2} \phi_j \}_{j=1}^{\infty} \) be a CONS for \( H^\otimes_r \), then \( \{ e_{j_1} \otimes \cdots \otimes e_{j_n} \}_{j_1, \ldots, j_n=1}^{\infty} \) is a CONS for \( H^\otimes_n \) and using (4.1.10) and (4.1.6)

\[
\sum_{j_1, \ldots, j_n=1}^{\infty} \| e_{j_1} \otimes \cdots \otimes e_{j_n} \|_r^2 \leq \sum_{j=1}^{\infty} \| e_j \|_r^2 \leq \theta_1^m < \infty
\]

Thus the injection from \( H^\otimes_n \) into \( H^\otimes_r \) is a Hilbert-Schmidt map.

Q.E.D.

The next proposition will be useful in studying multiple Wiener integrals. It gives the \( n \)th tensor quadratic form of a continuous positive definite bilinear (c.p.d.b.) form on \( \Phi \times \Phi \).

Proposition 4.1.2 Let \( Q(\cdot, \cdot) \) be a continuous positive definite bilinear form on \( \Phi \times \Phi \). Define

\[
Q^\otimes_n(\psi_1 \otimes \cdots \otimes \psi_n, \eta_1 \otimes \cdots \otimes \eta_n) = Q(\psi_1, \eta_1) \cdots Q(\psi_n, \eta_n) \quad \psi_i, \eta_i \in \Phi.
\]

Then \( Q^\otimes_n \) can be extended to a continuous positive definite bilinear form on \( \Phi^\otimes_n \times \Phi^\otimes_n \). Moreover, there exist \( \theta_2 > 0 \) and \( r_2 > 0 \) such that

\[
Q^\otimes_n(\eta, \eta) \leq \theta_2 \| \eta \|_r^2 \quad \eta \in \Phi^\otimes_n.
\]

Proof The first part follows as in the construction of a tensor product inner product (see Proposition 1, page 49 of Reed and Simon (1980)). To
prove (4.1.13), since $Q$ is a c.p.d.b. form on $\phi \times \phi$, by the nuclear theorem there exist $\theta_2 > 0$ and $r_2 > 0$ such that

$$Q(\phi, \phi) \leq \theta_2 \|\phi\|_{r_2}^2 \quad \phi \in \phi.$$  

Then using (4.1.12), for $\psi_i \in \phi \ i=1,\ldots,n$

$$Q^{\otimes n}(\psi_1 \otimes \cdots \otimes \psi_n, \psi_1 \otimes \cdots \otimes \psi_n) \leq \theta_2^{\otimes n} \|\psi_1 \otimes \cdots \otimes \psi_n\|_{r_2}^{2n}$$

and (4.1.13) follows from the extension of $Q^{\otimes n}$.

Q.E.D.

We finish this section by showing how the nuclear spaces $S(\mathbb{R}^d)$ and $S(\mathbb{E}^d)$ may be obtained in the framework of Example 4.1.1.

**Example 4.1.2** ($S(\mathbb{R}^d)$) (Itô (1978a)). Let $H_0 = L^2(\mathbb{R}, m)$, $m$ Lebesgue measure in $\mathbb{R}$, $-L = d^2/dx^2 - x^2/4$ be the harmonic oscillator (Reed and Simon (1980)), $L\phi_n = \lambda_n \phi_n \ n=1,2,\ldots$ where $\{\phi_n\}_{n \geq 1}$ are Hermite functions, $\lambda_n = n - \frac{1}{2}$ $n \geq 1$

$$\phi_{k+1}(x) = (g(x))^{1/2} h_k(2^{-k}x) (2^k k!(\pi)^{-1/2})^{-1/2} \quad k \geq 0$$

$$g(x) = (2\pi)^{-1/2} e^{-x^2/2} \quad \text{and} \ h_k \ \text{are Hermite polynomials defined by}$$

$$h_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} \left( e^{-x^2} \right) \quad k=0,1,2,\ldots.$$  

Then (see Itô (1978a)) $r_1 = 1$, $\theta_1 = \pi/2$ and $\phi$ as defined in Example 4.1.1 is the space $S(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$, with the topology defined by the family of Hilbertian norms

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} (n+1) 2p \phi_n^2 \quad p \geq 0, \quad \langle \phi, \phi \rangle_0 = \int_{\mathbb{R}} \phi(x) \phi(x) dx.$$  

In a similar fashion Itô (1978a) constructs the space $S(\mathbb{R}^d)$ $d \geq 1$ which can be seen as the d-fold tensor product nuclear space $(S(\mathbb{R}))^d$.  


Example 4.1.3 \((S(\mathbb{Z}^d))\). Let \(H_0 = \ell_2\) be the Hilbert space of all real sequences \(x = (x_n)\) such that \(\sum_{n=1}^{\infty} x_n^2 < \infty\), \(L x = \{nx_n\}\). Then \(\lambda_n = n\) and \(\phi_n = e_n\) \(n \geq 1\), where \(\{e_n\}_{n=1}^{\infty}\) is the canonical basis in \(\ell_2\). Thus \(r_1 = 1\), \(\theta_1 = \pi/6\) and the space \(\Phi\) defined by

\[ \Phi = \{x \in \mathbb{R}^\infty : \sum_{n=1}^{\infty} (n+1)^{2p} x_n^2 < \infty \quad \text{all } p \geq 0\} \]

and topologized by the family of Hilbertian norms

\[ \|x\|_p^2 = \sum_{n=1}^{\infty} (n+1)^{2p} x_n^2 \quad p \geq 0 \]

is the space \(S(\mathbb{Z})\) of all rapidly decreasing sequences. The space \(S(\mathbb{Z}^d)\) may be constructed as the \(d\)-fold tensor product nuclear space of \(S(\mathbb{Z})\).

4.1.2 \(\Phi\)'-valued Wiener process

Throughout this section we assume the hypotheses and notation of Example 4.1.1. Let \((\Omega, F, \mathbb{P})\) be a fixed but arbitrary complete probability space. All \(\Phi\)'-valued random elements and \(\Phi\)'-valued stochastic processes considered in this section are defined on this probability space. We denote by \(B(\Phi')\) the \(\sigma\)-field on \(\Phi\)' generated by the sets

\[ E_{\phi, a} = \{\xi \in \Phi' : \xi(\phi) < a\} \quad a \in \mathbb{R}, \phi \in \Phi \]

which is the \(\sigma\)-field generated by the open sets in the weak topology. Measures on \(\Phi\)', \(\Phi\)'-valued random elements and \(\Phi\)'-valued stochastic processes are defined with respect to the \(\sigma\)-field \(B(\Phi')\). Thus a mapping

\[ X_t(\omega) : [0, \infty) \times \Omega \to \Phi' \]

is a \(\Phi\)'-valued stochastic process if and only if \(X_t(\cdot)[\phi]\) is a real valued stochastic process for all \(\phi \in \Phi\).
A \( \phi' \)-valued stochastic process \((X_t)_{t \geq 0} \) is called an \( H_{-r} \)-valued process if for every \( t \geq 0 \) \( X_t \) is an \( H_{-r} \)-valued element.

From now on we will write \( \mathbb{R}_+ = [0, \infty) \) and \( \mathcal{B}(\mathbb{R}_+) \) will denote its Borel sets.

**Definition 4.1.1**  A sample continuous \( \phi' \)-valued stochastic process \( W = (W_t)_{t \in \mathbb{R}_+} \) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) is called a (centered) \( \phi' \)-valued Wiener process with covariance \( Q(\cdot, \cdot) \) if

a) \( W_0 \equiv 0 \).

b) \( W_t \) has independent increments.

c) For each \( \phi \in \Phi \) and \( t \geq 0 \)

\[
E(e^{i W_t(\phi)}) = \exp(-t/2 \, Q(\phi, \phi))
\]

where \( Q(\cdot, \cdot) \) is a continuous positive definite bilinear (c.p.d.b.) form on \( \phi \times \phi \).

From the above definition we see that the system

\[
\{W_t(\phi); \phi \in \Phi, t \geq 0\}
\]

is a Gaussian system of random variables and that if \( \phi, \psi \in \Phi \), the real valued processes \( W_t(\phi) \) and \( W_t(\psi) \) are independent on non-overlapping increments. Moreover, for each \( \phi, \psi \in \Phi \) and \( s, t \in \mathbb{R}_+ \)

\[
E(W_s(\phi)W_t(\psi)) = \min(s, t) \, Q(\phi, \psi).
\]

If \( Q(\cdot, \cdot) = \langle \cdot, \cdot \rangle_0 \), following Itô (1978a), \( W_t \) may be called a standard \( \phi' \)-valued Wiener process. If \( Q(\cdot, \cdot) = \langle \cdot, \cdot \rangle_r \) for some \( r \in \mathbb{R} \), then \( \{\phi_j\}_{j \geq 1} \) the system of eigenvectors of the generator \( L \), diagonalize \( Q \) (see (4.1.10)). In general we will not assume that \( Q \) is diagonalized by the system \( \{\phi_j\}_{j \geq 1} \).
Several examples of $\phi'$-valued Wiener processes with different covariances $Q$ are presented in Section 4.1.3.

The existence of a $\phi'$-valued Wiener process with an $H_q$ continuous version is now established.

**Theorem 4.1.1** Let \( \{Y(t,\phi): \phi \in \phi, t \in \mathbb{R}_+\} \) be a centered Gaussian system of random variables such that

\[
E(Y(t,\phi)Y(s,\psi)) = \min(s,t)Q(\phi,\psi) \quad \phi, \psi \in \phi, s, t \in \mathbb{R}_+
\]

where $Q$ is a c.p.d.b. form on $\phi \times \phi$. Then there exists a $\phi'$-valued Wiener process $(W_t) \ t \in \mathbb{R}_+$ with c.p.d.b. form $Q$ such that $Y(t,\phi) = W_t[\phi]$ a.s. for all $\phi \in \phi, t \in \mathbb{R}_+$, and $W_t$ has an $H_q$-valued continuous version for some $q \geq r_1 + r_2$, where $r_2$ is such that for some $\theta_2 > 0$

\begin{equation}
Q(\phi,\phi) \leq \theta_2 ||\phi||^2_{r_2} \quad \forall \ \phi \in \phi.
\end{equation}

**Proof** Using the bilinearity of $Q$, for $t \geq 0$ fixed and $c_1, c_2 \in \mathbb{R}$, $\phi_1, \phi_2 \in \phi$ we obtain that

\[
E(Y(t,c_1\phi_1 + c_2\phi_2) - Y(t,c_1\phi_1) - Y(t,c_2\phi_2))^2 = 0.
\]

By the Kernel theorem for CHNS's (Gelfand and Vilenkin (1964)), there exist $\theta_2 > 0$ and an integer $r_2 > 0$ such that (4.1.14) is satisfied. Therefore

\begin{equation}
E|Y(t,\phi)|^2 \leq \theta_2 t ||\phi||^2_{r_2} \quad \phi \in \phi.
\end{equation}

Fix $t$ throughout the argument. Then $Y_t: \phi \mapsto Y(t,\phi)$ is a bounded linear operator from the pre-Hilbert space $(\phi, ||\cdot||_{r_2})$ into $L^2 = L^2(\mathbb{R}, F, P)$ and hence extends uniquely to a bounded linear operator from $H_{r_2}$ into $L^2$, denoted also by $Y_t$.

Let \((1+\lambda_j)^{-1} (r_1 + r_2) \phi_j\}_{j \geq 1} be a CONS for $H_{r_1 + r_2}$. Write $\tilde{\phi}_j = (1 + \lambda_j)^{-1} (r_1 + r_2) \phi_j$. 

and let \( \{f_j\}_{j=1}^{\infty} \) be a CONS for \( H_{-(r_1+r_2)} \) dual to \( \{\phi_j\}_{j=1}^{\infty} \), i.e.

\[
\langle f_k, \phi_j \rangle - (r_1+r_2) = \delta_{kj}. \quad \text{Set } X_t^j = Y(t, \phi_j). \text{ Then from (4.1.15)}
\]

\[
E(\sum_{j=1}^{\infty} (X_t^j)^2) \leq \theta_2 t \sum_{j=1}^{\infty} \|\phi_j\|_{r_2}^2 = \theta_2 t \sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1} = \theta_1 \theta_2 t > 0.
\]

Next let \( \Omega_1 = \{\omega \in \Omega : \sum_{j=1}^{\infty} (X_t^j(\omega))^2 < \infty\} \), then \( P(\Omega_1) = 1 \). Define

\[
W_t(\omega) = \begin{cases} 
0 & \omega \notin \Omega_1 \\
\sum_{j=1}^{\infty} X_t^j(\omega) f_j & \omega \in \Omega_1
\end{cases}.
\]

Then for each \( t \geq 0 \) \( W_t(\omega) \in H_q = H_{-q} \) a.s. for \( q \geq r_1+r_2 \) and

\[
E\|W_t\|_{-q}^2 = E(\sum_{j=1}^{\infty} (X_t^j)^2) \leq \theta_1 \theta_2 t.
\]

Next for \( \phi \in H_q \)

\[
(4.1.16) \quad W_t[\phi] = \sum_{j=1}^{\infty} X_t^j(\omega) f_j[\phi] = \sum_{j=1}^{\infty} X_t^j \langle \phi, \phi_j \rangle_q
\]

and for \( 0 \leq s \leq t \)

\[
W_t[\phi] - W_s[\phi] = \sum_{j=1}^{\infty} (X_t^j - X_s^j) \langle \phi, \phi_j \rangle_q.
\]

But

\[
E(X_t^j - X_s^j)^2 = (t-s)Q(\phi_j, \phi_j) = (t-s)(1+\lambda_j)^{-2(r_1+r_2)} Q(\phi_j^2) \leq (t-s)(1+\lambda_j)^{-2(r_1+r_2)} \theta_2 \|\phi_j\|_{r_2}^2 = \theta_2 (t-s)(1+\lambda_j)^{-2r_1}.
\]

Then

\[
E\|W_t - W_s\|_{-q}^2 = E \sup_{\|\phi\|_{q \leq 1}} |W_t[\phi] - W_s[\phi]|^2
\]

\[
= E \sup_{\|\phi\|_{p \leq 1}} \left( \sum_{j=1}^{\infty} (X_t^j - X_s^j)^2 \langle \phi, \phi_j \rangle_q \right)^2 \leq E \sup_{\|\phi\|_{q \leq 1}} \left( \sum_{j=1}^{\infty} (X_t^j - X_s^j)^2 \right) (\sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle_q^2)^{-2r_1} \leq E(\sum_{j=1}^{\infty} (X_t^j - X_s^j)^2) \leq \theta_2 (t-s) \sum_{j=1}^{\infty} (1+\lambda_j)^{-2r_1} = \theta_2 \theta_1 (t-s) < \infty.
\]
Thus

\[(4.1.17) \quad E\|W_t - W_s\|_{q}^2 \leq \theta_1 \theta_2 |t-s|\]

and applying Kolmogorov continuity theorem we have that there exists an $H_{-q}$-valued continuous version of $(W_t)_{t \in \mathbb{R}^+}$ for $q \geq r_1 + r_2$.

Q.E.D.

**Corollary 4.1.1** If $(W_t)_{t \in \mathbb{R}^+}$ is a standard $\phi'$-valued Wiener process, then it has an $H_{-r_1}$-valued continuous version, where $r_1$ is as in (4.1.4).

**Proof** Since for a standard $\phi'$-valued Wiener process $Q(\cdot, \cdot) = \langle \cdot, \cdot \rangle_0$, then we have equality in (4.1.14) with $\theta_2 = 1$ and $r_2 = 0$.

Q.E.D.

If $(W_t)_{t \geq 0}$ is a $\phi'$-valued Wiener process with c.p.d.b. form $Q$ on $\phi \times \phi$, then $W(t, \phi) = W_t[\phi]$ is a centered Gaussian system and therefore by the last theorem $W_t$ has an $H_{-q}$-valued continuous version, also denoted by $W_t$, for $q \geq r_1 + r_2$.

**Assumption 4.1.2** From now on we will assume that $(W_t)_{t \in \mathbb{R}^+}$ is a $\phi'$-valued Wiener process with c.p.d.b. form $Q$ on $\phi \times \phi$ and an $H_{-q}$-valued continuous version for $q \geq r_1 + r_2$, given by Theorem 4.1.1, where $r_2$ and $\theta_2$ are as in (4.1.4). Moreover, assume that for each $t \geq 0$ $F_t^W = \sigma(W_s[\phi] : 0 \leq s \leq t, \phi \in \phi)$ with $F_0$ containing all $P$-null sets of $F$.

**Lemma 4.1.2** Let $q \geq r_1 + r_2$. Then for each $\phi \in H_q$ $(W_t[\phi], F_t^W)$ is a continuous martingale with quadratic variation process

\[(4.1.18) \quad <W[\phi]>_t = tQ(\phi, \phi) \quad t \geq 0.

Moreover, the cross predictable quadratic variation of $W_t[\phi]$ and $W_t[\psi]$ for $\phi, \psi \in H_q$ is
(4.1.19) \( <W[\phi], W[\psi]>_t = tQ(\phi, \psi) \quad t \geq 0. \)

**Proof** The martingale property follows since \( (W_t)_{t \geq 0} \) is a \( \Phi' \)-valued process with independent increments and for \( \phi \in H_q \) and \( t \geq 0 \) \( E(W_t[\phi]) = 0 \).

Next since for each \( t \geq 0 \)

\[
<W[\phi], W[\psi]>_t = \frac{1}{2} \left( <W[\phi]+W[\psi], W[\phi]+W[\psi]>_t - <W[\phi], W[\psi]>_t - <W[\psi], W[\psi]>_t \right)
\]

then we only have to prove (4.1.18). But from (4.1.16) since \( Q \) has a continuous extension to \( H_q \times H_q \) for \( q \geq r_1 + r_2 \) (see Proposition 4.1.3) if \( \phi \in H_q \) then

\[
E(W_t[\phi])^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} <\phi_j, \phi_k> Q(\phi_j, \phi_k) = tQ(\phi, \phi)
\]

and hence \( E((W_t[\phi] - W_s[\phi])^2 | F_s) = (t-s)Q(\phi, \phi) \quad s < t \quad \phi \in \Phi. \)

For \( \phi \in H_q \) \( q \geq r_1 + r_2 \) \( (W_t[\phi], F_t^W) \) has a continuous version since Theorem 4.1.1 \( W_t \) has an \( H_q \)-continuous version.

Q.E.D.

**Corollary 4.1.2** Let \( (W_t)_{t \geq 0} \) be a \( \Phi' \)-valued Wiener process with a c.p.d.b. form \( Q \) on \( \Phi \times \Phi \). Then if \( q \geq r_1 + r_2 \)

\[
E(W_t[\phi]W_s[\psi]) = \min(s, t)Q(\phi, \psi) \quad \phi, \psi \in H_q
\]

where \( Q(\phi, \psi) \) \( \phi, \psi \in H_q \) is a c.p.d.b. form on \( H_q \times H_q \) that extends the c.p.d.b. form \( Q \) on \( \Phi \times \Phi \).

**Proof** It follows from the proof of the last lemma by observing that if \( s < t \)
\[ E(W_t[\phi]W_s[\psi]) = E(W_s[\psi]E(W_t[\phi]|F^W_s)) = E(W_s[\phi]W_s[\psi]) \]

\[ = \frac{1}{2} \{E(W_s[\phi+\psi])^2 - E(W_s[\phi])^2 - E(W_s[\psi])^2 \}. \]

Q.E.D.

We now consider several concepts associated with a \( \phi' \)-valued Wiener process \((W_t)_{t \geq 0}\) with a c.p.d.b. form \(Q\) on \(\phi \times \phi\) and an \(H\)-continuous version for \(q \geq r_1 + r_2\).

**Rigged Hilbert space associated with a \(\phi'\)-valued Wiener process**

Let \((W_t)_{t \geq 0}\) be a \(\phi'\)-valued Wiener process with a c.p.d.b. form \(Q\) on \(\phi \times \phi\). Then \(Q\) is an inner product on \(\phi \times \phi\). Denote by \(H_Q\) the completion of \(\phi\) with respect to \(Q\) and by \(\langle \cdot, \cdot \rangle_Q\) or \(Q(\cdot, \cdot) (||\cdot||_Q)\) the corresponding inner product (norm) on \(H\). Then

\[
(4.1.20) \quad \phi \in H_s \subset H_Q \equiv H_{s} \subset H_{-s} \subset \phi' \quad s \geq r_2.
\]

The system (4.1.20) is called the **Rigged Hilbert Space** (see Gelfand and Vilenkin (1964)) associated with the \(\phi'\)-valued Wiener process \((W_t)_{t \geq 0}\) with c.p.d.b. form \(Q\) on \(\phi \times \phi\).

**Proposition 4.1.3**

a) For \(s \geq r_2\) and \(\phi \in H_s\)

\[
(4.1.21) \quad Q(\phi, \phi) \leq \theta_2 \|\phi\|_s^2
\]

and therefore (4.1.20) makes sense, and \(Q\) has a continuous extension to \(H_s \times H_s\).

b) For \(s \geq r_1 + r_2\) the injection of \(H_s\) into \(H_Q\) is a Hilbert-Schmidt map.

**Proof**

a) From the Kernel theorem (see (4.1.14))

\[
Q(\phi, \phi) \leq \theta_2 \|\phi\|_r^2 \quad \forall \phi \in \phi
\]
Next if $\psi \in H_s$ since $\phi$ is dense in $H_s$ there exists a sequence $\{\psi_n\}$ in $\phi$ such that $\psi_n \to \psi$ in $H_s$. Then $Q(\psi_n - \psi_m, \psi_n - \psi_m) \leq \theta_2 \|\psi_n - \psi_m\|^2_s + 0$ which implies that $Q(\psi_n, \psi_n) + Q(\psi, \psi) \to \infty$ and then (4.1.21) follows.

b) Let $\{\phi_j = (1+\lambda_j)^s \phi_j\}_{j \geq 1}$ be a CONS for $H_s$, then

$$\sum_{j=1}^{\infty} Q(\phi_j, \phi_j) = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2s} Q(\phi_j, \phi_j) \leq \theta_2 \sum_{j=1}^{\infty} (1+\lambda_j)^{-2s} \|\phi_j\|^2_{r_2}$$

$$= \theta_2 \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(s-r_2)} \leq \theta_2 \theta_1 < \infty$$

and then the injection of $H_s$ into $H_Q$ is a Hilbert-Schmidt map for $s \geq r_1 + r_2$.

Q.E.D.

In a similar way, for each $n \geq 1$ the Rigged Hilbert Space

$$\phi^n \subset H^n_s \subset H^n_Q = (H^n_Q)^\prime \subset (\phi^n)^\prime,$$

may be constructed where

$$Q^n(n, n) \leq \theta^n_s \|n\|^2_{s^n} \quad \eta \in H^n_s \quad s \geq r_2$$

and the injection from $H^n_s$ into $H^n_Q$ is a Hilbert-Schmidt map for $s \geq r_1 + r_2$.

Wiener integral and the Gaussian space $H$ of $W_t$. Let $C = C([0, \infty) \to \phi)$ be the linear manifold of all measurable step functions on $\mathbb{R}_+$ with values in $\phi$, i.e. $f \in C$ iff there exists a finite collection of positive real numbers $0 = t_0 < t_1 < t_2 < \ldots < t_k$ and $\alpha_i \in \phi$ for $i = 1, \ldots, k$ such that

$$f(t) = \sum_{i=1}^{k} \alpha_i 1_{(t_{i-1}, t_i]}(t).$$

For $f \in C$ define the Wiener integral

$$I_1(f) = \sum_{i=1}^{k} (W_{t_i} - W_{t_{i-1}}) [\alpha_i].$$

(4.1.22)
Then $I_1(\cdot)$ has the following properties.

**Lemma 4.1.3** Let $f, g \in C$. Then

a) $I_1(cf) = cI_1(f)$, $c \in \mathbb{R}$.

b) $I_1(f+g) = I_1(f) + I_1(g)$.

c) $E(I_1(f)) = 0$.

d) $E(I_1(f)I_1(g)) = \int_{\mathbb{R}^+} Q(f(t), g(t)) dt$.

e) $E(I_1(f))^2 = \int_{\mathbb{R}^+} \|f(t)\|^2 dt$.

**Proof** (a) and (b) are proved as in the real valued case using the fact that for each $t \geq 0$ $W_t \in \Phi'$. The proof of (c) follows since $E(W_t[\phi]) = 0$ $\forall \phi \in \Phi$ and $t \geq 0$. To prove (d) write $A_i = (t_{i-1}, t_i]$ and $W(A_i) = W_{t_i} - W_{t_{i-1}}$.

Then if

$$f(t) = \sum_{i=1}^{k} \alpha_i 1_{A_i}(t)$$

and

$$g(t) = \sum_{i=1}^{k} \beta_i 1_{A_i}(t)$$

for $0 \leq t_0 < t_1 < \ldots < t_k$, $\alpha_i, \beta_i \in \Phi$, $i = 1, \ldots, k$, by definition of $I_1$

$$I_1(f) = \sum_{i=1}^{k} W(A_i)[\alpha_i]$$

$$I_1(g) = \sum_{i=1}^{k} W(A_i)[\beta_i]$$

Then if $m(\cdot)$ denotes the Lebesgue measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$

$$E(I_1(f)I_1(g)) = \sum_{i=1}^{k} \sum_{j=1}^{k} E(W(A_i)[\alpha_i]W(A_j)[\beta_j])$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} m(A_i \cap A_j)Q(\alpha_i, \beta_j) = \sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathbb{R}^+} 1_{A_i \cap A_j}(t)Q(\alpha_i, \beta_j) dt$$
\[ \int_{\mathbb{R}^+} \left( \sum_{i=1}^{k} a_i A_i(t) \right) \left( \sum_{j=1}^{k} b_j A_j(t) \right) dt = \int_{\mathbb{R}^+} Q(f(t), g(t)) dt \]

which proves (d) and (e). Q.E.D.

To extend \( I_1 \) to \( L^2(\mathbb{R}^+ \to H_Q) \) we first prove the following result, where \( L^2(\mathbb{R}^+ \to H_Q) \) is the Hilbert space of \( H_Q \)-valued measurable functions \( f \) on \( \mathbb{R}^+ \) (identifying those which are equal a.e. \( dt \)) such that

\[ \int_{\mathbb{R}^+} \| f(t) \|_Q^2 dt < \infty. \]

**Lemma 4.1.4** \( C \) is a dense linear manifold in \( L^2(\mathbb{R}^+ \to H_Q) \).

**Proof** Let \( f \in L^2(\mathbb{R}^+ \to H_Q) \), then for each \( \varepsilon > 0 \) there exists an \( H_Q \)-valued step function \( f^\varepsilon \), i.e.

\[ f^\varepsilon(t) = \sum_{i=1}^{k} a_i A_i(t) \]

\( a_i \in H_Q \), \( A_i = (t_{i-1}, t_i) \) \( i = 1, \ldots, k \), \( 0 \leq t_0 < t_1 < \ldots < t_k \), \( k \geq 1 \), such that

\[ \int_{\mathbb{R}^+} \| f(t) - f^\varepsilon(t) \|_Q^2 dt < \varepsilon/2. \]

Since \( \Phi \) is dense in \( H_Q \), there exist \( \alpha_i \in \Phi \) \( i = 1, \ldots, k \) such that

\[ \| a_i - \alpha_i \|_Q^2 < \frac{\varepsilon}{2k(t_i - t_{i-1})} \quad i = 1, \ldots, k. \]

Define

\[ g^\varepsilon(t) = \sum_{i=1}^{k} \alpha_i A_i(t) \]

then \( g^\varepsilon \in C \) and

\[ \int_{\mathbb{R}^+} \| f^\varepsilon(t) - g^\varepsilon(t) \|_Q^2 dt < \varepsilon/2. \]

Then from the last expression and (4.1.24)

\[ \int_{\mathbb{R}^+} \| f(t) - g^\varepsilon(t) \|_Q^2 dt < \varepsilon \]
i.e. $C$ is a dense linear manifold in $L^2(\mathbb{R}^+ + H_Q) \cong L^2(\mathbb{R}^+ \circ H_Q)$.

Q.E.D.

The **Gaussian space** (linear space) associated with the $\phi'$-valued Wiener process $W_t$ is defined as

\[(4.1.25) \quad H = L_1(W) = \overline{\text{sp}\{W_t[\phi] : \phi \in \Phi, \ t \geq 0\}}\]

where the closure is taken with respect to $L^2(\Omega, F^W, \mathbb{P})$, where $F^W = F^\infty$. From Proposition 7.3. in Neveu (1968)

\[(4.1.26) \quad L^2(\Omega, F^W, \mathbb{P}) \cong \sum_{n \geq 0} H^\otimes n \]

where

$$n_1(\exp \Theta(h)) = \exp(h - E(h^2)) \quad h \in H \quad \text{and} \quad \exp \Theta(h) = (1, h, \frac{1}{\sqrt{2!}} h^2, \frac{1}{\sqrt{3!}} h^3, ...)$$

as it was shown in Chapter III (see also Kallianpur (1980) Chapter VI).

Then for all $n \geq 0$ $H^\otimes n$ may be seen as a closed subspace of $L^2(\Omega, F^W, \mathbb{P})$.

This fact will be used in Sections 5.1 and 5.2 together with the next definition.

**Definition 4.1.2** Lemma 4.1.3 shows that $I_1$ is an isometry from the linear space $C$ of $\phi'$-valued step functions into $H$. Hence from Lemma 4.1.4 this isometry can be extended uniquely to an isometry from $L^2(\mathbb{R}^+ + H_Q)$ onto $H$, also denoted by $I_1$ and called the **Wiener integral**. It has the properties (a)-(e) of Lemma 4.1.3.

**$\phi'$-valued Gaussian random measure** Let $T = \mathbb{R}^+$ and $A = B(\mathbb{R}^+)$. A $\phi'$-valued set function $W(\cdot)$ on $(T, A)$ is said to be a $\phi'$-valued Gaussian random measure if for each $\phi \in \Phi$, $W(\cdot)[\phi]$ is a real valued Gaussian random measure on $(T, A)$. For $r \in \mathbb{R}$ we denote by $L^2(\Omega + H_r)$ the Hilbert space (identifying
elements which are equal a.e. $dP$ of $H_r$-valued random elements $G$ such that $E\|G\|_r^2 < \infty$.

**Proposition 4.1.4** Let $(W_t)_{t \geq 0}$ be a $\Phi'$-valued Wiener process as in Assumption 4.1.2. Define for $A = [s, t]$, $0 < s < t$

$$W(A) = W_t - W_s .$$

Then $W(\cdot)$ can be extended to a $\Phi'$-valued Gaussian random measure on $(T, A)$, which is an orthogonally scattered measure in $L^2(\Omega \times H_{-q})$, for $q \geq r_1 + r_2$, and control measure $\mu(\cdot) = \theta_m(\cdot)$ where $m$ denotes the Lebesgue measure on $(T, A)$ and

$$\theta_q = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} Q(\phi_j, \phi_j) < \infty .$$

Moreover, if $A \in A$, $m(A) < \infty$ and $\phi \in H_q$, $W(A)[\phi] \in H$, and if $B \in A$, $m(B) < \infty$ then for all $\phi$, $\psi \in H_q$

$$(4.1.27) \quad E(W(A)[\phi]W(B)[\psi]) = m(\cap \psi).$$

**Proof** From Theorem 4.1.1 $(W_t)_{t \in T}$ has an $H_{-q}$-valued continuous version for $q \geq r_1 + r_2$. Let $(1+\lambda_j)^q \phi_j$, $j \geq 1$ be a CONS for $H_{-q}$. Then if $A = [s, t]$, $W(A) = W_t - W_s$

$$E\|W(A)\|_{-q}^2 = E(\sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} \langle W(A), \phi_j \rangle^2)$$

$$= \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} E(W(A)[\phi_j])^2 = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} m(A) Q(\phi_j, \phi_j) .$$

Let

$$\theta_q = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} Q(\phi_j, \phi_j)$$

then

$$\theta_q \leq \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q} \theta_2 \|\phi_j\|_{-r_2}^2 \leq \theta_2 \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q-r_2} \leq \theta_2 \theta_1 < \infty$$

and hence

$$E\|W(A)\|_{-q}^2 = \theta_q m(A) = \mu(A) .$$
Thus $W(\cdot)$ can be extended to an orthogonally scattered measure on $(T,A)$ with values in the Hilbert space $L^2(\Omega; H_q)$ and control measure $\mu(\cdot) = q m(\cdot)$.

Next if $A \in A, m(A) < \infty$ and $\{A_n\}_{n \geq 1}$ is a sequence of disjoint sets in $A$ with $m(A_n) < \infty$ all $n \geq 1$ and $A = \bigcup_{n=1}^{\infty} A_n$, then for all $\phi \in H_q$

$$E(W(A)[\phi] - \sum_{j=1}^{n} W(A_j)[\phi])^2 \leq ||\phi||^2_q E||W(A) - \sum_{j=1}^{n} W(A_n)||_{-q}^2 \rightarrow 0 \text{ as } m \rightarrow \infty$$

i.e., $W(\cdot)[\phi]$ is a real-valued Gaussian random measure with values in $H_q$. Let $\{\tilde{\phi}_j = (1+\lambda_j)^{-q} \phi_j\}$ be a CONS for $H_q$. Taking $A \in A, m(A) < \infty$, $W(A)[\phi_j]_{j \geq 1}$ and using a similar argument to the one used in the proof of Theorem 4.1.1 one shows that $W$ is a $\tilde{\phi}'$-valued Gaussian random measure. Finally, (4.1.27) follows by applying the approximation theorem to $A, B \in A, m(A) < \infty, m(B) < \infty$ and using Corollary 4.1.2.

Q.E.D.

Now let $T = [0,T_0], T_0 > 0$ and $A = B(T)$. For $q \geq r_1 + r_2$ let $\{e_k\}_{k \geq 1}$ be a CONS for $H_q$. Then from the above proposition we are able to obtain an infinite system of independently scattered Gaussian random measures $\{W(\cdot)[e_k]\}_{k \geq 1}$ on $(T,A)$ with values in $H_q$ that are mutually independent over disjoint sets, each one with control measure $u_k(\cdot) = m(\cdot) Q(e_k,e_k)$ and such that for all $k,j \geq 1$ and $A, B \in A$.

$$E(W(A)[e_k] W(B)[e_j]) = m(A \cap B) Q(e_j,e_k)$$

(4.1.28)

Then for each set of index $k_1, \ldots, k_n$ in $\{1,2,\ldots\}$ we may construct the symmetric tensor product measure of $W(\cdot)[e_{k_1}], \ldots, W(\cdot)[e_{k_n}]$, denoted by $\otimes_{i=1}^{n} W[e_{k_i}]$, as in Sections 2.2 and 3.1 of this work, and construct multiple integrals with respect to it. In the next result we summarize the main properties of $\otimes_{i=1}^{n} W[e_{k_i}]$ and their integrals. It will be used in
Propositions 5.1.3 and 5.1.4.

Lemma 4.1.5. Let $T = [0, T_0]$, $T_0 > 0$, $A = B(T)$ and $H$ be the Gaussian space defined in (4.1.25). For $q \geq r_1 + r_2$ let $\{e_k\}_{k \geq 1}$ be a CONS in $H_q$. For a finite collection of index $k_1, \ldots, k_n$ in $\{1,2,\ldots\}$ denote by $\otimes_{i=1}^{n} W[e_{k_i}]$ the symmetric tensor product measure of $W(\cdot)[e_{k_1}], \ldots, W(\cdot)[e_{k_n}]$ given by Theorem 2.2.1. Then

\begin{align*}
&\text{(a)} \quad \otimes_{i=1}^{n} W[e_{k_i}] \text{ is an } H^\otimes -\text{valued measure on } (T^n, \mathcal{A}^n). \\
&\text{(b)} \quad E(\otimes_{i=1}^{n} W[e_{k_i}](A)) = 0 \quad \text{for each } A \in \mathcal{A}^n. \\
&\text{(c)} \quad \text{A function } f: T^n \to \mathbb{R} \text{ is } \otimes_{i=1}^{n} W[e_{k_i}]\text{-integrable (see Definition 2.3.1) if and only if } f \in L^2(T^n, \mathcal{A}^n, m^\otimes). \text{ Denote by } \int_{T^n} f(t) \otimes_{i=1}^{n} W[e_{k_i}](t) \text{ this integral.} \\
&\text{(d)} \quad \text{Let } f \in L^2(T^n, \mathcal{A}^n, m^\otimes) \text{ and } g \in L^2(T^n, \mathcal{A}^n, m^\otimes). \\
&\text{Then for each collection of index } j_1, \ldots, j_n, k_1, \ldots, k_n \text{ in } \{1,2,\ldots\} \\
&\quad E(\int_{T^n} f(t) \otimes_{i=1}^{n} W[e_{k_i}](t) \int_{T^n} g(t) \otimes_{i=1}^{n} W[e_{k_i}](t)) \\
&\quad = \delta_n n! Q^\otimes (e_{j_1} \otimes \ldots \otimes e_{j_n} \otimes e_{k_1} \otimes \ldots \otimes e_{k_n}) \int_{T^n} \tilde{f}(t) \tilde{g}(t) dt \\
&\quad \text{where } \tilde{f}(t) = \frac{1}{n!} \sum \frac{f(t_{\pi})}{\pi}. \\
\end{align*}

Proof. Since for each $k \geq 1$ $W(\cdot)[e_k]$ is a zero mean $H$-valued independently scattered measure then (a) follows by using Theorem 2.2.1. The proof of (b) follows since $H^\otimes$ is orthogonal to $\overline{\sigma_p(1)}$ in $L^2(\Omega, \mathcal{F}^W, P)$ and by using (a). Next since $\mu_k(\cdot) = m(\cdot) Q(e_k, e_k)$ where $m$ is the Lebesgue measure on $(T, \mathcal{A})$, the proof of (c) follows by Theorem 2.3.2.

Finally, the proof of (d) is similar to the proof of Theorem 2.3.3,
first using $A^n$-simple functions and then using approximation arguments.

Q.E.D.

If $T_0 = 1$ and $Q(e_k, e_k) = 1$, all $k \geq 1$, \{W(\cdot)[e_k]\}_{k=1}^{\infty}$ is a sequence of orthogonally scattered measures in $H$ of the kind considered in Theorem 2.1.4. Then we can construct the infinite tensor product measure $\bigotimes_{k=1}^{\infty} W[e_k]$ on $(T^\omega, A^\omega, \mu^\omega)$ with values in $\bigotimes_{i=1}^{\omega} H$, $\mu = (W(T)[e_k])_{k \geq 1}$ (see Theorem 2.1.4). However, the construction of the infinite symmetric tensor product measure is a problem that remains open.

4.1.3 Examples

We now consider some examples of $\Phi'$-valued Wiener processes.

Example 4.1.4 (Kallianpur and Wolpert (1984)). Let $\phi$ be defined in Example 4.1.1 where $H_0 = L^2(\Gamma, d\Gamma)$ for $\Gamma$ a $\sigma$-finite measure on $(X, E)$. Let $\mu$ be a $\sigma$-finite measure on $\mathbb{R} \times X$ such that the bilinear form

$$Q(\phi, \psi) = \int_{\mathbb{R} \times X} a^2 \phi(x) \psi(x) \mu(da, dx)$$

on $\Phi \times \Phi$ is continuous. In connection with neurophysiology applications, Kallianpur and Wolpert (1984) give several examples of $Q$ in which the measure $\mu$ is of the form

$$\mu(A \times B) = \sum_{k=1}^{P_1} \gamma^k e_i \gamma^k(B) + \sum_{l=1}^{P_2} \delta^l a_i \delta^l(B), A \in B(\mathbb{R}), B \in E$$

where $\{a^k, a^l\}$ are positive real numbers and $\{\gamma^k, \delta^l\}$ are finite measures on $(X, E)$. The authors consider a $W$-path-continuous $\Phi'$-valued independent increments process with characteristic functional

$$E(e^{i W_t[\phi]} - 1) = e^{-\frac{1}{2} \int_0^t Q(\phi, \phi)}$$

for $\phi \in \Phi$, $t \geq 0$. 

where \( m \in \Phi' \). Theorem 4.1.1 enables us to construct such a process with continuous paths lying in \( H_{-q} \) for any \( q \geq r_1 + r_2 \) if \( m \) and \( Q \) satisfy

\[
|m(\phi)|^2 + Q(\phi, \phi) \leq \theta_2 \|\phi\|^2 \quad \phi \in \Phi
\]

for some positive constants \( \theta_2 \) and \( r_2 \).

The continuous positive definite bilinear form \( Q \) on \( \Phi \times \Phi \) is not necessarily diagonalized by the system \( \{\phi_j\}_{j \geq 1} \) of eigenvectors of \(-L\). If for example

\[
\Gamma(dx) = \int a^2 \mu(da, dx)
\]

then \( Q(\phi_i, \phi_j) = \int a^2 \phi_i(x)\phi_j(x) \mu(da, dx) = 0 \) for \( i \neq j \); but in general this is not the case.

Example 4.1.5 (Itô (1978a)). Let \( \Phi \) be as in Example 4.1.2, i.e. \( \Phi = S(\mathbb{R}^d) \) \( d \geq 1 \). Itô (1978a) gives the following examples:

1) The Wiener \( \Phi' \)-valued process corresponding to \( Q_1(\phi, \psi) = \langle \phi, \psi \rangle_0 \) is called a standard \( \Phi' \)-valued process and it is denoted by \( (b_t)_{t \in \mathbb{R}^+} \). By Corollary 4.1.1 (b) is an \( H_{-1} \)-valued Wiener process, since for the Example 4.1.2 \( r_1 = 1 \).

(ii) Let \( \Delta = \sum_{i=1}^{d} \beta_i^2 \) and define \( \Delta b_t = \Delta b_t[\phi] = b_t(\Delta \phi) \). Then \( W_t = \Delta b_t \) is an \( H_{-2} \)-valued Wiener process for which

\[
Q_2(\phi, \psi) = \langle \Delta \phi, \Delta \psi \rangle_0 \quad \phi, \psi \in \Phi.
\]

(iii) Let \( b_t = (b^1_t, b^2_t, \ldots, b^d_t) \) where the component processes \( b^i_t \) \( i = 1, \ldots, d \) are independent standard \( \Phi' \)-valued Wiener processes. Then

\[
W_t = \sum_{i=1}^{d} \beta_i b^i_t
\]

is an \( H_{-3} \)-valued Wiener process with
where $\Delta$ is defined in (ii).

We observe that while $Q_1$ is diagonalized by $\{\phi_j\}_{j \geq 1}$, $Q_2$ and $Q_3$ are not.

**Example 4.1.6** $\phi'$-valued Wiener processes arise in a natural way in the following manner. Let $A$ be a relatively compact subset of $\mathbb{R}^d$ $d \geq 1$.

Let $\tilde{W}(t, x)$ $t \in T$, $x \in A$ be a two parameter sample continuous centered Gaussian system of random variables such that

$$E(\tilde{W}(t, x)\tilde{W}(s, y)) = \min(s, t)V(x, y) \quad t, s \in \mathbb{R}_+, x, y \in A$$

where $V$ is square integrable over $A \times A$. Define for $s, t \in \mathbb{R}_+$

$$W_t[\phi] = \int_A \tilde{W}(t, x)\phi(x)dx$$

for $\phi$ in a suitable class, $\phi = S(\mathbb{R}^d)$ for example. Then from Theorem 4.1.1 we have that $\{W_t\}_{t \in \mathbb{R}_+}$ is a $\phi'$-valued Wiener process such that for $s, t \in \mathbb{R}_+$ and $\phi, \psi \in \phi$

$$E(W_t[\phi]W_s[\psi]) = \min(s, t)Q(\phi, \psi)$$

where

$$Q(\phi, \psi) = \int_A \int_A V(x, y)\phi(x)\phi(y)dxdy.$$ 

is not necessarily diagonalized by the system $\{\phi_j\}_{j \geq 1}$, which in the case of $\phi = S(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^d} \phi_i(x)\phi_j(x)dx = 0 \quad \text{for } i \neq j.$$ 

**Example 4.1.7** (Cylindrical Brownian motion) Let $K$ be a Hilbert space, $(\Omega, F, P)$ a complete probability space and $(F_t)_{t \geq 0}$ be an increasing family...
of sub $\sigma$-fields of $F$. A measurable mapping

$$B_t(k, w) = [0, \infty) \times K \times \Omega \to \mathbb{R}$$

is called an $F_t$-cylindrical Brownian motion on $K$ (Yor (1974)) (c.B.m.) if it satisfies the following two conditions:

a) For each $k \in K$, $k \neq 0$, $B_t(k)/\|k\|_K$ is a one dimensional $F_t$-Brownian motion.

b) $B_t(k)$ is linear in $k \in K$.

Two well-known observations (Miyahara (1981)) are the following:

1) A c.B.m. $(B_t)$ cannot be regarded as a process on $K$, i.e. it is not a $K$-valued process; 2) if $k_1$ and $k_2$ are orthogonal elements in $K$, then $\{B_t(k_1)\}$ and $\{B_t(k_2)\}$ are independent.

Using the notation of Example 4.1.1 we now study the following case considered by Miyahara (1981). Let $H_0 = L^2([0, \pi])$, $L = \Delta$, $\hat{\omega} = \sqrt{-\Delta}$ where $\Delta$ is the Laplacian ($\Delta = d/dx^2$) on $H_0$. Hence for $j=1, 2, \ldots$, $\lambda_j = j$ and

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \cos(jx).$$

Then construct $\phi$ as in Example 4.1.1. Miyahara (1981) considers a cylindrical Brownian motion $B_t$ on $H_0$. Then $\{B_t(\phi) : \phi \in \Phi, t \geq 0\}$ is a centered Gaussian system of random variables such that for $\phi$, $\psi \in \Phi$ and $s, t \in \mathbb{R}_+$

$$EB_t(\phi)B_s(\psi) = \min(s, t) Q(\phi, \psi)$$

where $Q(\phi, \psi) = \langle \phi, \psi \rangle_0 \phi, \psi \in \Phi$. Then $r_2 = 0$ and since

$$\sum_{j=1}^{\infty} (1+j)^{-2r} < \infty$$

for $r > \frac{1}{2}$ then $r_1 = \frac{1}{2}$. Thus, applying Theorem 4.1.1 there exists a
\( \phi' \)-valued Wiener process \( W_t \) such that

\[
W_t[\phi] = B_t(\phi) \quad \text{a.s.}
\]

for \( t \geq 0 \) and \( \phi \in \phi \). Moreover, using Corollary 4.1.1 \( W_t \) has an \( H_{-1} \)-valued continuous version.

Note that in this particular example the continuous positive definite bilinear form \( Q \) on \( \phi \times \phi \) is diagonalized by the eigenvector system \( \{ \phi_j \}_{j \geq 1} \).

**Example 4.1.8** (Independent system of one dimensional Brownian motions (Hitsuda and Watanabe (1978))). Let \( \phi \) be the countably Hilbert nuclear space defined in Example 4.1.3, i.e. \( \phi = S(\mathbb{E}) \). Let \( \{ B^{(i)}_t \}_{t \geq 0} \) be a system of independent one dimensional Brownian motions on a complete probability space \( (\Omega, \mathcal{F}, P) \). For \( \phi \in \phi \), \( \phi = (\phi^1, \phi^2, \ldots) \) define

\[
W_t[\phi] = \sum_{i=1}^{\infty} B_t^{(i)} \phi^i.
\]

Then since

\[
E\left( \sum_{i=1}^{\infty} B_t^{(i)} \phi^i \right)^2 = t \sum_{i=1}^{\infty} (\phi^i)^2 < \infty \quad t \geq 0
\]

we obtain that since \( <\phi, \psi>_0 = \sum_{i=1}^{\infty} \phi^i \psi^i \)

\[
E(W_t[\phi]W_s[\psi]) = \min(s, t) <\phi, \psi>_0.
\]

Hence, using Theorem 4.1.1 \( W_t \) has a version which is a \( S(\mathbb{E}) \)-valued Wiener process with c.p.d.b. form \( Q(\cdot, \cdot) = <\cdot, \cdot>_0 \). Moreover, using Corollary 4.1.1 \( W_t \) has an \( H_{-1} \)-valued continuous version.

**Example 4.1.9** (Finite dimensional case). Suppose \( \phi \) is finite dimensional, then \( \phi = H = \phi' = \mathbb{R}^m \) say. Then \( W_t = (W_t^{(1)}, \ldots, W_t^{(n)}) \) \( t \geq 0 \), where each \( W_t^{(i)} \) is a Gaussian process with independent increments and
\[ E(W_t^{(i)}W_s^{(j)}) = r_{ij} \min(s,t) \]

where \( R = (r_{ij}) \) is an \( m \times m \) positive definite matrix. Then if \( \phi = \sum_{j=1}^{m} \phi_j e_j, \)
\[ e_j = (0,...,1,...,0) \quad \phi_j \in \mathbb{R} \quad j=1,...,m \]
\[ Q(\phi,\psi) = \sum_{i=1}^{m} \sum_{j=1}^{m} \phi_i \psi_j r_{ij}. \]

Stochastic processes of this type have been considered in Chapter III.

4.2 Stochastic integrals

Throughout this section we will assume that \( (W_t)_{t \geq 0} \) is a \( \Phi' \)-valued Wiener process with a c.p.d.b. form \( Q \) on \( \Phi \times \Phi \), defined on a complete probability space \( (\Omega,F,P) \), and that for each \( t \geq 0 \) \( F_t = F^{W_t}_t = \sigma(W_s : 0 \leq s \leq t) \), with \( F_0 \) containing all \( P \)-null sets of \( F \). Also we make Assumptions 4.1.1 and 4.1.2 of Section 4.1. We recall that from Theorem 4.1.1 \( (W_t)_{t \geq 0} \) has an \( H_q \)-continuous version for \( q \geq r_1 + r_2 \).

Stochastic integrals with respect to \( S(\mathbb{R}^d)' \)-valued Wiener processes and \( E' \)-valued (\( E \) is a CHNS) processes have been discussed in Ikeda (1978a) and Mitoma (1981b) respectively. They propose to use the theory of stochastic integration on Hilbert spaces, as presented for example in Kunita (1970) or Kuo (1975), to construct stochastic integrals for the \( H_q \)-valued Wiener process \( (W_t)_{t \geq 0} \). Here we construct "weak" stochastic integrals similar to the case of a cylindrical Brownian motion as presented in Yor (1974). However, we do not work with a cylindrical Brownian motion but rather with an \( H_q \)-valued Wiener process. Secondly, if \( \{e_k\}_{k \geq 1} \) is any CONS in \( H_q \), then \( \{W_t[e_k]\}_{k \geq 1} \) is not necessarily a system of independent random variables (see Corollary 4.1.2), as it would be required in the case of a cylindrical Brownian motion (see Example 4.1.7). Moreover, we do not assume that the common orthogonal system in \( H_q, t \geq 0 \) \( \{\phi_j\}_{j \geq 1} \) (the
eigenvectors of the infinitesimal generator $L$ diagonalizes the c.p.d.b. form $Q$. The case when $Q(\cdot,\cdot) = \langle \cdot, \cdot \rangle_0$, and then $\{\phi_j\}_{j \geq 1}$ diagonalizes $Q$, has been considered by Daletskii (1967) and Miyahara (1981) (see Example 4.1.7). Nevertheless, the nuclearity of the space $(\Phi, \| \cdot \|_r, r \geq 0)$ enables us to construct "weak" integrals even when $\{W_t[\phi_j]\}_{j \geq 1}$ is not an independent system of random variables.

We present real valued (Section 4.2.1) and $\Phi'$-valued (Section 4.2.2) stochastic integrals with respect to $W_t$. They will be useful in Chapter V in studying real valued and $\Phi'$-valued nonlinear functionals of $W$. We make extensive use of the c.p.d.b. form $Q$ on $\Phi \times \Phi$ and its associated Rigged Hilbert Space (see (4.1.20)). This is motivated from Definition 4.1.2 which suggests that $L^2(\mathbb{R}_+^*) \otimes H_Q$ should play the role of the Reproducing Kernel Hilbert space, a concept that has been useful in studying nonlinear functionals of Gaussian processes (see Chapter VI of Kallianpur (1980)).

4.2.1 Stochastic integrals for $\Phi$-valued random integrands (Real valued stochastic integrals)

The aim of this section is to define stochastic integrals for $\Phi$-valued non-anticipative functions.

**Definition 4.2.1** Let $K$ be a real separable Hilbert space. A function $f: [0, \infty) \times \Omega \to K$ is said to belong to the class $M(W,K)$ if $f$ is an $F_t$-adapted measurable (non-anticipative) function on $\mathbb{R}_+ \times \Omega$ to $K$ such that for each $t > 0$

$$\int_0^t E\| f(s) \|_K^2 \, ds < \infty.$$  

(4.2.1)

The special classes we will be concerned with are $M_q = M(W, H_q)$, $q \geq r_1 + r_2$ and $M_Q = M(W, H_Q)$. 


Stochastic integrals for elements in $M_q \ q \geq r_1 + r_2$

**Definition 4.2.2** Let $q \geq r_1 + r_2$. For $g \in M_q$ and $t > 0$ define the stochastic integral $\int_o^t <g_s, dW_s>_q$ as

$$ (4.2.2) \quad \int_o^t <g_s, dW_s>_q = \sum_{i=1}^{\infty} \int_o^t <g_s, e_i>_q dW_s[e_i] $$

where $\{e_i\}_{i \geq 1}$ is a CONS for $H_q$ and the integrals on the right hand side of (4.2.2) are Itô integrals.

**Proposition 4.2.1** Let $g \in M_q \ q \geq r_1 + r_2$. Then the integral (4.2.2) is a well defined element in $L^2(\Omega,F^W_t,P)$. If $q_1 \geq r_1 + r_2$ and $g \in M_q$, then this integral is independent of $q$ and $q_1$. Moreover the following properties are satisfied for $f, g \in M_q$.

a) For $a, b \in \mathbb{R}$ and $t > 0$

$$ \int_o^t <af_s + bg_s, dW_s>_q = a \int_o^t <f_s, dW_s>_q + b \int_o^t <g_s, dW_s>_q \quad \text{a.s.} $$

b) $E(\int_o^t <g_s, dW_s>_q) = 0 \quad t > 0$.

c) $E(\int_o^t <g_s, dW_s>_q)^2 \leq E(\int_o^t Q(f_s, g_s) ds) \quad t > 0$.

d) $E(\int_o^t <f_s, dW_s>_q)^2 \leq E(\int_o^t Q(f_s, f_s) ds) \leq E(\int_o^t \|f_s\|_q^2 ds) \leq \infty$.

**Proof** We first prove that for $t > 0 \int_o^t <g_s, dW_s>_q$ is a well defined element in $L^2(\Omega,F^W_t,P)$. Let $\{e_i\}_{i \geq 1}$ be any CONS for $H_q \ q \geq r_1 + r_2$. Then for each $t > 0$

$$ g(t, \omega) = \sum_{j=1}^{\infty} <g_t(\omega), e_j>_q e_j $$

and for $n, m \geq 1$, using Lemma 4.1.2
\[
E(\sum_{j=m}^{n} \int_{0}^{t} \langle g_{s}, e_{j}\rangle q \, dW_{s}[e_{j}] )^{2} = \sum_{j=m}^{n} \sum_{k=m}^{n} E(\int_{0}^{t} \langle g_{s}, e_{j}\rangle q \, dW_{s}[e_{j}] \int_{0}^{t} \langle g_{s}, e_{k}\rangle q \, dW_{s}[e_{k}] ) = \sum_{j=m}^{n} \sum_{k=m}^{n} E(\int_{0}^{t} \langle g_{s}, e_{j}\rangle \langle g_{s}, e_{k}\rangle q Q(e_{j}, e_{k}) \, ds).
\]

Then since \( Q \) is a bilinear form, using (4.1.21) we obtain

\[
(4.2.3) \quad E(\sum_{j=m}^{n} \int_{0}^{t} \langle g_{s}, e_{j}\rangle q \, dW_{s}[e_{j}] )^{2} = E(\int_{0}^{t} Q(\sum_{j=m}^{n} \langle g_{s}, e_{j}\rangle q e_{j}, \sum_{j=m}^{n} \langle g_{s}, e_{j}\rangle q e_{j}) \, ds) \leq \theta_{2} E(\sum_{j=m}^{n} \langle g_{s}, e_{j}\rangle q e_{j})^{2} \, ds \quad (e_{j}'s \text{ are orthonormal})
\]

\[
\rightarrow 0 \quad \text{as } n, m \rightarrow \infty \quad \text{since } g \in M_{q}.
\]

Thus \( \int_{0}^{t} \langle g_{s}, dW_{s}\rangle q \) is an element of \( L^{2}(\Omega, F^{W}_{t}, P) \) defined as the \( L^{2}(\Omega) \)-limit of the Cauchy sequence \( \{\sum_{j=1}^{n} \int_{0}^{t} \langle g_{s}, e_{j}\rangle q \, dW_{s}[e_{j}] \}_{n \geq 1} \).

The next argument will also show that (4.2.2) is independent of the \( \text{CONS} \ \{e_{j}\}_{j=1}^{\infty} \) in \( H_{q} \). Let \( q_{1,2} \geq r_{1,2} q_{1} \geq q \) and \( \{\psi_{j}\}_{j=1}^{\infty} \) be a \( \text{CONS} \) for \( H_{q_{1}} \).

Then \( ||\cdot||_{r_{2}} \leq ||\cdot||_{r_{1}} \leq ||\cdot||_{q_{1}} \) \( H_{q} \subset H_{q_{1}} \subset H_{r_{2}} \) and by Theorem 4.1.1, \( W_{t} \) has an \( H_{q_{1}} \)-valued continuous version. Hence using Lemma 4.1.2, if \( g \in M_{q_{1}} \)

\[
E(\sum_{j=1}^{n} \int_{0}^{t} \langle g_{s}, e_{j}\rangle q \, dW_{s}[e_{j}] - \int_{0}^{t} \langle g_{s}, \psi_{j}\rangle q \, dW_{s}[\psi_{j}] )^{2} =
E(\int_{0}^{t} Q(\sum_{j=1}^{n} \langle g_{s}, e_{j}\rangle q e_{j} - \sum_{j=1}^{n} \langle g_{s}, \psi_{j}\rangle q_{1} e_{j}, \sum_{j=1}^{n} \langle g_{s}, e_{j}\rangle q e_{j} - \sum_{j=1}^{n} \langle g_{s}, \psi_{j}\rangle q_{1} e_{j}) \, ds) \leq \theta_{2} E(\int_{0}^{t} || \sum_{j=1}^{n} \langle g_{s}, \psi_{j}\rangle q_{1} e_{j} - \sum_{j=1}^{n} \langle g_{s}, e_{j}\rangle q e_{j} ||_{q_{1}}^{2} \, ds
\]
+ 0 as $n \to \infty$ by dominated convergence theorem since

$$\left\| \sum_{j=1}^{n} \langle g_s, \psi_j \rangle q_{1j} - \sum_{j=1}^{n} \langle g_s, e_j \rangle q \right\|^2_2$$

$$\leq 2 \left( \left\| \sum_{j=1}^{n} \langle g_s, \psi_j \rangle q_{1j} \right\|^2_2 + \left\| \sum_{j=1}^{n} \langle g_s, e_j \rangle q_j \right\|^2_2 \right)$$

$$\leq 2 \left( \left\| g_s \right\|^2_2 + \left\| g_s \right\|^2_2 \right) \text{ all } n \geq 1.$$ 

Hence the integral (4.2.2) is independent of $q$ and $q_1$.

The proof of (a) follows by the linearity property of the ordinary Itô integral and the proof of (b) follows since for each $j \geq 1$

$$E(\int_0^T <f_s, e_j>_q d\mathcal{W}_s[e_j]) = 0.$$ 

The proof of (c) follows by using Lemma 4.1.2:

$$E(\int_0^{t_1} <g_s, d\mathcal{W}_s>_q \int_0^{t_2} <f_s, d\mathcal{W}_s>_q) =$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} E(\int_0^{t_1} <g_s, e_j>_q d\mathcal{W}_s[e_j] \int_0^{t_2} <f_s, e_k>_q d\mathcal{W}_s[e_k]) =$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_0^{t_1 \wedge t_2} <g_s, e_j>_q <g_s, e_k>_q Q(e_j, e_k) ds =$$

$$E(\int_0^{t_1 \wedge t_2} Q(f_s, g_s) ds) \quad t_1, t_2 \geq 0.$$ 

Finally (d) is obtained by (c) and (4.1.21).

Q.E.D.

**Proposition 4.2.2** Let $f \in M_q$ for $q \geq r_1 + r_2$. Then the real valued process

$$\{\int_0^t <f_s, d\mathcal{W}_s>_q \}_{t \geq 0}$$

is an $F_t$-martingale with associated increasing process

$$E(\int_0^t Q(f_s, f_s) ds).$$

(4.2.4)

For $t \in T = [0, T_0], T_0 > 0$, it is a square integrable martingale with
a continuous modification.

Proof For \( t \geq 0 \) and \( \{ e_i \}_{i \geq 1} \) a CONS for \( H_q \), let

\[
Y_t = \sum_{j=1}^{\infty} \int_0^t <f_{s_j}, e_j>_q \, dW_s[e_j]
\]

and for \( n \geq 1 \) write

\[
Y^n_t = \sum_{j=1}^{n} \int_0^t <f_{s_j}, e_j>_q \, dW_s[e_j].
\]

In the proof of Proposition 4.2.1 we have shown that for each \( t > 0 \)

\[
Y^n_t \to Y_t
\]

in mean square. Next since for each \( i \geq 1 \)

\[
\int_0^t <f_{s_j}, e_j>_q \, dW_s[e_j]
\]

is an Itô integral, then for each \( n \geq 1 \)

\[
E(Y^n_s | F_s) = Y^n_s \quad \text{a.s.} \quad s < t
\]

and

\[
E(Y^n_s - E(Y^n_s | F_s))^2 = E(E(Y^n_s | F_s) - E(Y^n_s | F_s))^2
\]

\[
= E(E(Y^n_t - Y^n_t | F_s))^2 \leq E(Y^n_t - Y^n_t | F_s) = E(Y^n_t - Y^n_t)^2 \to 0, \quad n \to \infty.
\]

Thus for \( s < t \) \( E(Y^n_s | F_s) = Y_s \) a.s. and from Proposition 4.2.1(c) for

\( t \in T = [0, T_0], \quad (Y^n_t)_{t \in T} \)

is a square integrable martingale with increasing process \( E\int_0^t <f_s, f_s> \, ds \).

Next since each \( Y^n_t \) has a continuous modification, then for each \( n,m \geq 1 \)

\(|Y^n_t - Y^m_t|\)

is a continuous non-negative submartingale and by Doob's inequality

\[
E[\sup_{t \in T} |Y^n_t - Y^m_t|^2] \leq 4 \, E |Y^n_{T_0} - Y^m_{T_0}|^2 \to 0, \quad \text{as } n,m \to \infty.
\]

Hence there exists a subsequence \( \{Y^n_k\} \) that converges uniformly a.s. on \( T \)
to a continuous version of \( Y_t \).

Q.E.D.

Corollary 4.2.1 For \( q \geq r_1 + r_2 \) let \( f : [0, \infty) \times \Omega \to H_q \) be a non-anticipative

\( H_q \)-valued process such that
Define \( \int_0^\infty <f_s, dW_s>_q \) as the mean square limit of \( \int_0^t <f_s, dW_s>_q \) as \( t \to \infty \). Then this integral is well defined and has the properties (a)-(d) of Proposition 4.2.1 writing \( \infty \) instead of \( t \). Moreover, for all \( t > 0 \)

\[
E(\int_0^\infty <f_s, dW_s>_q | F_t) = \int_0^t <f_s, dW_s>_q \quad \text{a.s.}
\]

and \( (\int_0^t <f_s, dW_s>_q, F_t)_{t \geq 0} \) is a square integrable martingale with increasing process (4.2.4) and a continuous version on \( \mathbb{R}_+ \).

**Proof** First we observe that from (4.2.5) we obtain that \( f \in M_Q \) and therefore for each \( t > 0 \) the integral \( Y_t = \int_0^t <f_s, dW>_q \) is well defined. Next by (4.1.21) in Proposition 4.1.3 and (4.2.5)

\[
\int_0^\infty E(Q(f_s, f_s))ds < \infty
\]

and therefore from Proposition 4.2.4 \((Y_t, F_t)_{t \geq 0}\) is a square integrable martingale. Then \( Y_\infty = \int_0^\infty <f_s, dW>_q \) can be defined as the mean square limit of \( Y_t \) as \( t \to \infty \) and it is such that \( E(Y_\infty | F_t) = Y_t \) a.s. for each \( t > 0 \). The continuous version of \( Y_t \) is obtained as in the proof of Proposition 4.2.2 by writing \( Y_\infty \) instead of \( Y_t \).

Q.E.D.

We will see in Corollary 5.1.5 of Chapter V that the stochastic integrals defined in the above corollary are dense in the space of real valued nonlinear functionals of \( (W_t)_{t \geq 0} \).

Stochastic integrals for elements in \( M_Q \) For \( f \in M_Q \) a stochastic integral of the form (4.2.1) cannot be defined since \( (W_t)_{t \geq 0} \) is not an \( H_Q \)-valued process. However, we are still able to define a stochastic integral for \( f \in M_Q \) with the help of the following lemma.
Lemma 4.2.1 Let \( q \geq r_1 + r_2 \) and \( f \in \mathcal{M}_Q \). Then there exists a sequence 
\[ \{ f_n \}_{n \geq 1} \] in \( \mathcal{M}_Q \) such that for each \( t > 0 \)
\[ \int_0^t E\| f(s) - f_n(s) \|_Q^2 \, ds \to 0 \] as \( n \to \infty \).

Proof Recall that \( f \in \mathcal{M}_Q \) implies that \( f \) is non-anticipative and that for each \( t > 0 \)
\[ \int_0^t \| f(s) \|_Q^2 \, ds < \infty . \]

Next let \( \{ e_i \}_{i \geq 1} \) be a CONS for \( H_Q \) and let \( P_n \) be the orthogonal projector onto the span of \( \{ e_1, \ldots, e_n \} \). For each \( t > 0 \) by monotone convergence theorem
\[ \int_0^t E\| f_s \|_Q^2 \, ds = \sum_{j=1}^\infty \int_0^t E< f_s, e_j >_Q^2 \, ds \]
and hence for each \( t > 0 \)
\[ \int_0^t E\| P_n f_s - f_s \|_Q^2 \, ds = \sum_{j=n+1}^\infty \int_0^t E< f_s, e_j >_Q^2 \, ds + 0 \] as \( n \to \infty \).

Next for all \( n \geq 1 \) there exists a sequence \( \{ \hat{a}_k^n \}_{k \geq 1} \) of non-anticipative step processes with values in the range of \( P_n \) (this is the finite dimensional case, see for example Lemma 4.3.2 in Strook and Varadhan (1979) or Lemma 1.1 in Ikeda and Watanabe (1981)) such that for each \( t > 0 \)
\[ \int_0^t E\| \hat{a}_k^n(s) - P_n f_s \|_Q^2 \, ds < \frac{1}{k} \quad k=1,2,\ldots . \]

Define the \( H_Q \)-valued step process
\[ \alpha_n(t)(\omega) = \hat{a}_n(s)(\omega) \quad 0 \leq t < \infty \quad \omega \in \Omega \quad n \geq 1. \]

Then for all \( t > 0 \)
\[ \int_0^t E\| \alpha_n(s) - f_s \|_Q^2 \, ds \leq \int_0^t E\| \alpha_n(s) - P_n f_s \|_Q^2 \, ds \]
Thus we have shown that if \( f \in \mathcal{H}_Q^2 \), for all \( \varepsilon > 0 \) there exists an \( \mathcal{H}_Q \)-valued step process \( \alpha(t, \omega) \) such that for each \( t > 0 \)

\[
\int_0^t \mathbb{E} \| \alpha(s) - f(s) \|^2_Q \, ds < \varepsilon/4
\]

where

\[
\alpha(s, \omega) = \alpha_{t_j}(\omega) \quad \text{a.s.} \quad t_j \leq s < t_{j+1} \quad j = 0, \ldots, n-1
\]

\[
= \alpha_{t_n}(\omega) \quad \text{a.s.} \quad s \geq t_n
\]

where \( 0 = t_0 < t_1 < \ldots < t_n < \infty \) and each \( \alpha_{t_j} \) takes values in a finite dimensional subspace \( B_j \) of \( \mathcal{H}_Q^2 \), it is \( F_{t_j} \)-measurable and \( \mathbb{E} \| \alpha_{t_j} \|^2_Q < \infty \) for \( j = 1, \ldots, n \).

Next for each \( j = 1, \ldots, n \) let \( \{e_1^j, \ldots, e_{k_j}^j\} \) be an orthogonal basis for \( B_j \). Since \( H_q^2 \) is dense in \( \mathcal{H}_Q^2 \) we can choose \( \{\psi_1^j, \ldots, \psi_{k_j}^j\} \) such that \( \psi_{k_j}^j \in H_q^2 \) and

\[
\|\psi_{k_j}^j - e_{k_j}^j\|^2_Q < \frac{\varepsilon}{2k_j(t_{j+1} - t_j)\mathbb{E} \| \alpha_{t_j} \|^2_Q}
\]

\( k_j = 1, \ldots, k_j \).

Each \( \alpha_{t_j} \) can be written as

\[
\alpha_{t_j}(\omega) = a_1^j(\omega) e_1^j + \ldots + a_{k_j}^j(\omega) e_{k_j}^j
\]

where

\[
\mathbb{E} \| \alpha_{t_j} \|^2_Q = \mathbb{E}((a_1^j)^2 + \ldots + (a_{k_j}^j)^2) < \infty.
\]

Define

\[
\alpha^*_j(\omega) = a_1^j(\omega)\psi_1^j + \ldots + a_{k_j}^j(\omega)\psi_{k_j}^j
\]

then

\[
\mathbb{E} \| \alpha^*_j \|^2_Q \leq \mathbb{E}((a_1^j)^2 + \ldots + (a_{k_j}^j)^2) \sum_{i=1}^{k_j} \| \psi_i^j \|^2 < \infty
\]

and

\[
\mathbb{E} \| \alpha_{t_j} - \alpha^*_j \|^2_Q = \mathbb{E} \| \sum_{i=1}^{k_j} a_i^j(e_i^j - \psi_i^j) \|^2_Q
\]
\[ \leq \mathbb{E} \left( \sum_{i=1}^{k_j} \left| a_i \right| \left\| e_i^j - \psi_i^j \right\| \right)^2 \leq \mathbb{E} \left( \sum_{i=1}^{k_j} \left( a_i \right)^2 \right) \left( \sum_{i=1}^{k_j} \left\| e_i^j - \psi_i^j \right\|_Q^2 \right) < \frac{\varepsilon}{2(t_{j+1} - t_j)}. \]

Finally define
\[
\alpha^*(s, \omega) = \begin{cases} 
\alpha^*_j(\omega) & t_j \leq s < t_{j+1}, j = 1, \ldots, n-1 \\
\alpha^*_{n}(\omega) & s > t_n 
\end{cases}
\]
which is an element of \( M^Q \). Then for each \( f \in M^Q \) and \( \varepsilon > 0 \) there exists \( \alpha^* \in M^Q \) such that for each \( t > 0 \)
\[
\int_0^t \mathbb{E} \left\| \alpha^*(t) - f(t) \right\|_Q^2 \, dt < \varepsilon
\]
and the existence of the required sequence follows. Q.E.D.

**Definition 4.2.3** Let \( f \in M^Q \), then from Lemma 4.2.1 there exists a sequence of functions \( \{f_n\}_{n \geq 1} \) in \( M^Q \) for \( q \geq r_1 + r_2 \), such that for each \( t > 0 \)
\[
\int_0^t \mathbb{E} \left\| f(s) - f_n(s) \right\|_Q^2 \, ds \to 0 \quad \text{as } n \to \infty.
\]
Then by Proposition 4.2.1 for each \( t > 0 \)
\[
\mathbb{E} \left( \int_0^t \langle f_n(s) - f_m(s), dW_s \rangle_Q \right)^2 = \int_0^t \mathbb{E} \left\| f_n(s) - f_m(s) \right\|_Q^2 \, ds \to 0 \quad n, m \to \infty.
\]
Define for each \( t > 0 \) the stochastic integral \( \int_0^t \langle f_n(s), dW_s \rangle_Q \) as the \( L^2(\Omega) \)-limit of the Cauchy sequence \( \left\{ \int_0^t \langle f_n(s), dW_s \rangle_Q \right\}_{n \geq 1} \). This integral satisfies (a)-(d) of Proposition 4.2.1 and Proposition 4.2.2 for elements in \( M^Q \). If in addition \( f \) is such that
\[
\int_0^\infty \mathbb{E} \left\| f_s \right\|_Q^2 \, ds < \infty
\]
then the stochastic integral \( \int_0^\infty <f_s, dW_s>_Q \) is defined (as in Corollary 4.2.1) as the mean square limit of \( \int_0^t <f_s, dW_s>_Q \) as \( t \to \infty \).

The main properties of the above integral are summarized in the next corollary.

**Corollary 4.2.2** Let \( f, g \in \mathcal{M}_Q \). Then

a) If \( a, b \in \mathbb{R} \) and \( t > 0 \)

\[
\int_0^t <af_s + bg_s, dW_s>_Q = a \int_0^t <f_s, dW_s>_Q + b \int_0^t <g_s, dW_s>_Q \text{ a.s.}
\]

b) \( E(\int_0^t <f_s, dW_s>_Q) = 0 \) all \( t > 0 \).

c) \( E(\int_0^1 <f_s, dW_s>_Q \int_0^2 <g_s, dW_s>_Q) = E(\int_0^{t_1} <f_s, g_s>_Q ds \text{ } t_1, t_2 > 0) \).

d) \( E(\int_0^t <f_s, dW_s>_Q)^2 = E(\int_0^t ||f_s||_Q^2 ds < \infty) \).

e) If \( E(\int_0^\infty ||f_s||_Q^2 ds < \infty \) then \( \{\int_0^t <f_s, dW_s>_Q, F_t\}_{t \geq 0} \) is a square integrable martingale with corresponding increasing process

\[
\int_0^t E||f_s||_Q^2 ds
\]

and a continuous modification on \( \mathbb{R}_+ \). Moreover, for \( t > 0 \)

\[
E(\int_0^t <f_s, dW_s>_Q | F_t) = \int_0^t <f_s, dW_s>_Q \text{ a.s.}
\]

and

\[
E(\int_0^\infty <f_s, dW_s>_Q)^2 = E(\int_0^\infty ||f_s||_Q^2 ds).
\]

The proof follows by the above definition, Propositions 4.2.1 and 4.2.2 and Corollary 4.2.1.
4.2.2 **Stochastic integrals for operator valued processes (Φ'-valued stochastic integrals)**

Let \( L(\Phi',\Phi') \) denote the class of continuous linear operators from \( \Phi' \) to \( \Phi' \). In this section we study the stochastic integrals of \( L(\Phi',\Phi') \)-valued non-anticipative processes with respect to a \( \Phi' \)-valued Wiener process with c.p.d.b. form \( Q \) on \( \Phi \times \Phi \).

**Definition 4.2.4** A function \( f: [0,\infty) \times \Omega \rightarrow L(\Phi',\Phi') \) is said to belong to the class \( Q(\Phi',\Phi') \) if \( f \) is an \( F_t \)-adapted measurable (non-anticipative) function on \( [0,\infty) \times \Omega \) to \( L(\Phi',\Phi') \) such that for each \( t > 0 \)

\[
(4.2.9) \quad E \int_0^t Q(f_s^*(\phi), f_s^*(\phi)) \, ds < \infty \quad \forall \phi \in \Phi
\]

where \( f_s^* : \Phi' \times \Phi \rightarrow \Phi' \), the adjoint of \( f_s \), i.e. \( f^* \) is defined by the relation

\[
f(\psi)[\phi] = \psi[f^*(\phi)] \quad \psi \in \Phi', \phi \in \Phi.
\]

**Lemma 4.2.2** Let \( f \in Q(\Phi',\Phi') \). Then for each \( t > 0 \) there exists \( q_{t,f} \geq r_1 + r_2 \) such that

\[
E \int_0^t ||f_s||^2 \sigma_2(\mathcal{Q} t_f, H_Q) \, ds = E \int_0^t ||f_s||^2 \sigma_2(H_Q, \mathcal{Q} t_f, H_Q) \, ds < \infty
\]

where \( \sigma_2(H_Q, \mathcal{Q} t_f, H_Q) \) denotes the Hilbert space of Hilbert-Schmidt operators from \( H_Q \) to \( H_Q \).

**Proof** For each \( t > 0 \) and \( \phi \in \Phi \) let

\[
(4.2.10) \quad V_{t}^2(\phi) = E \int_0^t Q(f_s^*(\phi), f_s^*(\phi)) \, ds.
\]

Then since \( f \in Q(\Phi',\Phi') \) for each \( t > 0 \)

\[
(4.2.11) \quad V_{t}^2(\phi) < \infty \quad \forall \phi \in \Phi.
\]
We first show that for \( t > 0 \) \( V_t(\phi) \) is a continuous function on \( \phi \). Let \( \phi_n \to \phi \) in \( \phi \), then since \( f_s \in L(\phi, \phi) \) and \( Q \) is \( \phi \)-continuous, using Fatou's lemma we have that

\[
V_t(\phi) = \left\{ \liminf_{0}^{t} Q(f_s(\phi_n), f_s(\phi_n))ds \right\}^{\frac{1}{2}}
\]

\[
\leq \left\{ \liminf_{0}^{t} Q(f_s(\phi_n), f_s(\phi_n))ds \right\}^{\frac{1}{2}} = \liminf_{0}^{t} V_t(\phi_n)
\]

which shows that \( V_t \) is a lower semicontinuous function on \( \phi \). Applying the triangle inequality we obtain that for \( \phi, \psi \in \phi \) \( V_t(\phi + \psi) \leq V_t(\phi) + V_t(\psi) \) and clearly \( V_t(a\phi) = |a| V_t(\phi) \) for \( a \in \mathbb{R} \). Then by Lemma 4.1.1 \( V_t(\phi) \) is a continuous function on \( \phi \) and there exist \( V_{t,f} > 0 \) and \( \theta_{t,f} > 0 \) such that

\[
(4.2.12) \quad V_{t,f}^2(\phi) \leq \theta_{t,f}^2 \|\phi\|^2_{r_{t,f}} \quad V \phi \in \phi.
\]

Next let \( \{\phi_j\}_j \geq 1 \) and \( \{\lambda_j\}_j \geq 1 \) be as in Assumption 4.1.1. Choose \( q_{t,f} \geq r_{t,f} + r_1 \) and write \( \phi_j = (1+\lambda_j)^{-q_{t,f}} \phi_j \) \( j \geq 1 \). Then \( \{\phi_j\}_j \geq 1 \) is a CONS for \( H \) and using (4.2.10) and (4.2.12) we have that

\[
q_{t,f} \sum_{j=1}^{\infty} V_t(\phi_j) \leq \theta_{t,f} \sum_{j=1}^{\infty} \|\phi_j\|_{r_{t,f}}^2 = \theta_{t,f} \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(q_{t,f}^{-r_{t,f}})}
\]

\[
\leq \theta_{t,f} \theta_1 < \infty. \quad \text{Thus for each} \ t > 0
\]

\[
(4.2.13) \quad E \int_0^t \|f_s\|^2_{2(H_{t,f}, H_Q)}ds = E \int_0^t \sum_{j=1}^{\infty} Q(f_s(\phi_j), f_s(\phi_j))ds < \infty.
\]

Q.E.D.

**Proposition 4.2.3** Let \( f \in Q(\phi', \phi') \). Then for each \( t > 0 \) there exists a \( \phi' \)-valued element \( Y_t(f) \) such that
(4.2.14) \( Y_t(f)[\phi] = \int_0^t \langle f^s(\phi), dW_s \rangle \) a.s. \( \forall \phi \in \Phi \)

where the RHS of (4.2.14) is the stochastic integral of Definition 4.2.3.

Moreover, for each \( T_0 > 0 \) there exists a positive integer \( q_{T_0} \) such that

\( Y_t(f) \in H_{-q_{T_0}} \) a.s. for \( 0 \leq t \leq T_0 \). \( Y_t(f) \) is called the \( \Phi' \)-valued stochastic integral of \( f \) w.r.t. \( W \) and is denoted by

\[ Y_t(f) = \int_0^t f_s dW_s. \]

**Proof** We first note that for each \( t > 0 \) and \( \phi \in \Phi \int_0^t \langle f^s(\phi), dW_s \rangle \) is defined in the sense of Definition 4.2.3 since \( f^s(\phi) \) is non-anticipative and

\[ \int_0^t Q_s(f^s(\phi), f^s(\phi)) ds < \infty \]

i.e. \( f^s(\phi) \in M_Q \).

Using the notation of the proof of Lemma 4.2.2, define

\[ Y_t(f)[\tilde{\phi}_j] = \int_0^t \langle f^s(\tilde{\phi}_j), dW_s \rangle \]

Then by Corollary 4.2.2 (d), (4.2.10) and (4.2.12)

\[ E(\sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j])^2) = \sum_{j=1}^{\infty} E(Y_t(f)[\tilde{\phi}_j])^2 = \sum_{j=1}^{\infty} v^2(t_{\phi}) \]

\[ \leq \theta^2 E(\sum_{j=1}^{\infty} ||\phi_j||^2) \leq \theta^2 \theta_1 < \infty. \]

Thus \( \sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j])^2 < \infty \) a.s. Let

\[ \Omega_1 = \{ \omega: \sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j](\omega))^2 < \infty \} \text{ then } P(\Omega_1) = 1. \]

Let \( \{\psi_j\}_{j \geq 1} \) be the CONS for \( H_{-q_{T_0}} \) dual to \( \{\phi_j\}_{j \geq 1} \) and define
Then for each $t > 0$ \( \tilde{Y}_t(f) \in H_{q_t,f} \) a.s. for \( q_{t,f} \geq r_t, t + r_1 \) and therefore \( \tilde{Y}_t(f) \in \Phi' \) a.s.

It remains to prove that \( \tilde{Y}_t \) satisfies (4.2.14). Let \( t > 0 \) and \( \phi \in \Phi \), then \( \phi \in H_{q_t,f} \) and

\[
\phi = \lim_{n \to \infty} \sum_{j=1}^{n} \langle \phi, \tilde{\phi}_j \rangle q_{t,f} \quad \text{(limit in } H_{q_t,f})
\]

and therefore

\[
V_t \left( \sum_{j=1}^{n} \langle \phi, \tilde{\phi}_j \rangle q_{t,f} \tilde{\phi}_j - \phi \right) \to 0 \quad \text{as } n \to \infty
\]

which implies from (4.2.10) that

\[
E \int_0^T \left( \sum_{j=1}^{n} \langle \phi, \tilde{\phi}_j \rangle q_{t,f} \right) ds = 0 \quad \text{as } n,m \to \infty.
\]

On the other hand, since \( \psi_j[\phi] = \langle \phi, \tilde{\phi}_j \rangle q_{t,f} \) then

\[
\tilde{Y}_t(f)[\phi] = \sum_{j=1}^{\infty} Y_t[\tilde{\phi}_j] \psi_j[\phi] = \sum_{j=1}^{\infty} Y_t[\tilde{\phi}_j] \langle \phi, \tilde{\phi}_j \rangle q_{t,f}
\]

Thus if \( g_n(s) = \sum_{j=1}^{n} \langle \phi, \tilde{\phi}_j \rangle q_{t,f} \)

\[
\tilde{Y}_t(f)[\phi] = \lim_{n \to \infty} \int_0^T \langle g_n(s), dW_s \rangle Q \quad \text{a.s.}
\]

and from (4.2.16) and Definition 4.2.3

\[
\int_0^T \langle g_n(s), dW_s \rangle Q + \int_0^T \langle f^*(\phi), dW_s \rangle Q \quad \text{in } L^2(\Omega).
\]

Thus for each \( t > 0 \) \( \tilde{Y}_t(f)[\phi] = \int_0^T \langle f^*(\phi), dW_s \rangle Q \quad \text{a.s. } \forall \phi \in \Phi. \) From now on
We write $Y_t(f)$ instead of $\tilde{Y}_t(f)$.

Q.E.D.

The $\Phi'$-valued stochastic integral $Y_t(f) = \int_0^t f(s) dW_s$ has the following properties.

**Proposition 4.2.4** Let $f, g \in \mathcal{O}_Q(\Phi', \Phi')$.

a) If $a, b \in \mathbb{R}$ then for each $t > 0$

$$Y_t(af + bg) = aY_t(f) + bY_t(g) \text{ a.s.}$$

b) $E(Y_t(f)[\phi]) = 0$ $\forall \phi \in \Phi$ $t > 0$.

c) $E(Y_t(f)[\phi]Y_t(f)[\psi]) = E \int_0^t Q(f_s(\phi), f_s(\psi)) ds$ $\forall \phi, \psi \in \Phi$.

d) $E ||Y_t(f)||^2_{\Gamma_t,f} = E \int_0^t ||f_s||^2_{\Gamma_t,f} Q_s(\Phi', H_{-q_t,f}) ds < \infty$ $\forall t > 0$.

e) $(Y_t(f), F_t)_{t \geq 0}$ is a $\Phi'$-valued martingale. If $T = [0, T_0], T_0 > 0$ then

$(Y_t(f), F_t)_{t \in T}$ is a $\Phi'$-valued square integrable martingale with an $H_{-q_{T_0}}$ continuous version for some $q_{T_0} > 0$.

**Proof** (a), (b) and (c) follow from Proposition 4.2.3 and Corollary 4.2.2.

To prove (d) let $\{\phi_j = (1+\lambda_j)^{q_t,f}\phi_j\}_{j \geq 1}$ be a CONS for $H_{-q_t,f}$. Then using monotone convergence theorem, (4.1.8), (4.2.14) and (c) above we have that

$$E ||Y_t(f)||^2_{\Gamma_t,f} = E(\sum_{j=1}^{\infty} <Y_t(f), \phi_j>^2_{\Gamma_t,f}) = \sum_{j=1}^{\infty} E<Y_t(f), (1+\lambda_j)^{q_t,f}\phi_j>^2_{\Gamma_t,f}$$

$$= \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q_t,f}E(Y_t(f)[\phi_j])^2 = \sum_{j=1}^{\infty} (1+\lambda_j)^{-2q_t,f}E \int_0^t Q(\phi_j, \phi_j) ds$$

$$= E \int_0^t \sum_{j=1}^{\infty} Q(\phi_j, (1+\lambda_j)^{-q_t,f}\phi_j, (1+\lambda_j)^{-q_t,f}\phi_j) ds$$

$$= E \int_0^t ||\phi||^2_{\Gamma_t,f} Q_s(\Phi', H_{-q_t,f}) ds$$

$\{(1+\lambda_j)^{-q_t,f}\phi_j\}_{j \geq 1}$ is a CONS for $H_{-q_t,f}$.
\[ = E \int_0^t \| f_s \|^2 \sigma_s(\mathcal{H}_0^f, \mathcal{H}_t^f) ds < \infty \quad \text{(by Lemma 4.2.2)}. \]

e) Let \( V_t(\phi) \) be as in (4.2.10). Then for each \( \phi \in \Phi \), \( V_t(\phi) \) is a non-decreasing function of \( t \). Hence from (4.2.12) there exist \( \theta_{T_0,f} > 0 \) and \( r_{T_0,f} > 0 \) such that for \( \phi \in \Phi \)

\[ V_t^2(\phi) \leq \theta_{T_0,f}^2 \| \phi \| ^2 r_{T_0,f} \quad \forall \ t \in T = [0,T_0) \quad T_0 > 0. \]

Then for each \( t \in T \), \( V_t(\phi) \in \mathbb{H}_{-q_{T_0,f}}^f \) a.s. for \( q_{T_0,f} \geq r_{T_0,f} + r_1 \) (see proof of Proposition 4.2.3). Then using (4.2.14), from Corollary 4.2.2 we have that for each \( \phi \in \Phi \), \( (Y_t(\phi), F_t)_{t \geq 0} \) is a martingale, i.e. \( (Y_t(\phi), F_t)_{t \geq 0} \) is a \( \Phi' \)-valued martingale. Moreover, it is a square integrable martingale in \( T = [0,T_0] \) with associated increasing process

\[ E \int_0^t Q(f^*(\phi), f^*(\phi)) ds. \]

Next, using the notation as in the proof of Proposition 4.2.3

\[ Y_t(f) = \sum_{j=1}^n Y_t(\tilde{\phi}_j) \psi_j \quad \text{a.s.} \]

Therefore, from (4.2.14) and Corollary 4.2.2 for each \( j \geq 1 \), \( Y_t(\tilde{\phi}_j) \) has a continuous version on \( T = [0,T_0] \) and therefore for each \( n \geq 1 \)

\[ M_n(t) = \sum_{j=1}^n Y_t(\tilde{\phi}_j) \psi_j \]

is an \( \mathbb{H}_{-q_{T_0,f}}^f \)-valued martingale with a continuous version. Then using the usual argument, since \( \| M_n(t) - M_m(t) \|_{-q_{T_0,f}} \) is a continuous non-negative submartingale, by Doob's inequality

\[ E(\sup_{t \in T} \| M_n(t) - M_m(t) \|_{-q_{T_0,f}}^2) \leq 4 E \| M_n(T_0) - M_m(T_0) \|_{-q_{T_0,f}}^2. \]
Therefore, there exists a subsequence \( M'_{n_k} \) that converges on \( T = [0, T_0) \) uniformly to an \( \mathbb{H}^{-q_{T_0},f} \)-continuous version of \( Y_t(f) \).

Q.E.D.

We now extend the definition of \( Y_t(\cdot) \) to functions which are integrable in \([0, \infty) \times \Omega\). Lemma 4.2.2 and Proposition 4.2.4 (d) suggest that it is enough to construct stochastic integrals for functions of the form

\[
f: [0, \infty) \times \Omega \to \sigma_2(\mathbb{H}_q, \mathbb{H}_r)
\]

for \( r > 0 \) as we now do.

Let \( r > 0 \). A function \( f: [0, \infty) \times \Omega \to \sigma_2(\mathbb{H}_q, \mathbb{H}_r) \) is said to belong to the class \( \mathcal{O}(\mathbb{H}_q, \mathbb{H}_r) \) if \( f \) is an \( F_t \)-adapted measurable function on \( \mathbb{R} \times \Omega \) to \( \sigma_2(\mathbb{H}_q, \mathbb{H}_r) \) such that

\[
\int_0^\infty \mathbb{E} \left\| f_s \right\|_{\sigma_2(\mathbb{H}_q, \mathbb{H}_r)}^2 ds < \infty.
\]

(4.2.17)

Proposition 4.2.5 Let \( r \geq r_1 + r_2 \) and \( f \in \mathcal{O}(\mathbb{H}_q, \mathbb{H}_r) \). Then there exists an \( \mathbb{H}_r \)-valued element \( Y(f) \), called the stochastic integral for elements in \( \mathcal{O}(\mathbb{H}_q, \mathbb{H}_r) \), such that

\[
Y(f)[\phi] = \int_0^\infty f_s(\phi) d\mathbb{W}_s \quad \text{a.s.} \quad \forall \phi \in \mathbb{H}_r
\]

(4.2.18)

where the RHS is the stochastic integral of Definition 4.2.3. We denote this integral by
It has the following properties: If \( f, g \in \mathcal{O}(H_0'; H_{-r}) \)

a) For \( a, b \in \mathbb{R} \), \( Y(af + bg) = aY(f) + bY(g) \) a.s.

b) \( E(Y(f)[\phi]) = 0 \quad \forall \, \phi \in H_r \).

c) \( E(Y(f)[\phi]Y(g)[\psi]) = E \int_0^\infty Q(f^*_s(\phi), g^*_s(\psi)) ds \quad \phi, \psi \in H_r \).

d) \( E||Y(f)||_{-r}^2 = E \int_0^\infty ||f_s||_{\sigma_2(H_0'; H_{-r})}^2 ds < \infty \).

e) If \( Y_0(f) = \int_0^t f(s) dW_s \), then \( (Y_t(f), f^0_t) \) \( t \geq 0 \) is a \( \phi' \)-valued square integrable martingale with an \( H_{-r} \) continuous version.

**Proof**

Taking \( r_2 = r_{t,f} \) and \( r = q_{t,f} \) all \( t \geq 0 \) as in the proof of Proposition 4.2.3 one shows that for each \( t > 0 \) the stochastic integral \( Y_t(f) \in H_{-r} \) a.s. and it is such that

\[
Y_t(f)[\phi] = \int_0^t \langle f^*_s(\phi), dW_s \rangle \quad \text{a.s.} \quad \forall \, \phi \in H_r.
\]

Then using Proposition 4.2.4 (d) and (4.2.17)

\[
E||Y_{t'} - Y_t||_{-r}^2 = E \int_t^{t'} ||f_s||_{\sigma_2(H_0'; H_{-r})}^2 ds + 0 \quad \text{as } t' > t + \infty.
\]

Therefore there exists \( Y(f) = Y_\infty(f) \) with the required property (4.2.18).

From (4.2.18) and Corollary 4.2.9 (a), (b) and (c) are proved. The proofs of (d) and (e) are similar to the proofs of (d) and (e) in Proposition 4.2.4 writing \( r_2 = r_{T_0,f} \) and \( r = q_{T_0,f} \).

Q.E.D.
CHAPTER V

MULTIPLE WIENER INTEGRALS FOR A NUCLEAR
SPACE VALUED WIENER PROCESS

In this chapter we construct real valued (Section 5.1.1) and
\( \Phi' \)-valued (Section 5.2.1) multiple Wiener integrals with respect to a
\( \Phi' \)-valued Wiener process \( (W_t : t \in \mathbb{R}_+) \) with a continuous positive definite bi-
linear form \( Q \) on \( \Phi \times \Phi \), where \( \Phi \) is the countably Hilbert Nuclear space of
Section 4.1.1. We consider multiple Wiener integral expansions and stochas-
tic integral representations for real valued (Section 5.1.2) and
\( \Phi' \)-valued (Section 5.2.2) nonlinear functionals of \( W \). The Wiener decom-
position of the space of \( \Phi' \)-valued nonlinear functionals is obtained
(Theorem 5.2.2) as well as representation theorems for real valued (The-
orem 5.1.2) and \( \Phi' \)-valued (Theorem 5.2.4) square integrable martingales.

Throughout the chapter we will assume that \((\Omega, F, P)\) is a complete prob-
ability space on which there is defined a \( \Phi' \)-valued Wiener process \( (W_t : t \in \mathbb{R}_+) \)
with a c.p.d.b. form \( Q \) on \( \Phi \times \Phi \), and for \( t \geq 0 \), \( F_t = \sigma(W_s : 0 \leq s \leq t) \) with
\( F_0 \) containing all \( P \)-null sets of \( F \). Also we assume that \( \theta_1, r_1, \{ \phi_j \}_{j \geq 1},
\{ \lambda_j \}_{j \geq 1}, H_{r_2}, \) and \( \theta_2, r_2 \) are as in Assumptions 4.1.1 and 4.1.2.

5.1 Real valued multiple Wiener integrals

For \( n \geq 1 \) let \( \Phi^n \) denote the \( n \)-fold tensor product of \( \Phi \) (see Section
4.1.1). The aim of this section is to construct real valued multiple
Wiener integrals for \( \Phi^n \)-valued functions (Subsection 5.1.1) and then
use them to study real valued nonlinear functionals of $W$ (Subsection 5.1.2).

5.1.1 Multiple Wiener integrals for $\phi^n$-valued functions

Throughout this subsection we will assume, unless otherwise stated, that $n \geq 1$ and $T = [0, T_0], T_0 > 0$ are fixed but arbitrary. Denote by $L_Q(T^n \to \phi^n)$ the class of $\phi^n$-valued measurable functions $f$ on $T^n$ such that

$$\int_{T^n} Q^n \langle \xi(t), f(t) \rangle dt < \infty \quad t = (t_1, \ldots, t_n)$$

where $Q^n$ is the c.p.d.b. form on $\phi^n \times \phi^n$ which is the $n$th tensor product of $Q$ (see Section 4.1.1).

We shall define multiple Wiener integrals for elements in the class $L_Q(T^n \to \phi^n)$.

A useful concept in the theory of finite dimensional multiple Wiener integrals is that of symmetric real valued functions on $T^n$ (see Theorem 2.3.3 and (2.3.10) and (2.3.11)). We now introduce the analogous concept of symmetrization of $\phi^n$-valued and $K^n$-valued multivariate functions, where $K$ is a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$ and norm $\| \cdot \|_K$.

Definition 5.1.1. Let $n \geq 1$ and $f: \mathbb{R}^n_+ \to K^n$. Denote by $\tilde{f}$ the symmetrization of $f$ defined by

$$\tilde{f}(t) = \frac{1}{n!} \sum_{\Pi} f_{\Pi}(t)$$

where the sum is taken over all permutations $\Pi = (\Pi(1), \ldots, \Pi(n))$ of $(1, \ldots, n)$ and
(5.1.3) \[ f_{\mathbf{\Pi}}(t) = \sum_{j_1 \cdots j_n = 1}^{\infty} \langle f(t_{\Pi(1)}, \ldots, t_{\Pi(n)}), e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}} \rangle_{KN} \cdot e_{j_1} \otimes \cdots \otimes e_{j_n} \]

for \( \{e_i\}_{i \geq 1} \) a CONS of K.

**Proposition 5.1.1.** Let \((K_1, \langle \cdot, \cdot \rangle_{K_1})\) be another separable Hilbert space such that \( K \subseteq K_1 \). Then \( f_{\mathbf{\Pi}}(t) \) is well defined for all permutations \( \mathbf{\Pi} = (\Pi(1), \ldots, \Pi(n)) \) and therefore independent of the CONS \( \{e_i\}_{i \geq 1} \) in K.

Before giving the proof of the above proposition we shall define the symmetrization of a \( \phi_{\mathbf{\Omega}_\mathbf{n}} \)-function.

**Corollary 5.1.1.** Let \( n \geq 1 \) and \( f: \mathbb{R}^n \rightarrow \phi_{\mathbf{\Omega}_\mathbf{n}} \). Then the symmetrization \( \tilde{f} \) of \( f \) defined as the symmetrization of \( f \) on any of the Hilbert spaces \( H_{\mathbf{\Omega}_r} \), \( r \geq 1 \) is well defined.

The proof of the corollary follows since \( \phi_{\mathbf{\Omega}_n} = \bigcap_{r=1}^{\infty} H_{\mathbf{\Omega}_r} \) and \( H_{\mathbf{\Omega}_r} \supset H_{\mathbf{\Omega}_s} \) for \( s > r \) and by using Proposition 5.1.1.

**Proof of Proposition 5.1.1** Suppose \( K \subseteq K_1 \) and let \( \{\psi_i\}_{i \geq 1} \) and \( \{e_i\}_{i \geq 1} \) be CONS for \( K_1 \) and \( K \) respectively and assume that for each \( t \in \mathbb{R}^n_+ \) \( f(t) \) is \( K_{\mathbf{\Omega}_n} \) and \( K_{\mathbf{\Omega}_n} \)-valued. Then

\[ f^{K}_{\mathbf{\Pi}}(t) = \sum_{j_1 \cdots j_n = 1}^{\infty} \langle f(t_{\Pi(1)}), e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}} \rangle_{\mathbf{\Omega}_n} e_{j_1} \otimes \cdots \otimes e_{j_n} \]

and

\[ f^{K_1}_{\mathbf{\Pi}}(t) = \sum_{j_1 \cdots j_n = 1}^{\infty} \langle f(t_{\Pi(1)}), \psi_{j_{\Pi(1)}} \otimes \cdots \otimes \psi_{j_{\Pi(n)}} \rangle_{\mathbf{\Omega}_n} \psi_{j_1} \otimes \cdots \otimes \psi_{j_n} \]

Thus

\[ f^{K}_n(t) = \sum_{l_1 \cdots l_n = 1}^{\infty} \sum_{j_1 \cdots j_n = 1}^{\infty} \langle f(t_{\Pi(1)}), e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}} \rangle_{\mathbf{\Omega}_n} \cdot \langle \psi_{l_{\Pi(1)}}, \psi_{l_{\Pi(n)}} \rangle \cdot e_{j_{\Pi(1)}} \otimes \cdots \otimes e_{j_{\Pi(n)}} \rangle_{\mathbf{\Omega}_n} \cdot \psi_{l_1} \otimes \cdots \otimes \psi_{l_n} \]
But for all permutations $\Pi$

$$<\psi_1, \ldots, \psi_n, e_j \Pi(1), \ldots, e_j \Pi(n) \rangle_{\mathbb{K}_n} = <\psi_1, \ldots, \psi_n, e_j \Pi(1), \ldots, e_j \Pi(n) \rangle_{\mathbb{K}_n}$$

then

$$f_{\Pi}(t) = \sum_{j_1 \cdots j_n = 1}^{\infty} <f(t_1^n), e_j \Pi(1) \ldots e_j \Pi(n) \rangle_{\mathbb{K}_n} e_j \Pi(1) \ldots e_j \Pi(n)$$

The above argument proves that $f_{\Pi}(t)$ is independent of the CONS in $\mathbb{K}$.

Q.E.D.

For $T = [0, T_0]$, $T_0 > 0$ or $T = \mathbb{R}_+$ we denote by $L^2(T^n + \mathbb{K}_n)$ the Hilbert space of $\mathbb{K}_n$-valued measurable functions on $T^n$ such that

$$\int_{T^n} \| f(t) \|_{\mathbb{K}_n}^2 \, dt < \infty.$$

Proposition 5.1.2. Let $T = [0, T_0]$, $T_0 > 0$ or $T = \mathbb{R}_+$. For $f \in L^2(T^n + \mathbb{K}_n)$ let

$$(Sf)(t) = \tilde{f}(t) = \frac{1}{n!} \sum_{\Pi} f_{\Pi}(t).$$

Then $S$ is an orthogonal projection operator on $L^2(T^n + \mathbb{K}_n)$ whose range $S$ may be identified with the n-fold symmetric tensor product space $(L^2(T) \otimes \mathbb{K})^n$.

Proof. It is known (Reed and Simon (1980)) that $L^2(T^n + \mathbb{K}_n) \cong L^2(T^n) \otimes \mathbb{K}_n \cong (L^2(T) \otimes \mathbb{K})^n$. Then it is enough to consider functions of the form

$$f(t) = f(t_1, \ldots, t_n) = f_{i_1} (t_1) \cdots f_{i_n} (t_n) e_{i_1} \cdots e_{i_n}$$

where $\{f_{i_j}\}_{i \geq 1}$ and $\{e_{i_k}\}_{i \geq 1}$ are CONS for $L^2(T)$ and $\mathbb{K}$ respectively. Then since
for all \( \Pi \)

\[
\mathbf{f}_\Pi(t) = f_{i_1}(t) \cdot \ldots \cdot f_{i_n}(t^n) \sum_{j_1 \ldots j_n = 1}^{\infty} \langle e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle \, e_{j_1} \otimes \ldots \otimes e_{j_n}
\]

\[
= \frac{1}{n!} \sum_{\Pi^*} \langle f_{i_1}(t), e_{j_1} \rangle \otimes \ldots \otimes \langle f_{i_n}(t^n), e_{j_n} \rangle
\]

is an element of \( (L^2(T) \otimes K)^n \).

Next for each permutation \( \Pi \), from (5.1.3) it is seen that the operator

\[
\mathcal{D}_\Pi(f) = \mathbf{f}_\Pi
\]

is linear with adjoint equal to its inverse. Then

\[
P^* = \frac{1}{n!} \sum_{\Pi} \mathcal{D}_\Pi^* = \frac{1}{n!} \sum_{\Pi} \mathcal{D}_\Pi^{-1} = P.
\]

Also from (5.1.3) we have that for each permutation \( \Pi \)

\[
(P(f))_{\Pi} = \frac{1}{n!} \sum_{\Pi^*} \langle f_{\Pi^*}(t), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle
\]

where the first sum is taken over all permutations \( \Pi^* \) of \( \Pi \). Then for all permutations \( \Pi \) of \( (1, \ldots, n) \) \( (Pf)_\Pi = (Pf) \) and hence

\[
P^2(f) = P(P(f)) = \frac{1}{n!} \sum_{\Pi} (Pf)_\Pi = Pf.
\]

Thus \( P \) is a projection operator whose range can be identified with the symmetric tensor product space \( (L^2(T) \otimes K)^n \).

Q.E.D.

**Corollary 5.1.2**  Let \( T, K \) and \( f \) be as in the last proposition. Then

\[
\int_T \| \tilde{f}(t) \|_K^2 \, dt \leq \int_T \| f(t) \|_K^2 \, dt.
\]

The proof follows by using the fact that \( P f = \tilde{f} \) is a projection operator.
Remark Multiple Wiener integrals on a Hilbert space have been defined by Miyahara (1981) for the case of a cylindrical Brownian motion on $H_0$. In his case $Q(\cdot, \cdot) = \langle \cdot, \cdot \rangle_0$ and the CONS \( \{\phi_j\}_{j \geq 1} \) of eigenvectors of $L$ diagonalizes $Q$ (see Example 4.1.7). In our case we do not consider a cylindrical Brownian motion but rather a $\Phi'$-valued Wiener process with an $H_{-q}$ continuous version for $q \geq r_1 + r_2$. Moreover, the c.p.d.b. form $Q$ on $\Phi \times \Phi$ is not assumed to be diagonalized by $\{\phi_j\}_{j \geq 1}$. This leads to finite dimensional multiple integrals with dependent integrators of the type studied in Chapters II and III (see also Lemma 4.1.5).

In order to define real valued multiple Wiener integrals for elements in $L_Q(T^n + \Phi^m_\infty)$ we shall first construct multiple integrals for $H^m_q$-valued functions for $q \geq r_1 + r_2$ (as in the case of stochastic integrals in Chapter IV) and then apply density arguments to define them on the spaces $L_Q(T^n + \Phi^m_\infty)$ and $L^2(T^n + H^m_\infty)$.

Multiple Wiener integrals for elements in $L^2(T^n + H^m_q)$

Definition 5.1.2 Let $n \geq 1$ fixed and $q \geq r_1 + r_2$. For $f \in L^2(T^n + H^m_q)$ define the real valued multiple Wiener integral of $f$ with respect to the $\Phi'$-valued Wiener process $W_t$ by

\[
I_{n,T}(f) = \sum_{j_1, \ldots, j_n = 1}^{\infty} \int_{T^n} \langle f(t), e_{j_1} \otimes \ldots \otimes e_{j_n} \rangle_{H^m_q} d \Phi^m W[e_{j_i}](t)
\]

where $\{e_{j_i}\}_{i \geq 1}$ is a CONS for $H_q$ and each multiple integral in the RHS of (5.1.4) is an integral with respect to the symmetric tensor product measure $d \Phi^m W[e_{j_i}]$ defined in Sections 2.3 and 3.1 (see also Lemma 4.1.5).

Proposition 5.1.3 Let $H$ be the linear (Hilbert) space of $(W_t)_{t \in \mathbb{R}_+}$ defined...
in (4.1.25). Let \( n \geq 1, q \geq r_1 + r_2 \) and \( f \in L^2(T^n + H^q) \). Then the multiple integral (5.1.4) is a well defined element in \( H^q \) (the \( n \)-fold symmetric tensor product of \( H \)) and of \( L^2(\Omega, F^W, P) \).

**Proof**  Let \( \{e_i\}_{i \geq 1} \) be a CONS for \( H^q \). Then for each \( j_1, \ldots, j_n \)

\[
\int_{T^n} \langle f(t), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle^2 \, dt \leq \int_{T^n} \| f(t) \|^2 \, q_{j_1} \otimes \cdots \otimes q_{j_n} \, dt < \infty
\]

and by Lemma 4.1.5 and Theorem 2.3.2 \( \langle f(t), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle \) is \( W[\{e_i\}] \)-integrable and each integral in the RHS of (5.1.4) is an element in \( H^q \).

Next by Theorem 2.3.3 (b)

(5.1.5)  \[
\mathbb{E} \left( \int_{T^n} \langle f(t), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle \, q_{j_1} \otimes \cdots \otimes q_{j_n} \, dt \right)^2
\]

\[
\leq \int_{T^n} \langle f(t), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle^2 \, q_{j_1} \otimes \cdots \otimes q_{j_n} \, Q(e_{j_1}, e_{j_1}) \cdots Q(e_{j_n}, e_{j_n}) \, dt
\]

\[
= \int_{T^n} Q(q_{j_1}) \langle f(t), e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle \, q_{j_1} \otimes \cdots \otimes q_{j_n} \, e_{j_1} \otimes \cdots \otimes e_{j_n} \, dt
\]

Then if \( \{e_j = (1+\lambda_j)^{-q} \} \) is the CONS in \( H^q \), by applying Cauchy-Schwartz inequality and (5.1.5) above we have

(5.1.6)  \[
\mathbb{E}(I_{n, f})^2 \leq \left\{ \sum_{j_1 \ldots j_n = 1}^{\infty} (1+\lambda_{j_1}) \cdots (1+\lambda_{j_n}) \right\}^{-2(q-r_2)} \cdot \cdot \cdot (1+\lambda_{j_n})^{-2(q-r_2)}
\]

\[
\sum_{j_1 \ldots j_n = 1}^{\infty} (1+\lambda_{j_1}) \cdots (1+\lambda_{j_n})^{2r_1} \cdots (1+\lambda_{j_n})^{2r_2}
\]

\[
\mathbb{E} \left( \int_{T^n} \langle f(t), \phi_{j_1} \otimes \cdots \otimes \phi_{j_n} \rangle \, q_{j_1} \otimes \cdots \otimes q_{j_n} \, dt \right)^2
\]

\[
\leq \sum_{j_1 \ldots j_n = 1}^{\infty} \Pi_{i=1}^{n} (1+\lambda_{j_i})^{2r_1} \int_{T^n} Q(q_{j_1}) \langle f(t), \phi_{j_1} \otimes \cdots \otimes \phi_{j_n} \rangle \, e_{j_1} \otimes \cdots \otimes e_{j_n} \, dt
\]

\[
\langle f(t), \phi_{j_1} \otimes \cdots \otimes \phi_{j_n} \rangle \, q_{j_1} \otimes \cdots \otimes q_{j_n} \, e_{j_1} \otimes \cdots \otimes e_{j_n} \, dt
\]
(Proposition 4.1.2)

\[
\sum_{j_1 \ldots j_n = 1}^n \prod (1 + \lambda_{j_i}) \| e_{j_1} \ldots e_{j_n} \|^2 \int_T \langle f(t), \phi_{j_1} \ldots \phi_{j_n} \rangle^2 d\phi_{j_n} q_n \]

\[
= \sum_{j_1 \ldots j_n = 1}^n \int_T \langle f(t), e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n \quad (\| e_{j_1} \ldots e_{j_n} \| = 1)
\]

\[
= \sum_{j_1 \ldots j_n = 1}^n \int_T \| f(t) \|^2 d\phi_{j_n} q_n < \infty.
\]

Therefore \( E(I_n, T(f))^2 < \infty \) and the multiple series (5.1.4) converges in mean square. Then the linearity of \( I_n, T(\cdot) \) follows.

The next step will also show that \( I_n, T(\cdot) \) is independent of the choice of the CONS in \( H_q \).

Let \( q_1 = r_1 + r_2 \), \( q = q_1 \) and assume that \( f \) belongs to \( L^2(T^n, H_q) \) and \( \{ \psi_i \}_{i=1}^m \) is a CONS in \( H_{q_1} \). Then using Lemma 4.1.5, the bilinearity of \( Q_{on}^{\phi_{j_1}} \) in \( H_q \times H_q \) and Corollary 5.1.2, we have that for all \( m \geq 1 \)

\[
E( \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n \quad i=1 \quad (e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n )^2
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), \psi_{j_1} \ldots \psi_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), \psi_{j_1} \ldots \psi_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), \psi_{j_1} \ldots \psi_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), \psi_{j_1} \ldots \psi_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), e_{j_1} \ldots e_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]

\[
= \sum_{j_1 \ldots j_n = 1}^m \int_T \langle f(t), \psi_{j_1} \ldots \psi_{j_n} \rangle^2 d\phi_{j_n} q_n e_{j_1} \ldots e_{j_n} q_n e_{j_1} \ldots e_{j_n}
\]
\[ \leq c^2 \int_T \left| \sum_{j_1 \cdots j_n=1}^m \langle f(t), e_{j_1} \cdots e_{j_n} \rangle \right|^2 dt \]

which goes to zero as \( m \to \infty \) by dominated convergence theorem since

\[ \left\| \sum_{j_1 \cdots j_n=1}^m \langle f(t), e_{j_1} \cdots e_{j_n} \rangle \right\|^2 \leq 2\left( \|f(t)\| \|e_{j_1} \cdots e_{j_n}\|_q \right)^2 \quad \text{all } m \geq 1. \]

Then the proof of the proposition is complete. Q.E.D.

The multiple integral \( I_{n,T}(\cdot) \) has several properties analogous to those of the multiple Wiener integral for a real valued Wiener process of Itô (1951). We now present them.

**Proposition 5.1.4** Let \( n \geq 1, q \geq r_1 + r_2 \) and \( f \in L^2(T^n_{\mathbb{H}^q}) \). Then if \( \tilde{f} \) denotes the symmetrization of \( f \) (Definition 5.1.1).

a) \( I_{n,T}(\tilde{f}) = I_{n,T}(f) \).

b) \( E(I_{n,T}(f)) = 0 \).

c) \( E(I_{n,T}(f))^2 \leq n! \delta_2^n \|f\|_2^2 L^2(T^n_{\mathbb{H}^q}) < \infty. \)

d) If \( g \in L^2(T^n_{\mathbb{H}^q}) \) for \( m \geq 1 \), then

\[ E(I_{n,T}(f)I_{m,T}(g)) = \delta_{nm} n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(T^n_{\mathbb{H}^q})}. \]

e) \( E(I_{n,T}(f))^2 = E(I_{n,T}(\tilde{f}))^2 = n! \left\| \tilde{f} \right\|_2^2 L^2(T^n_{\mathbb{H}^q}) \leq n! \left\| f \right\|_2^2 L^2(T^n_{\mathbb{H}^q}). \)
Proof  

a) Since for all $j_1 \ldots j_n$ and $t \in T$

\begin{equation}
\langle \tilde{f}(t), e_{j_1} \ldots e_{j_n} \rangle = \frac{1}{n!} \sum_{j_1 \ldots j_n = 1}^{\infty} \prod_{k=1}^{n} \langle e_{j_k}, e_{j_{k+1}} \rangle \langle e_{j_{k+2}} \ldots e_{j_n} \rangle \langle e_{j_1} \rangle
\end{equation}

\begin{equation}
\langle f(t), e_{j_1} \ldots e_{j_n} \rangle = \frac{1}{n!} \sum_{j_1 \ldots j_n = 1}^{\infty} \prod_{k=1}^{n} \langle e_{j_k}, e_{j_{k+1}} \rangle \langle e_{j_{k+2}} \ldots e_{j_n} \rangle \langle e_{j_1} \rangle
\end{equation}

it follows from (5.1.4) that $I_{n,T}(\tilde{f}) = I_{n,T}(f)$.

b) By Lemma 4.1.5, for each $j_1 \ldots j_n$

\begin{equation}
E(\int_{T^n} f(t), e_{j_1} \ldots e_{j_n} \rangle \prod_{i=1}^{n} \langle e_{j_i} \rangle d \circ W[e_{j_i}]{(t)}) = 0
\end{equation}

and hence $E(I_{n,T}(f)) = 0$.

d)  

\begin{equation}
E(I_{n,T}(f)I_{m,T}(g)) = \sum_{j_1 \ldots j_n = 1}^{\infty} \sum_{k_1 \ldots k_n = 1}^{\infty} \prod_{j=1}^{n} \langle e_{j_k}, e_{j_{k+1}} \rangle \langle e_{j_{k+2}} \ldots e_{j_n} \rangle \langle e_{j_1} \rangle
\end{equation}

Then by Lemma 4.1.5 $E(I_{n,T}(f)I_{m,T}(g)) = 0$ if $n \neq m$ and if $n = m$

\begin{equation}
E(\int_{T^n} f(t), e_{j_1} \ldots e_{j_n} \rangle \prod_{i=1}^{n} \langle e_{j_i} \rangle d \circ W[e_{j_i}]{(t)}) = 0
\end{equation}

\begin{equation}
\int_{T^n} g(t), e_{k_1} \ldots e_{k_n} \rangle \prod_{i=1}^{m} \langle e_{k_i} \rangle d \circ W[e_{k_i}]{(t)})
\end{equation}

\begin{equation}
= \int_{T^n} \frac{1}{n!} \prod_{i=1}^{n} \langle f(t), e_{j_i} \rangle \langle e_{j_{i+1}} \rangle \langle e_{j_{i+2}} \ldots e_{j_n} \rangle \langle e_{j_1} \rangle d t
\end{equation}
Therefore, using the continuity of $Q^m$ on $H^m_q \times H^m_q$

$$E(I_n, T(f) I_n, T(g)) = \int_{T^n} Q^{\otimes n} (\tilde{f}(t), \tilde{g}(t)) \, dt$$

$$= \langle \tilde{f}, \tilde{g} \rangle_{L^2(T^n \to H^m_Q)}$$

and (d) is proved.

The proof of (e) follows from (d), (a) and Corollary 5.1.2. The proof of (c) follows from (d) and since for $\psi \in H^m_q$

$$|| \psi ||^2_{Q^m} \leq \sigma^m \frac{2}{\eta^m} \frac{|| \psi ||^2_{Q^m}}{q^m}$$

\[ q \geq r_1 + r_2. \]

Q.E.D.

We will extend the definition of $I_n, T(\cdot)$ to functions in $L^2(\mathbb{R}^n_+ \to H^m_q)$, denoted by $I_n$, and show (Corollary 5.1.4) that orthogonal series of $I_n$ are dense in $L^2(\Omega, F^W, P)$.

**Multiple Wiener integrals for elements in $L^2(T^n \to H^m_Q)$**

**Lemma 5.1.1** Let $f \in L^2(T^n \to H^m_Q)$, then for all $q \geq r_1 + r_2$ there exists a sequence $\{f_m\}_{m \geq 1}$, $f_m \in L^2(T^n \to H^m_Q)$, $m \geq 1$, such that $f_m \to f$ in $L^2(T^n \to H^m_Q)$.

**Proof** Let $f \in L^2(T^n \to H^m_Q)$, then given $\epsilon > 0$ there exists an $H^m_Q$-valued step function $g^\epsilon$ such that

$$\int_{T^n} \left\| f(t) - g^\epsilon(t) \right\|^2_{Q^m} \, dt < \epsilon/2$$

where $g^\epsilon(t) = \sum_{i=1}^{k} a_i^\epsilon(t)$, $a_i^\epsilon \in H^m_Q$, $A_i \in \mathcal{B}(T^n)$ and $m^\otimes(A_i) < \infty$ for $i = 1, \ldots, k$.

Next, since for $q \geq r_1 + r_2$ $H^m_Q$ is dense in $H^m_Q$, then there exist $b_i \in H^m_Q$ such that
Define \( f^e(t) = \sum_{i=1}^{k} b_i \mathbb{1}_{A_i}(t) \), then \( f^e \in L^2(T^n + H_q^\infty) \) and

\[
\int_{T^n} \| f^e(t) - g^e(t) \|_{Q^\infty}^2 dt < \epsilon/2.
\]

Then from the last expression and (5.1.10)

\[
(5.1.11) \quad \int_{T^n} \| f(t) - f^e(t) \|_{Q^\infty}^2 dt < \epsilon
\]

and the existence of the required sequence follows.

Q.E.D.

Note that the above result holds if \( T = \mathbb{R}_+ \).

**Definition 5.1.3** By Propositions 5.1.3 and 5.1.4 we have that \( I_{n,T} \) defined in (5.1.4) is a bounded linear operator from \( L^2(T^n + H_q^\infty) \) to \( L^2(\Omega) \) (in fact to \( H_q^\infty \)). Hence by Proposition 5.1.4 (e) and Lemma 5.1.1, \( I_{n,T} \) has a unique extension to \( L^2(T^n + H_q^\infty) \). We denote this extension by \( I_{n,T} \) and call it the \( n^{th} \) real valued multiple Wiener integral for elements in \( L^2(T^n + H_q^\infty) \).

We summarize the main properties of \( I_{n,T} \) in the following lemma whose proof follows by Proposition 5.1.3 and 5.1.4 and the above definition.

**Lemma 5.1.2** Let \( f \in L^2(T^n + H_q^\infty) \). Then if \( \tilde{f} \) denotes the symmetrization of \( f \) on \( H_q^\infty \) (see Definition 5.1.1)

- a) \( I_{n,T}(\tilde{f}) = I_{n,T}(f) \in H_q^\infty \).
- b) \( E(I_{n,T}(f)) = 0 \).
- c) If \( g \in L^2(T^m + H_q^m) \) for \( m \geq 1 \) then

  \[
  E(I_{n,T}(f)I_{n,T}(g)) = \delta_{nm} n! \int_{T^n} Q^\infty(\tilde{f}(t),\tilde{g}(t)) dt.
  \]

- d) \( E(I_{n,T}(f)I_{n,T}(f)) \leq n! \int_{T^n} Q^\infty(\tilde{f}(t),\tilde{f}(t)) dt \leq n! \int_{T^n} Q^\infty(f(t),f(t)) dt \).
The proof follows by Definition 5.1.3 and Propositions 5.1.3 and 5.1.4.

**Iterated stochastic integrals** We now define the real valued iterated stochastic integral $J_n(\cdot)$ for elements in $L^2(\mathbb{R}^n + H_q^\otimes n)$, $q \geq r_1 + r_2$, and show its relationship with the multiple Wiener integral $I_{n, T}(\cdot)$, $T = [0, t]$, $t \geq 0$, of Definition 5.1.2 for elements in $L^2(T^n + H_q^\otimes n)$. This connection will allow us to extend the definition of $I_{n, T}(\cdot)$ to functions in the space $L^2(\mathbb{R}^n + H_q^\otimes n)$.

In what follows $L^2(\mathbb{R}^n_+) = L^2(\mathbb{R}^n_+ B(\mathbb{R}^n), dt)$ where $dt$ stands for the Lebesgue measure on $(\mathbb{R}^n_+, B(\mathbb{R}^n))$.

Let $\{f_k\}_{k \geq 1}$ and $\{e_k\}_{k \geq 1}$ be complete orthonormal sets for $L^2(\mathbb{R}_+)$ and $H_q$ respectively where $q \geq r_1 + r_2$. Then (see Reed and Simon (1980))

\[
\begin{align*}
\{f_{k_1}(t_1) \ldots f_{k_n}(t_n)e_{k_1} \otimes \ldots \otimes e_{k_n} \mid k_1 \geq 1, \ldots, k_n \geq 1\} \\
\end{align*}
\]

is a CONS for $L^2(\mathbb{R}^n_+) \otimes H_q^\otimes n \approx L^2(T^n + H_q^\otimes n)$.

For $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$ and $n$ fixed, let

\[
\begin{align*}
(f(t) = (f_{k_1}(t_1)e_{k_1}) \otimes \ldots \otimes (f_{k_n}(t_n)e_{k_n}) \quad t = (t_1, \ldots, t_n)
\end{align*}
\]

and for $t > 0$ define

\[
\begin{align*}
J_0(f)_{t_1} &= f_{k_1}(t_1)e_{k_1} \\
J_{m-1}(f)_{t_m} &= \int_{0}^{t} J_{m-2}(f)_{t_{m-1}}, dW_{t_{m-1}} > f_{k_m}(t_m)e_{k_m} \quad 2 \leq m \leq n \\
J_{n, t}(f) &= \int_{0}^{t} J_{n-1}(f)_{t_{n}}, dW_{t_{n}} >
\end{align*}
\]

where the stochastic integrals of the RHS are in the sense of Definition 4.2.2. If $f$ and $g$ are two $H_q^\otimes n$-valued functions as in (5.1.13) define

\[
J_{n, t}(f+g) = J_{n, t}(f) + J_{n, t}(g)
\]

Then $J_{n, t}$ is extended to the linear mani-
fold \( S_n \) generated by \( H^m \)-valued functions of the form (5.1.13).

**Proposition 5.1.5** If \( f \in S_n \), then for each \( t > 0 \) \( J_{n,t}(f) \) is well defined and

\[
E(J_{n,t}(f))^2 < \int_{\mathbb{R}^n} \| f(t) \|_Q^2 \, dt < \infty.
\]

**Proof** By the preceding paragraph to this proposition, it is enough to prove the result for functions of the form (5.1.13). We first show that for \( m = 1, \ldots, n \) \( J_{m-1}(f) t_m \) belongs to \( M_q \) (see Definitions 4.2.1 and 4.2.2). Using Fubini's theorem

\[
\int_{\mathbb{R}^n} \left\{ \prod_{i=1}^n \left\| f_{k_i} \left(t_i, e_i, k_i \right) \right\|_Q^2 dt \right\} = \int_{\mathbb{R}^n} \left\{ \prod_{i=1}^n \left\| f_{k_i} \left(t_i, e_i, k_i \right) \right\|_Q^2 dt \right\} < \infty
\]

and therefore for each \( 1 \leq m \leq n \)

\[
\int_{\mathbb{R}^m} \prod_{i=1}^m \left\{ \prod_{i=1}^m \left\| f_{k_i} \left(t_i, e_i, k_i \right) \right\|_Q^2 dt \right\} < \infty \quad a.e. \quad dt_{m+1} \ldots dt_n.
\]

Then \( J_{m}(f) t_m \) is \( H_q \)-valued and from Propositions 4.2.1 and 4.2.2 it is non-anticipative such that for each \( t_3 > 0 \)

\[
E \int_{\mathbb{R}^3} \left\{ J_{m}(f) t_m \right\}^2 \, dt < \infty.
\]

Proceeding in the same way we have that for \( 2 \leq m < n \), \( J_{m-1}(f) t_m \) is \( H_q \)-valued, non-anticipative and

\[
E \int_{\mathbb{R}^m} \left\{ J_{m-1}(f) t_m \right\}^2 \, dt = \int_{\mathbb{R}^m} \left\{ \prod_{i=1}^m \left\| f_{k_i} \left(t_i, e_i, k_i \right) \right\|_Q^2 dt \right\} < \infty,
\]

i.e., \( J_{m-1}(f) t_m \) is \( M_q \) and hence the stochastic integral \( \int_{\mathbb{R}^m} \left\{ J_{m-1}(f) t_m \right\}^2 \, dt < \infty \).
can be defined as in Definition 4.2.2. Thus for each $t > 0$ $J_{n,t}(f)$ is well defined and

$$E(J_{n,t}(f))^2 = \int_0^t E\left\| J_{n-1}(f) \right\|_{H_Q}^2 dt = \int_0^t \int_0^{n-1} \int_0^n \prod_{i=1}^n \left\| f_{\xi_i}(t_i)e_{k_i} \right\|_{Q}^2 dt_i \ldots dt_n \leq \int_{\mathbb{R}_+^n} \left\| f(u) \right\|_{\mathbb{Q}}^2 du \leq \int_{\mathbb{R}_+^n} \left\| f(u) \right\|_{\mathbb{Q}}^2 du$$

Q.E.D.

**Definition 5.1.4** Since elements of the form (5.1.13) generate $L^2(\mathbb{R}_+^{n} \otimes \mathbb{Q})$ and from Proposition 5.1.5 for each $t > 0$ $J_{n,t}(\cdot)$ is a bounded linear transformation from $S_n$ to $L^2(\mathbb{R}_+^{n} \otimes \mathbb{Q})$, then it can be extended to $L^2(\mathbb{R}_+^{n} \otimes \mathbb{Q})$. This extension is also denoted by $J_{n,t}(\cdot)$ and called the $n$th iterated stochastic integral. Moreover, using Lemma 5.1.1 $J_{n,t}(\cdot)$ is also extended to $L^2(\mathbb{R}_+^{n} \otimes \mathbb{Q})$.

We now present the relation between $J_{n,t}(\cdot)$ and $I_{n,t}(\cdot)$.

**Proposition 5.1.6** Let $q \geq r_1 + r_2$ and $n \geq 1$. For any $t > 0$ let $T = [0,t]$ and $I_{n,T}(\cdot)$ be the multiple Wiener integral of Definition 5.1.2 for elements in $L^2(T^n \otimes \mathbb{Q})$. Let $f \in L^2(\mathbb{R}_+^{n} \otimes \mathbb{Q})$, then for each $t > 0$ if $f$ is restricted to $T = [0,t]$

(5.1.16) \[ I_{n,T}(f) = I_{n,T}(\tilde{f}) = n! J_{n,t}(\tilde{f}) \]

and

(5.1.17) \[ I_{n,T}(f) = \int_0^t <g(s),dW_s> \]

where $g \in \mathcal{M}_Q$ and the RHS of (5.1.17) is the stochastic integral of Definition 4.2.3. Moreover, $E \int_0^t \|g(s)\|_{Q}^2 ds < \infty$.

**Proof** Let $\{f_k\}_{k \geq 1}$ and $\{e_k\}_{k \geq 1}$ be CONS for $L^2(\mathbb{R}_+)$ and $\mathbb{H}$ respectively.
and \( f(t) \) be as in (5.1.3). Then from Definition 5.1.1

\[
\tilde{f}(t) = \frac{1}{n!} \sum_{\Pi} (f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}})
\]

and since for each \( t > 0 \) \( J_{n,t}(f) \) and \( I_{n,T}(f) \) are linear on \( f \)

\[
J_{n,t}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} J_{n,t}((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}}))
\]

and

\[
I_{n,T}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} I_{n,T}((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}})).
\]

But from (5.1.4) in Definition 5.1.2 and Lemma 2.3.1 we have that for all permutations \( \Pi = (\Pi_1, \ldots, \Pi_n) \) of \( (1, \ldots, n) \) and \( T = [0, t], \ t > 0 \)

\[
I_{n,T}((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}})) = \int_{T^{n}} f_{\Pi}^{(1)}(t_{\Pi}^{(1)}) \ldots f_{\Pi}^{(n)}(t_{\Pi}^{(n)}) \, d\mathbb{W}[e_k]((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}}))
\]

where \( \mathbb{W}[e_k]((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}})) \) is the isometric integral of \( f_{\Pi}^{(1)} \) w.r.t. the o.s.m. \( \mathbb{W}[e_k] \)
(see Lemma 4.1.5 and Theorem 2.1.1). Then from (5.1.18)

\[
(5.1.19) \quad I_{n,T}(\tilde{f}) = \frac{1}{n!} \sum_{\Pi} I_{\Pi}((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}}))
\]

On the other hand, from the definition of \( J_{n,t} \) and (4.2.2) in Definition 4.2.2, for all \( t > 0 \)

\[
J_{n,t}((f_{\Pi}^{(1)}(t_{\Pi}^{(1)})e_{k_{\Pi}^{(1)}}) \ldots (f_{\Pi}^{(n)}(t_{\Pi}^{(n)})e_{k_{\Pi}^{(n)}})) = \int_{0}^{t} \int_{0}^{t} \ldots \int_{0}^{t} f_{\Pi}^{(1)}(t_{\Pi}^{(1)}) \ldots f_{\Pi}^{(n)}(t_{\Pi}^{(n)}) \, d\mathbb{W}_t[e_k] \ldots d\mathbb{W}_t[e_k].
\]
By the last expression, (5.1.19) and Theorem 3.3.3 it follows that if $f$ is as in (5.1.13) then for each $t > 0$ $I_{n,T}(\tilde{f}) = n! J_{n,t}(\tilde{f})$. Next, since for each $t > 0$ $I_{n,T}(\cdot)$ and $J_{n,t}(\cdot)$ are bounded linear operators from $L^2(\mathbb{R}^n + H^\otimes_q)$ to $L^2(\Omega)$, that agree (up to a constant $n!$) on a dense linear manifold $S_n$ of $L^2(\mathbb{R}^n + H^\otimes_q)$, then (5.1.16) follows for all $f$ in $L^2(\mathbb{R}^n + H^\otimes_q)$.

Now we shall prove (5.1.17). Let $f$ be as in (5.1.13) and $g(s,\omega) = n! J_{n-1}(\tilde{f})_s(\omega) s \in \mathbb{R}_+, \omega \in \Omega$. In the proof of Proposition 5.1.5 we have shown that $J_{n-1}(f)$, defined in (5.1.14), belongs to $M_q$ and moreover

$$E \int_0^\infty ||J_{n-1}(f)_s||_Q^2 \, ds < \infty.$$ 

Therefore $g \in M_q, E \int_0^\infty ||g(s)||_Q^2 \, ds < \infty$ and from (5.1.15) and (5.1.16) for each $t > 0$, $T = [0,t]$

$$I_{n,T}(f) = \int_0^t <g_s, dW_s>_q.$$ 

The above result extends if $f$ belongs to the linear manifold $S_n$.

Next, if $f \in L^2(\mathbb{R}^n + H^\otimes_q)$, there exists a sequence of functions $\{f_m\}_{m \geq 1}$ in $S_n$ such that $f_m$ converges to $f$ in $L^2(\mathbb{R}^n + H^\otimes_q)$. Then there exists a sequence of functions $\{g_m\}_{m \geq 1}$ in $M_q$ such that

$$E \int_0^\infty ||g_m(s)||_Q^2 \, ds < \infty, \quad m \geq 1$$

and for each $t > 0$

$$I_{n,T}(f_m) = \int_0^t <g_m(s), dW_s>_q.$$ 

Then by Proposition 4.2.1 (d) for each $t > 0$

$$E(I_{n,T}(f_m - f_k))^2 = E(\int_0^t <g_m(s) - g_k(s), dW_s>_q)^2$$

$$= E \int_0^t ||g_m(s) - g_k(s)||_Q^2 \, ds.$$
On the other hand, by Proposition 5.1.4 (c), for each \( t > 0 \), \( T = [0, t] \)

\[
(5.1.22) \quad E(I_n, T(f - f_m)) \leq n! \theta_2^n \| f_m - f_k \|_{L^2(\mathbb{R}^n \rightarrow H^q)}^2 \leq n! \theta_2^n \| f_m - f_k \|_{L^2(\mathbb{R}^n \rightarrow H^q)}^2 \to 0.
\]

Then

\[
\sup_{0 \leq t < \infty} E(I_n, T(f_m - f_k))^2 \to 0 \quad \text{as } m, k \to \infty
\]

and

\[
\sup_{0 \leq t < \infty} E \int_0^t \| g_m(s) - g_k(s) \|_Q^2 \, ds \to 0 \quad \text{as } m, k \to \infty.
\]

Thus, using (5.1.20) and dominated convergence theorem

\[
E \int_0^\infty \| g_m(s) - g_k(s) \|_Q^2 \, ds = \sup_{0 \leq t < \infty} E \int_0^t \| g_m(s) - g_k(s) \|_Q^2 \, ds \to 0 \quad \text{as } m \to \infty
\]

and then there exists an \( H_q \)-valued, \( \Omega \times \mathbb{R}_+ \) measurable function \( g \), \( g \in M_Q \)

such that

\[
E \int_0^\infty \| g_m(s) - g(s) \|_Q^2 \, ds \to 0 \quad \text{as } m \to \infty.
\]

Thus by Definition 4.2.3, for each \( t > 0 \)

\[
E(\int_0^t <g(s), dW_s>_Q - \int_0^t <g_m(s), dW_s>_q)^2 \to 0 \quad \text{as } m \to \infty
\]

and by (5.1.21) since for each \( t > 0 \)

\[
E(I_n, T(f) - I_n, T(f_m))^2 \to 0 \quad \text{as } m \to \infty
\]

then

\[
I_n, T(f) = \int_0^t <g(s), dW_s>_Q \quad \text{for each } t > 0.
\]

Q.E.D.

**Corollary 5.1.3** Let \( f \in L(\mathbb{R}^n \rightarrow H^q) \). Then if \( H \) is the Gaussian space of \( \mathcal{W}_t \) (see 4.1.25),
a) For each $t > 0$ \[ J_{n,t}(\tilde{f}) \in H^q_{\mathbb{R}^n}. \]

b) $(J_{n,t}(\tilde{f}), F^W_t)_{t \in \mathbb{R}^+_t}$ is a square integrable martingale with a continuous modification and increasing process

\[ \mathbb{E} \int_0^t \| g(s) \|^2_Q \, ds = \mathbb{E} \int_{0}^{t} \| \tilde{f}(u) \|^2_Q \, du \]

for some function $g \in \mathcal{M}_Q$ with $\mathbb{E} \int_0^\infty \| g(s) \|^2_Q \, ds < \infty$.

The proof follows from the last proposition, Definition 4.2.3 and Proposition 4.2.2.

Remark Using Lemma 5.5.1 one can show that Propositions 5.1.6 and Corollary 5.1.3 hold if $f$ belongs to $L^2(\mathbb{R}^n + H^q_{\mathbb{Q}})$. However, we have left them the way they are to show the role played by the bilinear form $Q$ and the Hilbert space $H_{\mathbb{Q}}$.

Multiple Wiener integrals for elements in $L^2(\mathbb{R}^n + H^q_{\mathbb{Q}})$.

**Definition 5.1.5** In Definition 5.1.2 we have given the multiple Wiener integral $I_{n,T}(\cdot)$ for elements in the space $L^2(T^n + H^q_{\mathbb{Q}})$ where $q \geq r_1 + r_2$ and $T = [0,t]$, $t > 0$ is a finite interval of the real line. Now let

$f \in L^2(\mathbb{R}^n + H^q_{\mathbb{Q}})$ and $I_{n,t}(f)$ be the multiple integral of $f$ restricted to $L^2(T^n + H^q_{\mathbb{Q}})$. Proposition 5.1.16 shows that for all $t > 0$, $I_{n,t}(f) = n! J_{n,t}(\tilde{f})$ and from Corollary 5.1.3 we are able to define $J_{n,\infty}(\tilde{f})$ as the $L^2(\Omega)$-limit of $J_{n,t}(\tilde{f})$ such that $\mathbb{E}(J_{n,\infty}(\tilde{f}) | F^W_t) = J_{n,t}(\tilde{f})$ a.s. Write $I_n(f) = n! J_{n,\infty}(\tilde{f})$ and call it the multiple Wiener integral for elements in $L^2(\mathbb{R}^n + H^q_{\mathbb{Q}})$. Then using Lemma 5.5.1 and density arguments as before, $I_n$ is also defined for elements in $L^2(\mathbb{R}^n + H^q_{\mathbb{Q}})$. Moreover, it is a linear operator from each of the above spaces to $L^2(\Omega, F^W, \mathbb{P})$.

The main properties of $I_n(\cdot)$ are summarized in the next result.
Lemma 5.1.3  Let $f \in L^2(\mathbb{R}^+_n + H^m_Q)$. Then

a) $I_n(f) = I_n(\tilde{f}) \in H^m$ where $H$ is the linear space of $(W_t)_{t \in \mathbb{R}_+}$.

b) $E(I_n(f)) = 0$.

c) If $g \in L^2(\mathbb{R}^+_m + H^m_Q)$ for $m \geq 1$ then

$$E(I_n(f) I_m(g)) = \delta_{nm} n! \int_{\mathbb{R}^+_n} \langle \tilde{f}(s), \tilde{g}(s) \rangle d\omega_n ds.$$  

d) $E(I_n(f))^2 = n! \int_{\mathbb{R}^+_n} ||\tilde{f}(s)||^2 d\omega_n ds \leq n! \int_{\mathbb{R}^+_n} ||f(g)||^2 d\omega_n ds$.

e) For $T = [0, t], t > 0$; if $I_{n,T}(f)$ is the multiple integral of $f$ restricted to $T^n$ then

$$E(I_{n,T}(f) | F^W_t) = I_{n,T}(f).$$

f) $I_{n}(f) = \int_0^\infty \langle g(s), dW_s \rangle_Q$ for $g \in H_Q$. $E \int_0^\infty ||g(s)||^2_Q ds < \infty$ and the random variable $\int_0^\infty \langle g(s), dW_s \rangle_Q$ is defined as the limit of the square integrable martingale $\int_0^T \langle g(s), dW_s \rangle_Q$.

The proof follows from the above definition, Lemma 5.1.2, Proposition 5.1.6 and Corollary 5.1.3.

5.1.2 Real valued nonlinear functionals

By a real valued nonlinear functional of $(W_t)_{t \in \mathbb{R}_+}$ we mean an element of the space $L^2(\Omega, F^W_t, P)$ where $F^W_t = F^W_\infty$. In this subsection we use the techniques developed in the last section to obtain multiple Wiener integral expansions and stochastic integral representations for elements in $L^2(\Omega, F^W_t, P)$. We follow the same ideas as for the one dimensional Wiener
process, as presented, for example, in Chapter VI of Kallianpur (1980).

**Multiple Wiener integral orthogonal expansions** Let $S_I$ be the subspace of $L^2(\Omega) = L^2(\Omega, f^W, \rho)$ spanned by the multiple Wiener integrals $I_n(f_n)$ $f_n \in L^2(\mathbb{R}_+^n \rightarrow H_Q^{\otimes n})$ $n \geq 1$ where $I_n$ is as in Definition 5.1.5 for $q \geq r_1 + r_2$, that is

$$S_I = \overline{sp} L^2(\Omega) \{ I_n(f_n): f_n \in L^2(\mathbb{R}_+^n \rightarrow H_Q^{\otimes n})$ $n \geq 1 \}. $$

By the definition of $I_n(\cdot)$ for elements in $L^2(\mathbb{R}_+^n \rightarrow H_Q^{\otimes n})$, we have that $S_I$ is also equal to

$$\overline{sp} L^2(\Omega) \{ I_n(f_n): f_n \in L^2(\mathbb{R}_+^n \rightarrow H_Q^{\otimes n})$ $n \geq 1 \}. $$

The next result shows the role played by the space $L^2(\mathbb{R}_+) \otimes H_Q$.

**Proposition 5.1.7** Let $\text{EXP}(L^2(\mathbb{R}_+) \otimes H_Q)$ denote the Exponential Hilbert Space of $L^2(\mathbb{R}_+) \otimes H_Q$. Then

$$\text{EXP}(L^2(\mathbb{R}_+) \otimes H_Q) \ni S_I$$

where for $g \in \text{EXP}(L^2(\mathbb{R}_+) \otimes H_Q)$, $g = (g_0, g_1, \ldots)$ $g_n \in (L^2(\mathbb{R}_+) \otimes H_Q)^{\otimes n}$ $n \geq 0$

$$n(g) = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} I_n(g_n). \quad I_0(\cdot) = 1.$$

**Proof** Let $g = \exp \Theta(f)$ $f \in L^2(\mathbb{R}_+) \otimes H_Q$, i.e.

$$(5.1.23) \exp \Theta(f) = (f^0, f, \frac{1}{\sqrt{2!}} f^2, \frac{1}{\sqrt{3!}} f^3, \ldots) .$$

Then by Proposition 5.1.2 and Lemma 5.1.3 (c) and (d)

$$E(n(\exp \Theta(f)))^2 = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} E(I_n(f^{\otimes n}))^2 = \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} ||f^{\otimes n}||^2_{L^2(\mathbb{R}_+) \otimes H_Q}$$

$$= \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \left( \frac{||f||^2}{L^2(\mathbb{R}_+) \otimes H_Q} \right)^n = ||\exp \Theta(f)||^2_{\text{EXP}(L^2(\mathbb{R}_+) \otimes H_Q)}$$
and the proof follows since elements of the form (5.1.23) span
\( \text{EXP}(L^2(\mathbb{R}_+ \otimes Q)) \) (see Proposition 2.2 in Guichardet (1972)).

Q.E.D.

**Proposition 5.1.8** Let \( F \in L^2(\Omega, F^W, P) \). Then

\[
F - E(F) = \sum_{n=1}^{\infty} I_n(f_n) \quad \text{a.s.}
\]

where \( f_n \in (L^2(\mathbb{R}_+ \otimes Q)^\otimes n) \approx L^2(\mathbb{R}_+ \otimes H_Q^n) \) and \( I_n(\cdot) \) \( n \geq 1 \) are the multiple Wiener integrals of Definition 5.1.5.

**Proof** From Definition 4.1.2 \( H = L^1(W) \approx L^2(\mathbb{R}_+ \otimes H_Q) \) and by (4.1.26) \( L^2(\Omega, F^W, P) \) can be identified with \( \text{EXP}(H) \). Then

\[
\text{EXP}(H) \approx \text{EXP}(L^2(\mathbb{R}_+ \otimes Q))
\]

where

\[
\eta_2(\exp \oplus (I_1(f))) = \exp \oplus (f) \quad f \in L^2(\mathbb{R}_+ \otimes Q).
\]

Then by Proposition 5.1.7 \( L^2(\Omega, F^W, P) \) can be identified with \( S_I \) in such a way that if \( F \in L^2(\Omega, F^W, P) \)

\[
F - E(F) = \sum_{n=1}^{\infty} I_n(f_n) \quad \text{a.s.}
\]

where \( f_n \in L^2(\mathbb{R}_+ \otimes H_Q^n) \) and \( I_n(\cdot) \) \( n \geq 1 \) are the multiple Wiener integrals of Definition 5.1.5.

Q.E.D.

The following result shows the density of orthogonal expansions of multiple Wiener integrals of elements in \( L^2(\mathbb{R}_+ \otimes H_Q^n) \) \( q \geq r_1 + r_2 \).

**Corollary 5.1.4** Let \( F \in L^2(\Omega, F^W, P) \), \( E(F) = 0 \) and \( q \geq r_1 + r_2 \). Then for all \( \varepsilon > 0 \) there exist \( g_n \in L^2(\mathbb{R}_+ \otimes H_Q^n) \) \( n \geq 1 \) such that

\[
E(F - \sum_{n=1}^{\infty} I_n(g_n))^2 < \varepsilon.
\]
Proof By Proposition 5.1.8 there exist $f_n \in L^2(\mathbb{IR}^n \rightarrow H^0_\Omega)$ $n \geq 1$ such that

$$F = \sum_{n=1}^{\infty} I_n(f_n) \quad (L^2(\Omega)-\text{convergence}).$$

Let $q \geq r_1 + r_2$ and $\varepsilon > 0$. Then by Definition 5.1.5 and Lemma 5.1.1 for each $n \geq 1$ there exists $g_n \in L^2(\mathbb{IR}^n \rightarrow H^0_\Omega)$ such that

$$E(I_n(g_n) - I_n(f_n)) < \varepsilon/2^n.$$

Then by the orthogonality of $I_n$ for $n \neq m$

$$E\left(\sum_{n=1}^{\infty} I_n(g_n) - I_n(f_n)\right)^2 = \sum_{n=1}^{\infty} E(I_n(g_n) - I_n(f_n))^2 \leq \sum_{n=1}^{\infty} \varepsilon/2^n.$$

and hence

$$E(F - \sum_{n=1}^{\infty} I_n(g_n))^2 < \varepsilon.$$ Q.E.D.

Stochastic integral representations The next result is the analog of Theorem 6.7.1 in Kallianpur (1980) for the one-dimensional Wiener process.

Theorem 5.1.1 Let $F \in L^2(\Omega,F^W,P)$, $E(F) = 0$. Then

$$F(\omega) = \int_0^{\infty} <g(t,\omega),dW_t>_Q$$

where $g \in M_Q \quad E \int_0^{\infty} ||g(t)||^2_Q dt < \infty$ and the RHS is the stochastic integral of Definition 4.2.3.

Proof Since $F \in L^2(\Omega,F^W,P)$, by Proposition 5.1.8

$$F = \sum_{n=1}^{\infty} I_n(f_n) \quad (L^2(\Omega)-\text{convergence})$$

where $f_n \in L^2(\mathbb{IR}^n \rightarrow H^0_\Omega)$ $n \geq 1$. Then by Lemma 5.1.3 (f)

$$I_n(f_n) = \int_0^{\infty} <g_n(t),dW_t>_Q \quad \text{a.s.}$$

where $g_n \in M_Q$ and $E \int_0^{\infty} ||g_n(t)||^2_Q dt < \infty$ $n \geq 1$. Write $g^*_n = \sum_{i=1}^{n} g_i$. Then by
linearity of the integral \( \int_0^\infty <\cdot, dW_t>^Q \)

\[
\sum_{i=1}^n I_i (f_i) = \int_0^\infty <g^*_n(t), dW_t>^Q
\]

and therefore \( E(F - \int_0^\infty <g^*_n(t), dW_t>^Q)^2 \to 0 \) as \( n \to \infty \). Then using Corollary 4.2.2 we obtain that

\[
E \int_0^\infty ||g^*_n(t)-g^*_m(t)||^2_Q dt = E[ \int_0^\infty <g^*_n(t)-g^*_m(t), dW_t>^Q ] + 0 \text{ as } n,m \to \infty.
\]

Therefore there exists \( g \in M_Q, E \int_0^\infty ||g(s)||^2_Q ds < \infty \) such that

\[
E \int_0^\infty ||g^*_n(t)-g(t)||^2_Q dt \to 0 \text{ as } n \to \infty \text{ and }
E[ \int_0^\infty <g^*_n(t), dW_t>^Q - \int_0^\infty <g(t), dW_t>^Q ]^2 \to 0.
\]

Hence

\[
F(\omega) = \int_0^\infty <h_t, dW_t>^Q \quad \text{a.s.}\.
\]

Q.E.D.

The next result shows the density of the stochastic integrals \( \int_0^\infty <\cdot, dW_t>^Q \) in the space of nonlinear functionals \( L^2(\Omega, F^W_t, P) \).

**Corollary 5.1.5** Let \( F \in L^2(\Omega, F^W_t, P), E(F) = 0 \). Then for all \( q \geq r_1 + r_2 \) and \( \varepsilon > 0 \) there exists \( g \in M_q, E \int_0^\infty ||g(s)||^2_q ds < \infty \) such that

\[
E(F - \int_0^\infty <g_t, dW_t>^Q)^2 < \varepsilon.
\]

The proof follows from the last theorem and Definition 4.2.3.

**Representation of real valued square integrable martingales**

**Theorem 5.1.2** Let \( (M_t^W)_{t \in \mathbb{R}^+_0} \) be a square integrable martingale, with \( M_0 = 0 \). Then \( (M_t^W) \) has a continuous modification, say \( (\tilde{M}_t^W) \), which is given by the stochastic integral
\[ \tilde{M}_t(\omega) = \int_0^t \langle g(s, \omega), dW_s \rangle_Q \quad \text{a.s.} \]

for every \( t > 0 \), where \( g \in M_Q \), i.e. \( g(s, \omega) \) is jointly measurable, \( g(s, \cdot) \) is \( F^W_s \)-adapted and

\[ E \int_0^\infty \| g(s) \|^2_Q \, ds < \infty. \]

**Proof** Since \( (M_t)_{t \in \mathbb{R}^+} \) is a square integrable martingale, i.e. \( \sup_{0 \leq t < \infty} E(M_t^2) < \infty \), then \( M_\infty \) exists as the mean square limit (and also as the almost sure limit) of \( M_t \) \( t \to \infty \), and moreover, \( M_\infty \) is \( F^W_\infty \)-measurable and \( E(M_\infty^2) < \infty \), i.e. \( M_\infty \in L^2(\Omega, F, P) \).

Then by Theorem 5.1.1

\[ M_\infty = \int_0^\infty \langle g(t), dW_t \rangle_Q \quad \text{a.s.} \]

where \( g \in M_Q \), \( E \int_0^\infty \| g(s) \|^2_Q \, ds < \infty \). Then by Corollary 4.2.2

\[ M_t = E(M_\infty | F^W_t) = \int_0^t \langle g(t), dW_t \rangle_Q \quad \text{a.s.} \]

and the stochastic integral has a continuous version which is the required modification of \( M_t \).

Q.E.D.

5.2 \( \phi' \)-valued multiple Wiener integrals

Let \( \phi \) and \( \phi^{\otimes n} \), \( n \geq 1 \), be as in Section 4.1.1 and denote by \( L((\phi^{\otimes n})', \phi') \) the class of continuous linear operators from \( (\phi^{\otimes n})' \) to \( \phi' \). In this section we define multiple Wiener integrals of the form

\[ Y_{n, T}(f) = \int_{\mathbb{T}^n} f(t) dW_{t_1} \ldots dW_{t_n} \quad t = (t_1, \ldots, t_n) \]

where \( f(t) \) is a non-random element in \( L((\phi^{\otimes n})', \phi') \) and \( T = [0, T_0] \) for all \( n \geq 1 \) and \( T_0 > 0 \) (Section 5.2.1). Then we construct multiple stochastic integral expansions and stochastic integral representations for \( \phi' \)-valued
nonlinear functionals and \( \Phi' \)-valued square integrable martingales of
\((W_t)_{t \in \mathbb{R}_+} \) (Section 5.2.2). A Wiener type decomposition of the space
of \( \Phi' \)-valued nonlinear functionals is obtained in Theorem 5.2.2 as the
inductive limit of the spaces \( L^2(\Omega \to H_{\rho r}) \) \( r \geq 0 \).

5.2.1 Multiple Wiener integrals for \( L((\Phi^n)' , \Phi') \)-functions

Throughout this section \( n \geq 1 \) is fixed but arbitrary.

Definition 5.2.1 A measurable function \( f : \mathbb{R}^n \to L((\Phi^n)' , \Phi') \) is said
to belong to the class \( \Theta_Q((\Phi^n)' , \Phi') \) if for all \( T = [0,T_0] \), \( T_0 > 0 \)
\begin{equation}
\int_{[0,T_0]} Q^n(f^*(\phi), f^*(\phi)) \, dt < \infty \quad \forall \, \phi \in \Phi
\end{equation}
where \( f^* : \Phi \to \Phi^n \) is the adjoint of \( f \) and \( t = (t_1, \ldots, t_n) \).

Theorem 5.2.1 Let \( f \in \Theta_Q((\Phi^n)' , \Phi') \). Then for each \( T_0 > 0 \) there exists
a \( \Phi' \)-valued element \( Y_{n,T}(f) \), \( T = [0,T_0] \) such that
\begin{equation}
Y_{n,T}(f)[\phi] = I_{n,T}(f^*(\phi)) \quad \text{a.s.} \quad \forall \, \phi \in \Phi
\end{equation}
where \( I_{n,T} \) is the real valued multiple Wiener integral of Definition
5.1.3 for elements in \( L^2(T^n \to H_{\Phi}^n) \). \( Y_{n,T} \) is called the \( n \)-th \( \Phi' \)-valued
multiple Wiener integral.

Proof We prove this theorem in a very similar way to Proposition 4.2.3.
First note that for each \( \phi \in \Phi \) \( I_{n,T}(f^*(\phi)) \) is well defined since from
\begin{equation}
f^*(\phi) \in L^2(T^n \to H_{\Phi}^n).
\end{equation}
Next for \( \phi \in \Phi \) define \( V_{f}(\phi) = V_{f,T}(\phi) \) as
\begin{equation}
V_{f}^2(\phi) = \int_{[0,T_0]} Q^n(f^*(\phi), f^*(\phi)) \, dt.
\end{equation}
Since \( Q^n \) is \( \Phi^n \)-continuous, using Fatou's lemma one can show (as in
Proposition 4.2.3) that \( V_f \) is a lower semicontinuous function on \( \Phi \). Moreover, \( V_f \) is a non-negative function on \( \Phi \) that satisfies conditions (a), (b) and (c) of Lemma 4.1.1 (use triangle inequality to prove (a) and (5.2.1) to show (c)). Then this lemma implies that \( V_f \) is a continuous function on \( \Phi \) and therefore there exist \( \theta_f = \theta_{f,T} \) and \( r_f = r_{f,T} \) such that

\[
(5.2.4) \quad V_f^2(\phi) \leq \frac{\theta_f \|\phi\|^2}{r_f^2} \quad \forall \phi \in \Phi.
\]

Next let \( \{\phi_j\}_{j \geq 1} \) and \( \{\lambda_j\}_{j \geq 1} \) be as in Section 4.1.1, \( q_f = q_{f,T} \) such that

\[
q_f \geq r_f + r_1 \quad \text{and write} \quad \tilde{\phi}_j = (1 + \lambda_j)^{-q_f} \phi_j, \quad j \geq 1.
\]

Then \( \{\tilde{\phi}_j\}_{j \geq 1} \) is a CONS for \( \Phi \) and we denote by \( \{\psi_j\}_{j \geq 1} \) the CONS for \( \Phi \) dual to \( \{\phi_j\}_{j \geq 1} \), i.e.

\[
<\psi_k, \tilde{\phi}_j>_{q_f} = \delta_{kj}.
\]

Define \( Y_{n,T}(f)[\tilde{\phi}_j] = I_{n,T}(f^*(\tilde{\phi}_j)) \) for \( j \geq 1 \). Then by Definition 5.1.3 and Proposition 5.1.4 (e)

\[
\sum_{j=1}^{\infty} E(Y_{n,T}(f)[\tilde{\phi}_j])^2
\leq n! \sum_{j=1}^{\infty} \int I_{n} Q^n(f^*(\tilde{\phi}_j), f^*(\tilde{\phi}_j)) d\tau = n! \sum_{j=1}^{\infty} V_f^2(\tilde{\phi}_j)
\leq n! \theta_f \sum_{j=1}^{\infty} \frac{\|\tilde{\phi}_j\|^2}{r_f^2} = n! \theta_f \sum_{j=1}^{\infty} \frac{1}{(1 + \lambda_j)^{q_f - r_f}} \leq n \theta_f \theta_1 < \infty
\]

where \( \theta_1 \) is as in (4.1.4). Then \( \sum_{j=1}^{\infty} (Y_{n,T}(f)[\tilde{\phi}_j])^2 < \infty \) a.s. Let \( \Omega_1 = \{\omega: \sum_{j=1}^{\infty} (Y_{n,T}(f)[\tilde{\phi}_j](\omega))^2 < \infty\} \), then \( P(\Omega_1) = 1 \). Define

\[
(5.2.5) \quad \tilde{Y}_{n,T}(f)(\omega) = \begin{cases} 
\sum_{j=1}^{\infty} Y_{n,T}(f)[\tilde{\phi}_j](\omega) \psi_j & \omega \in \Omega_1 \\
0 & \omega \notin \Omega_1
\end{cases}
\]

Then \( \tilde{Y}_{n,T}(f) \in H_{-q_f} \) a.s. for \( q_f \geq r_f + r_1 \) and then \( \tilde{Y}_{n,T}(f) \in \Phi' \) a.s. From now on write \( Y_{n,T}(f) = \tilde{Y}_{n,T}(f) \).
Next if $\phi \in \phi$, then $\phi \in H_{q_f}$, $q_f \geq r_f + r_1$ and

$$Y_{n,T}(f)[\phi] = \sum_{j=1}^{\infty} Y_{n,T}(f)[\tilde{\phi}_j] \psi_j[\phi] = \sum_{j=1}^{\infty} Y_{n,T}(f)[\tilde{\phi}_j] <\phi, \tilde{\phi}_j> q_f$$

$$= \sum_{j=1}^{\infty} Y_{n,T}(f)[\phi, \tilde{\phi}_j] q_f \tilde{\phi}_j .$$

Thus for all $\phi \in \phi$, since $Y_{n,T}(f)[\tilde{\phi}_j] = I_{n,T}(f^*(\tilde{\phi}_j))$

(5.2.6) $Y_{n,T}(f)[\phi] = \lim_{m \to \infty} I_{n,T}(f^*(\sum_{j=1}^{m} <\phi, \tilde{\phi}_j> q_f \tilde{\phi}_j))$ a.s.

On the other hand, since $\sum_{j=1}^{m} <\phi, \tilde{\phi}_j> q_f \tilde{\phi}_j = \phi$ on $H_f$ then

$$V_{f}(\sum_{j=1}^{m} <\phi, \tilde{\phi}_j> q_f \tilde{\phi}_j - \phi) \to 0$$

which implies using (5.2.3) and Proposition 5.1.4 (e) that

$$E(I_{n,T}(f^*(\phi)) - I_{n,T}(f^*(\sum_{j=1}^{m} <\phi, \tilde{\phi}_j> q_f \tilde{\phi}_j)))^2 \to 0$$

as $m \to \infty$.

Then from (5.2.6) for all $T = [0,T_0]$, $T_0 > 0$

$$Y_{n,T}(f)[\phi] = I_{n,T}(f^*(\phi)) a.s. \forall \phi \in \phi .$$

Q.E.D.

The following properties of the $\phi^*$-valued multiple Wiener integral $Y_{n,T}(f)$ follow from the last theorem, (5.2.2), Proposition 5.1.4, Definition 5.1.3 and Corollary 5.1.1.

Proposition 5.2.1 Let $f, g \in \Theta_Q((\phi^m)', \phi')$. Then for each $T = [0,T_0]$, $T_0 > 0$

a) $Y_{n,T}(af+bg) = aY_{n,T}(f) + bY_{n,T}(g)$ a.s. $a, b \in \mathbb{R}$.

b) $E(Y_{n,T}(f)[\phi]) = 0 \forall \phi \in \phi$.

c) If $g \in \Theta_Q((\phi^m)', \phi')$

$$E(Y_{n,T}(f)[\phi] Y_{m,T}(g)[\psi])$$
where \( \tilde{f}_t^*(\phi) \) is the symmetrization (Corollary 5.1.1) of \( f_t^*(\phi) \) on \( \phi^\otimes n \).

d) \( E(Y_{n,T}(f)[\phi])^2 \leq n! \int_{\mathbb{T}} Q^\otimes n(f_t^*(\phi), f_t^*(\phi)) \, dt \).

**Proposition 5.2.2** Let \( n \geq 1 \) and \( g: \mathbb{R}_+ \to L((\phi^\otimes n)', \phi') \). Define the symmetrization \( \tilde{g} \) of \( g \) such that for each \( t \in \mathbb{R}_+ \) and \( \phi \in \phi \)

\[
\tilde{g}_t^*(\phi) = g_t^*(\phi) \quad \text{where} \quad g_t^*(\phi) \quad \text{is the symmetrization of} \quad g_t^*(\phi) \quad \text{on} \quad \phi^\otimes n \quad \text{of Corollary 5.1.1}.
\]

If \( f \in \Theta_Q((\phi^\otimes n)', \phi') \) then

a) \( \tilde{f} \in \Theta_Q((\phi^\otimes n)', \phi') \) and for each \( T = [0, T_0], T_0 > 0 \)

\[
Y_{n,T}(\tilde{f}) = Y_{n,T}(f) \quad \text{a.s.} \quad .
\]

b) For each \( T = [0, T_0], T_0 > 0 \) there exists \( q_f, T > 0 \) such that a.e.

\( \tilde{f} \in T^n \tilde{f}_t^* \) and \( \tilde{f}_t^* \) are Hilbert-Schmidt operators from \( H_{\phi^\otimes n} \) to \( H_{-q_f, T} \).

and

\[
(5.2.7) \quad E\|Y_{n,T}(f)\|^2_{-q_f, T} = n! \|\tilde{f}\|^2_{\mathbb{T}} \quad \text{for all} \quad \tilde{f} \in \tilde{f}_t^*(\phi) = f_t^*(\phi) \in \phi^\otimes n \quad \text{and} \quad \phi \in \phi \quad \text{with} \quad f \in \Theta_Q((\phi^\otimes n)', \phi') .
\]

Proof a) By Corollary 5.1.1 for each \( t \in \mathbb{R}_+ \) and \( \phi \in \phi \)

\[
\tilde{f}_t^*(\phi) = f_t^*(\phi) \in \phi^\otimes n
\]

and by Corollary 5.1.2 for each \( T = [0, T_0], T_0 > 0 \) and \( \phi \in \phi \)

\[
\int_{\mathbb{T}} Q^\otimes n(f_t^*(\phi), f_t^*(\phi)) \, dt \leq \int_{\mathbb{T}} Q^\otimes n(f_t^*(\phi), f_t^*(\phi)) \, dt
\]

which is finite for each \( T \) since \( f \in \Theta_Q((\phi^\otimes n)', \phi') \), proving that
\[ \tilde{f} \in Q((\phi^n)', \phi'). \]

Thus \( Y_{n,T}(\tilde{f}) \) is defined for each \( T = [0,T_0] \), \( T_0 > 0 \) and from Theorem 5.2.1, Definition 5.1.3, Proposition 5.1.4 (a) and (5.2.2) we have that

\[ Y_{n,T}(\tilde{f}) = Y_{n,T}(f) \quad \text{a.s. for each } T = [0,T_0], T_0 > 0. \]

b) For \( T = [0,T_0], T_0 > 0 \) let \( V_{f,T} \) and \( q_{f,T} \) be as in the proof of Theorem 5.2.1 and take \( \{(1+\lambda_j)^{-q_{f,T}\phi_j}\}_{j=1}^\infty \) a CONS in \( H^{-q_{f,T}} \). Then \( Y_{n,T}(f) \in H^{-q_{f,T}} \)

\[ \text{a.s. and} \]

\[ E\|Y_{n,T}(f)\|^2_{-q_{f,T}} = E \sum_{j=1}^\infty <Y_{n,T}(f), (1+\lambda_j)^{-q_{f,T}\phi_j}>^2_{-q_{f,T}} \]

\[ = \sum_{j=1}^\infty (1+\lambda_j)^{-q_{f,T}} E <Y_{n,T}(f), (1+\lambda_j)^{-q_{f,T}\phi_j}>^2_{-q_{f,T}} \]

\[ = \sum_{j=1}^\infty (1+\lambda_j)^{-q_{f,T}} E (Y_{n,T}(f)[\phi_j])^2 \quad \text{(by (4.1.8))} \]

\[ = n! \sum_{j=1}^\infty (1+\lambda_j)^{-q_{f,T}} \int_0^\infty \| f^*(\phi_j) \|^2_{Q_{\phi^n}} d\tau \quad \text{(Proposition 5.2.1 (d))} \]

\[ = n! \sum_{j=1}^\infty \int_0^\infty \| f^*((1+\lambda_j)^{-q_{f,T}\phi_j}) \|^2_{Q_{\phi^n}} d\tau = n! \int_0^\infty \sum_{j=1}^\infty \| f^*((1+\lambda_j)^{-q_{f,T}\phi_j}) \|^2_{Q_{\phi^n}} d\tau \]

\[ \leq n! \int_0^\infty \sum_{j=1}^\infty \| f^*((1+\lambda_j)^{-q_{f,T}\phi_j}) \|^2_{Q_{\phi^n}} d\tau \quad \text{(Corollary 5.1.2)} \]

\[ = n! \int_0^\infty V_{f,T}^2(1+\lambda_j)^{-q_{f,T}\phi_j} d\tau \]

\[ = n! \int_0^\infty (1+\lambda_j)^{-q_{f,T}\phi_j} d\tau \quad \text{(by 5.2.4)} \]

\[ \leq n! \theta_{f,T} \sum_{j=1}^\infty (1+\lambda_j)^{-2\tau_1} = n! \theta_{f,T} \theta_1 < \infty. \]

Then a.e. \( \tau \in T^n \)
\[ \| f_\ell \|_2^2 = \sum_{j=1}^{\infty} \| f_\ell((1+\lambda_j)^{-1} q_f, T \phi_j) \|_Q^{\infty} < \infty \]

and

\[ \| \tilde{f}_\ell \|_2^2 = \sum_{j=1}^{\infty} \| \tilde{f}_\ell((1+\lambda_j)^{-1} q_f, T \phi_j) \|_Q^{\infty} < \infty \]

where \( \{(1+\lambda_j)^{-1} q_f, T \phi_j\}_{j=1}^{\infty} \) is a CONS for \( H_{q_f,T} \). Then (5.2.7) follows since

\[ \| f \|_2 = \| f_\ell \|_2 = \| \tilde{f}_\ell \|_2 \]

Q.E.D.

We now extend the definition of \( Y_n \) to functions \( f: \mathbb{R}^n \to L((\phi_+^n), \phi^i) \).

Proposition 5.2.2 (b) suggests that it is enough to construct multiple Wiener integrals for functions \( f: \mathbb{R}^n \to (H_{q_f,T}^n, H_s) \) for \( s \geq r_1 + r_2 \), as we now do.

**Proposition 5.2.3** Let \( s \geq q_1 + q_2 \) and \( n \geq 1 \) be fixed but arbitrary. Let

\[ f \in L^2(\mathbb{R}^n \to \sigma_2(H_{q_f,T}^n, H_s)) \].

Then there exists an \( H_s \)-valued element \( Y_n(f) \) called the multiple Wiener integral for functions in \( L^2(\mathbb{R}^n \to \sigma_2(H_{q_f,T}^n, H_s)) \) such that

\[ Y_n(f)(\phi) = I_n(f^*(\phi)) \quad \text{a.s.} \quad \forall \phi \in H_s \]

where \( I_n \) is the multiple Wiener integral of Definition 5.1.5 for elements in \( L^2(\mathbb{R}^n \to H_Q^n) \). Moreover, \( Y_n(f) \) satisfies the following properties

a) If \( g \in L^2(\mathbb{R}^n \to \sigma_2(H_{q_f,T}^n, H_s)) \) and \( a, b \in \mathbb{R} \),

\[ Y_n(af + bg) = aY_n(f) + bY_n(g) \quad \text{a.s.} \]

b) If \( g \in L^2(\mathbb{R}^m \to \sigma_2(H_{q_f,T}^m, H_s)) \) then

\[ \langle Y_n(f), Y_m(g) \rangle_{-s} = \delta_{n,m} n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(\mathbb{R}^n \to \sigma_2(H_{q_f,T}^n, H_s))} \]
and
\[ E \| Y_n(f) \|_{-s}^2 = n! \| \tilde{f} \|_{L^2(\mathbb{R}^{n+\sigma_2}(H_0, H_{-s}))}^2 \leq n! \| f \|_{L^2(\mathbb{R}^{n+\sigma_2}(H_0, H_{-s}))}^2 < \infty. \]

\[ c) \] For \( T = [0, t], t > 0 \) if \( I_{n, T} \) is the real valued multiple Wiener integral for elements in \( L^2(T^n \rightarrow H_0^{\otimes n}) \) then
\[ E(Y_n(f)[\phi]|F_t^W) = I_{n, T}(f^*(\phi)) \text{ a.s. } \forall \phi \in H_s. \]

\[ d) \] For each \( \phi \in H_s \) there exists \( g_\phi \in M_Q, E \int_0^\infty \| g_\phi(s) \|_{L^2(Q)}^2 ds < \infty \) such that
\[ Y_n(f)[\phi] = \int_0^\infty <g_\phi(s), dW_s>_Q \text{ a.s.} \]

where the RHS is the real valued stochastic integral of Definition 4.2.3.

Proof
The first part is proved as in Theorem 5.2.1 using Definition 5.1.5 for elements in \( L^2(\mathbb{R}^{n+\sigma_2}(H_0, H_{-s})) \). (a) follows from (5.2.8) and the linearity property on \( I_{n}(\cdot) \). (c) and (d) follow from (5.2.8) and Lemma 5.1.3 (e) and (d). The second part of (b) follows as in the proof of (b) in Proposition 5.2.2. To prove the first part of (b) let \( \{e_k = (1+\lambda_k)^{s} \phi_k\}_{k \geq 1} \) be a CONS for \( H_{-s} \), then using (5.2.8) and Lemma 5.1.3 (c)
\[ E<Y_n(f), Y_m(g)>_{-s} = \sum_{k=1}^\infty (1+\lambda_k)^2 \hat{E}(I_{n}(f^*(\phi_k))I_m(g^*(\phi_k))) \]
\[ = \delta_{nm} n! \int_{\mathbb{R}^n} \sum_{k=1}^\infty <f^*(e_k), g^*(e_k)>_Q\otimes dt \]
\[ = \delta_{nm} \tilde{f}, \tilde{g} \|_{L^2(\mathbb{R}^{n+\sigma_2}(H_0, H_{-s}))}. \] Q.E.D.
We will see in the next section that the multiple Wiener integrals

\( Y_n(\cdot) \) for elements in \( L^2(\mathbb{R}^n + \sigma_2(H_Q, H_{-s})) \) form a complete system in the space of \( \Phi' \)-valued nonlinear functionals of \((W_t)_{t \in \mathbb{R}_+}\).

The next result will be useful for the representation of \( \Phi' \)-valued nonlinear functionals. It relates the \( \Phi' \)-multiple Wiener integral \( Y_n \) with the \( \Phi' \)-stochastic integral of Proposition 4.2.3 and it is an infinite dimensional analog of Lemma 6.7.2 in Kallianpur (1980).

**Proposition 5.2.4** Let \( n \geq 1 \), \( s \geq q_1 + q_2 \) and \( f \in L^2(\mathbb{R}^n + \sigma_2(H_Q, H_{-s})) \). Then there exists a non-anticipative \( \sigma_2(H_Q, H_{-s}) \)-valued process \( h(t, \omega) \) such that

\[
E \int_0^\infty \| h(t, \omega) \|^2_{\sigma_2(H_Q, H_{-s})} dt < \infty
\]

and

\[
Y_n(f) = \int_0^\infty h(t, \omega) dW_t
\]

where the RHS of the last expression is the \( \Phi' \)-valued stochastic integral of Proposition 4.2.5.

**Proof** By Proposition 5.2.3 (d) for each \( \phi \in H_s \)

\[(5.2.9) \quad Y_n(f)[\phi] = \int_0^\infty \langle g_{\phi}(t), dW_t \rangle_Q = I_n(f^*(\phi)) \text{ a.s.} \]

where \( g_{\phi} \in M_Q \), \( E(\int_0^\infty \| g_{\phi}(s) \|^2_Q ds) < \infty \). Let \( \{e_k\}_{k \geq 1} \) be a CONS for \( H_s \) and define \( h^*(t, \omega)(e_k) = g_{e_k}(t, \omega) \) \( k \geq 1 \). Then \( h^*(t)(e_k) \) is \( H_Q \)-valued and \( h^*(t)(e_k) \in M_Q \) \( k \geq 1 \). Next

\[
E(\sum_{k=1}^\infty \| h^*(t)(e_k) \|^2_Q) dt = \sum_{k=1}^\infty E(\| g_{e_k}(t) \|^2_Q dt
\]

\[
= \sum_{k=1}^\infty E(\int_0^\infty \langle g_{e_k}(t), dW_t \rangle_Q^2)
\]

(by Corollary 4.2.1)
= \sum_{k=1}^{\infty} E(Y_n(f)[e_k])^2 \quad \text{(by (5.2.9))}

\leq n! \sum_{k=1}^{\infty} \| f^*(e_k) \|^2_{L^2(\mathbb{R}^n+H_Q^\otimes)} \quad \text{(by Lemma 5.1.3 (d))}

= n! \| f^* \|^2_{L^2(\mathbb{R}^n+\sigma_2(H_s,H_Q))} = n! \| f \|^2_{L^2(\mathbb{R}^n+\sigma_2(H_Q,H^-_s))} < \infty.

Then

h^*(t)(\cdot) = \sum_{k=1}^{\infty} \langle \cdot, e_k \rangle_{H_s} h^*(t)(e_k)

defines an a.s. dtdP linear operator from $H_s$ to $H_Q$. Moreover, from the above calculations $h^*(t,\omega) \in \sigma_2(H_s,H_Q)$ a.s. dtdP. Then

$$\int_0^\infty E \| h(t) \|^2_{\sigma_2(H_s,H^-_Q)} dt = \int_0^\infty E \| h^*(t) \|^2_{\sigma_2(H_s,H_Q)} dt < \infty$$

and the proposition follows by the definition of the $\Phi'$-valued stochastic integral $\int_0^\infty h(t,\omega) dW_t$ of Proposition 4.2.5. Q.E.D.

### 5.2.2 $\Phi'$-valued nonlinear functionals

Let $F^W = F^W_\infty$. By a $\Phi'$-valued nonlinear functional of $(W_t)_{t \in \mathbb{R}^+}$ we mean a $\Phi'$-valued random element $F: \Omega \to \Phi'$ such that $F$ is $F^W \to B(\Phi')$ measurable, and

$$E(F[\phi])^2 < \infty \quad \forall \phi \in \Phi.$$

We denote by $L^2(\Omega \to \Phi') = L^2((\Omega,F^W,P) \to \Phi')$ the linear space of all $\Phi'$-valued nonlinear functionals of $(W_t)_{t \in \mathbb{R}^+}$. Observe that it is not a Hilbert space.

For $r > 0$ let $L^2(\Omega \to H^-_r) = L^2((\Omega,F^W,P) \to H^-_r)$ be the Hilbert space of all $F^W$-measurable elements $F: \Omega \to H^-_r$ such that $E(\| F \|^2_{H^-_r}) < \infty$. The Hilbert space $L^2(\Omega \to H^-_r)$ is called the space of $H^-_r$-valued nonlinear
functionals of \((W_t)_{t \in \mathbb{R}_+}\).

Wiener decomposition of the space \(L^2(\Omega^+ \Phi')\)

Let \(H = L_1(W)\) be the Gaussian space of \((W_t)_{t \in \mathbb{R}_+}\) defined in (4.1.25) and \(H^{\otimes n}\) be its n-fold symmetric tensor product. For fixed \(s > 0\) and \(n \geq 1\) let

\[(5.2.10) \quad G_n(H_s) = \{ \eta \in L^2(\Omega \times H_s) : \eta(\phi) \in H^{\otimes n} \quad \forall \phi \in H_s \}.\]

Recall that for all \(n \geq 1\) \(H^{\otimes n}\) is a subspace of \(L^2(\Omega, \mathcal{F}, \mathbb{F}, P)\).

**Theorem 5.2.2** (Wiener decomposition of \(L^2(\Omega^+ \Phi')\)). The linear space \(L^2(\Omega^+ \Phi')\) is a complete locally convex space in the topology given by the strict inductive limit of the Hilbert spaces \(L^2(\Omega \times H_{-r})\), \(r \geq 0\) and

\[(5.2.11) \quad L^2(\Omega^+ \Phi') = \lim_{r \to \infty} \bigoplus_{n \geq 0} G_n(H_{-r}).\]

The proof of this theorem is based on the following lemmas.

**Lemma 5.2.1**

\[(5.2.12) \quad L^2(\Omega^+ \Phi') = \bigcup_{r=0}^{\infty} L^2(\Omega \times H_{-r}).\]

**Proof** Let \(F \in L^2(\Omega \times H_{-r})\), \(r \geq 0\). Then \(F[\phi]\) is \(\mathcal{F}^W\)-measurable for all \(\phi \in \Phi\) and \(E(F[\phi])^2 \leq \|\phi\|_r^2 E \|F\|_r^2 < \infty\), i.e. \(F \in L^2(\Omega^+ \Phi')\) and hence

\[(5.2.13) \quad \bigcup_{r=0}^{\infty} L^2(\Omega \times H_{-r}) \subset L^2(\Omega^+ \Phi').\]

Next let \(F \in L^2(\Omega^+ \Phi')\) and for all \(\phi \in \Phi\) define \(V^2(\phi) = E(F[\phi])^2\).

Then

\[(5.2.14) \quad V^2(\phi) < \infty \quad \forall \phi \in \Phi.\]

As in the proof of Proposition 4.2.3 and Theorem 5.2.1, using the continuity of \(F\) on \(\Phi\) and Fatou's lemma, one can show that \(V(\phi)\) is a lower
semicontinuous function of $\phi$. Moreover it is non-negative and satisfies conditions (a), (b) and (c) of Lemma 4.1.1. Then $V(\phi)$ is a continuous function on $\phi$ and hence there exist $\theta_F > 0$ and $r_F > 0$ such that

\[(5.2.15) \quad V^2(\phi) = E(F(\phi))^2 \leq \frac{\theta_F \|\phi\|^2}{r_F} \quad \forall \ \phi \in \Phi .\]

Let $r \geq r_F + r_1$, then the imbedding of $H_r$ into $H$ is a Hilbert-Schmidt map. Take $\tilde{\phi}_j = (1+\lambda_j)^{-r_1} \phi_j$, then $\{\tilde{\phi}_j\}_{j \geq 1}$ is a CONS in $H_r$ and

\[
E(\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2) = \sum_{j=1}^{\infty} E(F[\tilde{\phi}_j])^2 = \sum_{j=1}^{\infty} V^2(\tilde{\phi}_j) \\
\leq \theta_F \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|^2 = \theta_F \sum_{j=1}^{\infty} (1+\lambda_j)^{-2(r-r_F)} \leq \theta_F \theta_1 < \infty
\]

where $\theta_1$ is as in (4.1.4). Then $\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2 < \infty$ a.s., and if $\{\psi_j\}_{j \geq 1}$ is the CONS in $H_{-r}$ dual to $\{\tilde{\phi}_j\}_{j \geq 1}$

\[
P(\tilde{F}(\omega) = \sum_{j=1}^{\infty} F[\tilde{\phi}_j](\omega) \psi_j < \infty) = 1
\]

and $\tilde{F} \in H_{-r}$ a.s. Moreover,

\[
E\|\tilde{F}\|_r^2 = \sum_{j=1}^{\infty} E < \tilde{F}, (1+\lambda_j)^{-r_1} \phi_j >^2 = E(\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2) < \infty .
\]

It remains to show that for each $\phi \in \Phi F[\phi] = \tilde{F}[\phi]$. By using (5.2.15) since $\sum_{j=1}^{m} <\phi, \phi_j >_r \tilde{\phi}_j \rightarrow \phi$ in $H_r$

\[
E(F[\phi] - \sum_{j=1}^{m} F[\tilde{\phi}_j] \psi_j [\phi])^2 = E(F[\phi] - \sum_{j=1}^{m} <\phi, \tilde{\phi}_j > \tilde{\phi}_j]^2 \\
\leq \theta_F \|\phi\| - \sum_{j=1}^{m} <\phi, \tilde{\phi}_j > \tilde{\phi}_j \|r_{\phi_j}\|^2 \rightarrow 0 \quad \text{as} \ m \rightarrow \infty
\]

and therefore for each $\phi \in \Phi F[\phi] = \tilde{F}[\phi]$ a.s.

Thus if $F \in L^2(\Omega, \Phi')$ there exists $r \geq 0$ such that $F \in L^2(\Omega, H_{-r})$

which together with (5.2.13) implies (5.2.12).

Q.E.D.
Lemma 5.2.2 For \( r \geq 0 \) and \( n \geq 1 \) let \( G_n(\mathcal{H}_-r) \) be as in (5.2.10). Then for fixed \( r \geq 0 \) \( G_n(\mathcal{H}_-r) \) are Hilbert subspaces of \( L^2(\Omega + \mathcal{H}_-r) \) such that if \( n \neq m \) \( G_n(\mathcal{H}_-r) \) and \( G_m(\mathcal{H}_-r) \) are orthogonal in \( L^2(\Omega + \mathcal{H}_-r) \).

Proof We first prove that for each \( n \geq 1 \) \( G_n(\mathcal{H}_-r) \) is a Hilbert subspace of \( L^2(\Omega + \mathcal{H}_-r) \). Let \( \{\eta_k\}_{k \geq 1} \) be a Cauchy sequence in \( G_n(\mathcal{H}_-r) \), then \( \eta_k \to \eta \) in \( L^2(\Omega + \mathcal{H}_-r) \). Since for each \( \phi \in \mathcal{H}_-r \)

\[
|\eta_k[\phi] - \eta[\phi]| \leq ||\phi|| \cdot ||\eta_k - \eta||_{\mathcal{H}_-r}
\]

then for each \( \phi \in \mathcal{H}_-r \) \( \eta_k[\phi] \to \eta[\phi] \) in \( L^2(\Omega, \mathcal{F}_r, \mathcal{P}) \).

But by hypothesis \( \eta_k[\phi] \in \mathcal{H}_{\phi}^n \). Then since \( \mathcal{H}_{\phi}^n \) is a closed subspace of \( L^2(\Omega, \mathcal{F}_r, \mathcal{P}) \), \( \eta[\phi] \in \mathcal{H}_{\phi}^n \) for all \( \phi \in \mathcal{H}_-r \) i.e. \( \eta \in G_n(\mathcal{H}_-r) \), proving that \( G_n(\mathcal{H}_-r) \) is a closed subspace of \( L^2(\Omega + \mathcal{H}_-r) \) for \( n \geq 1 \).

Next if \( n \neq m \), let \( \eta_n \in G_n(\mathcal{H}_-r) \), \( \eta_m \in G_m(\mathcal{H}_-r) \) and \( \{e_k\}_{k \geq 1} = \{(1 + \lambda_k)^{\tau} \phi \}_{k \geq 1} \) a CONS for \( \mathcal{H}_-r \). Then

\[
E<\eta_n, \eta_m>_{\mathcal{H}_-r} = \sum_{j=1}^{\infty} E<\eta_n, e_k>_{\mathcal{H}_-r} \cdot E<\eta_m, e_k>_{\mathcal{H}_-r} = \sum_{k=1}^{\infty} (1 + \lambda_k) E(\eta_n[\phi] \cdot \eta_m[\phi]) = 0
\]

since \( \eta_n[\phi] \in \mathcal{H}_{\phi}^n \), \( \eta_m[\phi] \in \mathcal{H}_{\phi}^m \) all \( k \geq 1 \) and \( \mathcal{H}_{\phi}^n \) and \( \mathcal{H}_{\phi}^m \) are orthogonal in \( L^2(\Omega, \mathcal{F}_r, \mathcal{P}) \).

Q.E.D.

The following result is the Wiener decomposition of the space \( L^2(\Omega + \mathcal{H}_-r) \). A proof of it appears in Miyahara (1981) for a general Hilbert space \( K \), i.e. for \( L^2(\Omega + K) \).

Lemma 5.2.3 For each \( r \geq 0 \)

\[
L^2(\Omega + \mathcal{H}_-r) = \sum_{n \geq 0} \mathcal{G}_n(\mathcal{H}_-r).
\]
Proof Let \( \{e_k\}_{k \geq 1} \) be a CONS for \( H_{-r} \) and \( \eta \in L^2(\Omega + H_{-r}) \), i.e. \( \eta \) is \( F^W \)-measurable and \( E ||n||^2_{-r} < \infty \). Then

\[
\eta(\omega) = \sum_{k=1}^{\infty} <\eta(\omega), e_k>_{-r} e_k \quad (L^2(\Omega + H_{-r}) \text{ convergence})
\]

where \( E(<\eta, e_k>^2_{-r}) < \infty \) all \( k \geq 1 \), i.e. \( <\eta, e_k>_{-r} \in L^2(\Omega, F^W, P) \) and therefore

\[
<\eta, e_k>^2_{-r} = \sum_{n=1}^{\infty} x_n^k \quad (L^2(\Omega)\text{-convergence})
\]

where for each \( k \), \( x_n^k \in \mathbb{H}^{\otimes n} \) \( n \geq 1 \).

Define

\[
\eta_n = \sum_{k=1}^{\infty} x_n^k e_k.
\]

We now prove that for each \( n \geq 1 \), \( \eta_n \in \mathbb{G}_n(H_{-r}) \). Note that

\[
E \left| \sum_{k=m}^{\ell} x_n^k e_k \right|^2 = E \left( \sum_{k=m}^{\ell} |x_n^k|^2 \right) = \sum_{k=m}^{\ell} E |x_n^k|^2 \\
\leq \sum_{k=m}^{\ell} E(<\eta, e_k>^2_{-r}) \to 0 \quad \text{as } m, \ell \to \infty.
\]

Then \( \left\{ \sum_{k=1}^{m} x_n^k e_k \right\}_{k \geq 1} \) is a Cauchy sequence in \( L^2(\Omega + H_{-r}) \) and therefore converges to a limit denoted by \( \eta_n \).

Next if \( \phi \in H_{-r} \), \( \eta_n[\phi] = \sum_{k=1}^{\infty} x_n^k e_k[\phi] \in \mathbb{H}^{\otimes n} \) \( n \geq 1 \), i.e. \( \eta_n \in \mathbb{G}_n(H_{-r}) \) \( n \geq 1 \).

By construction of \( \eta_m \) \( m \geq 1 \) if \( \sum_{m=1}^{\infty} \eta_m \) converges in \( L^2(\Omega + H_{-r}) \) it must converge to \( \eta \). Thus it remains to prove that \( \sum_{m=1}^{n} \eta_m \) is a Cauchy sequence:

\[
E \left| \sum_{m=n}^{\ell} \eta_m \right|^2_{-r} = \sum_{m=n}^{\ell} E ||\eta_m||^2_{-r} = \sum_{k=1}^{\infty} \sum_{m=n}^{\infty} E |x_m^k|^2.
\]

But for each \( k \), \( \sum_{m=n}^{\ell} E |x_m^k|^2 \to 0 \) as \( n, \ell \to \infty \), therefore

\[
E \left| \sum_{m=n}^{\infty} \eta_m \right|^2_{-r} \to 0 \quad \text{as } n, \ell \to \infty.
\]

Then \( \sum_{n=1}^{\infty} \eta_n \) is an element of \( L^2(\Omega + H_{-r}) \) and of \( \sum_{n \geq 0} \mathbb{G}_n(H_{-r}) \), which is equal to \( \eta \) a.e..
Proof of Theorem 5.2.2 Since \( L^2(\Omega + H_{-r}) \subseteq L^2(\Omega + H_{-(r+1)}) \) for all \( r \geq 0 \) the theorem follows by Lemmas 5.2.1 and 5.2.3 and the following result (Theorem V.15 of Reed and Simon (1980)): Let \( X \) be a real vector space and \( X_n \) be a family of subspaces with \( X_n \subseteq X_{n+1} \), \( X = \bigcup_{n=1}^{\infty} X_n \). Suppose that each \( X_n \) has a locally convex topology so that the restriction of the topology of \( X_{n+1} \) to \( X_n \) is the given topology on \( X_n \). Let \( U \) be the collection of balanced, absorbing, convex sets \( O \) in \( X \) for which \( O \cap X_n \) is open in \( X_n \) for each \( n \). Then a) The topology generated by \( U \) is the strongest locally convex topology on \( X \) so that the injections \( X_n \rightarrow X \) are continuous; b) The restriction of the topology on \( X \) to each \( X_n \) is the given topology on \( X_n \); c) If each \( X_n \) is complete, so is \( X \). The locally convex space \( X \) is called the strict inductive limit of the spaces \( X_n \).

Q.E.D.

Define for \( n \geq 1 \) \( G_n(\phi') = \{ \eta \in L^2(\Omega + \phi'): \eta(\phi) \in H^{\otimes n} \quad \forall \phi \in \phi \} \). The following lemma can be proved in the same way as Theorem 5.2.2, using Lemma 5.2.2 and the fact that for each \( n \geq 1 \) \( H^{\otimes n} \) is a Hilbert subspace of \( L^2(\Omega, F^X, P) \).

**Lemma 5.2.4** For each \( n \geq 1 \), the linear space \( G_n(\phi') \) is a complete locally convex space in the topology given by the strict inductive limit of the Hilbert spaces \( G_n(H_{-r}) \) \( r \geq 0 \), i.e. \( G_n(\phi') = \lim_{r \to 0} G_n(H_{-r}) \).

**Multiple Wiener integral orthogonal expansions** Let

\[ S_\gamma = \{ Y_n(f_n): f_n \in L^2(\mathbb{R}^n + \sigma_2(H^{\otimes n}_Q, H^{-s}_Q)) \quad n \geq 1, \quad s \geq 0 \}. \]

We shall show that \( S_\gamma \) is a complete set in the space \( L^2(\Omega + \phi') \). We see from Theorem 5.2.2 and Lemma 5.2.1 that it is enough to study the completeness of the multiple Wiener integrals in each of the subspaces \( L^2(\Omega + H_{-r}) \) \( r \geq 0 \).
For \( n \geq 0 \) let \( S^r_Y \) be the closed subspace of \( L^2(\Omega + H_{n-r}) \) spanned by the multiple Wiener integrals \( Y_n(\cdot) \) of Proposition 5.2.3, for elements in 
\[ L^2((\mathbb{R}^+ \circ \sigma_2(H_Q^\infty, H_{n-r})), \text{ i.e.} \]
\[ S^r_Y = \overline{\text{sp}} \{ Y_n(f_n) : f_n \in L^2((\mathbb{R}^+ \circ \sigma_2(H_Q^\infty, H_{n-r})), n \geq 1 \} \]
where the closure is taken with respect to \( L^2(\Omega + H_{n-r}) \).

Although multiple Wiener integrals on Hilbert spaces have been studied before (Miyahara (1981)) an analog of the next result was not found in the literature.

**Proposition 5.2.5** For each \( n \geq 0 \)

\[(5.2.16) \quad \sigma_2(\text{EXP}(L^2(\mathbb{R}^+ \circ H_Q)), H_{n-r}) \cong S^r_Y \]

where for \( g \in \sigma_2(\text{EXP}(L^2(\mathbb{R}^+ \circ H_Q)), H_{n-r}) \), \( g^* = (g_0^*, g_1^*, \ldots) \)

\[ g_n^* \in \sigma_2(H_{n-r}(L^2(\mathbb{R}^+ \circ H_Q)^\infty)) \quad n \geq 1 \]

\[ \xi(g) = \sum_{n=1}^{\infty} Y_n(g_n) \quad \text{(convergence in } L^2(\Omega + H_{n-r})). \]

**Proof** Let \( g \in \sigma_2(\text{EXP}(L^2(\mathbb{R}^+ \circ H_Q)), H_{n-r}) \), then \( g^* \in \sigma_2(H_{n-r}, \text{EXP}(L^2(\mathbb{R}^+ \circ H_Q)), \text{ i.e. for each } \phi \in H_{n-r} \quad g^*(\phi) \in \text{EXP}(L^2(\mathbb{R}^+ \circ H_Q)), g^*(\phi) = (g_0^*(\phi), g_1^*(\phi), \ldots) \)

and

\[ \sum_{n=0}^{\infty} || g_n^*(\phi) ||^2_{(L^2(\mathbb{R}^+ \circ H_Q)^\infty)} < \infty. \]

We first show that for each \( n \geq 1 \) \( g_n^* \in \sigma_2(H_{n-r}(L^2(\mathbb{R}^+ \circ H_Q)^\infty)) \). Let \( \{ e_m \}_{m=1}^{\infty} \) be a CONS in \( H_{n-r} \), then

\[ \sum_{m=1}^{\infty} || g^*(e_m) ||^2_{\text{EXP}(L^2(\mathbb{R}^+ \circ H_Q))} < \infty \]
and hence
\[
\sum_{m=1}^{\infty} \| g^*(e_m) \|_{L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q)}^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| g^*_n(e_m) \|_{L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q)}^2 < \infty.
\]

Thus for each \( n \) and \( \{e_m\}_{m=1}^{\infty} \) a CONS for \( H_\tau \)
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| g^*_n(e_m) \|_{L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q)}^2 < \infty,
\]
i.e.
\[
g^*_n \in \sigma_2(H_\tau, (L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q))^n), \quad n \geq 1.
\]

Next, if \( g \in \sigma_2(EXP(L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q), H_{-\tau}) \) using Proposition 5.2.3 (b)
\[
E \| \xi(g) \|^2 = \sum_{n=1}^{\infty} E \| Y_n(g) \|^2
\]
\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| g^*_n(e_m) \|_{L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q)}^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| g^*_n(e_m) \|_{L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q)}^2 < \infty.
\]

Then the result follows since \( g \) as above is a typical element in \( \sigma_2(EXP(L^2(\mathbb{R}_+ \oplus \mathbb{H}_Q), H_{-\tau})). \)

Q.E.D.

The completeness of the multiple Wiener integrals \( Y_n(f_n) \),
\( f_n \in L^2(\mathbb{R}_+ \oplus \sigma_2(H\oplusn, H_{-\tau})) \) in \( L^2(\Omega \oplus H_{-\tau}) \) is now obtained.

**Proposition 5.2.6** Let \( r \geq 0 \) and \( F \in L^2(\Omega \oplus H_{-\tau}), \quad E(F) = 0. \) Then
\[
F = \sum_{n=1}^{\infty} Y_n(f_n) \text{ a.s. (convergence in } L^2(\Omega \oplus H_{-\tau}).
\]

where \( f_n \in L^2(\mathbb{R}_+ \oplus \sigma_2(H\oplusn, H_{-\tau})) \), \( n \geq 1. \)
Proof. By Lemma 5.2.3 \( L^2(\Omega \cup H_{-r}) = \bigoplus_{n \geq 0} G_n(H_{-r}) \) where for each \( n \geq 1 \)

\[ G_n(H_{-r}) = \{ \eta \in L^2(\Omega \cup H_{-r}) : \eta[\phi] \in H^{\otimes n} \ \forall \ \phi \in H_r \} . \]

Then by Proposition 5.2.5 it is enough to prove that

\[ \sigma_2((L^2(\mathbb{R}^+) \otimes H_Q)^{\otimes n}, H_{-r}) \overset{\gamma_n}{=} G_n(H_{-r}). \]

But the last isometry follows from Proposition 5.2.4 and since from Lemma 5.1.3 and Proposition 5.1.2

\[ (L^2(\mathbb{R}^+) \otimes H_Q)^{\otimes n} \overset{I_n}{=} H^{\otimes n} \]

where \( I_n(\cdot) \) is the real valued multiple Wiener integral of Definition 5.1.5 for elements in \( L^2(\mathbb{R}^n \rightarrow H_Q) \).

Q.E.D.

The above proposition and Theorem 5.2.2 yield the next result which gives multiple Wiener integral expansions for \( \phi' \)-valued nonlinear functionals.

**Theorem 5.2.3** Let \( F \in L^2(\Omega \rightarrow \phi') \), \( E(F[\phi]) = 0 \ \forall \ \phi \in \phi \). Then there exists \( r_F > 0 \) such that \( F \in H_{-r_F} \) a.s. and

\[ F = \sum_{n=1}^{\infty} \gamma_n(f_n) \ \text{a.s.} \ \text{(}L^2(\Omega \cup H_{-r_F})\text{-convergence)} \]

where \( f_n \in L^2(\mathbb{R}^n \rightarrow \sigma_2(H_Q, H_{-r_F})) \ n \geq 1 \).

**Corollary 5.2.1** For \( n \geq 1 \) let

\[ \sigma_2((L^2(\mathbb{R}) \otimes H_Q)^{\otimes n}, \phi') = \bigcup_{r=0}^{\infty} \sigma_2((L^2(\mathbb{R}) \otimes H_Q)^{\otimes n}, H_{-r}). \]

Then

\[ G_n(\phi') \overset{\gamma_n}{=} \sigma_2((L^2(\mathbb{R}) \otimes H_Q)^{\otimes n}, \phi'). \]
The proof follows from the last part of the proof of Proposition 5.2.5 and from Lemma 5.2.4.

Stochastic integral representations for \( \Phi' \)-valued nonlinear functionals

From Proposition 5.2.4 and Theorem 5.2.3 one obtains the following stochastic integral representation for elements in \( L^2(\Omega \rightarrow \Phi') \). This result is the \( \Phi' \)-valued analog of Theorem 6.7.1 in Kallianpur (1980), from which the idea of the proof is taken.

**Theorem 5.2.4** Let \( F \in L^2(\Omega \rightarrow \Phi') \). Then there exist \( r_F > 0 \) and a non-anticipative \( \sigma_2(H, Q, \Phi', H - r_F) \)-valued process \( h \) with

\[
(5.2.17) \quad \int_0^\infty E[|h(t, \omega)|^2] \sigma_2(H, Q, \Phi', H - r_F) dt < \infty
\]

such that

\[
F(\omega) = \int_0^\infty h(t, \omega) dW_t \quad \text{a.s.}
\]

where the RHS in the last expression is the \( \Phi' \)-valued stochastic integral of Proposition 4.2.5 with an \( H_{-r_F} \) continuous version.

**Proof** Since \( F \in L^2(\Omega \rightarrow \Phi') \) from Lemma 5.2.1 and Theorem 5.2.3 there exists \( r_F > 0 \) such that \( F \in H_{-r_F} \) a.s. and

\[
F = \sum_{n=1}^\infty Y_n(f_n) \quad \text{(} L^2(\Omega \rightarrow H_{-r_F}) \text{-convergence)}
\]

where \( f_n \in L^2(\mathbb{R}^+ \rightarrow \sigma_2(H, Q, \Phi', H - r_F)) \) \( n \geq 1 \).

Next, from Proposition 5.2.4, for each \( n \geq 1 \)

\[
Y_n(f_n) = \int_0^\infty h_n(t, \omega) dW_t \quad \text{a.s.}
\]
where \( h_n \) is non-anticipative and \( E \int_0^\infty \| h_n(t, \omega) \|_{\sigma_2(H_{Q/t}, H_{-R_F})}^2 \, dt < \infty \), and the \( \phi' \)-valued stochastic integral is defined in Proposition 4.2.5.

Define \( g_n = \sum_{k=1}^n h_k \), then

\[
\sum_{k=1}^n \chi_k(f_k) = \int g_n(t, \omega) \, dW_t
\]

and hence

\[
(5.2.18) \quad E\| F - \int g_n(t, \omega) \, dW_t \|_{-R_F}^2 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Then using Proposition 4.2.5 (d)

\[
E \int \| g_n(t) - g_m(t) \|_{\sigma_2(H_{Q/t}, H_{-R_F})}^2 \, dt = E\| \int (g_n - g_m)(t) \, dW_t \|_{-R_F}^2 \rightarrow 0 \quad \text{as} \quad n, m \rightarrow \infty
\]

and hence there exists a \( \sigma_2(H_{Q/t}, H_{-R_F}) \)-valued function \( g \), that satisfies (5.2.17), is non-anticipative and

\[
E\| \int h \, dW_t - \int g_n \, dW_t \|_{-R_F}^2 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Then from (5.2.18) \( F = \int h \, dW_t \) a.e. where \( h \) has the required properties.

Q.E.D.

Representation of \( \phi' \)-valued square integrable martingales

**Definition 5.2.2** A \( \phi' \)-valued stochastic process \((X_t)_{t \in \mathbb{R}_+}\) on \((\Omega, F, P)\) is said to be a \( \phi' \)-square integrable martingale with respect to an increasing family \((F_t)_{t \in \mathbb{R}_+}\) of sub \( \sigma \)-fields of \( F \) if:

For each \( \phi \in \phi \) \( (X_t[\phi], F_t)_{t \in \mathbb{R}_+} \) is a real valued square integrable martingale, i.e.

\[
(5.2.19) \quad \sup_{0 \leq t < \infty} E X_t^2[\phi] < \infty.
\]

Although the next proposition follows from condition (\( \dagger \)) in Mitoma
(1981b) (page 193), we shall establish and prove it here in a way that is more convenient for use in our next theorem on the representation of \( \phi' \)-valued square integrable martingales.

**Proposition 5.2.7** Let \((X_t, F_t) \in \mathbb{R}_+ \), \( X_0 = 0 \) be a \( \phi' \)-valued square integrable martingale. Then there exists \( r_X > 0 \) such that for each \( t \in \mathbb{R}_+ \)
\[ X_t \in H_{-r_X} \text{ a.s.} \]
Moreover, for each \( \phi \in \phi \) let \( X_{\infty}(\phi) \) be the mean square limit of \( X_t(\phi) \) as \( t \to \infty \). Then there exists \( \tilde{X}_{\infty} \in \phi' \) a.s. \( \tilde{X}_{\infty} \in L^2(\Omega + \phi') \), \( \tilde{X}_{\infty} \in L^2(\Omega + H_{-r_X}) \) such that \( \tilde{X}_{\infty}(\phi) = X_{\infty}(\phi) \) a.s. \( \forall \phi \in \phi \) and for \( t \geq 0 \)
\[ X_t[\phi] = E(\tilde{X}_{\infty}[\phi]|F_t) \text{ a.s. } \forall \phi \in \phi. \]

The proof is similar to the proof of Lemma 5.2.1 and therefore we will omit some details. In Lemma 5.2.1 we have proved that for each \( t \),
\[ (E(X_t[\phi]))^{1/2} \]
is a lower semicontinuous function of \( \phi \). Then by Lemma 1, page 5, of Gelfand and Vilenkin (1964)
\[ V(\phi) = \sup_{0 \leq t < \infty} (E(X_t[\phi]))^{1/2} \]
is also a lower semicontinuous convex function of \( \phi \). Hence, by Lemma 4.1.1, since by (5.2.19) \( V(\phi) < \infty \) \( \forall \phi \in \phi \), there exist \( \theta_X > 0 \) and \( s_X > 0 \) such that
\[ V^2(\phi) \leq \theta_X \| \phi \|^2 \]
where \( \| \phi \| \) is a CONS for \( H \) with dual \( \{ \psi_j \}_{j \geq 1} \) which is a CONS for \( H_{-r_X} \).

Then taking \( r_X \geq s_X + r_1 \) one can show that
\[ (5.2.20) \quad E(\sum_{j=1}^{\infty} (X_{\infty}(\phi_j))^2) \leq \theta_X \theta_1 < \infty \]
where \( \{ \phi_j = (1+\lambda_j)^{-r_X} \phi_j\}_{j \geq 1} \) is a CONS for \( H_{-r_X} \) with dual \( \{ \psi_j \}_{j \geq 1} \) which is a CONS for \( H_{-r_X} \). Define
(5.2.21) \[ \tilde{X}_\infty = \sum_{j=1}^{\infty} X_\infty(\tilde{\phi}_j)\psi_j \]

then \( \tilde{X}_\infty \) is \( H_{r_X} \)-valued a.s. and therefore \( \tilde{X}_\infty \in \Phi' \) a.s.. Note that from (5.2.20) and (5.2.21)

\[ E(\tilde{X}_\infty(\phi))^2 \leq \theta_X \theta_1 \| \phi \| r_X < \infty \]

and therefore \( \tilde{X}_\infty \in L^2(\Omega + \Phi'), \tilde{X}_\infty \in L^2(\Omega + H_{r_X}). \) Then it remains to prove that \( X_\infty(\phi) = \tilde{X}_\infty(\phi) \) a.s. \( \forall \phi \in \Phi. \)

From (5.2.21)

\[ \tilde{X}_\infty(\phi) = \sum_{j=1}^{\infty} X_\infty(\tilde{\phi}_j)\psi_j(\phi) \text{ a.s. } \phi \in H_{r_X} \]

and then

\[ \tilde{X}_\infty(\phi) = \lim_{n \to \infty} \sum_{j=1}^{n} X_\infty(\tilde{\phi}_j)\psi_j(\phi). \]

On the other hand, since \( X_\infty(\phi) \) is linear on \( \phi \) a.s.

\[ E(X_\infty(\phi) - X_\infty(\sum_{i=1}^{n} <\phi, \tilde{\phi}_j> \tilde{\phi}_j))^2 = E(X_\infty(\phi - \sum_{j=1}^{n} <\phi, \tilde{\phi}_j> \tilde{\phi}_j))^2 \]

\[ \leq \theta_X \| \phi - \sum_{j=1}^{n} <\phi, \tilde{\phi}_j> \tilde{\phi}_j \| r_X^2 \to 0 \text{ as } n \to \infty. \]

Then \( X_\infty(\phi) = \tilde{X}_\infty(\phi) \) a.s. \( \forall \phi \in \Phi. \quad \text{Q.E.D.} \)

The following theorem is the \( \Phi' \)-valued analog of Theorem 6.7.2 in Kallianpur (1980).

**Theorem 5.2.5** Let \( (X_t, F_t^W, t \in \mathbb{R}^+), X_0 = 0, \) be a \( \Phi' \)-valued square integrable martingale. Then there exists \( r_X > 0 \) such that \( X_t \) has an \( H_{r_X} \)-continuous version \( \tilde{X}_t \) given by the stochastic integral

(5.2.22) \[ \tilde{X}_t(\omega) = \int_{0}^{t} h(s, \omega) dW_s \text{ a.s.} \]
for every \( t \geq 0 \), where \( h(t,\omega) \) is non-anticipative and

\[
\int_0^\infty \mathbb{E} \| h(t,\omega) \|_2^2 \sigma_2(H_{Q^t\cdot H}^{-r_X}) \, dt < \infty
\]

and the RHS of (5.2.22) is the \( \Phi' \)-valued stochastic integral of Proposition 4.2.5.

**Proof** By Proposition 5.2.7 there exists \( r_X > 0 \) and \( \tilde{X}_\infty \in \Phi' \) such that

\( \tilde{X}_\infty \in H_{-r_X} \) a.s. and for each \( t > 0 \) \( X_t(\phi) = \mathbb{E}(\tilde{X}_\infty(\phi) | F_t^W) \). Then by Theorem 5.2.3 there exists a non-anticipative \( \sigma_2(H_{Q^t\cdot H}^{-r_X}) \)-valued process \( h(t,\omega) \) which satisfies (5.2.23) and

\[
\tilde{X}_\infty(\omega) = \int_0^\infty h(\mu,\omega) dW_\mu \quad \text{a.s.}
\]

Then the result follows using Proposition 4.2.5, from which the \( H_{-r_X} \) continuous version is also obtained.

Q.E.D.

**Remarks** Our results in Sections 4.2, 5.1 and 5.2 may be applied to the examples considered in Section 4.1.3 to construct stochastic integrals, multiple Wiener integrals, orthogonal expansions and stochastic integral representations for nonlinear functionals of a multiparameter Gaussian process (Example 4.1.6), a cylindrical Brownian motion (Example 4.1.7) or an infinite sequence of independent Brownian motions (Example 4.1.8) as well as to the finite dimensional Gaussian process with independent increments (Example 4.1.9).

Throughout Chapters IV and V we have made the assumption that \( \Phi \) is a countably Hilbert nuclear space of the kind defined in Example 4.1.1, which has the special property of having a common orthogonal set \( \{ \phi_j \}_{j \geq 1} \)
in $H_r \in R$. Although this assumption makes some computations easier it is not essential and all our results can be obtained using only the nuclearity property of $\phi$. 
APPENDIX

A. INFINITE TENSOR PRODUCTS OF HILBERT SPACES

We present here some material on infinite tensor products of Hilbert spaces following Guichardet (1972).

Let \( (H_i)_{i \in I} \) be a family of Hilbert spaces and for each \( i \in I \) a unit vector \( u_i \in H_i \). Let \( u = (u_i)_{i \in I} \) be fixed. Consider a family \( x_i \in H_i \) such that \( x_i = u_i \) for all but a finite number of \( i \)'s. Elements of this form are called Elementary Decomposable Vectors and are denoted by \( \otimes x_i \).

They form a pre-Hilbert space with inner product

\[
\langle \otimes x_i, \otimes y_i \rangle = \prod_{i \in I} \langle x_i, y_i \rangle_{H_i}
\]

whose completion \( \otimes H_i \) is called the Infinite Tensor Product Hilbert Space associated with the family of unit vectors \( u \) (Guichardet (1972)).

This space may also be constructed in the following manner: for each finite subset \( J \) of \( I \) construct the Hilbert tensor product

\[
H(J) = \otimes_{i \in J} H_i.
\]

For \( J \subseteq K \), \( J, K \) finite subsets of \( I \) define a mapping

\[
L_{J,K} : \otimes_{i \in K} H_i \to \otimes_{i \in I} H_i
\]

by writing \( \otimes_{i \in K} H_i = (\otimes_{i \in J} H_i) \otimes_{i \in I \setminus J} H_i \) as

\[
L_{J,K}(x) = x \otimes_{i \in I \setminus J} u_i.
\]
Then the mappings \( L_{J,K} \) are isometric and form an inductive system, i.e. for \( J \subset K \subset M \)

\[
L_{J,M} = L_{K,M} \circ L_{J,K}.
\]

Then \( \bigoplus_{i \in I} H_i \) is the inductive limit of the above system. Denote by \( L_J \)
the canonical injection

\[
H(J) \to \bigoplus_{i \in I} H_i.
\]

The next result identifies some elements of \( \bigoplus_{i \in I} H_i \).

**Proposition A1** Let \( (x_i)_{i \in I} \), \( x_i \in H_i \) be a family of vectors satisfying
the following two conditions:

1. \( \sum_i \| x_i \|_{H_i} - 1 \) \( < \infty \)

and

2. \( \sum_i |<x_i,u>_{H_i} - 1| \) \( < \infty \).

Then \( \Pi ||x_i||_{H_i} \) exists, and it is null if and only if one of the \( x_i \)'s
is null. The family of vectors

\[
L_J(\bigoplus_{i \in I} x_i)
\]

has a limit in \( \bigoplus_{i \in I} H_i \) denoted by \( x_i \) whose norm is

\[
\Pi ||x_i||_{H_i}.
\]

Moreover, \( \lim_{J \to \infty} ||x_i - \bigoplus_{i \in J} u_i|| = 0. \)

**Proof** See page 150 Guichardet (1972).
A family \((x_i)_{i \in I}\) satisfying (1) and (2) in the above proposition is called a Decomposable Vector. For any two decomposable vectors we have

\[
\sum_{i} |\langle x_i, y_i \rangle_{H_i} - 1| < \infty
\]

and

\[
\langle \otimes x_i, \otimes y_i \rangle = \prod \langle x_i, y_i \rangle_{H_i}.
\]
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