Line of Sight of an Aberrated Optical System

V. N. MAHAJAN
Electronics and Optics Division
Engineering Group
The Aerospace Corporation
El Segundo, Calif. 90245

14 January 1985

Final Report

APPROVED FOR PUBLIC RELEASE
DISTRIBUTION UNLIMITED

Prepared for
SPACE DIVISION
AIR FORCE SYSTEMS COMMAND
Los Angeles Air Force Station
P. O. Box 92960, Worldway Postal Center
Los Angeles, Calif. 90009-2960
This report was submitted by The Aerospace Corporation, El Segundo, CA 90245, under Contract No. F04701-83-C-0084 with the Space Division, P.O. Box 92960, Worldway Postal Center, Los Angeles, CA 90009-2960. It was reviewed and approved for The Aerospace Corporation by J. R. Parsons, Principal Director, Sensor Systems Subdivision, and J. R. Stevens, Principal Director, Advanced Programs. Capt. D. V. Thyfault was the Air Force project officer.

This report has been reviewed by the Public Affairs Office (PAS) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

D. V. Thyfault, Captain, USAF
Space Weapons Development Mgr
Strategic Defense Weapons SPO

J. W. Dettmer, Colonel, USAF
Director
Strategic Defense Weapons SPO

FOR THE COMMANDER

W. H. Crabtree, Colonel, USAF
Deputy Commander for Space Systems
**Line of Sight of an Aberrated Optical System**

<table>
<thead>
<tr>
<th>Line of Sight of an Aberrated Optical System</th>
</tr>
</thead>
</table>

**Title**

**Line of Sight of an Aberrated Optical System**

**Author**

Virendra N. Mahajan

**Performing Organization**

The Aerospace Corporation
El Segundo, Calif. 90245

**Report Date**

January 1985

**Number of Pages**

45

**Distribution Statement**

Approved for public release; distribution unlimited.

**Key Words**

Line of Sight, Centroid, Point Spread Function, Coma Aberration, Gaussian Illumination, Zernike Polynomials, Circular and Annular Apertures

**Abstract**

The line of sight of an aberrated optical system is defined in terms of the centroid of its point spread function (PSF). It is also expressed in terms of its optical transfer function as well as its pupil function. Although the PSFs obtained according to wave diffraction optics and ray geometrical optics...
are quite different from each other, they have the same centroid. For an aberration-free pupil, different amplitude distributions across it give the same centroid location; for an aberrated pupil, not only the phase but also the amplitude distribution affects the centroid location. If the amplitude across a pupil is uniform, the centroid may be obtained from the aberration along its perimeter only, without regard for the aberration across its interior regardless of its shape. Next, an optical system with aberrated but uniformly illuminated annular pupil is considered. The aberration function is expanded in terms of Zernike annular polynomials. It is shown that only those aberrations that vary with angle as \( \cos \theta \) or \( \sin \theta \) contribute to the line-of-sight. A simple expression is obtained for the line-of-sight in terms of the Zernike aberration coefficients. Similar results are obtained for annular pupils with radially symmetric illumination. Finally, specific results are discussed for annular pupils aberrated by classical primary and secondary coma. As an example of a radially symmetric illumination, we obtain numerical results for Gaussian illumination of aberrated annular pupils. It is emphasized that the centroid and the peak of the PSFs aberrated by coma are not coincident. Moreover, as the amount of the aberration increases, the separation of the centroid and peak locations also increases.
CONTENTS

I. INTRODUCTION ............................................. 5

II. THEORY .................................................. 6

III. APPLICATION TO SYSTEMS WITH ANNULAR PUPILS .......... 12
    A. Uniform Illumination .................................. 12
    B. Radially Symmetric Illumination ..................... 15

IV. NUMERICAL EXAMPLES .................................... 17
    A. Uniform Illumination .................................. 17
    B. Gaussian Illumination ................................. 33

V. DISCUSSION AND CONCLUSIONS .............................. 39

ACKNOWLEDGMENT ............................................. 41

APPENDIX. PSF FOR AN ANNULAR PUPIL WITH RADIALLY
           SYMMETRIC ILLUMINATION AND COMA
           ABERRATION ........................................ 43

REFERENCES .................................................... 45
**FIGURES**

1. PSF $I(x, \varepsilon)$ for several typical values of primary coma aberration $W_3$ in units of $\lambda$ .......................................................... 23

2. Variation of $I_m$, $I_p$, and $I_c$ with $W_3$, where the irradiances are in units of the aberration-free central irradiance and $W_3$ is in units of $\lambda$ .......................................................... 24

3. Variation of $x_m$, $x_p$, and $<x>$ with $W_3$ .......................... 25

4. Same as Figure 1 except that the aberration is secondary coma $W_5$ .......................................................... 28

5. Same as Figure 1 except that the aberration is a combination of primary and secondary coma given by Eq. (57) ...................... 31

6. Same as Figure 1 except that the illumination is Gaussian given by Eq. (61) with $\gamma = 1$ .................................................. 35

7. Same as Figure 6, except that the aberration is secondary coma $W_5$ .......................................................... 37

-2-
TABLES

1. Typical values of $x_m$, $x_p$, and $<x>$ in units of $\lambda F$, and the corresponding irradiances $I_m$, $I_p$, and $I_c$ in units of the aberration free central irradiance for PSF's aberrated by primary coma, $W(h,\theta) = W_3 \rho^3 \cos \theta$ ........................................... 26

2. Same as Table 1, except that the aberration is secondary coma, $W(h,\theta) = W_5 \rho^5 \cos \theta$ ........................................... 29

3. Same as Table 1, except that the aberration is a combination of primary and secondary coma given by Eq. (57) .................. 32

4. Same as Table 1, except that $A(h)$ is a Gaussian given by Eq. (61) with $\gamma = 1$ ........................................... 36

5. Same as Table 2, except that $A(h)$ is a Gaussian given by Eq. (61) with $\gamma = 1$ ........................................... 38
I. INTRODUCTION

The line-of-sight (LOS) of an aberrated optical system is defined in terms of the centroid of its point spread function (PSF). Using the Fourier transform relationship between the PSF and the optical transfer function (OTF), it is also expressed as the slope of the imaginary part of the OTF at the origin. Since the system OTF is equal to the autocorrelation of its pupil function, the centroid can also be written in terms of the amplitude and aberration across its pupil. Using the expression in terms of the pupil function, it is easy to show that the LOS obtained by wave diffraction optics is identical with that based on geometrical optics. It is shown that, whereas for an aberration-free pupil, different amplitude distributions across it give the same LOS, regardless of its shape; for an aberrated pupil both the aberration and the amplitude distribution affect the LOS. For a uniform amplitude distribution, the LOS depends only upon the aberration along the boundary of the pupil, i.e., it is independent of the aberration across its interior.

Next, an optical system with aberrated but uniformly illuminated annular pupil is considered. Its aberration function is expanded in terms of Zernike annular polynomials. It is shown that only those aberrations contribute to the LOS that vary with angle as \( \cos \theta \) or \( \sin \theta \), i.e., various orders of coma-type aberrations. A simple expression is obtained for the LOS in terms of the Zernike aberration coefficients. It is shown that two different orders of Zernike coma (including tilts) with the same standard deviation across the pupil do not contribute the same amount of LOS error; for a given standard deviation, a higher-order Zernike coma gives a larger LOS error. Finally, some numerical examples are considered. In particular, the PSF's aberrated by primary and secondary classical coma are discussed. It is shown that such aberrations shift the peak and centroid of the PSF (which are coincident in the aberration-free case) by significantly different amounts. Similar results are obtained for annular pupils with radially symmetric illumination. These results are illustrated by considering Gaussian illumination of an aberrated annular pupil. Numerical results for both circular and annular pupils are also given. In particular, it is shown that two PSF's aberrated by the same
amount of primary and secondary coma have identical centroids in the case of uniform illumination, but different centroids for Gaussian illumination.

The results obtained here are applicable to both imaging systems as well as laser transmitters. For a certain point object, the amplitude distribution across the pupil of an imaging system will generally be uniform. However, across a laser beam, the amplitude distribution is often given by a Gaussian function. Moreover, laser beams are often polarized. If the polarization is linear, then the scalar treatment of diffraction considered in this paper leads to a linearly polarized diffracted wave. If the beam is circularly or elliptically polarized, then the diffraction of the two orthogonal polarization components may be treated separately. The total diffracted field is obtained by taking a vector sum of the two diffracted components.

II. THEORY

The PSF of an optical imaging system, i.e., the irradiance distribution of the image of a certain point object, according to wave diffraction optics is given by

\[ I(x,y) = \left(\frac{1}{\lambda^2 R^2}\right) \int \int P(u,v) \exp[-2\pi i (xu + yv)/\lambda R] \, du \, dv \, |^2, \tag{1a} \]

where \( \lambda \) is the wavelength of the object radiation and \( i = \sqrt{-1} \). \( P(u,v) \) is the pupil function of the system corresponding to this point object, and if \( A(u,v) \) and \( W(u,v) \) represent the amplitude and aberration of the light wave at a point \( (u,v) \) on its exit pupil, then

\[ P(u,v) = A(u,v) \exp[2\pi i W(u,v)/\lambda], \text{ inside the pupil,} \]
\[ = 0, \text{ outside the pupil.} \tag{1b} \]

The integration in Eq. (1a) is carried out over the clear or illuminated region of the pupil. \( S \) represents the area of this region.

The aberration function \( W(u,v) \) represents the deviation of the optical wavefront at the exit pupil from a spherical wavefront, called the reference
sphere, measured along a ray passing through the point \((u,v)\). The aberration \(W(u,v)\) is considered positive if a ray from the point object passing through the point \((u,v)\) has to travel a longer optical path in reaching the reference sphere than a reference ray which passes through the center of the exit pupil. The optical wavefront and the reference sphere pass through the center of the exit pupil. The reference sphere has a radius of curvature \(R\). Its center of curvature is generally chosen to be at the Gaussian image of the point object. The center of curvature may also be chosen such that the variance of \(W(u,v)\) across the pupil is minimized. In either case, the center of curvature defines the origin of the \((x,y)\) image plane, which is parallel to the \((u,v)\) plane. The line joining the centers of the exit pupil and the reference sphere defines a reference axis. In a system with an axis of rotational symmetry, it may be considered as the reference axis.

The LOS for the point object as perceived by the optical system is determined by the centroid of its PSF. If \(<x>\) and \(<y>\) represent the coordinates of this centroid, then

\[
<x> = E^{-1} \int_{-\infty}^{\infty} x I(x,y) \, dx \, dy, \quad (2a)
\]

and

\[
<y> = E^{-1} \int_{-\infty}^{\infty} y I(x,y) \, dx \, dy, \quad (2b)
\]

where

\[
E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x,y) \, dx \, dy, \quad (3a)
\]

is the total power (energy) in the image. Applying Parseval's theorem to Eq. (1), we find that \(E\) is also given by

\[
E = \oint_S |P(u,v)|^2 \, du \, dv
\]

\[
= \oint_S I(u,v) \, du \, dv, \quad (3b)
\]
where
\[ I(u,v) = A^2(u,v) \]  \hspace{1cm} (4)

is the irradiance at a pupil point \((u,v)\). Equations (3a) and (3b) represent conservation of energy; i.e., the total energy in the image is equal to the total energy in the pupil. If \(<\alpha>\) and \(<\beta>\) represent the angular LOS, they are given by
\[ <\alpha> = <x>/R \]  \hspace{1cm} (5a)
and
\[ <\beta> = <y>/R. \]  \hspace{1cm} (5b)

where we have assumed that the angles are small so that they are approximately equal to their tangents.

The OTF of an imaging system is equal to the Fourier transform of its PSF. Thus if \(\tau(\xi,\eta)\) represents the OTF corresponding to a spatial frequency \((\xi,\eta)\), then
\[ \tau(\xi,\eta) = E^{-1} \int_{-\infty}^{\infty} I(x,y) \exp[2\pi i(\xi x + \eta y)] \, dx \, dy. \]  \hspace{1cm} (6)

Differentiating both sides of Eq. (6) with respect to \(\xi\) and evaluating the result at \(\xi = \eta = 0\), we find that
\[ <x> = \frac{1}{2\pi i} \left( \frac{\partial \tau}{\partial \xi} \right)_{\xi=\eta=0}. \]  \hspace{1cm} (7a)

Similarly,
\[ <y> = \frac{1}{2\pi i} \left( \frac{\partial \tau}{\partial \eta} \right)_{\xi=\eta=0}. \]  \hspace{1cm} (7b)

Thus the centroid of the PSF of an optical system is given by the slope of its corresponding OTF at the origin. However, since \(<x>\) and \(<y>\) are real,
only the slope of the imaginary part of the OTF at the origin contributes to the centroid.\textsuperscript{2} Thus, we may write

\[
<x> = \frac{1}{2\pi} \left( \frac{\partial \text{Im} T}{\partial \xi} \right)_{\xi = \eta = 0}, \quad (8a)
\]

and

\[
<y> = \frac{1}{2\pi} \left( \frac{\partial \text{Im} T}{\partial \eta} \right)_{\xi = \eta = 0}. \quad (8b)
\]

The OTF is also given by the autocorrelation of the pupil function as may be seen by substituting Eq. (1) into Eq. (6). Thus,

\[
\tau(\xi, \eta) = \frac{1}{E \Sigma} \iint P(u, v) P^*(u - \lambda R \xi, v - \lambda R \eta) \, du \, dv, \quad (9)
\]

where * indicates a complex conjugate, and \( \Sigma \) is the region of overlap of two pupils centered at \((0,0)\) and \((\lambda R \xi, \lambda R \eta)\). Substituting Eq. (9) into Eq. (8), we obtain\textsuperscript{3}

\[
<x> = -\frac{\lambda R}{2\pi E} \iint \text{Im} \left[ P(u, v) \frac{\partial P^*(u, v)}{\partial u} \right] \, du \, dv, \quad (10)
\]

and a similar equation for \(<y>\). Substituting Eq. (1b) into Eq. (10), we obtain

\[
<x> = \frac{R}{E} \iint I(u, v) \frac{\partial W(u, v)}{\partial u} \, du \, dv. \quad (11a)
\]

Similarly,

\[
<y> = \frac{R}{E} \iint I(u, v) \frac{\partial W(u, v)}{\partial v} \, du \, dv. \quad (11b)
\]

Since \( R(\partial W/\partial u) \) and \( R(\partial W/\partial v) \) represent the ray aberrations,\textsuperscript{4} i.e., the image-plane coordinates of a ray passing through the pupil point \((u,v)\), Eq. (11) shows that the centroid of the PSF according to wave diffraction optics is identical with that according to ray geometrical optics.\textsuperscript{5} (The PSF given by ray geometrical optics is called the spot diagram).
From Eq. (11), we also note that amplitude variations across the pupil affect the LOS only if it is aberrated. In the absence of aberrations, the PSF centroid lies at the center of curvature of the reference sphere regardless of the shape of the pupil and the amplitude distribution across it. This may also be seen from Eqs. (1) and (3). From Eq. (1), we note that if \( W(u,v) = 0 \), then \( I(x,y) = I(-x,-y) \). Hence the symmetry of the aberration-free PSF yields its centroid at the origin of the \((x,y)\) image plane. Similarly, since the aberration-free OTF is real (see Eq. (9)), Eq. (8) also gives the centroid at the origin.

It should be noted that the peak value of an aberrated PSF may or may not lie at its origin, depending on the magnitude and the type of aberration, whether or not the amplitude across the pupil is uniform. However, the peak value of an aberration-free PSF always lies at its origin, regardless of the amplitude variation across the pupil. This may be seen from Eq. (1). If we let

\[
f(u,v) = P(u,v) \exp[-2\pi i(xu + yv)/\lambda R],
\]

then, using Hölder's inequality, we find that

\[
\frac{I(x,y)}{I(0,0)} = \frac{\iint |f(u,v)|^2 \, du \, dv}{\iint |f(u,v)|^2 \, du \, dv} \leq 1.
\]

Thus, both the centroid and the peak value of an aberration-free PSF lie at its origin, regardless of the amplitude variations across the pupil.

Equations (3), (8), and (11) give the LOS of the system in terms of its PSF, OTF, and the aberration function. In practice, given an imaging system, the most convenient expression to use would be Eq. (3), since the PSF can be measured easily by using a photodetector array. In optical design and analysis, the simplest way to obtain the LOS would be to use Eq. (11), since the aberrations must be calculated even if the other two expressions were used. Thus one may trace rays all the way up to the image plane and determine...
the centroid of the ray distribution in this plane with appropriate weighting I(u,v) of each ray.

If the illumination across the pupil is uniform, e.g., if

\[ A(u,v) = A_0, \]

and

\[ I(u,v) = A_0^2 \]

\[ = I_0, \]

and, therefore,

\[ E = S I_0, \]

then Eq. (11) reduces to

\[ <x> = \frac{R}{S} \int \int \frac{\partial W(u,v)}{\partial u} du dv \]  

\[ \text{and} \]

\[ <y> = \frac{R}{S} \int \int \frac{\partial W(u,v)}{\partial v} du dv. \]

Using Stokes theorem, the surface integral in Eq. (14) involving the derivative of the aberration function can be written in terms of its line integral along the curve bounding the surface. Thus, we may write

\[ <x> = (R/S) \oint W(u,v) \hat{u} \cdot ds \]  

\[ \text{and} \]

\[ <y> = (R/S) \oint W(u,v) \hat{v} \cdot ds, \]

where \( \hat{u} \) and \( \hat{v} \) are unit vectors along the u and v axes, respectively, \( ds \) represents an element of arc length vector of the curve bounding the pupil.

It is evident from Eq. (15) that in the case of an aberrated but uniformly illuminated pupil, the centroid of the PSF can be obtained from the value of the aberration function only along the perimeter of the pupil. Accordingly, in that case, to calculate the centroid the knowledge of the aberration across the interior of the pupil is not needed.
III. APPLICATION TO SYSTEMS WITH ANNULAR PUPILS

A. Uniform Illumination

Consider an imaging system with an aberrated but uniformly illuminated annular pupil of inner and outer radii of \( \varepsilon a \) and \( a \), respectively, where \( 0 \leq \varepsilon \leq 1 \). The area of the pupil is given by

\[
S = \pi(1-\varepsilon^2) a^2. \tag{16}
\]

Owing to the circular boundary of the pupil, it is convenient to use plane polar coordinates \((h, \theta)\), where

\[
u = h \cos \theta \tag{17a}
\]

and

\[
v = h \sin \theta, \tag{17b}
\]

so that

\[
\theta = \tan^{-1}(v/u) \tag{17c}
\]

and

\[
h = (u^2 + v^2)^{1/2}. \tag{17d}
\]

Moreover, \( 0 \leq \theta < 2\pi \) and \( \varepsilon a \leq h \leq a \). A vector element of a circular arc with a center of curvature at \((0,0)\) and passing through a point \((u,v)\) is given by

\[
ds^+ = (u \, d\theta, v \, d\theta), \tag{18}
\]

Let the aberration function in polar coordinates be \( W(h, \theta; \varepsilon) \). Substituting Eq. (18) into Eq. (15), we obtain

\[
<x> = \int \left[ \frac{R}{\pi(1-\varepsilon^2)a} \int [W(a, \theta; \varepsilon) - \varepsilon W(\varepsilon a, \theta; \varepsilon)] \cos \theta \, d\theta \right] \cos \theta \, d\theta \tag{19a}
\]
\[
<y> = \left[ \frac{R \cdot \pi (1-\varepsilon^2)}{a} \right] \int_0^{2\pi} \left[ W(a, \theta; \varepsilon) - \varepsilon W(a, \theta; 0) \right] \sin \theta \, d\theta.
\]  

(19b)

Let us expand the aberration function \( W(h, \theta; \varepsilon) \) in terms of a complete set of Zernike annular polynomials \( R_m^m(\rho; \varepsilon) \cos m\theta \) and \( R_m^m(\rho; \varepsilon) \sin m\theta \) which are orthogonal over the annular pupil, where

\[
\rho = h/a.
\]

(20)

Thus, we may write

\[
W(h, \theta; \varepsilon) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} c_{nm} \sqrt{2(n+1)} R_n^m(\rho; \varepsilon) (c_n \cos m\theta + s_n \sin m\theta),
\]

(21)

where \( n \) and \( m \) are positive integers (including zero), \( n-m \geq 0 \) and even,

\[
\epsilon_m = \begin{cases} 
1/\sqrt{2}, & m = 0 \\
1, & m \neq 0,
\end{cases}
\]

(22)

and \( c_{nm} \) and \( s_{nm} \) are the expansion coefficients. Note that

\[
s_{n0} = 0.
\]

(23)

The radial polynomials obey the orthogonality relation

\[
\int_{0}^{1} \frac{1}{r} R_n^m(\rho; \varepsilon) R_{n'}^m(\rho; \varepsilon) \rho \, d\rho = \frac{(1-\varepsilon^2)}{2(n+1)} \delta_{nn'},
\]

(24)

where \( \delta_{nn'} \) is a Kronecker delta. The magnitude of each expansion coefficient (except \( c_{00} \)) represents the standard deviation of the corresponding aberration term across the annular pupil.
Substituting Eq. (21) into Eq. (19), and noting the orthogonality of trigonometric functions, we obtain

\[
<x> = \left[ \frac{R}{(1-\varepsilon^2)a} \right] \sum_{n=1}^{\infty} \frac{\sqrt{2(n+1)}}{n} \left[ R_n^1(1;\varepsilon) - \varepsilon R_n^1(\varepsilon;\varepsilon) \right] c_{n1} \quad (25a)
\]

and

\[
<y> = \left[ \frac{R}{(1-\varepsilon^2)a} \right] \sum_{n=1}^{\infty} \frac{\sqrt{2(n+1)}}{n} \left[ R_n^1(1;\varepsilon) - \varepsilon R_n^1(\varepsilon;\varepsilon) \right] s_{n1}, \quad (25b)
\]

where a prime on the summation sign indicates a summation over odd integral values of \( n \). Thus the only aberrations that contribute to the LOS error are those with \( m = 1 \). Aberrations of the type \( R_n^1(\rho;\varepsilon) \cos \theta \) contribute to \(<x>\) and those of the type \( R_n^1(\rho;\varepsilon) \sin \theta \) contribute to \(<y>\). This is also evident from the symmetry of the aberrations. Since \( R_n^1(\rho;\varepsilon) \) consists of terms in \( \rho^n, \rho^{n-2}, \ldots, \) and \( \rho \), therefore, for example, \( R_n^1(\rho;\varepsilon) \cos \theta \) is symmetric in \( v \) but not in \( u \). Hence the PSF is symmetric in \( y \) as may be seen from Eq. (1) noting that the amplitude across the pupil is uniform. Accordingly, \(<y> = 0\) for this aberration.

It is well known that, for small aberrations, the Strehl ratio for an optical system depends on the variance of its aberrations. Therefore, two aberration terms with \( m = 1 \), but different values of \( n \), affect the Strehl ratio in the same way if their coefficients are equal in magnitude. However, it is evident from Eq. (25) that their contribution to the LOS error is not the same; an aberration of higher order contributes a larger error for the same value of the coefficient. Hence, in tolerancing an optical system, one must be careful in allocating equal standard deviation to two aberration terms that also contribute to the LOS error.

In the case of a circular aperture \( (\varepsilon = 0) \),

\[
R_n^1(1;0) = 1. \quad (26)
\]
Therefore, Eq. (25) simplifies to

\[
<x> = \left(\frac{R}{a}\right) \sum_{n=1}^{\infty} \sqrt{2(n+1)} \frac{c_n}{n!}
\quad (27a)
\]

and

\[
<y> = \left(\frac{R}{a}\right) \sum_{n=1}^{\infty} \sqrt{2(n+1)} \frac{s_n}{n!}
\quad (27b)
\]

We note that as in the case of an annular pupil, two aberration terms with equal standard deviation but different order \(n\) do not contribute equally to the LOS error. For a given standard deviation, a higher-order aberration gives a larger LOS error compared to a lower-order aberration.

B. Radially Symmetric Illumination

Let \(A(h)\) and \(I(h)\) describe the radially symmetric amplitude and irradiance distributions across the aberrated annular pupil, where

\[
I(h) = A^2(h).
\quad (28)
\]

In polar coordinates, Eq. (11) for the LOS can be written

\[
<x> = \frac{R}{E} \int_a^2 \int_0^{2\pi} I(h) \left[ \cos \theta \frac{\partial W(h, \theta; \varepsilon)}{\partial h} - \frac{\sin \theta}{h} \frac{\partial W(h, \theta; \varepsilon)}{\partial \theta} \right] h \, dh \, d\theta
\quad (29a)
\]

and

\[
<y> = \frac{R}{E} \int_a^2 \int_0^{2\pi} I(h) \left[ \sin \theta \frac{\partial W(h, \theta; \varepsilon)}{\partial h} + \frac{\cos \theta}{h} \frac{\partial W(h, \theta; \varepsilon)}{\partial \theta} \right] h \, dh \, d\theta,
\quad (29b)
\]

where

\[
E = 2\pi \int_a^2 I(h) h \, dh \, d\theta.
\quad (30)
\]
We now expand the aberration function in terms of annular polynomials $S_n^m(\rho; \varepsilon) \cos m\theta$ and $S_n^m(\rho; \varepsilon) \sin m\theta$ that are orthogonal over the radially symmetric illuminated annular pupil. Thus we write

$$W(h, \theta; \varepsilon) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \frac{\sqrt{2(n+1)}}{\sqrt{2(n+1)}} S_n^m(\rho; \varepsilon)(c_{nm} \cos m\theta + s_{nm} \sin m\theta),$$

where

$$S_n^m(\rho; \varepsilon) = \frac{(-1)^{(n-m)/2}}{n!} \sum_{i \geq i}^{} (n-2i+1) <R_n^m(\rho; \varepsilon) S_{n-2i}^m(\rho; \varepsilon)> S_{n-2i}^m(\rho; \varepsilon)$$

is a radial polynomial with properties similar to those of $R_n^m(\rho; \varepsilon)$. It should be evident that the aberration coefficients $c_{nm}$ and $s_{nm}$ in Eq. (31) are different from those in Eq. (21). In Eq. (32)

$$<R_n^m(\rho; \varepsilon) S_{n-2i}^m(\rho; \varepsilon)> = \int_{\varepsilon}^{2\pi} \int_{\varepsilon}^{2\pi} R_n^m(\rho; \varepsilon) S_{n-2i}^m(\rho; \varepsilon) A(\rho) \rho \, d\rho / \int_{\varepsilon}^{2\pi} A(\rho) \rho \, d\rho,$$

and $M_n^m$ is a normalization constant such that

$$\int_{\varepsilon}^{2\pi} S_n^m(\rho; \varepsilon) S_n^m(\rho; \varepsilon) A(\rho) \rho \, d\rho / \int_{\varepsilon}^{2\pi} A(\rho) \rho \, d\rho = \frac{1}{n+1} \delta_{nn'}.$$

Substituting Eq. (31) into Eq. (29) we find that

$$<x> = \frac{\pi a R}{E} \sum_{n=1}^{\infty} \sqrt{2(n+1)} c_{n1} \int_{\varepsilon}^{2\pi} I(\rho) \frac{\partial}{\partial \rho} \left[ \rho \frac{S_n^1(\rho; \varepsilon)}{S_n^1(\rho; \varepsilon)} \right] d\rho$$

and

$$<y> = \frac{\pi a R}{E} \sum_{n=1}^{\infty} \sqrt{2(n+1)} s_{n1} \int_{\varepsilon}^{2\pi} I(\rho) \frac{\partial}{\partial \rho} \left[ \rho \frac{S_n^1(\rho; \varepsilon)}{S_n^1(\rho; \varepsilon)} \right] d\rho.$$
Once again the only aberrations that contribute to the LOS error are those with \( m = 1 \). Aberrations of the type \( S^m_n(p; \varepsilon) \cos \theta \) contribute to \( <x> \) and those of the type \( S^m_n(p; \varepsilon) \sin \theta \) contribute to \( <y> \), and this can be explained from the symmetry properties of the aberrated PSF in a manner similar to the discussion following Eq. (25).

IV. **NUMERICAL EXAMPLES**

A. **Uniform Illumination**

Using polar coordinates

\[
x = r \cos \phi \\
y = r \sin \phi,
\]

it can be shown that for an aberration-free optical system with a uniformly illuminated annular pupil, Eq. (1) reduces to

\[
I(r; \varepsilon) = \frac{1}{(1-\varepsilon^2)^2} \left[ \frac{2J_1(\pi r_s)}{\pi r_s} - \varepsilon^2 \frac{2J_1(\pi r_s)}{\pi r_s} \right]^2 I(0; \varepsilon). \tag{37}
\]

Here

\[
r = (x^2 + y^2)^{1/2}
\]

is the radial distance of a point \((x, y)\) in the image plane from its origin, and

\[
r_s = r/\lambda F
\]

is a scaled radial distance of an image point. \( J_1(\cdot) \) is the first-order Bessel function of the first kind, and

\[
F = R/2a
\]

-17-
is the f-number of the system. The central value of the PSF is given by

\[ I(0;\varepsilon) = ES/\lambda^2 R^2, \]  

(41)

where the power \( E \) in the image is given by Eq. (13c).

The line joining the center of the pupil and the centroid of the PSF for a given point object defines the LOS of the optical system in its image space for that point object. Since the aberration-free PSF is circularly symmetric about the origin of the \((x,y)\) plane, its centroid also lies at this origin, i.e., it lies at the center of curvature of the reference sphere with respect to which the aberration for the point object under consideration is zero. Hence, the line joining the origins of the \((u,v)\) and \((x,y)\) planes defines the LOS.

When aberrations are introduced into the system such that the centroid of its PSF shifts, the position of the point object as perceived by the system changes, i.e., there is a LOS error. We have shown that in the case of optical systems with pupils having circular boundaries and uniform or radially symmetric amplitude distributions, the only aberrations that contribute to the LOS error are of the type \( R_n^1(\rho;\varepsilon)\cos\theta \) and \( R_n^1(\rho;\varepsilon)\sin\theta \). Since the radial polynomials \( R_n^1(\rho;\varepsilon) \) consist of terms in \( \rho^n, \rho^{n-2}, \ldots, \) and \( \rho \), with their coefficients varying with \( \varepsilon \), there is no loss of generality if we consider aberrations of the type \( \rho^n\cos\theta \) (or \( \rho^n\sin\theta \)), where \( n \) is an odd integer, to determine their contribution to the LOS error. Thus, we consider an aberration

\[ W(h,\theta) = W_n (h/a)^n \cos\theta, \quad \varepsilon a \leq h \leq a \]  

(42a)

\[ = W_n \rho^n \cos\theta, \quad \varepsilon \leq \rho \leq 1, \]  

(42b)

where \( W_n \) is the peak aberration at the edge of the pupil relative to a value of zero at its center. Since \( \langle y \rangle \) and correspondingly \( \langle \beta \rangle \) are both zero for this type of an aberration, we shall not explicitly so state from now on. We shall, for example, refer to the centroid as simply \( \langle x \rangle \).
Substituting Eq. (42) into Eq. (20), we obtain

\[
\langle x \rangle = 2w_n P \sum_{i=0}^{(n-1)/2} e^{2i\theta}.
\]  

(43)

The corresponding angular LOS error is given by

\[
\langle \alpha \rangle = 2(w_n/D) \sum_{i=0}^{(n-1)/2} e^{2i\theta},
\]  

(44)

where

\[
D = 2a
\]  

(45)

is the outer diameter of the annular pupil. We note from Eq. (44), for example, that for circular pupils (\( \epsilon = 0 \)), one wave (\( w_n = 1\lambda \)) of aberration of the type given by Eq. (42) produces an angular LOS error of \( 2\lambda/D \). This is quite large considering that the angular radius of the (aberration-free) Airy disc is only \( 1.22 \lambda/D \).

It is interesting to note that when \( \epsilon = 0 \), the LOS error depends only on the value of \( w_n \) but not on \( n \), the power of \( \rho \) in Eq. (42b). This is consistent with our earlier observation [following Eq. (15)] that, for a uniformly illuminated pupil, the centroid of its aberrated PSF depends only on the aberration along its perimeter. In the case of a circular pupil, the variation of the aberration along its perimeter as given by Eq. (42) is the same for different orders of the aberration. Hence, for a given value of \( w_n \), even though aberrations such as tilt (\( n = 1 \)), primary coma (\( n = 3 \)), secondary coma (\( n = 5 \)), etc., which are completely different from each other across the interior of the pupil and, therefore, give completely different PSF's, nevertheless give the same centroid since the aberrations are identical on the perimeter of the pupil. This is not true for an annular pupil, in which case, although the aberration along the outer perimeter is the same for different values of \( n \), it is different along the inner perimeter. Hence, for a given value of \( w_n \), an annular pupil gives aberrated PSF's with different centroids for different orders of the aberration.
We may add that similar observations hold for balanced aberrations
represented by Zernike polynomials $R_n^1(\rho; \phi) \cos \theta$ and $R_n^1(\rho; \phi) \sin \theta$. For
example, for a given value of $W_n$, aberrations $W_n R_n^1(\rho; 0) \cos \theta$ across a
circular pupil give PSF's with the same centroid since $R_n^1(1; 0) = 1$, i.e.,
since the aberration along the perimeter of the pupil is independent of the
aberration order $n$. In the case of an annular pupil, $R_n^1(1; \phi)$ as well as
$R_n^1(\phi; \epsilon)$ depend on the value of $n$. Hence, PSF's with different centroids are
obtained for different aberration orders $n$.

Wavefront Tilt

If we let $n = 1$ in Eq. (42), the aberration is simply a tilt of the
optical wavefront through the center of the pupil. The PSF shifts and its
centroid moves from $(0,0)$ to

$$<x> = 2W_1 F$$

(46a)
corresponding to

$$<\alpha> = 2W_1/D.$$

(46b)

Thus, for example, one wave of wavefront tilt, produces an angular LOS error
of $2\lambda/D$, regardless of the value of $\epsilon$. Of course, in this case, the
position of the peak value of the PSF is coincident with the position of its
centroid.

Primary Coma

If we let $n = 3$, in Eq. (42) the aberration obtained is called classical
primary coma. The centroid in this case is given by

$$<x> = 2W_3 F(1 + \epsilon^2).$$

(47)

Thus, for a given value of $W_3$, the centroid for an annular pupil shifts by a
factor of $(1 + \epsilon^2)$ larger than that for a circular pupil.
For small values of \( W_3 \), the peak value of the aberrated PSF occurs at a point such that if the aberration is measured with respect to a reference sphere centered at this point, the variance of the aberration across the annular pupil is minimum. From the properties of the Zernike annular polynomials, we find that the polynomial \( R_3^1(\rho; \epsilon)\cos \theta \) gives the optimum combination of \( \rho^3 \cos \theta \) and \( \rho \cos \theta \) terms leading to a minimum variance. Since

\[
R_3^1(\rho; \epsilon) = \frac{3(1+\epsilon^2)\rho^3 - 2(1+\epsilon^2+\epsilon^4)\rho}{(1-\epsilon^2)[(1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}},  \tag{48}
\]

we note that, for small values of \( W_3 \), the peak value of the aberrated PSF occurs at

\[
x_m = 4W_3 \frac{F}{(1 + \epsilon^2 + \epsilon^4)/3(1 + \epsilon^2)},  \tag{49}
\]

where the subscript \( m \) refers to the point corresponding to minimum aberration variance. From the form of the aberration, it is understood that \( y_m = 0 \).

Thus, an amount \( W_3 \) of primary u-coma shifts the centroid and peak of the PSF by different amounts, the movement of the peak being \( 2(1 + \epsilon^2 + \epsilon^4)/3(1 + \epsilon^2)^2 \) of the movement of the centroid. As an example, a circular pupil aberrated by a quarter wave of primary u-coma (\( W_3 = \lambda/4 \)) gives an aberrated PSF with a centroid at \( \langle x \rangle = \lambda F/2 \) and a peak value at \( x_m = \lambda F/3 \).

For large values of \( W_3 \), the peak of the aberrated PSF does not occur at the point corresponding to minimum aberration variance. For example, in the case of circular pupils, the peak lies approximately at the point corresponding to \( W_1 = (2/3)W_3 \) only when \( W_3 \leq 0.7\lambda \). For larger values of \( W_3 \), the peak occurs closer to the origin than the point corresponding to minimum aberration variance. For \( W_3 \geq 1.6\lambda \), the distance of the peak from the origin does not increase monotonically, but fluctuates as \( W_3 \) increases. Since, according to Eq. (47), the distance of the centroid increases linearly with \( W_3 \), it is clear that the separation between the locations of the centroid and peak increases as \( W_3 \) increases.

-21-
If we let \( \phi = 0 \) and \( A(h) = A_0 \) in Eq. (A6), we find that, along the \( x \)-axis, the PSF aberrated by primary coma of the type \( W_3 \beta^3 \cos \theta \) can be written

\[
I(x;\varepsilon) = \frac{[I(0;\varepsilon)/(1-\varepsilon^2)]}{[\int_{\varepsilon^2}^1 J_0(\pi B) \, dt]^2}, \tag{50}
\]

where

\[
B = (2tW_3 - x_s)t^{1/2}, \tag{51}
\]

\[
x_s = x/\lambda F, \tag{52}
\]

and \( W_3 \) is in units of \( \lambda \). Figure 1 shows how \( I(x;\varepsilon) \) normalized by the aberration-free central value \( I(0;\varepsilon) \) given by Eq. (41) varies with \( x \) for several typical values of \( W_3 \) varying from 0 to 2\( \lambda \), and \( \varepsilon = 0 \) and \( \varepsilon^2 = 0.5 \). Figure 2 shows how the irradiance \( I_m(W_3;\varepsilon) \) at \( x_m \), the peak irradiance \( I_p(W_3;\varepsilon) \) and the irradiance \( I_c(W_3;\varepsilon) \) at \( <x> \) vary with \( W_3 \). Figure 3 shows how \( x_m \), \( x_p \) (the point at which peak irradiance occurs), and \( <x> \) vary with \( W_3 \). The observations made above about the PSF's aberrated by primary coma are evident from these figures. Several typical values of \( x_m \), \( x_p \), and \( <x> \) and the corresponding irradiances \( I_m \), \( I_p \), and \( I_c \) are noted in Table 1, where the numbers without parentheses are for \( \varepsilon = 0 \) and those with parenthesis are for \( \varepsilon^2 = 0.5 \). The aberrated central irradiance \( I(0) \) is also given in this Table. The irradiance values \( I(0) \) and \( I_m \) are the Strehl ratios calculated for primary and balanced primary coma, respectively, in an earlier paper.\(^{10}\)

Secondary Coma

If we let \( n = 5 \) in Eq. (42), the aberration obtained is called classical secondary coma. The centroid in this case is given by

\[
<x> = 2W_5F(1 + \varepsilon^2 + \varepsilon^4). \tag{53}
\]
Figure 1. PSF $I(x_0; \varepsilon)$ for several typical values of primary coma aberration $W_3$ in units of $\lambda$. The amplitude $A(u,v)$ across the pupil is uniform. The PSF's are normalized by the aberration-free central value given by Eq. (41). $x_0$ represents $x$ in units of $\lambda F$.

(a) $\varepsilon = 0$, (b) $\varepsilon^2 = 0.5$. 

-23-
Figure 2. Variation of $I_m$, $I_p$, and $I_c$ with $W_3$, where the irradiances are in units of the aberration-free central irradiance and $W_3$ is in units of $\lambda$. (a) $\epsilon = 0$, (b) $\epsilon^2 = 0.5$. 

-24-
Figure 3. Variation of $x_m$, $x_p$, and $<x>$ with $W_3$. The $x$-values are in units of $\lambda F$ and $W_3$ is in units of $\lambda$. (a) $\varepsilon = 0$, (b) $\varepsilon^2 = 0.5$
Table 1. Typical values of $x_m$, $x_p$, and $\langle x \rangle$ in units of $\lambda F$, and the corresponding irradiances $I_m$, $I_p$, and $I_c$ in units of the aberration-free central irradiance for PSF's aberrated by primary coma, $W(h,\theta) = W_3 p^3 \cos \theta$. The units of $W_3$ are $\lambda$. The aberrated central value $I(0)$ is also given here. The amplitude $A(u,v)$ across the pupil is uniform. The numbers without parentheses are for a circular pupil ($\epsilon = 0$) and those with parentheses are for an annular pupil with $\epsilon^2 = 0.5$.

<table>
<thead>
<tr>
<th>$W_3$</th>
<th>$x_m$</th>
<th>$x_p$</th>
<th>$\langle x \rangle$</th>
<th>$I_m$</th>
<th>$I_p$</th>
<th>$I_c$</th>
<th>$I(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.67</td>
<td>0.66</td>
<td>1.00</td>
<td>0.8712</td>
<td>0.8712</td>
<td>0.6535</td>
<td>0.3175</td>
</tr>
<tr>
<td></td>
<td>(0.78)</td>
<td>(0.78)</td>
<td>(1.50)</td>
<td>(0.9283)</td>
<td>(0.9283)</td>
<td>(0.0524)</td>
<td>(0.0403)</td>
</tr>
<tr>
<td>1.0</td>
<td>1.33</td>
<td>1.30</td>
<td>2.00</td>
<td>0.5708</td>
<td>0.5717</td>
<td>0.1445</td>
<td>0.0791</td>
</tr>
<tr>
<td></td>
<td>(1.56)</td>
<td>(1.55)</td>
<td>(3.00)</td>
<td>(0.7410)</td>
<td>(0.7412)</td>
<td>(0.1357)</td>
<td>(0.0319)</td>
</tr>
<tr>
<td>1.5</td>
<td>2.00</td>
<td>1.80</td>
<td>3.00</td>
<td>0.2715</td>
<td>0.2844</td>
<td>0.0004</td>
<td>0.0618</td>
</tr>
<tr>
<td></td>
<td>(2.33)</td>
<td>(2.32)</td>
<td>(4.50)</td>
<td>(0.5063)</td>
<td>(0.5064)</td>
<td>(0.0139)</td>
<td>(0.0010)</td>
</tr>
<tr>
<td>2.0</td>
<td>2.67</td>
<td>1.57</td>
<td>4.00</td>
<td>0.0864</td>
<td>0.1978</td>
<td>0.0061</td>
<td>0.0341</td>
</tr>
<tr>
<td></td>
<td>(3.11)</td>
<td>(3.07)</td>
<td>(6.00)</td>
<td>(0.2936)</td>
<td>(0.2946)</td>
<td>(0.0160)</td>
<td>(0.0000)</td>
</tr>
</tbody>
</table>
The variance of an aberration of the type $\rho^5 \cos \theta$ is reduced if an amount $-[(1+\epsilon^2+\epsilon^4+\epsilon^6)/2(1+\epsilon^2)]W_5$ of $\rho \cos \theta$ aberration is introduced. Hence, the corresponding value of $x_m$ is given by

$$x_m = W_5 F (1+\epsilon^2+\epsilon^4+\epsilon^6)/(1+\epsilon^2).$$ \hspace{1cm} (54)

Thus, for example, in the case of a circular pupil, the variance is reduced by a factor of 4 if $W_5 = -0.5W_5$. Accordingly, for small values of $W_5$, the peak of the corresponding aberrated PSF occurs at $x_m = W_5 F$, while the centroid occurs at $<x> = 2W_5 F$. The aberrated PSF along the x-axis in this case is given by Eq. (50) where

$$B = (2t^2 W_5 - x_s)t^{1/2},$$ \hspace{1cm} (55)

and $W_5$ is in units of $\lambda$. Figure 4 shows how $I(x;\epsilon)$ varies with $x$ for several values of $W_5$, and $\epsilon = 0$ and $\epsilon^2 = 0.5$. The values of $x_m$, $x_p$, and $<x>$, and the corresponding irradiances $I_m$, $I_p$, and $I_c$ for the values of $W_5$ considered are noted in Table 2.

The variance of the aberration $\rho^5 \cos \theta$ is reduced even further if an appropriate amount of $\rho^3 \cos \theta$ aberration is also introduced. For a given value of $W_5$, the appropriate amounts of $W_3$ and $W_1$ that give minimum variance may be obtained from the radial Zernike annular polynomial $R_3^1(\rho;\epsilon)$, where

$$R_3^1(\rho;\epsilon) = \frac{10(1+4\epsilon^2+\epsilon^6)\rho^5 - 12(1+4\epsilon^2+\epsilon^4+\epsilon^6)\rho^3 + 3(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)\rho}{(1-\epsilon^2)^2 ((1+4\epsilon^2+\epsilon^6)(1+9\epsilon^2+9\epsilon^4+\epsilon^6))^{1/2}}.$$ \hspace{1cm} (56)

Thus, in the case of a circular pupil, the variance is reduced by a factor of 100 if we introduce $\rho \cos \theta$ and $\rho^3 \cos \theta$ aberrations with $W_1 = 0.3W_5$ and $W_3 = -1.2W_5$. Hence the peak value of the PSF for a circular pupil aberrated by a small value of $W_5$ and $W_3 = -1.2W_5$ occurs at $x_m = -0.6W_5 F$. According to Eq. (43), the corresponding centroid occurs at $<x> = -0.4W_5 F$. Therefore, the
Figure 4. Same as Figure 1 except that the aberration is secondary coma $W_5$. 

-28-
Table 2. Same as Table 1, except that the aberration is secondary coma, \( W(h, \theta) = W_5 \rho^5 \cos \theta \), where \( W_5 \) is in units of \( \lambda \).

<table>
<thead>
<tr>
<th>( W_5 )</th>
<th>( x_m )</th>
<th>( x_p )</th>
<th>( &lt;x&gt; )</th>
<th>( I_m )</th>
<th>( I_p )</th>
<th>( I_c )</th>
<th>( I(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.50</td>
<td>0.49</td>
<td>1.00</td>
<td>0.8150</td>
<td>0.8153</td>
<td>0.4114</td>
<td>0.4955</td>
</tr>
<tr>
<td>(0.63)</td>
<td>(0.62)</td>
<td>(1.75)</td>
<td>(0.8400)</td>
<td>(0.8402)</td>
<td>(0.0760)</td>
<td>(0.1768)</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>0.83</td>
<td>2.00</td>
<td>0.4464</td>
<td>0.4664</td>
<td>0.0025</td>
<td>0.2332</td>
</tr>
<tr>
<td>(1.25)</td>
<td>(1.21)</td>
<td>(3.50)</td>
<td>(0.4948)</td>
<td>(0.4966)</td>
<td>(0.0282)</td>
<td>(0.0002)</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>1.50</td>
<td>0.81</td>
<td>3.00</td>
<td>0.1685</td>
<td>0.3237</td>
<td>0.0098</td>
<td>0.1873</td>
</tr>
<tr>
<td>(1.88)</td>
<td>(1.74)</td>
<td>(5.25)</td>
<td>(0.2003)</td>
<td>(0.2196)</td>
<td>(0.0130)</td>
<td>(0.0009)</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>2.00</td>
<td>1.11</td>
<td>4.00</td>
<td>0.0420</td>
<td>0.2523</td>
<td>0.0073</td>
<td>0.1389</td>
</tr>
<tr>
<td>(2.50)</td>
<td>(1.71)</td>
<td>(7.00)</td>
<td>(0.0573)</td>
<td>(0.1478)</td>
<td>(0.0074)</td>
<td>(0.0065)</td>
<td></td>
</tr>
</tbody>
</table>
separation between the peak and the centroid is $0.2W_5^*$. For large values of $W_5$, minimization of variance with respect to $W_3$ and $W_1$ does not lead to a maximum of the PSF.

As an example, we consider the PSF aberrated by an aberration

$$W(h, \theta) = (W_5 \theta^5 + W_3 \theta^3) \cos \theta,$$  \hspace{1cm} (57a)

where

$$W_3 = -1.2 W_5 (1+4\varepsilon^2+4\varepsilon^4+\varepsilon^6)/(1+4\varepsilon^2+\varepsilon^4).$$  \hspace{1cm} (57b)

According to Eq. (56), the point in the image plane with respect to which the aberration variance is minimized is given by

$$x_m = -0.6 W_5 F(1+4\varepsilon^2+10\varepsilon^4+4\varepsilon^6+\varepsilon^8)/(1+4\varepsilon^2+\varepsilon^4).$$  \hspace{1cm} (58)

Substituting Eq. (57) into Eq. (19), we obtain the centroid

$$\langle x \rangle = -W_5 F(0.4+2\varepsilon^2+7.2\varepsilon^4+2\varepsilon^6+0.4\varepsilon^8)/(1+4\varepsilon^2+\varepsilon^4).$$  \hspace{1cm} (59)

The aberrated PSF along the x-axis is obtained by substituting Eq. (57) into Eq. (A6). We find that it is given by Eq. (50), where

$$B = (2t^2 W_5 + 2tW_3 - x_0)^{-1/2}.$$  \hspace{1cm} (60)

Figure 5 shows the aberrated PSF $I(x; \varepsilon)$ for several values of $W_5$ with $W_3$ given by Eq. (57), and $\varepsilon = 0$ and $\varepsilon^2 = 0.5$. The values of $x_m$, $x_p$, and $\langle x \rangle$, and the corresponding irradiances $I_m$, $I_p$, $I_c$ are given in Table 3. Note that $x_m$, $x_p$, and $\langle x \rangle$ are all negative. Moreover, their magnitude for the values of $W_5$ considered, especially in the case of $\varepsilon^2 = 0.5$, is very large. Therefore, in Figure 5, the horizontal coordinate is chosen to be $x_s - x_m$. 

-30-
Figure 5. Same as Figure 1 except that the aberration is a combination of primary and secondary coma given by Eq. (57). Note that in this figure the horizontal coordinate is $x_s - x_m$. 
Table 3. Same as Table 1, except that the aberration is a combination of primary and secondary coma given by Eq. (57).

<table>
<thead>
<tr>
<th>$W_5$</th>
<th>$x_m$</th>
<th>$x_p$</th>
<th>$\langle x \rangle$</th>
<th>$I_m$</th>
<th>$I_p$</th>
<th>$I_c$</th>
<th>$I(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.60</td>
<td>-0.59</td>
<td>-0.40</td>
<td>0.9676</td>
<td>0.9682</td>
<td>0.8763</td>
<td>0.3721</td>
</tr>
<tr>
<td>(5.0)</td>
<td>(-5.60)</td>
<td>(-5.60)</td>
<td>(-5.35)</td>
<td>(0.8832)</td>
<td>(0.8832)</td>
<td>(0.0005)</td>
<td>(0.0039)</td>
</tr>
<tr>
<td>2.0</td>
<td>-1.20</td>
<td>-1.18</td>
<td>-0.80</td>
<td>0.8765</td>
<td>0.8784</td>
<td>0.5870</td>
<td>0.0030</td>
</tr>
<tr>
<td>(10.0)</td>
<td>(-11.19)</td>
<td>(-11.23)</td>
<td>(-10.69)</td>
<td>(0.6101)</td>
<td>(0.6128)</td>
<td>(0.0000)</td>
<td>(0.0014)</td>
</tr>
<tr>
<td>3.0</td>
<td>-1.80</td>
<td>-1.77</td>
<td>-1.20</td>
<td>0.7429</td>
<td>0.7459</td>
<td>0.2981</td>
<td>0.0014</td>
</tr>
<tr>
<td>(15.0)</td>
<td>(-16.79)</td>
<td>(-16.94)</td>
<td>(-16.04)</td>
<td>(0.3353)</td>
<td>(0.3558)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>4.0</td>
<td>-2.40</td>
<td>-2.37</td>
<td>-1.60</td>
<td>0.5886</td>
<td>0.5914</td>
<td>0.1173</td>
<td>0.0465</td>
</tr>
<tr>
<td>(20.0)</td>
<td>(-22.38)</td>
<td>(-22.80)</td>
<td>(-21.38)</td>
<td>(0.1493)</td>
<td>(0.2296)</td>
<td>(0.0000)</td>
<td>(0.0008)</td>
</tr>
</tbody>
</table>

-32-
B. Gaussian Illumination

As an example of a radially symmetric illumination, we consider a Gaussian annular pupil, i.e. one for which

\begin{align}
A(h) &= A_0 \exp(-\gamma h^2/a^2) \quad (61a) \\
&= A_0 \exp(-\gamma r^2), \quad (61b)
\end{align}

where \( \gamma > 0 \). The aberration-free PSF for such a pupil may be obtained by substituting Eq. (61) into Eq. (1) or Eq. (A6) [Appendix A] and letting \( W(h,0) = 0 \). Thus, we obtain

\begin{equation}
I(r;\gamma;\varepsilon) = \left[ \gamma/\left(e^{-\gamma} - e^{-\gamma \varepsilon^2}\right) \right]^2 I(0,\gamma;\varepsilon) \left[ \int_{\varepsilon^2}^{1} \exp(-\gamma t) J_0(\pi \varepsilon t^{1/2}) dt \right]^2, \quad (62)
\end{equation}

where

\begin{equation}
I(0,\gamma;\varepsilon) = \left( \pi a^2 A_0^2/\lambda R \right)^2 \left[ e^{-\gamma} - e^{-\gamma \varepsilon^2} \right] / \gamma^2 \quad (63)
\end{equation}

is the central value of the PSF.

Primary Coma

If we let \( \phi = 0 \) and substitute Eq. (61) into Eq. (A6), we find that, along the x-axis, the aberrated PSF in the presence of primary coma of the type \( W_3 P \cos \theta \) may be written

\begin{equation}
I(x;\gamma;\varepsilon) = \left[ \gamma/\left(e^{-\gamma} - e^{-\gamma \varepsilon^2}\right) \right]^2 I(0,\gamma;\varepsilon) \left[ \int_{\varepsilon^2}^{1} \exp(-\gamma t) J_0(\pi \varepsilon t^{1/2}) dt \right]^2, \quad (64)
\end{equation}

where \( B \) is given by Eq. (51).

Substituting Eq. (42) with \( n = 3 \) and Eq. (61) into Eq. (29), we find that the centroid of the PSF is given by

\begin{equation}
<x> = 4 W_3 P \left( \frac{1}{2\gamma} + \frac{e^2 e^{-2\gamma \varepsilon^2} - e^{-2\gamma}}{e^{-2\gamma \varepsilon^2} - e^{-2\gamma}} \right) \quad (65)
\end{equation}
From the radial polynomial $S^1_3(p;\varepsilon)$ for the Gaussian illumination\(^7\), we find that the point in the image plane with respect to which the aberration variance across the Gaussian pupil is minimized is given by

$$x_m = 2\, W_3 F \left[ \frac{2}{Y} + \frac{\gamma (e^+ e^{-2\gamma^2} - e^{-\gamma})}{e^{-2\gamma^2 (1+\gamma^2)} - e^{-\gamma(1+\gamma)}} \right].$$

(66)

For small values of $W_3$, the peak value of the PSF occurs at $x_m$.

We now consider some numerical results for $\gamma = 1$, which corresponds to a Gaussian illumination with an irradiance of $e^{-2}$ at the edge of a circular pupil relative to the irradiance at its center. Figure 6 shows how $I(x; l; \varepsilon)$ varies with $x$ for several values of $W_3$ and $\varepsilon^2 = 0$ and 0.5. The values of $x_m$, $x_p$, and $\langle x \rangle$, and the corresponding irradiances $I_m$, $I_p$, and $I_c$ for these values of $W_3$ are given in Table 4.

Secondary Coma

The aberrated PSF along the $x$-axis in the presence of secondary coma of the type $W_5 p \cos \theta$ is given by Eq. (64) where $B$ is given by Eq. (55). Substituting Eq. (42) with $n = 5$ and Eq. (61) into Eq. (29), we find that the centroid of the aberrated PSF is given by

$$\langle x \rangle = 12 W_5 F \frac{e^{-2\gamma^2 (e^+ e^{2/\gamma} + 1/2\gamma^2)} - e^{-2\gamma (1 + 1/\gamma + 1/2\gamma^2)}}{e^{-2\gamma^2} - e^{-2\gamma}}.$$

(67)

Some typical numerical results are presented for $\gamma = 1$ in Figure 7 and Table 5.

It is evident from the data given in Tables 4 and 5 that the centroids of two PSF's for nonuniformly illuminated circular pupils aberrated by equal amounts of primary coma and secondary coma are different. For example, when $W_3 = W_5 = 1\lambda$, $\langle x \rangle_3 = 1.37$ and $\langle x \rangle_5 = 1.12$ where $\langle x \rangle$ is in units of $\lambda F$.

In the case of a uniformly illuminated circular pupil, $\langle x \rangle_3 = \langle x \rangle_5$ for $W_3 = W_5$ as may be seen from Tables 1 and 2. Of course, for an annular pupil, the centroids $\langle x \rangle_3$ and $\langle x \rangle_5$ for $W_3 = W_5$ are also different whether the pupil is uniformly or nonuniformly illuminated.
Figure 6. Same as Figure 1 except that the amplitude across the pupil is Gaussian given by Eq. (61) with $\gamma = 1$. The PSF's are normalized by the aberration-free value $I(0;1,\epsilon)$ given by Eq. (63).
Table 4. Same as Table 1, except that $A(h)$ is a Gaussian given by Eq. (61) with $\gamma = 1$. The irradiances given here are normalized by the aberration-free central irradiance $I(0;1;\varepsilon)$ given by Eq. (63).

<table>
<thead>
<tr>
<th>$W_3$</th>
<th>$x_m$</th>
<th>$x_p$</th>
<th>$&lt;x&gt;$</th>
<th>$I_m$</th>
<th>$I_p$</th>
<th>$I_c$</th>
<th>$I(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
</tr>
<tr>
<td>0.50</td>
<td>0.61</td>
<td>0.60</td>
<td>0.69</td>
<td>0.8805</td>
<td>0.8806</td>
<td>0.8670</td>
<td>0.4567</td>
</tr>
<tr>
<td>(0.76)</td>
<td>(0.76)</td>
<td>(1.42)</td>
<td>(0.9288)</td>
<td>(0.9288)</td>
<td>(0.1126)</td>
<td>(0.0602)</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.22</td>
<td>1.15</td>
<td>1.37</td>
<td>0.6013</td>
<td>0.6062</td>
<td>0.5590</td>
<td>0.1708</td>
</tr>
<tr>
<td>(1.51)</td>
<td>(1.51)</td>
<td>(2.84)</td>
<td>(0.7435)</td>
<td>(0.7435)</td>
<td>(0.1273)</td>
<td>(0.0348)</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>1.82</td>
<td>1.40</td>
<td>2.06</td>
<td>0.3205</td>
<td>0.3672</td>
<td>0.2479</td>
<td>0.1199</td>
</tr>
<tr>
<td>(2.27)</td>
<td>(2.24)</td>
<td>(4.25)</td>
<td>(0.5112)</td>
<td>(0.5122)</td>
<td>(0.0014)</td>
<td>(0.0033)</td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>2.43</td>
<td>1.46</td>
<td>2.75</td>
<td>0.1305</td>
<td>0.2947</td>
<td>0.0624</td>
<td>0.0773</td>
</tr>
<tr>
<td>(3.03)</td>
<td>(2.93)</td>
<td>(5.67)</td>
<td>(0.3005)</td>
<td>(0.3065)</td>
<td>(0.0399)</td>
<td>(0.0000)</td>
<td></td>
</tr>
</tbody>
</table>
Figure 7. Same as Figure 6, except that the aberration is secondary coma $W_5$. 

-37-
Table 5. Same as Table 2, except that $A(h)$ is a Gaussian given by Eq. (61) with $\gamma = 1$. The irradiances given here are normalized by the aberration-free central irradiance $I(0; 1; \varepsilon)$ given by Eq. (63).

<table>
<thead>
<tr>
<th>$W_5$</th>
<th>$x_p$</th>
<th>$&lt;x&gt;$</th>
<th>$I_p$</th>
<th>$I_c$</th>
<th>$I(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
<td></td>
<td>(1.0000)</td>
<td>(1.0000)</td>
<td>(1.0000)</td>
</tr>
<tr>
<td>0.50</td>
<td>0.41</td>
<td>0.56</td>
<td>0.8452</td>
<td>0.8105</td>
<td>0.6322</td>
</tr>
<tr>
<td>(0.59)</td>
<td>(1.57)</td>
<td></td>
<td>(0.8451)</td>
<td>(0.0123)</td>
<td>(0.2253)</td>
</tr>
<tr>
<td>1.00</td>
<td>0.64</td>
<td>1.12</td>
<td>0.5659</td>
<td>0.4161</td>
<td>0.3793</td>
</tr>
<tr>
<td>(1.14)</td>
<td>(3.14)</td>
<td></td>
<td>(0.5144)</td>
<td>(0.0026)</td>
<td>(0.0025)</td>
</tr>
<tr>
<td>1.50</td>
<td>0.63</td>
<td>1.68</td>
<td>0.4541</td>
<td>0.1147</td>
<td>0.3083</td>
</tr>
<tr>
<td>(1.49)</td>
<td>(4.70)</td>
<td></td>
<td>(0.2595)</td>
<td>(0.0075)</td>
<td>(0.0011)</td>
</tr>
<tr>
<td>2.00</td>
<td>0.74</td>
<td>2.24</td>
<td>0.3824</td>
<td>0.0075</td>
<td>0.2476</td>
</tr>
<tr>
<td>(1.67)</td>
<td>(6.27)</td>
<td></td>
<td>(0.1892)</td>
<td>(0.0043)</td>
<td>(0.0084)</td>
</tr>
</tbody>
</table>
V. DISCUSSION AND CONCLUSION

We have defined the LOS of an optical system in terms of the centroid of its PSF. Since the imaging properties of an optical system are determined by its pupil function, the centroid of the PSF is no exception. By expressing the centroid in terms of the pupil function, it is easy to show that the wave diffraction optics and ray geometrical optics give identical expressions for the centroid, regardless of the shape of the pupil or the amplitude and phase distributions across it. This is in spite of the fact that the two PSFs are quite different from each other. Although the LOS of an optical system can be obtained from the centroid of its PSF, or from the slope of the imaginary part of its OTF evaluated at the origin, in optical design and analysis, the simplest way to obtain the LOS would be to determine the centroid of the ray spot diagram. The idea is that, since ray tracing would be needed to calculate the aberrations of the system any way, one might as well trace the rays up to the image plane and calculate their centroid without calculating the diffraction PSF. Of course, the precision with which the centroid would be calculated would depend on the number of rays used, just as it would depend on the number of rays used to calculate the aberration function and the number of points used to calculate the aberrated PSF, for example, using an FFT algorithm.

In the case of an aberrated system with an annular pupil and a radially symmetric illumination, e.g., Gaussian, the LOS may be determined from the aberration coefficients of the appropriate orthogonal Zernike polynomials \( S_n^m(\rho; \varepsilon) \cos \theta \) and \( S_n^m(\rho; \varepsilon) \sin \theta \). When the amplitude across the pupil is uniform, these polynomials reduce to polynomials \( R_n^m(\rho; \varepsilon) \cos \theta \) and \( R_n^m(\rho; \varepsilon) \sin \theta \), respectively. The results for a circular pupil may be obtained by letting \( \varepsilon = 0 \). In practice, however, given a certain optical system, the simplest way to determine the centroid would be to use its point spread function as measured, for example, by a photodetector array.

We have shown that for a uniformly illuminated pupil, since the centroid of an aberrated PSF depends only on the aberration along the perimeter of the
pupil, aberrations such as $W_n \rho^n \cos \theta$ and $W_n R_n^1(\rho; 0) \cos \theta$ (and similarly
$W_n \rho^n \sin \theta$ and $W_n R_n^1(\rho; 0) \sin \theta$) across a circular pupil with different
aberration orders $n$, give aberrated PSF's with centroids that depend only on
the value of $W_n$. Thus, for a given value of $W_n$, different aberration orders
give PSF's with the same centroid.

We have obtained numerical results on the PSF's aberrated by primary and
secondary coma for circular as well as annular pupils with uniform and
Gaussian illuminations. In the case of uniform illumination, for a given
value of $W_3$, the peak value, is higher for $\varepsilon^2 = 0.5$ than for $\varepsilon = 0$. The peak
value as well as the centroid occur at larger value of $x$ when $\varepsilon$ is no zero
compared to when it is zero. The peak value corresponds to minimum aberration
variance $(I_p - I_m)$ for $W_3 \leq 1.5\lambda$ when $\varepsilon = 0$ and for $W_3 \leq 2.5\lambda$ when $\varepsilon^2 = 0.5$.
Except when $W_3 \approx 0$, $I_c$ is quite small compared to $I_p$. As $W_3$ increases, $<x>$ and
$x_p$ increase linearly with it. However, $x_p$ first increases monotonically with $W_3$ and then fluctuates in a series of maxima and minima. For $\varepsilon^2 = 0.5$ the
maxima and minima are widely spaced and are less pronounced compared to those
for $\varepsilon = 0$.

In the case of secondary coma, the peak value does not occur for larger
and larger values of $x$ as $W_5$ increases, i.e., $x_p$ does not increase monotonically
with $W_5$. For example, when $\varepsilon = 0$, $x_p = 0.83$ for $W_5 = 1\lambda$ and $x_p = 0.81$ for
$W_5 = 1.5\lambda$. Similarly, when $\varepsilon^2 = 0.5$, $x_p = 1.74$ when $W_5 = 1.5\lambda$ and $x_p = 1.71$
when $W_5 = 2\lambda$. As in the case of primary coma, $<x>$ increases with $\varepsilon$ for a given
value of $W_5$.

In the case of Gaussian illumination numerical results are obtained for
$\gamma = 1$. $I_m$ and $I_p$ are higher, and $x_m$, $x_p$ and $<x>$ are smaller for Gaussian
illumination than those for uniform ($\gamma = 0$) illumination. It should be noted
that $I_m$ and $I_p$ are normalized by the aberration-free peak irradiance, and it is
only these normalized values that are higher for Gaussian illumination. [This
normalization is different for Gaussian and uniform illuminations as may be
seen by comparing Eqs. (41) and (63)]. Otherwise, for a given circular or
annular aperture and for a given total power $E$, an unaberrated pupil gives
maximum central irradiance when it is uniformly illuminated. 

-40-
The results given here are applicable to both imaging systems, e.g., those used for optical surveillance, as well as to laser transmitters used for active illumination of a target. In both cases, the LOS of the optical system is extremely important. A LOS error of a surveillance system will produce an error in the location of the target. In the case of a laser transmitter, a large LOS error may cause the laser beam to miss the target altogether. Whereas for static aberrations, we may be able to calibrate the LOS, for dynamic aberrations it is the analysis given here that will determine tolerances on aberrations of the type $\rho^m \cos \theta$ and $\rho^n \sin \theta$ in the case of annular or circular pupils.

Although we have defined the LOS of an optical system in terms of the centroid of its PSF, it could have been defined in terms of the peak of the PSF (assuming that the aberrations are small enough so that the PSF has a distinguishable peak). As pointed out in Section II, for an aberration-free PSF, its peak value and its centroid both lie at its origin, regardless of the amplitude variations across its pupil. The two are not coincident when aberrations are present. The precise definition of the LOS will perhaps depend on the nature of the application of the optical system. Moreover, in practice, only a finite central portion of the PSF will be sampled to measure its centroid, and the precision of this measurement will be limited by the noise characteristics of the photodetector array.

ACKNOWLEDGMENT

The author gratefully acknowledges helpful discussions with R. Boucher and W. Swantner, and computational and plotting assistance from A. Compito. The manuscript was carefully typed by Regina Wallace.

Some of the work presented here was reported at the 1984 Annual Meeting of the Optical Society of America held in San Diego, California. The full paper will appear in the June 1985 issue of the Journal of Optical Society of America, A.
Consider an optical system having an annular pupil with radially symmetric illumination \( A(h) \) and coma aberration

\[
W(h, \theta) = \sum_n^\prime \rho^n (W_n \cos \theta + W'_n \sin \theta). \tag{A1}
\]

where, as in Eq. (25), the prime on the summation sign indicates a summation over odd integral values of \( n \). Note that \( h = \rho a \) and \( \epsilon \leq \rho \leq 1 \). In polar coordinates, Eq. (1) for the aberrated PSF may be written

\[
I(r, \phi; \varepsilon) = (a^2/\lambda R)^2 \int_0^{2\pi} A(h) \exp\{\pi i [2W(h, \theta) - r_s \rho \cos(\theta - \phi)]\} \rho \, d\rho \, d\theta, \tag{A2}
\]

where we have assumed that \( W(h, \theta) \) and, therefore, \( W_n \) and \( W'_n \) are in units of \( \lambda \). Integration over \( \theta \) in Eq. (A2) can be carried out if we let

\[
2W(h, \theta) - r_s \rho \cos(\theta - \phi) = B \cos(\theta - \psi), \tag{A3}
\]

where

\[
B^2 = \left( \sum_n^\prime 2W_n \rho^n - r_s \rho \cos \phi \right)^2 + \left( \sum_n^\prime 2W'_n \rho^n - r_s \rho \sin \phi \right)^2 \tag{A4}
\]

and

\[
\tan \psi = \left( \sum_n^\prime 2W'_n \rho^n - r_s \sin \phi \right) / \left( \sum_n^\prime 2W_n \rho^n - r_s \cos \phi \right) \tag{A5}
\]

Thus, Eq. (A2) becomes

\[
I(r, \phi; \varepsilon) = (a^2/\lambda R)^2 \int_{\varepsilon^2}^{1} A(h) J_0(\pi B \rho) \, dt \left( \pi a^2/\lambda \right)^2, \tag{A6}
\]

where we have let

\[
t = \rho^2. \tag{A7}
\]
REFERENCES


2. The slope of the real part of the OTF at the origin is discontinuous, since its sign depends upon whether the origin is approached from the first or the third quadrant of the \( \xi \eta \) plane. However, directional derivatives at the origin have been used in optical analysis, e.g., see B. Tatian, "Asymptotic expansions for correcting truncation error in transfer-function calculations," J. Opt Soc. Am. 61, 1214-1224 (1971); V. N. Mahajan, "Asymptotic behavior of diffraction images," Can. J. Phys. 57, 1426-1431 (1979).

3. It is not difficult to show that Eq. (10) may also be obtained by substituting Eq. (1) into Eq. (2); see V. I. Tatarski, *The Effects of the Turbulent Atmosphere on Wave Propagation*, p. 287, (Translated by IPST Staff); Available from the U.S. Dept. of Commerce, Washington, D.C.

4. See Ref. 1, p. 206. Some authors define wave and ray aberrations with a sign that is opposite to the convention used in Ref. 1. See, for example, W. T. Welford, *Aberrations of the Symmetrical Optical System*, (Academic, New York, 1974), Sec. 6.2.


8. Ref. 1, p. 469.


