BIAXIAL BUCKLING OF SPECIALLY ORTHOTROPIC, SYMMETRIC COMPOSITE RECTANGULAR PLATES

THESIS

James P. McFadden

SCHOOL OF ENGINEERING

WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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COMPOSITE RECTANGULAR PLATES

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Aeronautical Engineering

James P. McFadden, B.S.

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Preface

My basic interest and job-related experience in the analysis of composite structures compelled me to explore this wide area for a research topic. I also possessed a working understanding of the theory of isotropic uniaxial plate buckling theory, so I merged these two concepts into a challenging task.

The main individual who provided considerable guidance and patience throughout this project is my thesis advisor, Dr. E. J. Brunelle. Without his help and indeed his initial idea and groundwork, this report would never have reached a satisfactory conclusion.

James P. McFadden
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\[\{(K_{y_o}/2m^2)^2 + K_{x_o} (a_o/mb_o)^2 - 1\} = 0\]

\[K_{y_o} < 0\]

\[K_{y_o} = 0\]

\[K_{y_o} > 0\]

\[\{(K_{y_o}/2m^2)^2 + K_{x_o} (a_o/mb_o)^2 - 1\} > 0\]

\(K_{y_o}\) Ranges from a Relatively Large Positive Number to Positive Infinity

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\[\{(K_{y_o}/2m^2)^2 + K_{x_o} (a_o/mb_o)^2 - 1\} = 0\]

\[K_{y_o} < 0\]

\[K_{y_o} = 0\]

\[K_{y_o} > 0\]
\[(K_{vo}/2m^2)^2 + K_xo (a_o/mb_o)^2 - 1 \]  

\[K_{vo}\] Ranges from a Relatively Large Positive Number to Positive Infinity  

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\[
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\]

\[
\{(K_{y_o}/2m^2)^2 + K_{x_o} (a_o/m_b)^2 - 1 \} = 0
\]

\[
K_{y_o} < 0
\]

\[
K_{y_o} = 0
\]

\[
K_{y_o} > 0
\]

\[
\{(K_{y_o}/2m^2)^2 + K_{x_o} (a_o/m_b)^2 - 1 \} > 0
\]

$K_{y_o}$ Ranges from a Relatively Large Positive Number to Positive Infinity

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and \( m \) increases as \( a_o/b_o \) increases.

Again, transition points represent quantities of special interest. First consideration is given to the determination of the point where the \( K_{x_o} \) versus \( a_o/b_o \) curve for \((m=1,(n+1))\) intersects the \( K_{x_o} \) versus \( a_o/b_o \) curve for \((m=1,n)\). Both curves utilize a fixed value of \( K_{y_o} \). Equating the right-hand side of equation (20) for \((m=1,(n+1))\) to an equivalent right-hand side of equation (20) for \((m=1,n)\) determines the value of \( a_o/b_o \) at the transition point.

\[
-K_{y_o} (n+1)^2 + 1.0/(a_o/b_o)_{tr_2}^2 + (n+1)^4 (a_o/b_o)_{tr_2}^2 =
-K_{y_o} (n)^2 + 1.0/(a_o/b_o)_{tr_2}^2 + (n)^4 (a_o/b_o)_{tr_2}^2
\]

\[
(a_o/b_o)_{tr_2} = \left[ K_{y_o} \left( (n+1)^2 - n^2 \right) / [(n+1)^4 - n^4] \right]^{1/2}
= [K_{y_o} / [(n+1)^2 + n^2]]^{1/2}
\]

(27)

where

\[
(a_o/b_o)_{tr_2} = \text{value of characteristic plate aspect ratio at transition from } (m=1,(n+1)) \text{ to } (m=1,n)
\]

Equation (28) gives the value of \( a_o/b_o \) for any fixed \( K_{y_o} \) greater than zero for the intersection of the \( K_{x_o} \) versus \( a_o/b_o \) curve for \((m=1,(n+1))\) with the \( K_{x_o} \) versus \( a_o/b_o \) curve for \((m=1,n)\). Furthermore, substitution of the quantity \( (a_o/b_o)_{tr_2} \) into equation (20) returns \( K_{x_o} \) for the transitional value of \( a_o/b_o \).
Of course since \((a_o/b_o)_{tr_1}\) and \(K_{x_0, tr_1}\) signify coordinates of a mutual point of the \(m\) and \((m+1)\) curves, either equation presented in equation (26) yields the correct answer.

Table I gives selected \(a_o/b_o\), \(K_{y_0}\), and \(K_{x_0}\) ordered triplets based upon equations (21), (25), and (26). The values of \(m\) and \(n\) (=1) are clearly marked for each point, and each entry which corresponds to a transition point is superscripted with a star (*). Note that \(K_{y_0}\) is less than or equal to zero for all points because compressive \(K_{y_0}\) will be considered separately in the next section.

Two key observations can be made after review of the data in Table I. First, the transition values of \(a_o/b_o\) increase as \(K_{y_0}\) becomes less tensile. This trend is most pronounced for the initial transition points, and its effect diminishes as \(a_o/b_o\) becomes large. Second, regardless of the tensile magnitude of \(K_{y_0}\), \(K_{x_0}\) attains a limiting value of 2.0 as \(a_o/b_o\) approaches infinity. Therefore, \(K_{x_0}\), for a tensile \(K_{y_0}\), is effectively independent of \(K_{y_0}\) for large \(a_o/b_o\).

**Compression Applied in \(y_o\)-Direction**

For \(K_{y_0}\) greater than zero, a value of \(n=1\) no longer uniformly satisfies equation (20) for minimum \(K_{x_0}\). Small \(a_o/b_o\) quantities in fact dictate that \(m\) achieve a value of unity and \(n\) increase as \(a_o/b_o\) decreases. As \(a_o/b_o\) becomes relatively large, however, the pattern demonstrated in the section for \(K_{y_0}\) less than or equal to zero reappears -- \(n=1\)
\[ \left( \frac{a_o}{b_o} \right)_{tr_1}^4 \frac{(m+1)^2 - m^2}{m(m+1)}^2 = -\left( \frac{a_o}{b_o} \right)_{tr_1}^2 \frac{K_y}{(m+1)^2 - m^2} \]

where

\( \left( \frac{a_o}{b_o} \right)_{tr_1} = \text{value of characteristic plate aspect ratio at transition from } m \text{ to } (m+1) \)

Application of the quadratic equation fixes

\( \left( \frac{a_o}{b_o} \right)_{tr_1} = \text{Only one root is applicable since the second yields a negative aspect ratio.} \)

\[ \left( \frac{a_o}{b_o} \right)_{tr_1}^2 = 0.5 \left\{ \frac{m(m+1)}{2} + \frac{K_y}{(m+1)^2} + \left\{ \frac{K_y^2}{(m+1)^2} + 4.0/(m+1)^2 \right\}^{1/2} \right\} \]

\[ \left( \frac{a_o}{b_o} \right)_{tr_1} = (0.5)^{1/2} \left\{ K_y + \left\{ K_y^2 + 4.0/(m+1)^2 \right\}^{1/2} \right\}^{1/2} \]

Equation (25) gives the value of \( a_o/b_o \), for any fixed \( K_y \) less than or equal to zero, for the intersection of the \( K_x \) versus \( a_o/b_o \) curve for \( m \) with the \( K_x \) versus \( a_o/b_o \) curve for \( (m+1) \). Furthermore, substitution of \( \left( \frac{a_o}{b_o} \right)_{tr_1} \) into equation (21) returns \( K_x \) for the transitional value of \( a_o/b_o \).

\[ K_{x_{tr_1}} = -\frac{K_y}{(m+1)^2} + \left( m+1 \left( \frac{a_o}{b_o} \right)_{tr_1} \right)^2 
+ \left( \frac{a_o}{b_o} \right)_{tr_1}^2 \]

\[ K_{x_{tr_1}} = -\frac{K_y}{m^2} + \left( \frac{m}{a_o/b_o} \right)_{tr_1}^2 
+ \left( \frac{a_o}{b_o} \right)_{tr_1}^2 \]

where

\( K_{x_{tr_1}} = x_o\)-buckling coefficient at transition from \( m \) to \( m+1 \)

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buckling.

Consider two cases of equation (20) -- the first for \( K_{y_0} \) less than or equal to zero and the second for positive \( K_{y_0} \). In other words, the first case reflects buckling behavior in the presence of tension or no loading in the \( y_o \)-direction, and the second case illustrates such characteristics for compression in the transverse direction.

**Tension or Zero Loading Applied in the \( y_o \)-Direction**

For \( K_{y_0} \) less than or equal to zero, it is obvious that the minimum \( K_{x_0} \) in all situations corresponds to the lowest value of \( n \) (\( n=1 \)). Thus, equation (20) becomes:

\[
K_{x_0} = -\frac{K_{y_0}}{m^2} + \left( \frac{m b_o}{a_o} \right)^2 + \left( \frac{a_o}{m b_o} \right)^2
\]

(21)

For small values of the characteristic ratio \( a_o/b_o \), \( m=1 \) produces the smallest \( K_{x_0} \), yet as \( a_o/b_o \) increases, \( m \) greater than one yields the minimum \( K_{x_0} \). Remember that \( m \) is constrained to be an integer quantity. Of special interest, therefore, is the location of those transition points where the \( K_{x_0} \) versus \( a_o/b_o \) curve for \( m \) intersects the \( K_{x_0} \) versus \( a_o/b_o \) curve for \( (m+1) \). Both curves utilize a fixed value of \( K_{y_0} \). Equating the right-hand side of equation (21) for \( m \) to an equivalent right-hand side of equation (21) for \( (m+1) \) determines the value of \( a_o/b_o \) at the transition point.

\[
-\frac{K_{y_0}}{(m+1)^2} + \left[ \frac{(m+1)}{(a_o/b_o)_{tr_1}} \right]^2 + \left[ \frac{(a_o/b_o)_{tr_1}}{(m+1)} \right]^2 = -\frac{K_{y_0}}{m^2} + \left[ \frac{m}{(a_o/b_o)_{tr_1}} \right]^2 + \left[ \frac{(a_o/b_o)_{tr_1}}{m} \right]^2
\]

(22)
Upon substitution of the relations in equation (17) into equation (14), the buckling equation produces the following sequence:

\[ F \sin(mn \frac{x_0}{a_0}) \sin(nn \frac{y_0}{b_0}) \left\{ (mn/a_0)^4 \right. \\
+ (nn/b_0)^4 - K_{x_0} \left[ \frac{(mn^2)}{(a_0b_0)} \right] \right. \\
- K_{y_0} \left[ \frac{(nn^2)}{(a_0b_0)} \right]^2 = 0 \]  \hspace{1cm} (18)

\[ (m/a_0)^4 + (n/b_0)^4 - K_{x_0} \left[ \frac{m}{(a_0b_0)} \right]^2 \\
- K_{y_0} \left[ \frac{n}{(a_0b_0)} \right]^2 = 0 \]  \hspace{1cm} (19)

\[ K_{x_0} = -K_{y_0} \left( \frac{n}{m} \right)^2 + \left( \frac{mb_o}{a_0} \right)^2 \\
+ \left[ \frac{(a_0n^2)}{(mb_o)} \right]^2 \]  \hspace{1cm} (20)

As shown in equation (20), an exact solution for \( K_{x_0} \) exists for any choice of \( K_{y_0} \). Of course, the minimum value of \( K_{x_0} \) returned from trials with various values of \( m \) and \( n \) is of the greatest significance. This minimum \( K_{x_0} \) represents the smallest level of the coefficient necessary to initiate
II. Flat Rectangular Composite Laminate Simply Supported On All Four Sides

The boundary conditions for a laminate simply supported on all sides specify that the vertical displacement along each edge and the normal component of moment to each edge must vanish in the affine space. In equation form, the following must hold:

\begin{align*}
\text{on edge } x_0 = 0, \quad w &= 0 \quad ; \quad w_{x_0 x_0} = 0 \\
\text{on edge } x_0 = a_0, \quad w &= 0 \quad ; \quad w_{x_0 x_0} = 0 \\
\text{on edge } y_0 = 0, \quad w &= 0 \quad ; \quad w_{y_0 y_0} = 0 \\
\text{on edge } y_0 = b_0, \quad w &= 0 \quad ; \quad w_{y_0 y_0} = 0
\end{align*}

The sine series shown below, which models the displacement \( w \), satisfies each stipulation of equation (15) and will be applied to the general equation (14).

\[ w = F \sin \left( \frac{m \pi x_0}{a_0} \right) \sin \left( \frac{n \pi y_0}{b_0} \right) \quad (16) \]

where

- \( F \) = arbitrary coefficient
- \( m \) = integer which can take on any of the values 1, 2, 3, ...
- \( n \) = integer which can take on any of the values 1, 2, 3, ...

Correct partial differentiation of equation (16) yields the following:
the range zero to one \((1)\). Furthermore, the character of the solution is rather invariant of this ratio over the range zero to one. In particular, a zero value of the starred bending stiffness ratio constitutes a close approximation to plate buckling behavior \((1)\). This null value will be employed throughout the report.

Application of all notational simplifications gives the final form of the biaxial buckling equation:

\[
\frac{w_{xx}x_0x_0}{y_0y_0y_0} + \frac{w_{yy}y_0y_0}{x_0x_0} + K_{x_0} \left(\frac{n}{b_0}\right)^2 \frac{w_{xx}x_0}{x_0} + K_{y_0} \left(\frac{n}{a_0}\right)^2 \frac{w_{yy}y_0}{y_0} = 0 \tag{14}
\]
straightforward manner:

\[ W_{rx} = (1/A)w_{rx_0} \]
\[ W_{ry} = (1/B)w_{ry_0} \]
\[ W_{rz} = w_{rz_0} \] (8)

Combining equations (5), (6), and (8) yields the buckling equation expressed in the desired doubly affine space:

\[
\begin{align*}
\left( \frac{D_{11}}{A^4} \right) w_{rx_0x_0x_0x_0} &+ 2 \left( D_{12} + 2D_{68} \right) / (AB)^2 w_{rx_0x_0y_0y_0} \\
+ \left( \frac{D_{22}}{B^4} \right) w_{ry_0y_0y_0} &- \left( \frac{N_x}{A^2} \right) w_{rx_0x_0} \\
&- \left( \frac{N_y}{B^2} \right) w_{ry_0y_0} = 0
\end{align*}
\] (9)

For maximum simplicity equation (9) dictates the choice of the arbitrary constants, A and B. For coefficients of unity for the first and third terms \( A = D_{11}^{1/4} \) and \( B = D_{22}^{1/4} \). The buckling equation then becomes:

\[
\begin{align*}
W_{rx_0x_0x_0x_0} &+ 2 \left( D_{12} + 2D_{68} \right) / (D_{11} D_{22})^{1/2} w_{rx_0x_0y_0y_0} \\
+ w_{ry_0y_0y_0} &- N_x / (D_{11})^{1/2} w_{rx_0x_0} \\
&- N_y / (D_{22})^{1/2} w_{ry_0y_0} = 0
\end{align*}
\] (10)

Further notational simplifications are:

\[
D^* = 2 \left( D_{12} + 2D_{68} \right) / (D_{11} D_{22})^{1/2}
\] (11)

\[
-N_x / (D_{11})^{1/2} = K_{x_0} \left( n/b_0 \right)^2
\] (12)

\[
-N_y / (D_{22})^{1/2} = K_{y_0} \left( n/a_0 \right)^2
\] (13)

where

\[
n = 3.1415927
\]

\( K_{x_0} \) = buckling coefficient in \( x_0 \)-direction

\( K_{y_0} \) = buckling coefficient in \( y_0 \)-direction

\( D^* \) = starred bending stiffness ratio

This starred bending stiffness ratio \( D^* \) must lie within
Differentiating the moment quantities given by equation (2) and substitution into equation (1) finally yields the buckling equation for a symmetric, specially orthotropic composite laminate expressed in Cartesian coordinates:

\[ D_{11} w_{xxxx} + 2(D_{12} + 2D_{66}) w_{xyxy} + D_{22} w_{yyyy} - N_x w_{xx} - N_y w_{yy} = 0 \]  

(5)

The coordinate system for this equation is now transformed from Cartesian to doubly affine space (1). The equations relating the coordinates are straightforward:

\[ A x_o = x \]
\[ B y_o = y \]
\[ z_o = z \]  

(6)

where

\[ x_o = \text{longitudinal plate coordinate in affine space} \]
\[ y_o = \text{transverse plate coordinate in affine space} \]
\[ z_o = z = \text{vertical plate coordinate in affine/real space} \]
\[ A = \text{arbitrary constant coefficient} \]
\[ B = \text{arbitrary constant coefficient} \]

In addition, if the rectangular plate measures \( a \) units in the longitudinal and \( b \) units in the transverse direction, the following must hold:

\[ a_o = a/A \]
\[ b_o = b/B \]
\[ a_o/b_o = (a/b)(B/A) \]  

(7)

From equations (6) differentiation in the affine space compares to differentiation in Cartesian space again in a
I. **Derivation of Buckling Equation for Specially Orthotropic Laminated Plates, Symmetric about Middle Surface, Expressed in Doubly Affine Space Coordinates**

The general buckling equation for a rectangular plate under biaxial loading is given by (2.245):

\[ M_{xx} + 2M_{xy}F_{xy} + M_{yy} + Nxw_{xx} + Nyw_{yy} = 0 \] (1)

where

- \( M \) = moment per unit length
- \( N \) = normal force per unit length positive in tension
- \( w \) = displacement of middle surface of the laminate

The moments are related to the displacement field of a symmetric laminate by the matrix equation (1.149-156):

\[
\begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{pmatrix}
-w_{xx} \\
-w_{yy} \\
-2w_{xy}
\end{pmatrix}
\] (2)

where

- \( D_{ij} \) = component of bending stiffness array
- \( u \) = displacement in x-direction of plate
- \( v \) = displacement in y-direction of plate

Now consideration is narrowed from that of a symmetric anisotropic laminate to one both specially orthotropic and symmetric about the middle surface. These restrictions cause the following terms in the stiffness array to vanish:

\[ D_{16} = 0 \] (3)
The biaxial plate buckling problem for specially orthotropic, symmetric laminates is transformed from Cartesian to doubly affine space. Setting the starred bending stiffness ratio D*(which ranges from zero to one) to the null value enables ready and very accurate solution of the buckling problem. Seven sets of boundary restraint configurations are examined, and corresponding buckling surfaces are presented. The character of these results vary widely between the strongest and weakest sets of support conditions. In order to prevent buckling for the weakest conditions, considerable tension must be provided on parallel edges for just small amounts of compression applied on the opposite set of edges. Additional keywords: buckling equations, flat rectangular composite laminate.
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<td>88</td>
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\[ K_{x_{0_{tr2}}} = -K_{y_{o}} (n+1)^2 + 1.0/(a_o/b_o)_{tr2}^2 \]
\[ + (n+1)^4 (a_o/b_o)_{tr2}^2 \]

or

\[ K_{x_{0_{tr2}}} = -K_{y_{o}} (n)^2 + 1.0/(a_o/b_o)_{tr2}^2 \]
\[ + (n)^4 (a_o/b_o)_{tr2}^2 \]

where

\[ K_{x_{0_{tr2}}} = x_o\text{-buckling coefficient at transition from} \]
\[ (m=1,(n+1)) \text{ to } (m=1,n) \]

Of course, since \((a_o/b_o)_{tr2}\) and \(K_{x_{0_{tr2}}}\) signify coordinates of a mutual point of the \((m=1,n+1)\) and \((m=1,n)\) curves, either equation presented in equation (29) yields the correct answer.

Consideration now is directed solely at relatively large aspect ratios, \(a_o/b_o\). The prior work in this section showed that for comparatively small \(a_o/b_o\), \(m=1\) returns the minimum \(K_{x_{o}}\). In contrast, setting \(n\) to its smallest integer value, one, produces the smallest \(K_{x_{o}}\) as \(a_o/b_o\) grows. This progression mirrors the logic presented for the case \(K_{y_{o}}\) less than or equal to zero. As a result, the identical equations (25) and (26) which defined the transitional quantities \((a_o/b_o)_{tr1}\) and \(K_{x_{0_{tr1}}}\) for \(K_{y_{o}}\) in tension apply for the present compression conditions. No further manipulation is required.

Table II gives selected \(a_o/b_o\), \(K_{y_{o}}\), and \(K_{x_{o}}\) ordered triplets based upon equations (20), (25), (26), (28), and (29). The values of \(m\) and \(n\) are clearly marked for each point, and each entry which corresponds to a transition point
is superscripted in the $a_o/b_o$ column with a star (*). Note that $K_{y_0}$ is greater than zero for all points since Table I presents buckling coefficient data for laminates under tension and zero $K_{y_0}$ loading.

The statistics presented in Table II expose three important buckling characteristics of laminates compressed in the $y_0$-direction. First, the transition values of $a_o/b_o$ increase as $K_{y_0}$ grows. As the pattern for negative $K_{y_0}$ values showed, this trend is most pronounced for the initial transition points, and its effect diminishes as $a_o/b_o$ becomes large. Second, irrespective of the compressive magnitude of $K_{y_0}$, $K_{x_0}$ attains a limiting value of 2.0 as $a_o/b_o$ approaches infinity. Therefore, $K_{x_0}$ is effectively independent of the magnitude of $K_{y_0}$ -- positive or negative -- because the same asymptote of 2.0 is approached by tensile $K_{y_0}$. Finally, for $K_{y_0}$ algebraically large, the $K_{x_0}$ value drops into the tensile range for certain $a_o/b_o$ values. Furthermore, as $K_{y_0}$ increases, a tension load in the $x_0$-direction produces buckling for larger spans of the $a_o/b_o$ dimension.

Figure 1 represents a plot of $K_{x_0}$ versus $a_o/b_o$ for eleven distinct values of $K_{y_0}$. The lowest curve characterizes $K_{y_0}=5.0$; whereas, the highest depicts $K_{y_0}=-5.0$. The nine other curves differ from each other by increments of one. This graph reinforces the concept that $K_{x_0}$ for a constant $K_{y_0}$ is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. Also, the tensile nature of $K_{x_0}$ for
large \( K_0 \) is illustrated by those portions of the curves which fall below \( K_0 = 0 \). Another key visual reinforcement of data which this graph provides is the merging of all curves to the \( K_0 = 2.0 \) asymptote as \( a_o/b_o \) becomes very large.

Figure 2 plots in three dimensions the same information as Figure 1. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 1; however, the quantitative aspect of Figure 2 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Rearrangement of equation (20) produces a relation which allows visualization of selected two-dimensional slices through Figure 2 (3:356-360).

\[
K_{x_0} m^2 + K_{y_0} n^2 = \left[ m^2 / (a_o/b_o) \right]^2 + \left[ n^2 (a_o/b_o) \right]^2
\]

Thus, for constant \( a_o/b_o \), each unique set of \( (m,n) \) yields a straight-line variation between \( K_{y_0} \) and \( K_{x_0} \). The locus of the lowest sections of each line comprises the complete two-dimensional graph.

Table III gives selected \( a_o/b_o \), \( K_{y_0} \), and \( K_{x_0} \) ordered triplets based upon equation (30). Small, intermediate, and relatively large values of \( a_o/b_o \) -- 0.8, 1.6, and 2.6, respectively -- constitute the three \( a_o/b_o \) quantities of interest. Also included is the corresponding pair \( (m,n) \) for each point. Each entry which constitutes a point of
intersection is subscripted in the $K_{y_0}$ column with a star (*).

Figures 3, 4, and 5 represent plots of $K_{x_0}$ versus $K_{y_0}$ for the constant $a_o/b_o$ values of 0.8, 1.6, and 2.6, respectively. The graphs constitute the minimum ordinates of intersecting straight line segments. Note especially that $K_{x_0}$ declines as $K_{y_0}$ increases and that the rate of decline of $K_{x_0}$ for an increase in $K_{y_0}$ jumps markedly for small $a_o/b_o$. 

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**TABLE I**

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported on All Sides (for $K_{y_0}$ tensile or zero)

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>$a_0/b_0$</th>
<th>$K_{y_0}$</th>
<th>$K_{x_0}$</th>
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### TABLE II

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported on All Sides (for $K_y$, compressive)

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<tr>
<th>m</th>
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<th>$a_o/b_o$</th>
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</table>
FOR THE RANGE $K_T = -5.0 \text{ TO } -5.0$ $\text{C}$ C EACH VARIANCE OF $1.0$ $K_T = -5.0$ $\text{HIGHEST CURVE}$

**Figure 1**

$X_0$-Buckling Coefficient versus Affine Aspect Ratio for an $S-S-S$ Laminate

For various constant $X_0$-Buckling Coefficient Values
FIGURE 2

SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-S-S-L LAMINATE
FOR THE RANGE
\( KYO = -5.0 \) TO \( +5.0 \)

Figure 5
Xo-Buckling Coefficient versus Yo-Buckling Coefficient at a constant affine aspect ratio of 2.6 for an S-S-S-S laminate.
III. *Flat Rectangular Composite Laminate Clamped on All Sides*

The boundary conditions for a laminate clamped on all sides specify that the vertical displacement and the slope of the vertical displacement with respect to the normal to each edge must vanish along each edge in the affine space. In equation form, the following must hold:

\[
\begin{align*}
\text{on edge } x_0 = -a_o/2, & \quad w = 0; \quad w,_{x_0} = 0 \\
\text{on edge } x_0 = a_o/2, & \quad w = 0; \quad w,_{x_0} = 0 \\
\text{on edge } y_0 = -b_o/2, & \quad w = 0; \quad w,_{y_0} = 0 \\
\text{on edge } y_0 = b_o/2, & \quad w = 0; \quad w,_{y_0} = 0
\end{align*}
\]

(31)

Note that the origin of coordinates in the affine space is taken to be at the center of the laminate. This choice of origin location allows maximum simplicity in manipulations with the doubly symmetric boundary conditions.

For this case the general buckling equation (14) is most efficiently solved by the separation of variables technique. Therefore, the displacement \( w \) is defined by a function \( X \), which depends only upon \( x_o \), multiplied by a different function \( Y \), which depends only upon \( y_o \).

\[
w = X(x_o) \ Y(y_o)
\]

(32)

Substitution of this assumed form of \( w \) into equation (14) yields:

\[
X''''(x_o) \ Y(y_o) + X(x_o) \ Y''''(y_o) + K_{x_o} (n/b_o)^2 X'''(x_o) \ Y(y_o)
+ K_{y_o} (n/a_o)^2 X(x_o) \ Y'''(y_o) = 0
\]

(33)
where

\[ X^IV(x_o) = \text{the fourth standard derivative of } X \text{ with respect to } x_o. \]

\[ X''(x_o) = \text{the second standard derivative of } X \text{ with respect to } x_o. \]

\[ Y^IV(y_o) = \text{the fourth standard derivative of } Y \text{ with respect to } y_o. \]

\[ Y''(y_o) = \text{the second standard derivative of } Y \text{ with respect to } y_o. \]

Division of equation (33) by the quantity \( X(x_o) \cdot Y(y_o) \) gives:

\[
\frac{X^IV(x_o)}{X(x_o)} + K_{x_o} \left( \frac{n}{b_o} \right)^2 \frac{X''(x_o)}{X(x_o)} + \frac{Y^IV(y_o)}{Y(y_o)} + K_{y_o} \left( \frac{n}{a_o} \right)^2 \frac{Y''(y_o)}{Y(y_o)} = 0 \quad (34)
\]

The terms enclosed in the first set of square brackets are functions solely of \( x_o \); whereas, the components bracketed by the second set depend only on \( y_o \). For these two groups to sum to zero, each individual group can be equal to no more than a constant. This concept is expressed by the following two equations:

\[
Y^IV(y_o)/Y(y_o) + K_{y_o} \left( \frac{n}{a_o} \right)^2 Y''(y_o)/Y(y_o) = k^4 \quad (35)
\]

\[
X^IV(x_o)/X(x_o) + K_{x_o} \left( \frac{n}{b_o} \right)^2 X''(x_o)/X(x_o) = -k^4 \quad (36)
\]

where

\[ k^4 = \text{constant. This quantity is raised to the fourth power so that fractional exponents may be avoided in the work to follow.} \]

The constant \( k \) is now expressed in terms of another constant, \( f_n \):
Figure 6

$K_{0}$-buckling coefficient versus affine aspect ratio for a C-C-C-C laminate for various constant $K_{0}$-buckling coefficient values.

For the range:

$K_{0} = -5.0$ to $+5.0$

11 curves, each varies by $K_{0}$ increment of 1.0

$k_{0} = -5.0$ highest curve.
TABLE V

$K_{x_0}$ Versus $K_{y_0}$ for Various Plate Aspect Ratios
for a Laminate Clamped on All Sides

(the first column denotes the symmetric or antisymmetric
nature of $f_n$, and the second the symmetric or antisymmetric
nature of $K_{x_0}$.)

<table>
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<th>$f_n$</th>
<th>$x$</th>
<th>$a_o/b_o$</th>
<th>$K_{y_0}$</th>
<th>$K_{x_0}$</th>
</tr>
</thead>
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### TABLE IV

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Clamped on All Sides

(The first column denotes the symmetric or antisymmetric nature of $f_n$, and the second the symmetric or antisymmetric nature of $K_{X_0}$.)

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<th>$K_{X_0}$</th>
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Figures 8, 9, and 10 represent two-dimensional plots at constant $a_o/b_o$ slices of 1.2, 2.4, and 3.6, respectively. These graphs, very similar to those obtained in the simply supported on all sides case, are composed of very nearly straight line segments. Note especially that $K_{x_o}$ declines as $K_{y_o}$ increases and that the rate of decline of $K_{x_o}$ for an increase in $K_{y_o}$ jumps markedly for small $a_o/b_o$. 
eleven distinct values of \( K_{y_0} \). The lowest curve characterizes \( K_{y_0} = 5.0 \); whereas, the highest depicts \( K_{y_0} = -5.0 \). The nine other curves differ from each other by increments of one. This graph reinforces the concept that \( K_{x_0} \) for a constant \( K_{y_0} \) is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. In addition, the merging of all curves to a limiting value of \( K_{x_0} = 4.59 \) for \( a_o/b_o \) large is readily apparent.

Figure 7 plots in three dimensions the same information as Figure 6. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 6; however, the quantitative aspect of Figure 7 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table V gives selected coordinates of \( K_{y_0} \) and \( K_{x_0} \) for three distinct values of \( a_o/b_o = 1.2, 2.4, \) and \( 3.6 \). Also included are the states of symmetry or antisymmetry for \( K_{x_0} \) and the \( f_n \) which produces this minimum \( K_{x_0} \).
(a₀/b₀, K₀, Kₓ₀) based upon symmetric Xₛ(x₀).

Comparison is now made between the triplets (a₀/b₀, K₀, Kₓ₀) based upon antisymmetric and symmetric X(x₀). The final answer, or coordinate point, is simply the one for which the Kₓ₀ is lowest. Therefore, only after this final trial is the correct Kₓ₀ fixed for any a₀/b₀ and K₀ combination.

Table IV gives selected a₀/b₀, K₀, and Kₓ₀ ordered triplets. Also included are the states of symmetry or antisymmetry for Kₓ₀ and the fₙ which produces this minimum Kₓ₀. Furthermore, each entry which corresponds to a transition point from symmetric to antisymmetric buckling, or vice versa, is superscripted in the a₀/b₀ column with a star (*).

The statistics presented in Table IV expose three important characteristics of laminates under compression or tension in the y₀-direction. First, the transition values of a₀/b₀ increase as K₀ becomes algebraically larger (or less tensile). This trend is most pronounced for the initial transition points, and its effect diminishes as a₀/b₀ becomes large. Second, irrespective of the magnitude of K₀, Kₓ₀ attains a limiting value of 4.59 as a₀/b₀ approaches infinity. Finally, the fₙ which produces the minimum Kₓ₀ is in all cases determined by the lowest value of the symmetric Yₛ(y₀) separation equation. This phenomenon holds for symmetric and antisymmetric Kₓ₀ values.

Figure 6 represents a plot of Kₓ₀ versus a₀/b₀ for
of \( f_n \) determined through the antisymmetric or symmetric \( Y(y_o) \) equations. The \( f_n \) of course of primary interest is the one which returns the smallest value of \( K_{x_0} \). Thus, for any fixed set of \( a_o/b_o \) and \( K_{y_o} \), all possible values of \( f_n \) must be considered so that the lowest \( K_{x_0} \) is found to complete the triplet \((a_o/b_o, K_{y_o}, K_{x_0})\) based upon antisymmetric \( X_A(x_o) \).

In an identical manner, equations (68) and (69) are employed as boundary conditions for the symmetric portion of \( X(x_o) \), \( X_S(x_o) \). For non-trivial constants \( F_n \) and \( H_n \), the following determinental equation must hold:

\[
\begin{vmatrix}
\cos(\omega a_o/2) & \cos(\phi a_o/2) \\
-w \sin(\omega a_o/2) & -\phi \sin(\phi a_o/2)
\end{vmatrix} = 0 \quad (72)
\]

Expansion of the determinant and multiplication by the quantity \((-a_o/2)\) gives:

\[
(\phi a_o/2) \sin(\phi a_o/2) \cos(\omega a_o/2) - (\omega a_o/2) \sin(\omega a_o/2) \cos(\phi a_o/2) = 0 \quad (73)
\]

Solution of equation (73) for the infinite number of sets of values of \( \omega a_o/2 \) and \( \phi a_o/2 \) determines another infinite set of values of \( K_{x_0} \) for any choice of \( f_n \), \( a_o/b_o \), and \( K_{y_0} \).

Again, the \( f_n \) may be any member from those groups of \( f_n \) returned by the antisymmetric or symmetric \( Y(y_o) \) equations. Primary interest of course centers upon the value of \( f_n \) which yields the smallest \( K_{x_0} \). So as before, for any fixed set of \( a_o/b_o \) and \( K_{y_o} \), all possible values of \( f_n \) must be considered so that the minimum \( K_{x_0} \) is found to complete the triplet.

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\[ X(-a_o/2) \ Y(y_o) = 0 \quad ; \quad X'(-a_o/2) \ Y(y_o) = 0 \]  
(67)

\[ X( a_o/2) \ Y(y_o) = 0 \quad ; \quad X'( a_o/2) \ Y(y_o) = 0 \]

The two lines of equation (67) each express identical information when the \( X(x_o) \) function is broken down into its components \( X_A(x_o) \) and \( X_S(x_o) \). Thus, only the bottom line of information of equation (67) will be manipulated. For equation (67) to hold for non-trivial \( Y(y_o) \), the following boundary conditions on the function \( X(x_o) \) must hold:

\[ X(a_o/2) = 0 \quad (68) \]
\[ X'(a_o/2) = 0 \quad (69) \]

Application of equations (68) and (69) first is made to the antisymmetric portion of \( X(x_o) \), \( X_A(x_o) \). For non-trivial constants \( E_n \) and \( G_n \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(wa_o/2) & \sin(\phi a_o/2) \\
\omega \cos(wa_o/2) & \phi \cos(\phi a_o/2)
\end{vmatrix} = 0 \quad (70)
\]

Expansion of the determinant and multiplication by the quantity \( a_o/2 \) gives:

\[
(\phi a_o/2) \sin(wa_o/2) \cos(\phi a_o/2) \\
- (wa_o/2) \sin(\phi a_o/2) \cos(wa_o/2) = 0 \quad (71),
\]

Solution of equation (71) for the infinite number of sets of values of \( wa_o/2 \) and \( \phi a_o/2 \) equivalently yields an infinite set of values of \( K_{x_o} \) for any choice of \( f_n, a_o/b_o \), and \( K_{y_o} \). Note that the \( f_n \) may be any member from the groups.
In a similar fashion, choice of the minus sign just before the first square bracket in equation (60) yields the final two values of \( g \). For this choice of the negative sign, the entire quantity in the curly brackets is again constrained to be greater than zero. So the final two values of \( g \) are:

\[
g_{3,4} = \pm \left( \frac{n}{b_o} \right) \left\{ \frac{K_{x_0}}{2} - \left[ \left( \frac{K_{x_0}}{2} \right)^2 - 16 f_n^4 \right]^{1/2} \right\}^{1/2}
\]  

(62)

By the theory of linear homogeneous equations, the function \( X(x_o) \) can be easily determined.

\[
X(x_o) = N_n e^{\omega x_0} + P_n e^{\omega x_0} + Q_n e^{\omega x_0} + R_n e^{\omega x_0}
\]  

(63)

where

\[ N_n, P_n, Q_n, R_n = \text{arbitrary constants} \]

Equivalently, equation (63) can be expressed as:

\[
X(x_o) = E_n \sin(\omega x_o) + F_n \cos(\omega x_o) + G_n \sin(\phi x_o)
\]  

+ \[ H_n \cos(\phi x_o) \]  

(64)

where

\[ \omega = \left( \frac{n}{b_o} \right) \left\{ \frac{K_{x_0}}{2} + \left[ \left( \frac{K_{x_0}}{2} \right)^2 - 16 f_n^4 \right]^{1/2} \right\}^{1/2} \]

\[ \phi = \left( \frac{n}{b_o} \right) \left\{ \frac{K_{x_0}}{2} - \left[ \left( \frac{K_{x_0}}{2} \right)^2 - 16 f_n^4 \right]^{1/2} \right\}^{1/2} \]

\[ E_n, F_n, G_n, H_n = \text{another set of arbitrary constants} \]

The function \( X(x_o) \) likewise can be further simplified by reduction into its antisymmetric and symmetric parts.

\[
X_A(x_o) = E_n \sin(\omega x_o) + G_n \sin(\phi x_o)
\]  

(65)

\[
X_S(x_o) = F_n \cos(\omega x_o) + H_n \cos(\phi x_o)
\]  

(66)

Consider the boundary conditions, equations (31), for this case of a laminate clamped on all sides. The initial two, expressed in the separation functions, become:
equation (37) into equation (36) and slight rearrangement yields:

\[ X^{IV}(x_0) + K_{x_0} (n/b_o)^2 X''(x_0) + 16 f_n^4 (n/b_o)^4 X(x_0) = 0 \quad (56) \]

Similar to equation (38), equation (55) is a linear, homogeneous differential equation with constant coefficients. Assume the following form of \( X(x_0) \):

\[ X(x_0) = e^{g x_0} \quad (57) \]

where

\[ g = \text{constant} \]

Substitution of equation (57) into equation (56) gives:

\[ \{ g^4 + K_{x_0} (n/b_o)^2 g^2 + 16 f_n^4 (n/b_o)^4 \} e^{g x_0} = 0 \quad (58) \]

For equation (58) to hold, the terms inside the brackets must sum to zero. This fact allows determination of the four values of \( g \) which satisfy equation (58). First, \( g^2 \) can be determined by the quadratic equation.

\[ g^2 = 0.5 \left\{ -K_{x_0} (n/b_o)^2 \pm \left[ K_{x_0}^2 (n/b_o)^4 - 64 f_n^4 (n/b_o)^4 \right]^{1/2} \right\} \quad (59) \]

\[ g^2 = (n/b_o)^2 i^2 \left\{ K_{x_0} / 2 \pm \left[ (K_{x_0} / 2)^2 -16 f_n^4 \right]^{1/2} \right\} \quad (60) \]

The first two values of \( g \) can be determined by choice of the positive, as opposed to the negative, sign just before the first square bracket in equation (60). For this choice of the positive sign, the entire quantity in the curly brackets is constrained to be greater than zero. Therefore, the initial two of the four desired values of \( g \) are:

\[ g_{1,2} = \pm i (n/b_o) \left\{ K_{x_0} / 2 + [(K_{x_0} / 2)^2 -16 f_n^4 \right\}^{1/2} \quad (61) \]
Expansion of the determinant and multiplication by the quantity $b_0/2$ gives:

$$(\rho b_0/2) \sin(n b_0/2) \cosh(\rho b_0/2)$$

$$- (n b_0/2) \cos(n b_0/2) \sinh(\rho b_0/2) = 0 \quad (53)$$

Solution of equation (53) for the infinite number of sets of values $n b_0/2$ and $\rho b_0/2$ equivalently yields an infinite set of constants $f_n$, $n=1,2,...$ for any choice of $K_y$ and $a_o/b_o$. These values of $f_n$ will be utilized in the $X(x_o)$ functional equations.

In an identical manner, equations (50) and (51) are employed as boundary conditions for the symmetric portion of $Y(y_o)$, $Y_s(y_o)$. For non-trivial constants $B_n$ and $D_n$, the following determinantal equation must hold:

$$\begin{vmatrix}
\cos(n b_0/2) & \cosh(\rho b_0/2) \\
- n \sin(n b_0/2) & \rho \sinh(\rho b_0/2)
\end{vmatrix} = 0 \quad (54)$$

Expansion of the determinant and multiplication by the quantity $b_0/2$ gives:

$$(\rho b_0/2) \cos(n b_0/2) \sinh(\rho b_0/2)$$

$$+ (n b_0/2) \sin(n b_0/2) \cosh(\rho b_0/2) = 0 \quad (55)$$

Solution of equation (55) for the infinite number of values $n b_0/2$ and $\rho b_0/2$ determines another infinite set of constants $f_n$, $n=1,2,...$ for any choice of $K_y$ and $a_o/b_o$. The importance of this group will be evident shortly.

Consider now equation (36), the differential equation which will render the $X(x_o)$ function. Substitution of
An, Bn, Cn, Dn = another arbitrary set of constants

The function \( Y(y_0) \) can be further simplified by reduction into its antisymmetric and symmetric parts.

\[
Y_A(y_0) = A_n \sin(\eta y_0) + C_n \sinh(\rho y_0) \quad (47)
\]
\[
Y_S(y_0) = B_n \cos(\eta y_0) + D_n \cosh(\rho y_0) \quad (48)
\]

Consider the boundary conditions, equations (31), for this case of a laminate clamped on all sides. The final two, expressed in the laminate separation functions, become:

\[
X(x_0) Y(-b_0/2) = 0 \quad X(x_0) Y'(-b_0/2) = 0 \quad (49)
\]
\[
X(x_0) Y(b_0/2) = 0 \quad X(x_0) Y'(b_0/2) = 0
\]

The two lines of equation (49) each express identical information when the function \( Y(y_0) \) is broken down into its components \( Y_A(y_0) \) and \( Y_S(y_0) \). Thus, only the bottom line of information of equation (49) will be manipulated. For equation (49) to hold for non-trivial \( X(x_0) \), the following boundary conditions on the function \( Y(y_0) \) must hold:

\[
Y(b_0/2) = 0 \quad (50)
\]
\[
Y'(b_0/2) = 0 \quad (51)
\]

Application of equations (50) and (51) first is made to the antisymmetric portion of \( Y(y_0) \), \( Y_A(y_0) \). For non-trivial constants \( A_n \) and \( C_n \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(\eta b_0/2) & \sinh(\rho b_0/2) \\
\eta \cos(\eta b_0/2) & \rho \cosh(\rho b_0/2)
\end{vmatrix}
= 0 \quad (52)
\]

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the first square bracket in equation (42). For this choice of the positive sign, the entire quantity in the curly brackets is constrained to be greater than zero. Therefore, the initial two of four desired values of \( r \) are:

\[
r_{1,2} = \pm (n/a_o) \sqrt{K_y/2 + \left[ (K_y/2)^2 + 16 f_n^4 (a_o/b_o)^4 \right]^{1/2}}
\]

(43)

In a similar fashion, choice of the minus sign just before the first square bracket in equation (42) yields the final two values of \( r \). For this selection, however, the quantity contained in the curly brackets is negative. Multiplication of the factor \( i^2 \) outside the curly brackets with each term inside those braces rectifies the situation and allows a square root of the entire equation (42) to be taken. This procedure yields the last two values of \( r \):

\[
r_{3,4} = \pm (n/a_o) \sqrt{-K_y/2 + \left[ (K_y/2)^2 + 16 f_n^4 (a_o/b_o)^4 \right]^{1/2}}
\]

(44)

By the theory of linear homogeneous equations, the function \( Y(y_o) \) can be easily determined.

\[
Y(y_o) = J_n e^{iy_o} + K_n e^{i2y_o} + L_n e^{i3y_o} + M_n e^{i4y_o}
\]

(45)

where

\[ J_n, K_n, L_n, M_n = \text{arbitrary constants} \]

Equivalently, equation (45) can be expressed as:

\[
Y(y_o) = A_n \sin(ny_o) + B_n \cos(ny_o) + C_n \sinh(\rho y_o) + D_n \cosh(\rho y_o)
\]

(46)

where

\[
n = \left( n/a_o \right) \sqrt{K_y/2 + \left[ (K_y/2)^2 + 16 f_n^4 (a_o/b_o)^4 \right]^{1/2}}^{1/2}
\]

\[
\rho = \left( n/a_o \right) \sqrt{-K_y/2 + \left[ (K_y/2)^2 + 16 f_n^4 (a_o/b_o)^4 \right]^{1/2}}^{1/2}
\]
Substitution of equation (37) into equation (35) and slight rearrangement yields:

\[ Y^{IV}(y_o) + K_{y_o}(n/a_o)^2 Y'''(y_o) - 16 f_n^4(n/b_o)^4 Y(y_o) = 0 \] (38)

Equation (38) is merely a linear, homogeneous differential equation with constant coefficients. The classic solution procedure is to predict that \( Y(y_o) \) fits the following:

\[ Y(y_o) = e^{r y_o} \] (39)

where

\[ e = \text{natural base} = 2.71828... \]
\[ r = \text{constant} \]

Substitution of equation (39) into equation (38) gives:

\[ \{ r^4 + K_{y_o}(n/a_o)^2 r^2 - 16 f_n^4(n/b_o)^4 \} e^{r y_o} = 0 \] (40)

For equation (40) to hold, the terms in the brackets must sum to zero. This fact allows determination of the four values of \( r \) which satisfy equation (40). First, \( r^2 \) can be determined by the quadratic equation.

\[ r^2 = 0.5 \left\{ -K_{y_o}(n/a_o)^2 + [K_{y_o}(n/a_o)^4 + 64 f_n^4(n/b_o)^4]^{1/2} \right\} \] (41)

\[ r^2 = (n/a_o)^2 i^2 \left\{ K_{y_o}/2 + [ (K_{y_o}/2)^2 + 16 f_n^4(a_o/b_o)^4]^{1/2} \right\} \] (42)

where

\[ i = (-1)^{1/2} \]

The first two values of \( r \) can be determined by choice of the positive, as opposed to the negative, sign just before
FIGURE 7
SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR A C-C-C-C LAMINATE
FIGURE 9

X0-BUCKLING COEFFICIENT VERSUS X0-BUCKLING COEFFICIENT AT A CONSTANT RATIO OF 1.2 FOR A C-C-C-LAMINATE FOR THE RANGE X0 = 5.0 TO 5.0.
FIGURE 9
XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.4 FOR A C-C-C-C LAMINATE

FOR THE RANGE
$K_YO = -5.0$ TO $5.0$
FOR THE RANGE $K_{10} = -5.0$ TO $5.0$

FIGURE 10

XO-BUCKLING COEFFICIENT VERSUS Y0-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 3.6 FOR A C-C-C-LAMINATE
IV. Flat Rectangular Composite Laminate Simply Supported
in the $x_o$-Direction and Clamped in the $y_o$-Direction

The boundary conditions for a laminate simply supported in the $x_o$-direction and clamped in the $y_o$-direction are not the same on each edge as in the previous work. For the two edges which have normals parallel to the $x_o$-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. On the other hand, for the two edges which have normals parallel to the $y_o$-axis, the vertical displacement and the slope of the vertical displacement with respect to $y_o$ must vanish along each edge. In equation form, the following must hold:

\begin{align*}
on edge x_o = -a_o/2, \quad w &= 0 \quad ; \quad w'|x_o = 0 \\
on edge x_o = a_o/2, \quad w &= 0 \quad ; \quad w'|x_o = 0 \\
on edge y_o = -b_o/2, \quad w &= 0 \quad ; \quad w'|y_o = 0 \\
on edge y_o = b_o/2, \quad w &= 0 \quad ; \quad w'|y_o = 0
\end{align*}

(74)

Note that the origin of coordinates in the affine space is taken to be the center of the laminate. This choice of origin location, in general, allows maximum simplicity in manipulations with the doubly symmetric boundary conditions. For one set of derivations, however, a different origin location will be used and of course will be noted.

The general buckling equation (14) is most efficiently solved for this case by the separation of variables
technique. Therefore, the displacement \( w \) is defined by a function \( X \), which depends only upon \( x_0 \), multiplied by a different function \( Y \), which depends only upon \( y_0 \).

\[
w = X(x_0) Y(y_0) \tag{75}
\]

An admissible function \( X(x_0) \) can be easily determined for a laminate simply supported along both edges perpendicular to the \( x_0 \)-direction. The following function identically satisfies the first two conditions of equations (74):

\[
X(x_0) = \sin\left(\frac{mn x_0}{a_0}\right) \tag{76}
\]

Substitution of equation (76) into equation (75) yields:

\[
w = Y(y_0) \sin\left(\frac{mn x_0}{a_0}\right) \tag{77}
\]

Likewise, substitution of this assumed form of \( w \) into equation (14) gives:

\[
sin\left(\frac{mn x_0}{a_0}\right) \left\{ \left(\frac{mn}{a_0}\right)^4 Y(y_0) + Y^{IV}(y_0) \right\} - \left(\frac{mn}{a_0}\right)^2 k_{x_0} (n/b_0)^2 Y(y_0) + k_{y_0} \left(\frac{n}{a_0}\right)^2 Y^{II}(y_0) \right\} = 0 \tag{78}
\]

For equation (78) to hold in general, the terms inside the curly brackets must sum to zero. Therefore, equation (78) is merely a linear homogeneous differential equation with constant coefficients. The form of \( Y(y_0) \) is predicted to be that of equation (39). Insertion of this value of \( Y(y_0) \) into equation (78) yields:

\[
\left[ \left(\frac{mn}{a_0}\right)^4 + r^4 - \left(\frac{mn}{a_0}\right)^4 k_{x_0} \left(\frac{a_0}{mb_0}\right)^2 + \left(\frac{k_{y_0}}{m^2}\right) \left(\frac{mn}{a_0}\right)^2 r^2 \right] e^{iy_0} = 0 \tag{79}
\]

Just as above, for the right-hand side of equation (79) to vanish, the terms in the square brackets must cancel. This fact enables determination of \( r^2 \) by use of the quadratic
equation.

\[ r^2 = 0.5 \{ -(K_{yo}/m^2)^2 (mn/a_o)^2 + [(K_{yo}/m^2)^2 (mn/a_o)^4 + 4 (mn/a_o)^4 (K_{xo} (a_o/mb_o)^2 - 1)]^{1/2} \} \quad (80) \]

\[ r^2 = (mn/a_o)^2 \{ -(K_{yo}/2m^2)^2 + [(K_{yo}/2m^2)^2 + K_{xo} (a_o/mb_o)^2 - 1]^{1/2} \} \quad (81) \]

The next step in the solution process depends exclusively upon the algebraic sign of the quantity in the square brackets in equation (81). Real square roots, of course, can only be taken of non-negative numbers. As a result, a separate investigation is now made for each of the three possible signs of this governing term---negative, zero, and positive. In addition, consideration of the cases in just this sequence is vital, for the desired value of \( K_{xo} \) for any \( K_{yo} \) and \( a_o/b_o \) combination is the smallest possible \( x_o \)-buckling coefficient. So for \( m \) constant, a \( K_{xo} \) root found in the region \( \{ (K_{yo}/2m^2)^2 + K_{xo} (a_o/mb_o)^2 - 1 \} < 0 \) supercedes any root found in the zero or positive regions.

\[ \{ (K_{yo}/2m^2)^2 + K_{xo} (a_o/mb_o)^2 - 1 \} < 0 \]

For the quantity \( \{ (K_{yo}/2m^2)^2 + K_{xo} (a_o/mb_o)^2 - 1 \} < 0 \) equation (81) becomes:

\[ r^2 = (mn/a_o)^2 \{ -K_{yo}/2m^2 + i \{ 1 - K_{xo} (a_o/mb_o)^2 - (K_{yo}/2m^2)^2 \}^{1/2} \} \quad (82) \]

The first two values of \( r \) stem from choice of the positive sign before the imaginary number \( i \) in equation (82), while the final two can be obtained through selection of the
negative sign.

\[ r_{1,2}^2 = (mn/a_0)^2 \left\{ -K_{y_0}/2m^2 \right. \]
\[ + i \left[ 1 - K_{x_0} (a_0/mb_0)^2 - (K_{y_0}/2m^2)^2 \right]^{1/2} \} \quad (83) \]

\[ r_{3,4}^2 = (mn/a_0)^2 \left\{ -K_{y_0}/2m^2 \right. \]
\[ - i \left[ 1 - K_{x_0} (a_0/mb_0)^2 - (K_{y_0}/2m^2)^2 \right]^{1/2} \} \quad (84) \]

Consider first equation (83). It is desired to express the right-hand side of this equation in polar form.

\[ r_{1,2}^2 = M e^{i\theta} \quad (85) \]

where

- \( M = \) the modulus of the complex number
- \( \theta = \) angle which is the amplitude of the complex number

\[ M = (mn/a_0)^2 \left\{ (-K_{y_0}/2m^2)^2 \right. \]
\[ + \left[ 1 - K_{x_0} (a_0/mb_0)^2 - (K_{y_0}/2m^2)^2 \right]^{1/2} \} \quad (86) \]

\[ M = (mn/a_0)^2 \left\{ 1 - K_{x_0} (a_0/mb_0)^2 \right. \]
\[ - (K_{y_0}/2m^2)^2 \right]^{1/2} / (-K_{y_0}/2m^2) \} \quad (87) \]

\[ \theta = \arctan \left\{ \left[ 1 - K_{x_0} (a_0/mb_0)^2 \right. \]
\[ - (K_{y_0}/2m^2)^2 \right]^{1/2} / (-K_{y_0}/2m^2) \} \quad (88) \]

Note especially that the modulus \( M \) is independent of \( K_{y_0} \).

For later use, note the following:

\[ -\tan(\theta) = \tan(-\theta) = \left\{ - \left[ 1 - K_{x_0} (a_0/mb_0)^2 \right. \right. \]
\[ - (K_{y_0}/2m^2)^2 \right]^{1/2} / (-K_{y_0}/2m^2) \} \quad (89) \]

\[ -\theta = \arctan \left\{ -\left[ 1 - K_{x_0} (a_0/mb_0)^2 \right. \right. \]
\[ - (K_{y_0}/2m^2)^2 \right]^{1/2} / (-K_{y_0}/2m^2) \} \quad (90) \]
Now consider equation (84). From equations (86) and (90) it is easy to see that the equation specifying the third and fourth roots of $r$ can be written in polar form as:

$$r_{3,4}^2 = M e^{-i\phi}$$  \hspace{1cm} (91)

Since the quantity $\phi$ is merely an angle, $\phi$ has the same value as $(\phi + 2\pi)$. This fact allows separation of roots one and two and roots three and four.

$$r_1^2 = M e^{i\phi}$$  \hspace{1cm} (92)

$$r_2^2 = M e^{i(\phi + 2\pi)}$$  \hspace{1cm} (93)

$$r_3^2 = M e^{-i\phi}$$  \hspace{1cm} (94)

$$r_4^2 = M e^{-i(\phi + 2\pi)}$$  \hspace{1cm} (95)

Now, determination of the actual values of each $r$ can be carried out.

$$r_1 = M^{1/2} e^{i\phi/2} = M^{1/2} \{ \cos(\phi/2) + i \sin(\phi/2) \}$$  \hspace{1cm} (96)

$$r_2 = M^{1/2} e^{i(\phi + 2\pi)/2}$$

$$= M^{1/2} \{ \cos(\phi/2 + \pi) + i \sin(\phi/2 + \pi) \}$$  \hspace{1cm} (97)

$$r_2 = M^{1/2} \{ -\cos(\phi/2) - i \sin(\phi/2) \}$$  \hspace{1cm} (98)

$$r_3 = M^{1/2} e^{-i\phi/2} = M^{1/2} \{ \cos(-\phi/2) + i \sin(-\phi/2) \}$$  \hspace{1cm} (99)

$$r_3 = M^{1/2} \{ \cos(\phi/2) - i \sin(\phi/2) \}$$  \hspace{1cm} (100)

$$r_4 = M^{1/2} e^{-i(\phi + 2\pi)/2}$$

$$= M^{1/2} \{ \cos(-\phi/2 - \pi) + i \sin(-\phi/2 - \pi) \}$$  \hspace{1cm} (101)

$$r_4 = M^{1/2} \{ -\cos(\phi/2) + i \sin(\phi/2) \}$$  \hspace{1cm} (102)

Note especially that $r_1$ and $r_3$ are complex conjugate pairs, as are $r_2$ and $r_4$.
Define the following variables:

\[
\begin{align*}
  c &= M^{1/2} \cos(p/2) \quad (103) \\
  s &= N^{1/2} \sin(p/2) \quad (104)
\end{align*}
\]

Since the four roots of \( r \) have been fixed, the function \( Y(y) \) now reads:

\[
Y(y_o) = J_m e^{r_1y_o} + K_m e^{r_2y_o} + L_m e^{r_3y_o} + M_m e^{r_4y_o} \quad (105)
\]

where

\[
J_m, K_m, L_m, M_m = \text{arbitrary constants}
\]

Equivalently, equation (105), when combined with equations (103) and (104) can be expressed as:

\[
Y(y_o) = e^{c_yo} \{ A_m \sin(sy_o) + B_m \cos(sy_o) \}
+ e^{-c_yo} \{ C_m \sin(sy_o) + D_m \cos(sy_o) \} \quad (106)
\]

where

\[
A_m, B_m, C_m, D_m = \text{another set of arbitrary constants}
\]

Consider now the boundary conditions, equations (74), for this case of a laminate simply supported in the \( x_o \)-direction and clamped in the \( y_o \)-direction. As indicated previously, the origin location is subject to change, and the present situation dictates such a movement for maximum ease in manipulations. In particular, the \( x_o \)-origin remains the same, but the \( y_o \)-origin drops from the center of the plate to one edge. The final two equations of (74) then become:

\[
Y(0) \sin(mnx_0/a_o) = 0 \quad ; \quad Y'(0) \sin(mnx_0/a_o) = 0 \quad (107)
\]

\[
Y(b_o) \sin(mnx_0/a_o) = 0 \quad ; \quad Y'(b_o) \sin(mnx_0/a_o) = 0 \quad (108)
\]

For equations (107) and (108) to have meaning in the general case, the following conditions must hold:
\[
Y(0) = 0 \quad (109)
\]
\[
Y'(0) = 0 \quad (110)
\]
\[
Y(b_o) = 0 \quad (111)
\]
\[
Y'(b_o) = 0 \quad (112)
\]

First, apply equation (109) to equation (106). This stipulation fixes \(D_m\) in terms of \(B_m\) such that \(D_m = -B_m\).

Utilization of equation (110) on the \(Y(y_o)\) equation similarly determines a value \(C_m\) in terms of \(A_m\) and \(B_m\).

\[
sA_m + cB_m + SC_m - cD_m = 0 \quad (113)
\]

But since \(D_m = -B_m\)

\[
C_m = -A_m - 2B_m (c/s) \quad (114)
\]

For these values of \(C_m\) and \(D_m\), equation (106) takes on the following form:

\[
Y(y_o) = A_m \{ \sin(sy_o) \ (e^{cy_o} - e^{-cy_o}) \} \\
+ B_m \{-2(c/s) \ e^{-cy_o} \sin(sy_o) + \cos(sy_o) \ (e^{cy_o} - e^{-cy_o}) \} \quad (115)
\]

The first derivative of \(Y(y_o)\) with respect to \(y_o\) is then easy to obtain.

\[
Y'(y_o) = A_m \{ \sin(sy_o) \ (ce^{cy_o} + ce^{-cy_o}) \\
+ \cos(sy_o) \ (se^{cy_o} - se^{-cy_o}) \} \\
+ B_m \{ \sin(sy_o) \ (se^{-cy_o} - se^{cy_o} + 2(c^2/s)e^{-cy_o}) \\
+ \cos(sy_o) \ (ce^{cy_o} - ce^{-cy_o}) \} \quad (116)
\]

Substitution of equations (115) and (116) into equations (111) and (112) yield two homogeneous linear equations in coefficients \(A_m\) and \(B_m\). For non-trivial \(A_m\) and \(B_m\), the following determinental equation must hold:
\[
\begin{vmatrix}
\sin(sb_o) (e^{cb_o} - e^{-cb_o}) & \sin(sb_o) \left( -(2/c) e^{-cb_o} \right) \\
\sin(sb_o) (ce^{cb_o} + ce^{-cb_o}) & \sin(sb_o) (-se^{cb_o} + se^{-cb_o}) \\
\sin(sb ) (ce^{cb_o} + ce^{-cb_o}) & \sin(sb ) (-se^{cb_o} + se^{-cb_o}) \\
+ \cos(sb_o) (se^{cb_o} - se^{-cb_o}) & + \cos(sb_o) (ce^{cb_o} - ce^{-cb_o})
\end{vmatrix} = 0
\]

Expansion of the determinant gives:
\[e^{2cb_o} - 2 + e^{-2cb_o} - 4(c/s)^2 \sin^2(sb_o) = 0 \quad (118)\]

Slight rearrangement and multiplication of the final term on the left-hand side of the equation by \((b_o/b_o)^2\) yields a transcendental equation in the variables \(a_o/b_o, K_{yo}, \) and \(K_{xo} - \)
\[e^{2cb_o} + e^{-2cb_o} - 2\left\{ 1 + 2(cb_o/sb_o)^2 \sin^2(sb_o) \right\} = 0 \quad (119)\]

Unfortunately, for any combination of \(a_o/b_o\) and \(K_{yo}\), no value of \(K_{xo}\) satisfies equation (119). In other words, no possible solutions exist for the present set of boundary conditions for \(\{(K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1 \} < 0\)

\[\{(K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1 \} = 0 \]

For the quantity \(\{(K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1 \} = 0\)
equation (81) becomes:
\[r^2 = (mn/a_o)^2 (-K_{yo}/2m^2) = (n/a_o)^2 \left( -K_{yo}/2 \right) \quad (120)\]

Since the character of equation (120) differs drastically for the choice of algebraic sign of \(K_{yo}\), each possible range of \(K_{yo}\) --negative, zero, and positive--will be analyzed as separate subcases.
\( K_{y_0} < 0 \).

Two values of \( r \) which satisfy equation (120) for \( K_{y_0} < 0 \) are straightforward.

\[ r_{1,2} = \pm \left( \frac{n}{\alpha_o} \right) \left( -K_{y_0}/2 \right)^{1/2} \]  
(121)

For simplicity, make the following definition:

\[ T = \left( \frac{n}{\alpha_o} \right) \left( -K_{y_0}/2 \right)^{1/2} \]  
(122)

From equations (121) and (122), the theory of linear homogeneous equations, and the concept of repeated roots, the value of the function \( Y(y_o) \) can be determined.

\[ Y(y_o) = A_m \sinh(Ty_o) + B_m \cosh(Ty_o) + C_m y_o \sinh(Ty_o) \]
\[ + D_m y_o \cosh(Ty_o) \]  
(123)

where

\[ A_m, B_m, C_m, D_m = \text{set of arbitrary constants} \]

Equations (109) through (112) again comprise the group of boundary conditions for the \( Y(y_o) \) function. The enforcement of equation (109) on equation (123) dictates that \( B_m \) must vanish. In addition, application of equation (110) means that \( D_m = -A_m T \) so equation (123) takes the following form:

\[ Y(y_o) = A_m \{ \sinh(Ty_o) - Ty_o \cosh(Ty_o) \} + C_m y_o \sinh(Ty_o) \]  
(124)

The first derivative of \( Y(y_o) \) with respect to \( y_o \) is therefore easy to obtain.

\[ Y'(y_o) = -A_m T^2 y_o \sinh(Ty_o) + C_m \{ \sinh(Ty_o) \]
\[ + Ty_o \cosh(Ty_o) \} \]  
(125)

Substitution of equations (124) and (125) into equations (111) and (112) again yield two homogeneous linear equations in coefficients \( A_m \) and \( C_m \). For non-trivial \( A_m \) and \( C_m \), the
following determinental equation must hold:

\[
\begin{vmatrix}
\sinh(Tb_o) - Tb_o \cosh(Tb_o) & b_o \sinh(Tb_o) \\
-T^2 b_o \sinh(Tb_o) & \sinh(Tb_o) + Tb_o \cosh(Tb_o)
\end{vmatrix} = 0
\]  
(126)

Expansion of the determinant gives:

\[
\sinh^2(Tb_o) - (Tb_o)^2 = 0
\]  
(127)

No value of \( Tb_o \) greater than zero can satisfy equation (127). Therefore, no possible solutions exist for the present boundary conditions for

\[
\{(K_yo/2m^2) + K_xo (a_0/mb_0)^2 \} - 1 = 0 \quad \text{and} \quad K_yo < 0
\]

\( K_yo = 0. \)

One value of \( r \) which satisfies equation (120) for

\( K_yo = 0 \) is simply \( r = 0 \) From this simple result, the theory of linear homogeneous equations, and the concept of repeated roots, the value of the function \( Y(yo) \) can be determined.

\[
Y(yo) = A_m + B_m yo + C_m yo^2 + D_m yo^3
\]  
(128)

where

\( A_m, B_m, C_m, D_m = \) set of arbitrary constants

Equations (109) through (112) constitute the set of constraints for the \( Y(yo) \) function. Equations (109) and (110) imply that \( A_m = B_m = 0 \) Equation (111) and equation (112), with each side of (112) multiplied by \( b_o \), yield the following set of simultaneous equations.

\[
\begin{align*}
C_m b_o^2 + D_m b_o^3 &= 0 \\
2C_m b_o^2 + 3D_m b_o^3 &= 0
\end{align*}
\]  
(129)

Only \( C_m = D_m = 0 \) constitutes a valid solution for
**TABLE VI**

**Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the $x_o$-Direction and Clamped on the Two Edges Normal to the $y_o$-Direction**

(The second column denotes the symmetric or antisymmetric nature of $K_{x_o}$.)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x$</th>
<th>$a_z/b_o$</th>
<th>$K_{y_o}$</th>
<th>$K_{x_o}$</th>
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<td>-3.0</td>
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<tr>
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<td>S</td>
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<td>3.0323</td>
<td>3.0</td>
<td>4.4590</td>
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</tbody>
</table>
however, the quantitative aspect of Figure 12 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table VII gives selected coordinates of $K_{y_o}$ and $K_{x_o}$ for three distinct values of $a_o/b_o$ -- 1.2, 2.4, and 3.6. Also included are the states of symmetry or antisymmetry for $K_{x_o}$.

Figure 13, 14, and 15 represent two-dimensional plots at constant $a_o/b_o$ slices of 1.2, 2.4, and 3.6, respectively. These graphs, very similar to those obtained in the simply supported on all sides case, are composed of very nearly straight line segments. Note especially that $K_{x_o}$ declines as $K_{y_o}$ increases and that the rate of decline of $K_{x_o}$ for an increase in $K_{y_o}$ jumps markedly for small $a_o/b_o$. 

-65-
the state of symmetry for \( K_{xo} \) and the integer value of \( m \) which produces this minimum \( K_{xo} \). Furthermore, each entry point which corresponds to a transition point from the \( m \) curve to the \( (m+1) \) curve is superscripted in the \( a_0/b_0 \) column with a star (*).

The statistics presented in Table VI expose two important characteristics of laminates under compression or tension in the \( y_0 \)-direction. First, the transition values of \( a_0/b_0 \) increase as \( K_{y_0} \) becomes algebraically larger (or less tensile). This trend is most pronounced for the initial transition points, and its effect diminishes as \( a_0/b_0 \) approaches a large number. Second, irrespective of the magnitude of \( K_{y_0} \), \( K_{xo} \) attains a limiting value of 4.534 as \( a_0/b_0 \) approaches infinity.

Figure 11 represents a plot of \( K_{xo} \) versus \( a_0/b_0 \) for eleven distinct values of \( K_{y_0} \). The lowest curve characterizes \( K_{y_0} = 5.0 \); whereas, the highest depicts \( K_{y_0} = -5.0 \). The nine other curves differ from each other by increments of one. This graph reinforces the concept that \( K_{xo} \) for a constant \( K_{y_0} \) is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. In addition, the merging of all curves to a limiting value of \( K_{xo} = 4.534 \) for \( a_0/b_0 \) large is readily apparent.

Figure 12 plots in three dimensions the same information as Figure 11. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 11;
\[(\beta_m b_o/2) \sinh(\beta_m b_o/2) \cosh(\beta_m b_o/2) - (\beta_m b_o/2) \sinh(\beta_m b_o/2) \cosh(\beta_m b_o/2) = 0 \] (180)

Attempts at solution of this antisymmetric buckling equation (180) for any combination of \(a_o/b_o\) and \(K_{yo}\) do not yield the smallest values of \(K_{x_o}\). Therefore, further consideration of this equation is dropped.

In an identical fashion, equations (150) and (151) are employed as boundary conditions for the symmetric portion of \(Y(Y_o), Y_s(Y_o)\). For non-trivial constants \(B_m\) and \(D_m\), the following determinental equation must hold:

\[
\begin{vmatrix}
\cosh(\beta_m b_o/2) & \cosh(\beta_m b_o/2) \\
\beta_m \sinh(\beta_m b_o/2) & \beta_m \sinh(\beta_m b_o/2)
\end{vmatrix} = 0 \] (181)

Expansion of the determinant and multiplication by the quantity \(b_o/2\) gives:

\[(\beta_m b_o/2) \cosh(\beta_m b_o/2) \sinh(\beta_m b_o/2)
- (\beta_m b_o/2) \sinh(\beta_m b_o/2) \cosh(\beta_m b_o/2) = 0 \] (182)

Attempts at solution of this antisymmetric buckling equation (182) for any combination of \(a_o/b_o\) and \(K_{yo}\) do not yield the smallest values of \(K_{x_o}\). As a result, further considerations of this equation and this subcase as a whole are abandoned.

**Discussion of Results**

Table VI gives selected \(a_o/b_o\), \(K_{yo}\), and \(K_{x_o}\) ordered triplets as determined by equation (169). Also included is
\[ J_m, K_m, L_m, M_m = \text{set of arbitrary constants} \]

Equivalently, equation (174) can be written as:
\[
Y(y_0) = A_m \sinh(\epsilon_m y_0) + B_m \cosh(\epsilon_m y_0) + C_m \sinh(\beta_m y_0) + D_m \cosh(\beta_m y_0)
\]  
(175)

where
\[
\epsilon_m = (m\pi/a_0) \{ -K_{y_0} /2m^2 - [(K_{y_0} /2m^2)^2
\]
\[+ K_{y_0} (a_0/m b_0)^2 - 1 ]^{1/2} \}^{1/2}  
\]  
(176)
\[
\beta_m = ( \text{defined in equation (163) } )
\]
\[ A_m, B_m, C_m, D_m = \text{another set of arbitrary constants} \]

which depend upon the integer \( m \)

The function \( Y(y_0) \) can be further simplified by reduction into its antisymmetric and symmetric parts.
\[
Y_A(y_0) = A_m \sinh(\epsilon_m y_0) + C_m \sinh(\beta_m y_0)  
\]  
(177)
\[
Y_S(y_0) = B_m \cosh(\epsilon_m y_0) + D_m \cosh(\beta_m y_0)
\]  
(178)

As illustrated in the first of these subcases, the boundary conditions shown in equations (150) and (151) govern. Application of equations (150) and (151) first is made to the antisymmetric portion of \( Y(y_0) \), \( Y_A(y_0) \). For non-trivial constants \( A_m \) and \( C_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sinh(\epsilon_m b_0/2) & \sinh(\epsilon_m b_0/2) \\
\epsilon_m \cosh(\epsilon_m b_0/2) & \beta_m \cosh(\beta_m b_0/2)
\end{vmatrix} = 0
\]  
(179)

Expansion of the determinant and multiplication by the quantity \( b_0/2 \) gives:
Attempts at solution of equation (169) for any combination of \( a_0/b_o \) and \( K_yo \) do yield the smallest values of \( K_{xo} \). The buckling is always symmetric in nature. Further consideration of results generated by equation (169) is postponed until the last of the three subcases is presented.

\( K_yo \) **Ranges from a Relatively Large Negative Number to Negative Infinity.**

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, \( K_yo \) can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form then, the search for a solution for \( K_{xo} \) for this subcase is limited by the following inequalities:

\[
(K_{yo}/2m^2)^2 + K_{xo}(a_o/mbo)^2 - l 0\]  \( (170) \)

\[-K_{yo}/2m^2 - [(K_{yo}/2m^2)^2 + K_{xo}(a_o/mbo)^2 - l ]^{1/2} 0\]  \( (171) \)

As a result, all four values of \( r \) can be determined.

\[ r_{1,2} = \pm (mn/a_o) \{ -K_{yo}/2m^2 - [(K_{yo}/2m^2)^2 \\
+ K_{xo}(a_o/mbo)^2 - l ]^{1/2} \}^{1/2} \]  \( (172) \)

\[ r_{3,4} = \pm (mn/a_o) \{ -K_{yo}/2m^2 + [(K_{yo}/2m^2)^2 \\
+ K_{xo}(a_o/mbo)^2 - l ]^{1/2} \}^{1/2} \]  \( (173) \)

Since all four values of \( r \) are known, the desired function \( Y(yo) \) can be written as:

\[ Y(yo) = \Sigma_m e^{r_yo} + K_m e^{r^2yo} + L_m e^{r^3yo} + M_m e^{r^4yo} \]  \( (174) \)

where
antisymmetric portion of \( Y(y_0) \), \( Y_A(y_0) \). For non-trivial constants \( A_m \) and \( C_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(a_m b_o/2) & \sinh(\beta_m b_o/2) \\
a_m \cos(a_m b_o/2) & \beta_m \cosh(\beta_m b_o/2)
\end{vmatrix} = 0 \quad (166)
\]

Expansion of the determinant and multiplication by the quantity \( b_o/2 \) gives:

\[
(\beta_m b_o/2) \sin(a_m b_o/2) \cosh(\beta_m b_o/2) \\
- (a_m b_o/2) \cos(a_m b_o/2) \sinh(\beta_m b_o/2) = 0 \quad (167)
\]

Attempts at solution of this antisymmetric buckling equation (167) for any combination of \( a_o/b_o \) and \( K_{y_o} \) do not yield the smallest values of \( K_{x_o} \). Therefore, further consideration of this equation is dropped.

In an identical manner, equations (150) and (151) are employed as boundary conditions for the symmetric portion of \( Y(y_0) \), \( Y_S(y_0) \). For non-trivial constants \( B_m \) and \( D_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\cos(a_m b_o/2) & \cosh(\beta_m b_o/2) \\
- a_m \sin(a_m b_o/2) & \beta_m \sinh(\beta_m b_o/2)
\end{vmatrix} = 0 \quad (168)
\]

Expansion of the determinant and multiplication by the quantity \( b_o/2 \) gives:

\[
(\beta_m b_o/2) \cos(a_m b_o/2) \sinh(\beta_m b_o/2) \\
+ (a_m b_o/2) \sin(a_m b_o/2) \cosh(\beta_m b_o/2) = 0 \quad (169)
\]
As a result, all four values of $r$ can be determined.

$$r_{1,2} = \pm (mn/a_0) i \left\{ K_{yo}/2m^2 + [(K_{yo}/2m^2)^2 \\
+ K_{x_0}(a_0/mb_0)^2 - 1]^{1/2} \right\}^{1/2} \quad (159)$$

$$r_{3,4} = \pm (mn/a_0) \left\{ -K_{yo}/2m^2 + [(K_{yo}/2m^2)^2 \\
+ K_{x_0}(a_0/mb_0)^2 - 1]^{1/2} \right\}^{1/2} \quad (160)$$

Since all four values of $r$ are known, the desired function $Y(y_o)$ can be written as:

$$Y(y_o) = J_m e^{iy_o} + K_m e^{2iy_o} + L_m e^{3iy_o} + M_m e^{4iy_o} \quad (161)$$

where

$J_m, K_m, L_m, M_m$ = set of arbitrary constants

Equivalently, equation (161) can be expressed as:

$$Y(y_o) = A_m \sin(\alpha_m y_o) + B_m \cos(\alpha_m y_o) + C_m \sinh(\beta_m y_o) \\
+ D_m \cosh(\beta_m y_o) \quad (162)$$

where

$$\alpha_m = (\text{defined in equation (145)})$$

$$\beta_m = (mn/a_0) \left\{ -K_{yo}/2m^2 + [(K_{yo}/2m^2)^2 \\
+ K_{x_0}(a_0/mb_0)^2 - 1]^{1/2} \right\}^{1/2} \quad (163)$$

$A_m, B_m, C_m, D_m$ = another set of arbitrary constants

which depend upon the integer $m$

The function $Y(y_o)$ can be further simplified by reduction into its antisymmetric and symmetric parts.

$$Y_A(y_o) = A_m \sin(\alpha_m y_o) + C_m \sinh(\beta_m y_o) \quad (164)$$

$$Y_S(y_o) = B_m \cos(\alpha_m y_o) + D_m \cosh(\beta_m y_o) \quad (165)$$

As illustrated in the previous subcase, the boundary conditions shown in equations (150) and (151) govern. Application of equations (150) and (151) first is made to the
\[
\begin{vmatrix}
\cos(\alpha_m b_0/2) & \cos(\nu_m b_0/2) \\
-\alpha_m \sin(\alpha_m b_0/2) & -\nu_m \sin(\nu_m b_0/2)
\end{vmatrix} = 0 \quad (154)
\]

Expansion of the determinant and multiplication by the quantity \((-b_0/2)\) gives:

\[
(\nu_m b_0/2) \sin(\nu_m b_0/2) \cos(\alpha_m b_0/2)
- (\alpha_m b_0/2) \sin(\alpha_m b_0/2) \cos(\nu_m b_0/2) = 0 \quad (155)
\]

Likewise, attempts at solution of equation (155) for any combination of \(\alpha_0/b_0\) and \(\nu_{y_0}\) do not yield the smallest values of \(K_{x_0}\). As a result, further considerations of this equation and this subcase as a whole are abandoned.

\(K_{y_0}\) **Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.**

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, \(K_{y_0}\) can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, \(K_{y_0}\) can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that \(K_{y_0}\) range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution for \(-K_{x_0}\) for this subcase is limited by the following three inequalities:

\[
(K_{y_0}/2m^2)^2 + K_{x_0} (\alpha_0/m_{b_0})^2 - 1 > 0 \quad (156)
-\nu_{y_0}/2m^2 - [(K_{y_0}/2m^2)^2 + K_{x_0} (\alpha_0/m_{b_0})^2 - 1]^{1/2} < 0 \quad (157)
-\nu_{y_0}/2m^2 + [(K_{y_0}/2m^2)^2 + K_{x_0} (\alpha_0/m_{b_0})^2 - 1]^{1/2} > 0 \quad (158)
\]
information of equation (149) will be manipulated. For equation (149) to hold in general, the following boundary conditions must be obeyed:

\[ Y(b_o/2) = 0 \]  \hspace{1cm} (150)
\[ Y'(b_o/2) = 0 \]  \hspace{1cm} (151)

Application of equations (150) and (151) first is made to the antisymmetric portion of \( Y(y_o) \), \( Y_A(y_o) \). For non-trivial constants \( A_m \) and \( C_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(a_m b_o/2) & \sin(v_m b_o/2) \\
\cos(a_m b_o/2) & \cos(v_m b_o/2)
\end{vmatrix} = 0 \quad (152)
\]

Expanding the determinant and multiplication by the quantity \( b_o/2 \) gives:

\[
(v_m b_o/2) \sin(a_m b_o/2) \cos(v_m b_o/2) - (a_m b_o/2) \sin(v_m b_o/2) \cos(a_m b_o/2) = 0 \quad (153)
\]

Attempts at solution of equation (153) for any combination of \( a_o/b_o \) and \( K_{yo} \) do not yield the smallest values of \( K_{x_o} \). Therefore, further consideration of this equation is dropped.

In an identical manner, equations (150) and (151) are employed as boundary conditions for the symmetric portion of \( Y(y_o) \), \( Y_S(y_o) \). For non-trivial constants \( B_m \) and \( D_m \), the following determinental equation must hold:
where

\[ J_m, K_m, L_m, M_m = \text{set of arbitrary constants} \]

Equivalently, equation (143) can be expressed as:

\[ Y(y_0) = A_m \sin(a_m y_0) + B_m \cos(a_m y_0) + C_m \sin(v_m y_0) \]
\[ + D_m \cos(v_m y_0) \quad (144) \]

where

\[ a_m = (mn/a_0)\left[K_{y_0}/2m^2 + [(K_{y_0}/2m^2)^2 \right. \]
\[ + K_{x_0}(a_0/mb_0)^2 - 1 \left.]^{1/2} \right)^{1/2} \quad (145) \]

\[ v_m = (mn/a_0)\left[K_{y_0}/2m^2 - [(K_{y_0}/2m^2)^2 \right. \]
\[ + K_{x_0}(a_0/mb_0) - 1 \left.]^{1/2} \right)^{1/2} \quad (146) \]

\[ A_m, B_m, C_m, D_m = \text{another set of arbitrary constants} \]

which depend on the integer \( m \)

The function \( Y(y_0) \) can be further simplified by reduction into its antisymmetric and symmetric parts.

\[ Y_A(y_0) = A_m \sin(a_m y_0) + C_m \sin(v_m y_0) \quad (147) \]
\[ Y_S(y_0) = B_m \cos(a_m y_0) + D_m \cos(v_m y_0) \quad (148) \]

Consider the boundary conditions, equations (74), for this case of a laminate simply supported in the \( x_0 \)-direction and clamped in the \( y_0 \)-direction. The final two, expressed in the separation functions, become:

\[ Y(-b_0/2) \sin(mn x_0/a_0) = 0 \quad ; \quad Y'(-b_0/2) \sin(mn x_0/a_0) = 0 \] 
\[ (149) \]
\[ Y(b_0/2) \sin(mn x_0/a_0) = 0 \quad ; \quad Y'(b_0/2) \sin(mn x_0/a_0) = 0 \]

The two lines of equation (149) each express identical information when the function \( Y(y_0) \) is broken down into its components \( Y_A(y_0) \) and \( Y_S(y_0) \). Thus, only the bottom line of
imply not only different solution forms but different domains of $K_y$ for valid solutions. Three subcases again must be considered so that a solution for $K_x$ may be determined for any range of $K_y$.

$K_y$ Ranges from a Relatively Large Positive Number to Positive Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, $K_y$ can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, $K_y$ can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely this last quoted domain. In equation form the search for a solution for $K_x$ for this subcase is limited by the following two inequalities:

\[ \frac{(K_y/2m^2)^2}{2} + K_x (a_o/mb_o)^2 - 1 > 0 \quad (139) \]
\[ -K_y/2m^2 + [((K_y/2m^2)^2 + K_x (a_o/mb_o)^2 - 1)^{1/2} < 0 \quad (140) \]

As a result, all four values of $r$ can be determined.

\[ r_{1,2} = \pm (mn/a_o) \left\{ \frac{K_y}{2m^2} + \left[ \frac{(K_y/2m^2)^2}{2} + K_x (a_o/mb_o)^2 - 1 \right]^{1/2} \right\}^{1/2} \quad (141) \]
\[ r_{3,4} = \pm (mn/a_o) \left\{ \frac{K_y}{2m^2} - \left[ \frac{(K_y/2m^2)^2}{2} \right. \right.
\[ \left. + K_x (a_o/mb_o)^2 - 1 \right]^{1/2} \right\}^{1/2} \quad (142) \]

Since all four values of $r$ are known, the desired function $Y(y)$ can be written as:

\[ Y(y) = J_m e^{r_1y} + K_m e^{r_2y} + L_m e^{r_3y} + M_m e^{r_4y} \quad (143) \]
coefficients $A_m$ and $C_m$. For non-trivial $A_m$ and $C_m$, the following determinental equation must hold:

$$
\begin{vmatrix}
\sin(U_b) - U_b\cos(U_b) & b_o\sin(U_b) \\
U^2b_o\sin(U_b) & \sin(U_b) + U_b\cos(U_b)
\end{vmatrix} = 0
$$

Expansion of the determinant gives:

$$
\sin^2(U_b) - (U_b)^2 = 0 \quad (136)
$$

No value of $U_b$ greater than zero can satisfy equation (136). Therefore, no possible solutions exist for the present boundary conditions for

$$(K_y/2m^2) + K_x (a_o/m_b)^2 - l) = 0 \quad \text{and} \quad K_y > 0$$

$$(K_y/2m^2) + K_x (a_o/m_b)^2 - l) > 0$$

For the quantity $$(K_y/2m^2) + K_x (a_o/m_b)^2 - l) > 0$$
equation (81) does not need to be altered. The first two roots of $r$ stem from the selection of the negative sign before the left square bracket in equation (81), while the final two can be obtained by choice of the positive sign. 

$$r_{1,2}^2 = \left(\frac{mn}{a_o}\right)^2 \left[-\frac{K_y}{2m^2} - \left\{\left(\frac{K_y}{2m^2}\right)^2 \right. \\
\left. + K_x (a_o/m_b)^2 - l \right\}^{1/2}\right] \quad (137)$$

$$r_{3,4}^2 = \left(\frac{mn}{a_o}\right)^2 \left[-\frac{K_y}{2m^2} + \left\{\left(\frac{K_y}{2m^2}\right)^2 \right. \\
\left. + K_x (a_o/m_b)^2 - l \right\}^{1/2}\right] \quad (138)$$

The quantities of intense interest now are those contained in the curly brackets of equations (137) and (138). Positive or negative characters of each of these quantities
equations (129). So again no possible solutions exist for the present boundary conditions for

\[ \{(K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1\} = 0 \quad \text{and} \quad K_{yo} = 0 \]

\[ K_{yo} > 0. \]

Two values of r which satisfy equation (120) for \( K_{yo} > 0 \) are straightforward.

\[ r_{1,2} = \pm i \left(\frac{n/a_o}{K_{yo}/2}\right)^{1/2} \quad (130) \]

For simplicity, make the following definition:

\[ U = \left(\frac{n/a_o}{K_{yo}/2}\right)^{1/2} \quad (131) \]

From equations (130) and (131), the theory of linear homogeneous equations, and the concept of repeated roots, the value of the function \( Y(y_o) \) can be determined.

\[ Y(y_o) = A_m \sin(Uy_o) + B_m \cos(Uy_o) + C_m y_o \sin(Uy_o) + D_m y_o \cos(Uy_o) \quad (132) \]

where

\[ A_m, B_m, C_m, D_m = \text{set of arbitrary constants} \]

Equations (109) through (112) again comprise the body of boundary conditions for the \( Y(y_o) \) function. The enforcement of equation (109) on equation (132) dictates that \( B_m \) must vanish. Furthermore, application of equation (110) means that \( D_m = -A_m U \) So equation (132) takes the following form:

\[ Y(y_o) = A_m \{\sin(Uy_o) - Uy_o \cos(Uy_o)\} + C_m y_o \sin(Uy_o) \quad (133) \]

The first derivative of \( Y(y_o) \) with respect to \( y_o \) is therefore easy to obtain.

\[ Y'(y_o) = A_m U^2 y_o \sin(Uy_o) + C_m \{\sin(Uy_o) + Uy_o \cos(Uy_o)\} \quad (134) \]

Substitution of equations (133) and (134) into equations (111) and (112) yield two homogeneous linear equations in
TABLE VII

$K_{xo}$ Versus $K_{yo}$ for Various Plate Aspect Ratios for a Laminate Simply Supported in the $x_o$-Direction and Clamped in the $y_o$-Direction

(The second column denotes the symmetric or antisymmetric nature of $K_{xo}$.)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$x$</th>
<th>$a_0/b_0$</th>
<th>$K_{yo}$</th>
<th>$K_{xo}$</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>S</td>
<td>1.2000</td>
<td>-5.0</td>
<td>6.1580</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>1.2000</td>
<td>-3.0</td>
<td>5.5520</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>1.2000</td>
<td>-1.0</td>
<td>4.9381</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>1.2000</td>
<td>0.0</td>
<td>4.6277</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>1.2000</td>
<td>1.0</td>
<td>4.3148</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>1.2000</td>
<td>3.0</td>
<td>3.6796</td>
</tr>
<tr>
<td>1</td>
<td>S</td>
<td>1.2000</td>
<td>5.0</td>
<td>1.7018</td>
</tr>
<tr>
<td>4</td>
<td>S</td>
<td>2.4000</td>
<td>-5.0</td>
<td>5.0153</td>
</tr>
<tr>
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<td>2.4000</td>
<td>-3.0</td>
<td>4.8607</td>
</tr>
<tr>
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<td>S</td>
<td>2.4000</td>
<td>-1.0</td>
<td>4.7056</td>
</tr>
<tr>
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<td>S</td>
<td>2.4000</td>
<td>0.0</td>
<td>4.6277</td>
</tr>
<tr>
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<td>S</td>
<td>2.4000</td>
<td>1.0</td>
<td>4.5497</td>
</tr>
<tr>
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<td>S</td>
<td>2.4000</td>
<td>3.0</td>
<td>4.3933</td>
</tr>
<tr>
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<td>S</td>
<td>3.6000</td>
<td>-5.0</td>
<td>4.8005</td>
</tr>
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<td>S</td>
<td>3.6000</td>
<td>-3.0</td>
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</tr>
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<td>S</td>
<td>3.6000</td>
<td>-1.0</td>
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<td>3.6000</td>
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<td>4.5930</td>
</tr>
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<td>S</td>
<td>3.6000</td>
<td>1.0</td>
<td>4.5431</td>
</tr>
<tr>
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<td>S</td>
<td>3.6000</td>
<td>3.0</td>
<td>4.4432</td>
</tr>
<tr>
<td>5</td>
<td>S</td>
<td>3.6000</td>
<td>5.0</td>
<td>4.3431</td>
</tr>
</tbody>
</table>
FOR THE RANGE
KYO = -5.0 TO +5.0
11 CURVES, EACH VARIES
BY KYO INCREMENT OF 1.0
KYO = -5.0 HIGHEST CURVE

FIGURE 11
XD-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-C-S-C LAMINATE
FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES
FIGURE 12
SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-C-S-C LAMINATE
Figure 13

Xo-buckling coefficient versus yo-buckling coefficient at a constant affine aspect ratio of 1.2 for an s-c-s-c laminate.
Figure 14: X0-Buckling Coefficient Versus X0 for Constant Affine Respect Ratio of 2.4 for an S-C-S-C Laminate
FOR THE RANGE
KYO = -5.0 TO 5.0

FIGURE 15
XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT
RATIO OF 3.6 FOR AN S-C-S-C LAMINATE
The boundary conditions for a laminate simply supported in the \( x_o \)-direction and simply supported and clamped on the two edges perpendicular to the \( y_o \)-direction display no symmetry in the \( y_o \)-direction. For the two edges which have normals parallel to the \( x_o \)-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. Similarly, for that edge normal to the \( y_o \)-direction which is simply supported, these same edge conditions also hold. However, for the remaining edge, which is oriented perpendicular to the \( y_o \)-direction, the vertical displacement and the slope of the vertical displacement with respect to \( y_o \) must vanish. In equation form, the following must hold:

\[
\begin{align*}
\text{on edge } x_o = -a_o/2, \quad w &= 0 \quad w_{x_o x_o} = 0 \\
\text{on edge } x_o = a_o/2, \quad w &= 0 \quad w_{x_o x_o} = 0 \\
\text{on edge } y_o = 0, \quad w &= 0 \quad w_{y_o y_o} = 0 \quad (183) \\
\text{on edge } y_o = b_o, \quad w &= 0 \quad w_{y_o y_o} = 0
\end{align*}
\]

Note that the origin of coordinates in the affine space is taken to be at the center of the simply supported edge normal to the \( y_o \)-direction. This choice of origin location, in general, allows maximum simplicity in manipulations since a lack of symmetry is present in the boundary conditions in
the \( y_0 \)-direction.

Just as before, a displacement function \( w \) which satisfies the first two stipulations of equations (183) is given by equation (77). Furthermore, substitution of this relation for \( w \) into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable \( r \) is defined in equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states—negative, zero, and positive—of the quantity in the square brackets of equation (81). Again, as explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

\[
\{ \left( \frac{K_{y_0}}{2m^2} \right)^2 + K_{x_0} \left( \frac{a_o}{m b_o} \right)^2 - 1 \} < 0
\]

For the quantity \( \{ \left( \frac{K_{y_0}}{2m^2} \right)^2 + K_{x_0} \left( \frac{a_o}{m b_o} \right)^2 - 1 \} < 0 \) equations (82) through (105) explicitly show that the unknown function \( Y(y_0) \) must take the form shown in equation (106).

Consider now the boundary conditions, equations (183), for this case of a laminate simply supported on three sides and clamped on the fourth. When the chosen form of \( w \), equation (77), is substituted into the final two equations of equations (183), the following must hold:

\[
Y(0) \sin(mn x_o/a_o) = 0 ; \quad Y''(0) \sin(mn x_o/a_o) = 0 \quad (184)
\]

\[
Y(b_o) \sin(mn x_o/a_o) = 0 ; \quad Y'(b_o) \sin(mn x_o/a_o) = 0 \quad (185)
\]

For equations (184) and (185) to have meaning in the general case, the following conditions must hold:
\[
\begin{align*}
Y(0) &= 0 \quad \text{(186)} \\
Y''(0) &= 0 \quad \text{(187)} \\
Y(b_0) &= 0 \quad \text{(188)} \\
Y'(b_0) &= 0 \quad \text{(189)}
\end{align*}
\]

First, apply equation (186) to equation (106). This solution fixes \( D_m \) in terms of \( B_m \) such that \( D_m = -B_m \).

Utilization of equation (187) on the \( Y(y_0) \) equation similarly determines a value \( C_m \) in terms of \( A_m \).

\[
A_m(2cs) + B_m(c^2 - s^2) - C_m(2cs) + D_m(c^2 - s^2) = 0 \quad \text{(190)}
\]

But since \( D_m = -B_m \),

\[
C_m = A_m(2cs)/(2cs) = A_m \quad \text{(191)}
\]

For these values of \( C_m \) and \( D_m \), equation (106) takes on the following form:

\[
Y(y_0) = A_m \{ \sin(sy_0) (e^{cy_0} + e^{-cy_0}) \\
+ B_m \{ \cos(sy_0) (e^{cy_0} - e^{-cy_0}) \} \} \quad \text{(192)}
\]

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is easy to obtain.

\[
Y'(y_0) = A_m \{ \cos(sy_0) (se^{cy_0} + se^{-cy_0}) \\
+ \sin(sy_0) (ce^{cy_0} - ce^{-cy_0}) \} \\
+ B_m \{ \sin(sy_0) (-se^{cy_0} + se^{-cy_0}) \\
+ \cos(sy_0) (ce^{cy_0} + ce^{-cy_0}) \} \quad \text{(193)}
\]

Substitution of equations (192) and (193) into equations (188) and (189) yield two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinental equation must hold:
sin(sb_o)(e^{cb_o} + e^{-cb_o}) \quad \cos(sb_o)(e^{cb_o} - e^{-cb_o})

\sin(sb)(ce^{cb_o} - ce^{-cb_o}) \quad \sin(sb_o)(-se^{cb_o} + se^{-cb_o})

+ \cos(sb_o)(se^{cb_o} + se^{-cb_o}) \quad + \cos(sb_o)(ce^{cb_o} + ce^{-cb_o})

= 0 \quad (194)

Expansion of the determinant gives:

\[ e^{2cb_o} - e^{-2cb_o} - 4(cb_o/sb_o) \sin(sb_o) \cos(sb_o) = 0 \] \quad (195)

Unfortunately, for any combination of \( a_o/b_o \) and \( K_{yo} \), no value of \( K_{xo} \) satisfies equation (195). In other words, no possible solutions exist for the present set of boundary conditions for \((K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1 < 0\)

\[(K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1 = 0\]

For the quantity \[(K_{yo}/2m^2)^2 + K_{xo}(a_o/mb_o)^2 - 1 = 0\] equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically for the choice of algebraic sign of \( K_{yo} \), each possible range of \( K_{yo} \)--negative, zero, and positive--will be analyzed as different subcases.

\( K_{yo} < 0. \)

For \( K_{yo} < 0 \) equations (121) and (122) explicitly demonstrate that the unknown function \( Y(y_o) \) must take the form shown in equation (123). Equations (186) through (189) again comprise the group of boundary conditions for the \( Y(y_o) \) function. The enforcement of equation (186) on equation (123) dictates that \( B_m \) must vanish. In addition, application
of equation (187) leads one to the conclusion that \( C_m \) is zero. So equation (123) takes on the following form:

\[
Y(y_0) = A_m \sinh(Ty_0) + D_m \cosh(Ty_0)
\]  

(196)

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is therefore easy to obtain.

\[
Y'(y_0) = A_m T \cosh(Ty_0) + D_m \{ \cosh(Ty_0) + Ty_0 \sinh(Ty_0) \}
\]  

(197)

Substitution of equations (196) and (197) into equations (188) and (189) yield two homogeneous linear equations in coefficients \( A_m \) and \( D_m \). For non-trivial \( A_m \) and \( D_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sinh(Tb_0) & b_0 \cosh(Tb_0) \\
T \cosh(Tb_0) & \cosh(Tb_0) + Tb_0 \sinh(Tb_0)
\end{vmatrix} = 0
\]  

(198)

Expansion of the determinant gives:

\[
\sinh(Tb_0) \cosh(Tb_0) - Tb_0 = 0
\]  

(199)

No value of \( Tb_0 \) greater than zero can satisfy equation (199). Therefore, no possible solutions exist for the present boundary conditions for

\[
\{(K_{y_0}/2m^2)^2 + K_{x_0} (a_0/mb_0)^2 - 1 \} = 0 \text{ and } K_{y_0} < 0
\]

\( K_{y_0} = 0 \).

For \( K_{y_0} = 0 \) equation (128) constitutes the required shape of the unknown \( Y(y_0) \) function. In addition, equations (186) through (189) are the sets of constraints for this \( Y(y_0) \) function. Equations (186) and (187) imply that

\( A_m = C_m = 0 \) Equations (188) and (189), with each side of (189) multiplied by \( b_0 \), yield the following set of simultaneous
equations:

\[ B_m b_o + D_m b_o^3 = 0 \]  
\[ B_m b_o + 3D_m b_o^3 = 0 \]

(200)

Only \( B_m = D_m = 0 \) constitutes a valid solution for equations (200). So again no possible solutions exist for the present boundary conditions for

\[ \{(K_y/2m^2)^2 + K_x (a_o/mb_o)^2 - 1 \} = 0 \]  
\[ K_y > 0. \]

For \( K_y > 0 \) equations (130) and (131) explicitly demonstrate that the unknown function \( Y(y_o) \) must take the form shown in equation (132). Equations (186) through (189) again comprise the group of boundary conditions for the \( Y(y_o) \) function. The enforcement of equation (186) on equation (132) dictates that \( B_m \) must vanish. In addition, application of equation (187) leads one to the conclusion that \( C_m \) is zero. So equation (132) takes on the following form:

\[ Y(y_o) = A_m \sin(Uy_o) + D_m Y_o \cos(Uy_o) \]  
(201)

The first derivative of \( Y(y_o) \) with respect to \( y_o \) is therefore easy to obtain.

\[ Y'(y_o) = A_m U \cos(Uy_o) + D_m \{ \cos(Uy_o) - Uy_o \sin(Uy_o) \} \]  
(202)

Substitution of equations (201) and (202) into equations (188) and (189) yields two homogeneous linear equations in coefficients \( A_m \) and \( D_m \). For non-trivial \( A_m \) and \( D_m \), the following determinental equation must hold:
\[
\begin{vmatrix}
\sin(Ub_0) & b_0\cos(Ub_0) \\
U\cos(Ub_0) & \cos(Ub_0) - Ubo\sin(Ub_0)
\end{vmatrix} = 0
\] (203)

Expansion of the determinant gives:
\[
\sin(Ub_0)\cos(Ub_0) - Ub_0 = 0
\] (204)

No value of \(Ub_0\) greater than zero can satisfy equation (204). Therefore, no possible solutions exist for the present boundary conditions for \({(K_y/2m^2)}^2 + K_x (a_o/mb_o)^2 - 1 = 0\) and \(K_y > 0\)

\[{(K_y/2m^2)}^2 + K_x (a_o/mb_o)^2 - 1 > 0\]

For the quantity \({(K_y/2m^2)}^2 + K_x (a_o/mb_o)^2 - 1\) \(> 0\) equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown \(y(y_o)\) function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of \(K_y\) for valid solutions. Three subcases must be considered so that a solution for \(K_x\) may be determined for any range of \(K_y\).

**\(K_y\) Ranges from a Relatively Large Positive Number to Positive Infinity.**

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, \(K_y\) can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity
likewise bracketed in equation (138) cannot be positive, $K_{y_o}$ can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144). Equations (186) through (189) again comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (186) to equation (144). This stipulation fixes $D_m$ in terms of $B_m$ such that $D_m = -B_m$. Utilization of equation (187) on the $Y(y_o)$ equation, on the other hand, forces $B_m$ and hence $D_m$ to vanish. So equation (144) takes on the following form:

$$Y(y_o) = A_m \sin(a_m y_o) + C_m \sin(v_m y_o)$$  

(205)

The first derivative of $Y(y_o)$ with respect to $y_o$ is easy to obtain.

$$Y'(y_o) = A_m a_m \cos(a_m y_o) + C_m v_m \cos(v_m y_o)$$  

(206)

Substitution of equations (205) and (206) into equations (188) and (189) yields two homogeneous linear equations in the coefficients $A_m$ and $C_m$. For non-trivial $A_m$ and $C_m$, the following determinental equation must hold:

$$\begin{vmatrix} \sin(a_m b_o) & \sin(v_m b_o) \\ a_m \cos(a_m b_o) & v_m \cos(v_m b_o) \end{vmatrix} = 0$$  

(207)
BIAXIAL BUCKLING OF SPECIALLY ORTHOTROPIC SYMMETRIC COMPOSITE RECTANGULAR PLATES(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI. J P MCFADDEN UNCLASSIFIED DEC 84 AFIT/GAE/AA/84D-15
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS:1963-A
Expansion of the determinant and multiplication by the quantity $b_o$ gives:

$$Y_m b_o \sin(\alpha_m b_o) \cos(\gamma_m b_o) - \alpha_m b_o \sin(\gamma_m b_o) \cos(\alpha_m b_o) = 0 \quad (208)$$

Attempts at solution of equation (208) for any combination of $\alpha_o/b_o$ and $K_{yo}$ do not yield the smallest values of $K_{yo}$. As a result, further considerations of this equation and this subcase are abandoned.

*K_{yo} Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.*

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, $K_{yo}$ can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, $K_{yo}$ can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that $K_{yo}$ range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) explicitly demonstrate that $Y(y_0)$ must take the form shown in equation (162). Note also that equations (145) and (163) define the variables in equation (162). Equations (186) through (189) once more comprise the group of boundary conditions for the $Y(y_0)$ function. First, apply equation (186) to equation (162). This stipulation fixes $D_m$ in terms of $B_m$ such that
D_m = -B_m Utilization of equation (187) on the Y(y_o) equation, on the other hand, forces B_m and hence D_m to vanish. So equation (162) takes on the following form:

\[ Y(y_o) = A_m \sin(a_m y_o) + C_m \sinh(\beta_m y_o) \]  
(209)

The first derivative of Y(y_o) with respect to y_o is easy to obtain.

\[ Y'(y_o) = A_m a_m \cos(a_m y_o) + C_m \beta_m \cosh(\beta_m y_o) \]  
(210)

Substitution of equations (209) and (210) into equations (188) and (189) yields two homogeneous linear equations in coefficients A_m and C_m. For non-trivial A_m and C_m, the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(a_m b_o) & \sinh(\beta_m b_o) \\
a_m \cos(a_m b_o) & \beta_m \cosh(\beta_m b_o)
\end{vmatrix}
= 0
\]  
(211)

Expansion of the determinant and multiplication by the quantity b_o gives:

\[ \beta_m b_o \sin(a_m b_o) \cosh(\beta_m b_o) - a_m b_o \cos(a_m b_o) \sinh(\beta_m b_o) = 0 \]  
(212)

Attempts at solution of equation (212) for any combination of a_o/b_o and K_y_o do yield the smallest values of K_y_o. Further consideration of results generated by equation (212) is postponed until the last of the three subcases is presented.

K_y_o Ranges from a Relatively Large Negative Number to Negative Infinity.

If the quantity contained in the curly brackets of
equation (137) is constrained to be greater than zero, \( K_{y_0} \) can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited by the two inequalities expressed in equations (170) and (171). Moreover, equations (172), (173), and (174) sequentially illustrate that \( Y(y_0) \) must take the form shown in equation (175). Note also that equations (163) and (176) define the variables in equation (175). Equations (186) through (189) once more comprise the group of boundary conditions for the \( Y(y_0) \) function. First, apply equation (186) to equation (175). This combination fixes \( D_m \) in terms of \( B_m \) such that \( D_m = -B_m \). Utilization of equation (187) on the \( Y(y_0) \) equation, in contrast, forces \( B_m \) and hence \( D_m \) to vanish. So equation (175) takes on the following form:

\[ Y(y_0) = A_m \sinh(\varepsilon_m y_0) + C_m \sinh(\beta_m y_0) \]  

(213)

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is easy to obtain.

\[ Y'(y_0) = A_m \varepsilon_m \cosh(\varepsilon_m y_0) + C_m \beta_m \cosh(\beta_m y_0) \]  

(214)

Substitution of equations (213) and (214) into equations (188) and (189) yields two homogeneous linear equations in coefficients \( A_m \) and \( C_m \). For non-trivial \( A_m \) and \( C_m \), the following determinental equation must hold:
\[
\begin{vmatrix}
\sinh(e_m b_o) & \sinh(e_r b_o) \\
e_m \cosh(e_m b_o) & e_r \cosh(e_r b_o)
\end{vmatrix} = 0 \quad (215)
\]

Expansion of the determinant and multiplication by the quantity \( b_o \) gives:

\[
\begin{align*}
e_m b_o \sinh(e_m b_o) \cosh(e_r b_o) \\
- e_m b_o \sinh(e_r b_o) \cosh(e_m b_o) = 0
\end{align*} \quad (216)
\]

Attempts at solution of equation (216) for any combination of \( a_o/b_o \) and \( K_{y_o} \) do not yield the smallest values of \( K_{x_o} \). As a result, further considerations of this equation and this subcase as a whole are dropped.

**Discussion of Results**

Table VIII gives selected \( a_o/b_o, K_{y_o} \), and \( K_{x_o} \) ordered triplets as determined by equation (212). Also included is the integer value of \( m \) which produces this minimum \( K_{x_o} \). Furthermore, each entry point which corresponds to a transition point from the \( m \) curve to the \((m+1)\) curve is superscripted in the \( a_o/b_o \) column with a star (*).

The statistics presented in Table VIII expose two important characteristics of laminates under compression or tension in the \( y_o \)-direction. First, the transition values of \( a_o/b_o \) increase as \( K_{y_o} \) becomes algebraically larger (or less tensile). This trend is most pronounced for the initial transition points, and its effect diminishes as \( a_o/b_o \) approaches a large number. Second, irrespective of the magnitude of \( K_{y_o}, K_{x_o} \) attains a limiting value of 3.125 as
$a_o/b_o$ approaches infinity.

Figure 16 represents a plot of $K_{x_o}$ versus $a_o/b_o$ for eleven distinct values of $K_{y_o}$ . The lowest curve characterizes $K_{y_o} = 5.0$ ; whereas, the highest depicts $K_{y_o} = -5.0$ . The nine other curves differ from each other by increments of one. This graph reinforces the concept that $K_{x_o}$ for a constant $K_{y_o}$ is determined not by one continuous curve but by the lowest values of an infinite number of intersecting curves. In addition, the merging of all curves to a limiting value of $K_{x_o} = 3.125$ for $a_o/b_o$ large is readily apparent.

Figure 17 plots in three dimensions the same information as Figure 16. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 16; however, the quantitative aspect of Figure 17 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table IX gives selected coordinates of $K_{y_o}$ and $K_{x_o}$ for three distinct values of $a_o/b_o$ -- 1.6, 2.6, and 4.0 Figures 18, 19, and 20 represent two-dimensional plots at these constant $a_o/b_o$ slices of 1.6, 2.6, and 4.0, respectively.
These graphs, very similar to those obtained in the simply supported on all sides case, are composed of very nearly straight line segments. Note especially that $K_{x_0}$ declines as $K_{y_0}$ increases and that the rate of decline of $K_{x_0}$ for an increase in $K_{y_0}$ jumps markedly for small $a_0/b_0$. 
TABLE VIII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the $x_0$-Direction and Simply Supported and Clamped on the Two Edges Normal to the $y_0$-Direction

<table>
<thead>
<tr>
<th>m</th>
<th>$a_0/b_0$</th>
<th>$K_{y_0}$</th>
<th>$K_{x_0}$</th>
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<td>7.0176</td>
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<td>6.5847</td>
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<td>1.4000</td>
<td>-3.0</td>
<td>4.1016</td>
</tr>
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<td>1.7873*</td>
<td>-3.0</td>
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<td>( K_{x 0} )</td>
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FOR THE RANGE
KYO = -5.0 TO +5.0
11 CURVES, EACH VARIES
BY KYO INCREMENT OF 1.0
KYO = -5.0 HIGHEST CURVE

FIGURE 16
XO-BUCKLING COEFFICIENT VERSUS AFFINE ASPECT RATIO FOR AN S-O-S-S LAMINATE
FOR VARIOUS CONSTANT YO-BUCKLING COEFFICIENT VALUES
FIGURE 17
SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-C-S-S LAMINATE
Figure 18: Xo-buckling coefficient versus xo-buckling coefficient at a constant affine aspect ratio of 1.6 for an S-C-S-S laminate for the range KY0 = -5.0 to 5.0.
can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, \( K_{y_0} \) can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that \( Y(y_0) \) must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144). Equations (228) through (231) once more comprise the group of boundary conditions for the \( Y(y_0) \) function. First, apply equation (228) to equation (144).

This stipulation fixes \( D_m \) in terms of \( B_m \) such that \( D_m = -B_m \). Utilization of equation (229), on the other hand, yields a relation between \( C_m \) and \( A_m \) such that \( C_m = -(a_m/v_m)A_m \). So equation (144) takes on the following form:

\[
Y(y_0) = A_m \{ \sin(a_m y_0) - [a_m/v_m] \sin(v_m y_0) \} \\
+ B_m \{ \cos(a_m y_0) - \cos(v_m y_0) \} \\
(255)
\]

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'(y_0) = A_m \{ a_m \cos(a_m y_0) - a_m \cos(v_m y_0) \} \\
- B_m \{ a_m \sin(a_m y_0) - v_m \sin(v_m y_0) \} \\
(256)
\]

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = -A_m \{ a_m^2 \sin(a_m y_0) - a_m v_m \sin(v_m y_0) \} \\
- B_m \{ a_m^2 \cos(a_m y_0) - v_m^2 \cos(v_m y_0) \} \\
(257)
\]

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \)
quantity $b_0^4$ results in the following system of equations (the second equation a simplification of the first):

$$
\begin{align*}
\sin^2(Ub_o)\{-&(Ub_o)^6 - 3(Ub_o)^4 + Ky_o (nb_o/a_o)^2 [(Ub_o)^4 + (Ub_o)^2] \\
+&\cos^2(Ub_o)\{-&(Ub_o)^6 - 4(Ub_o)^4 + Ky_o (nb_o/a_o)^2 (Ub_o)^4\} = 0 \quad (253)
\end{align*}
$$

$$
Ky_o (nb_o/a_o)^2 \{(Ub_o)^4 + (Ub_o)^2 \sin^2(Ub_o)\}
- (Ub_o)^4 \{3 + \cos^2(Ub_o) + (Ub_o)^2\} = 0 \quad (254)
$$

No value of $(Ub_o)$ greater than zero can satisfy equation (254). Therefore, no possible solutions exist for the present boundary conditions for

$$
\{(Ky_o/2m^2)^2 + Kx_o (a_o/mb_o)^2 - 1\} = 0 \quad \text{and} \quad Ky_o > 0
$$

$$
\{(Ky_o/2m^2)^2 + Kx_o (a_o/mb_o)^2 - 1\} > 0
$$

For the quantity $\{(Ky_o/2m^2)^2 + Kx_o (a_o/mb_o)^2 - 1\} > 0$ equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown $Y(Y_o)$ function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of $Ky_o$ for valid solutions. Three subcases must be considered so that a solution for $Kx_o$ may be determined for any range of $Ky_o$.

$Ky_o$ **Ranges from a Relatively Large Positive Number to Positive Infinity.**

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, $Ky_o$
Equation (228) implies that \( B_m \) must vanish, and equation (229) can be satisfied only if \( D_m = -A_m \). So equation (132) takes on the following form:

\[
Y(y_0) = A_m\{\sin(Uy_0) - Uy_0 \cos(Uy_0)\} + C_m Y_0 \sin(Uy_0) \tag{248}
\]

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'(y_0) = A_m U^2 y_0 \sin(Uy_0) + C_m \{\sin(Uy_0) + Uy_0 \cos(Uy_0)\} \tag{249}
\]

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = A_m \{U^2 \sin(Uy_0) + U^3 y_0 \cos(Uy_0)\} \\
+ C_m \{2U \cos(Uy_0) - U^2 y_0 \sin(Uy_0)\} \tag{250}
\]

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = A_m \{2U^3 \cos(Uy_0) - U^4 y_0 \sin(Uy_0)\} \\
+ C_m \{-3U^2 \sin(Uy_0) - U^3 y_0 \cos(Uy_0)\} \tag{251}
\]

Substitution of equations (249), (250), and (251) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( C_m \). For non-trivial \( A_m \) and \( C_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(Uy_0) \{U^2\} & \sin(Uy_0) \{-U^2 y_0\} \\
+ \cos(Uy_0) \{U^3 y_0\} & + \cos(Uy_0) \{2U\} \\
\sin(Uy_0) \{-U^4 y_0\} & \sin(Uy_0) \{-3U^2\} \\
+ K_{y_0} (n/a_0)^2 U^2 y_0 & + K_{y_0} (n/a_0)^2 \{U^2\} \\
+ \cos(Uy_0) \{2U^3\} & + \cos(Uy_0) \{-U^3 y_0\} \\
& + K_{y_0} (n/a_0)^2 \{U y_0\}
\end{vmatrix} = 0
\tag{252}
\]

Expansion of the determinant and multiplication by the
\begin{align*}
\cosh^2(T_b) \left\{ -(T_b)^6 + 4(T_b)^4 - K_{y_0} \left( n_b/a_o \right)^2 (T_b)^4 \right\} \\
- \sinh^2(T_b) \left\{ -(T_b)^6 + 3(T_b)^4 \\
+ K_{y_0} \left( n_b/a_o \right)^2 \left[ (T_b)^2 - (T_b)^4 \right] \right\} = 0 \quad (246)
\end{align*}

\begin{align*}
(T_b)^4 \left\{ 3 + \cosh^2(T_b) - (T_b)^2 \right\} \\
- K_{y_0} \left( n_b/a_o \right)^2 \left\{ (T_b)^4 + (T_b)^2 \sinh^2(T_b) \right\} = 0 \quad (247)
\end{align*}

No value of \((T_b)\) greater than zero can satisfy equation (247). Therefore, no possible solutions exist for the present boundary conditions for
\[
\left\{ (K_{y_0}/2m^2)^2 + K_x \left( a_o/m_b \right)^2 - 1 \right\} = 0 \quad \text{and} \quad K_{y_0} < 0
\]
\[
K_{y_0} = 0.
\]

For \(K_{y_0} = 0\) equation (128) constitutes the required shape of the unknown \(Y(y_0)\) function. In addition, equations (228) through (231) are the sets of constraints for this \(Y(y_0)\) function. Moreover, notice that equation (231) reduces to \(Y'''(b_o) = 0\) in this instance since \(K_{y_0} = 0\) Equations (228) and (229) imply that \(A_m = B_m = 0\) Also, satisfaction of equation (231) necessitates that the constant \(D_m\) must likewise have the null value. Finally, and in this sequence, enforcement of equation (230) reveals that \(C_m\) too must vanish. Thus, the function \(Y(y_0)\) is nothing more than the trivial function for \(K_{y_0} = 0\) and shall be ignored.

\[
K_{y_0} > 0.
\]

For \(K_{y_0} > 0\) equations (130) and (131) show that the unknown function \(Y(y_0)\) must fit the relation given by equation (132). Equations (228) through (231) again comprise the group of boundary conditions for the \(Y(y_0)\) function.
\[ Y'(y_0) = -A_m t^2 y_0 \sinh(T_y) \]
\[ + C_m \{ \sinh(T_y) + T_y \cosh(T_y) \} \]  \hspace{1cm} (242)

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[ Y''(y_0) = A_m \{-T^2 \sinh(T_y) - T^3 y_0 \cosh(T_y) \} \]
\[ + C_m \{ 2T \cosh(T_y) + T^2 y_0 \sinh(T_y) \} \]  \hspace{1cm} (243)

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[ Y'''(y_0) = A_m \{-2T^3 \cosh(T_y) - T^4 y_0 \sinh(T_y) \} \]
\[ + C_m \{ 3T^2 \sinh(T_y) + T^3 y_0 \cosh(T_y) \} \]  \hspace{1cm} (244)

Substitution of equations (242), (243), and (244) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( C_m \). For non-trivial \( A_m \) and \( C_m \), the following determinant equation must hold:

\[
\begin{vmatrix}
\sinh(T_{b_0}) & \{-T^2\} & \sinh(T_{b_0}) & \{T^2 b_0\} \\
\cosh(T_{b_0}) & \{T^3 b_0\} & + \cosh(T_{b_0}) & \{2T\} \\
\sinh(T_{b_0}) & \{-T^4 b_0\} & \sinh(T_{b_0}) & \{3T^2\} \\
-K_y (\pi/a_o)^2 T^2 b_0 & +K_y (\pi/a_o)^2 \} & -\cosh(T_{b_0}) & \{2T^3\} \\
-\cosh(T_{b_0}) & \{2T^3\} & + \cosh(T_{b_0}) & \{T^3 b_0\} \\
& & +K_y (\pi/a_o)^2 T b_0 & \} \\
\end{vmatrix} = 0
\]  \hspace{1cm} (245)

Expansion of the determinant and multiplication by the quantity \( b_0^4 \) results in the following equations (the second equation a simplification of the first):
determination of minimum $K_{x_0}$ for the combination of any plate aspect ratio and $K_{y_0}$ greater than zero. In other words, the roots of equation (240) yield the smallest values of $K_{x_0}$ for any $a_o/b_o$ and compressive $K_{y_0}$. Consideration of results generated by equation (240) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

\[ \{(K_{y_0}/2m^2)^2 + K_{x_0}(a_o/mb_o)^2 - 1\} = 0 \]

For the quantity $\{(K_{y_0}/2m^2)^2 + K_{x_0}(a_o/mb_o)^2 - 1\} = 0$ equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically with the choice of algebraic sign of $K_{y_0}$, each possible range of $K_{y_0}$--negative, zero, and positive--will be analyzed as different subcases.

$K_{y_0} < 0$.

For $K_{y_0} < 0$ equations (121) and (122) lead to the conclusion that the unknown function $Y(y_o)$ must take the form shown in equation (123). Equations (228) through (231) again comprise the group of boundary conditions for the $Y(y_o)$ function. The enforcement of equation (228) on equation (123) dictates that $B_m$ must vanish. Application of equation (229), on the other hand, gives the relation that $D_m = -A_mT$.

Therefore, equation (123) takes on the following form:

\[ Y(y_o) = A_m \{ \sinh(Ty_o) - Ty_o \cosh(Ty_o) \} + C_m y_o \sinh(Ty_o) \]  

(241)

The first derivative of $Y(y_o)$ with respect to $y_o$ is:
quantity $b_0^5$ results in the following equations (the second equation a simplification of the first):

\[
\sin^2(sbo) \left[ e^{2cbo} \{ -(sb_0)^5 - 2(cb_0)^2(sb_0)^3 - (cb_0)^4(sb_0) \\
+ K_y (nb_0/a_0)^2 ( (cb_0)^2(sb_0) + (sb_0)^3 ) \} \\
+ e^{-2cbo} \{ -(sb_0)^5 - 2(cb_0)^2(sb_0)^3 - (cb_0)^4(sb_0) \\
+ K_y (nb_0/a_0)^2 ( (cb_0)^2(sb_0) + (sb_0)^3 ) \} \\
+ \{ -8(cb_0)^2(sb_0)^3 - 6(cb_0)^4(sb_0) + 2(sb_0)^5 \\
+ 4(cb_0)^6/(sb_0) + K_y (nb_0/a_0)^2 (4(cb_0)^4/(sb_0) \\
+ 2(cb_0)^2(sb_0) - 2(sb_0)^3 ) \} \right] \\
+ \cos^2(sb_0) \left[ e^{2cbo} \{ -(sb_0)^5 - 2(cb_0)^2(sb_0)^3 - (cb_0)^4(sb_0) \\
+ K_y (nb_0/a_0)^2 ( (cb_0)^2(sb_0) + (sb_0)^3 ) \} \\
+ e^{-2cbo} \{ -(sb_0)^5 - 2(cb_0)^2(sb_0)^3 - (cb_0)^4(sb_0) \\
+ K_y (nb_0/a_0)^2 ( (cb_0)^2(sb_0) + (sb_0)^3 ) \} \\
+ \{ -12(cb_0)^2(sb_0)^3 - 14(cb_0)^4(sb_0) + 2(sb_0)^5 \\
- K_y (nb_0/a_0)^2 (2(sb_0)^3 \\
+ 2(cb_0)^2(sb_0) ) \} \right] = 0 \tag{239}
\]

\[
(e^{2cbo} + e^{-2cbo}) \{ -(sb_0)^5 - 2(cb_0)^2(sb_0)^3 - (cb_0)^4(sb_0) \\
+ K_y (nb_0/a_0)^2 ( (cb_0)^2(sb_0) + (sb_0)^3 ) \} \\
- \{ 8(cb_0)^2(sb_0)^3 + 6(cb_0)^4(sb_0) - 2(sb_0)^5 \\
+ K_y (nb_0/a_0)^2 (2(sb_0)^3 ) \} \\
- \cos^2(sb_0) \{ 4(cb_0)^2(sb_0)^3 + 8(cb_0)^4(sb_0) \\
+ K_y (nb_0/a_0)^2 (2(cb_0)^2(sb_0) ) \} \}
+ \sin^2(sb_0) \{ 4(cb_0)^6/(sb_0) \\
+ K_y (nb_0/a_0)^2 [4(cb_0)^4/(sb_0) + 2(cb_0)^2(sb_0) ] \} = 0 \tag{240}
\]

Equation (240) constitutes the governing equation for
\[ Y'''(y_0) = A_m \left[ \left( (3c^2s - s^3)e^{cy_0} + (s^3 - 3c^2s)e^{-cy_0} \right) \cos(sy_0) \right] \\
+ \left[ (c^3 - 3cs^2)e^{cy_0} + (c^3 - 3cs^2)e^{-cy_0} \right] \sin(sy_0) \right] + B_m \left[ \left( (c^3 - 3cs^2)e^{cy_0} - (5c^3 + cs^2)e^{-cy_0} \right) \cos(sy_0) \right] \\
+ \left[ (s^3 - 3c^2s)e^{cy_0} + (2(c^4/s) - 3c^2 \right. \\
- s^3)e^{-cy_0} \right] \sin(sy_0) \right] \\
\]  
(237)

Substitution of equations (235), (236), and (237) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinantal equation must hold:

\[
\begin{vmatrix}
\sin(sb_o) \{(c^2-s^2)e^{cb_o} + (s^2-c^2)e^{-cb_o} \} & \sin(sb_o) \{-2cse^{cb_o} + (s^2-c^2)e^{-cb_o} \} \\
+ \cos(sb_o) (2cs) \{ e^{cb_o} + e^{-cb_o} \} & + \cos(sb_o) \{(c^2-s^2)e^{cb_o} + (3c^2 + s^2)e^{-cb_o} \}
\end{vmatrix} = 0
\]

\[
\begin{vmatrix}
\sin(sb_o) \{(c^3-3cs^2)e^{cb_o} + (c^3-3cs^2)e^{-cb_o} \} + K_{y_0} (n/a_o)^2 [ce^{cb_o} + ce^{-cb_o}] \\
+ \cos(sb_o) \{(3c^2s-s^3)e^{cb_o} + (s^3-3c^2s)e^{-cb_o} \} + K_{y_0} (n/a_o)^2 [se^{cb_o} - se^{-cb_o}] \\
\end{vmatrix} = 0
\]

Expansion of the determinant and multiplication by the
First, apply equation (228) to equation (106). This solution fixes $D_m$ in terms of $B_m$ such that $D_m = -B_m$. Utilization of equation (229) on the $Y(y_o)$ equation similarly determines a value $C_m$ in terms of $A_m$ and $B_m$.

$$sA_m + cB_m + sC_m - cD_m = 0$$  \hspace{1cm} (232)

But since $D_m = -B_m$

$$C_m = -A_m - 2B_m(c/s)$$  \hspace{1cm} (233)

For these values of $C_m$ and $D_m$, equation (106) takes on the following form:

$$Y(y_o) = A_m(e^{cy_o} - e^{-cy_o}) \sin(sy_o) + B_m\{(e^{cy_o} - e^{-cy_o}) \cos(sy_o) - 2(c/s)e^{-cy_o} \sin(sy_o)\}$$  \hspace{1cm} (234)

The first derivative of $Y(y_o)$ with respect to $y_o$ is therefore:

$$Y'(y_o) = A_m\{(se^{cy_o} - se^{-cy_o}) \cos(sy_o)$$

$$+ (ce^{cy_o} + ce^{-cy_o}) \sin(sy_o)\}$$

$$+ B_m\{(ce^{cy_o} - ce^{-cy_o}) \cos(sy_o)$$

$$+ (-se^{cy_o} + se^{-cy_o} + 2(c^2/s)e^{-cy_o}) \sin(sy_o)\}$$  \hspace{1cm} (235)

The second derivative of $Y(y_o)$ with respect to $y_o$ is:

$$Y''(y_o) = A_m\{(2cse^{cy_o} + 2cse^{-cy_o}) \cos(sy_o)$$

$$+ [(c^2 - s^2)e^{cy_o} + (s^2 - c^2)e^{-cy_o}] \sin(sy_o)\}$$

$$+ B_m\{[(c^2 - s^2)e^{cy_o} + (3c^2 + s^2)e^{-cy_o}] \cos(sy_o)$$

$$+ [-2cse^{cy_o} - 2(c^3/s)e^{-cy_o}] \sin(sy_o)\}$$  \hspace{1cm} (236)

The third derivative of $Y(y_o)$ with respect to $y_o$ is:
equation (61) by equation (39). Equation (81) is now analyzed for the three possible algebraic states—negative, zero, and positive—of the quantity in the square brackets of equation (81). As explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

\[
\{(K_yo/2m^2)^2 + K_{zo} (a_o/mb_o)^2 - 1 \} < 0
\]

For the quantity \( \{ (K_yo/2m^2)^2 + K_{zo} (a_o/mb_o)^2 - 1 \} < 0 \) equations (82) through (105) illustrate that the unknown function \( Y(y_o) \) must take the form shown in equation (106).

Consider now the boundary conditions, equations (225), for this case of a laminate simply supported on one pair of opposite sides, clamped on a third, and free on the fourth. When the chosen form of \( w \), equation (77), is substituted into the final three lines of equations (225), the following must hold:

\[
Y(0) \sin(mnx_o/a_o) = 0 \quad Y'(0) \sin(mnx_o/a_o) = 0 \quad (226)
\]

\[
Y''(b_o) \sin(mnx_o/a_o) = 0 \quad (227)
\]

\[
[Y'''(b_o) + K_{yo} (n/a_o)^2 Y'(b_o)] \sin(mnx_o/a_o) = 0
\]

For equations (226) and (227) to have meaning in the general case, the following conditions must hold:

\[
Y(0) = 0 \quad (228)
\]

\[
Y'(0) = 0 \quad (229)
\]

\[
Y''(b_o) = 0 \quad (230)
\]

\[
Y'''(b_o) + K_{yo} (n/a_o)^2 Y'(b_o) = 0 \quad (231)
\]
equation (11). The key assumption of this investigation is that $D^* = 0$ and this provision will continue to be enforced. Consequently, equation (223) reduces to the relatively simple result:

$$w_{y_0 y_0 y_0} + K_{y_0} (n/a_0)^2 w_{y_0} = 0$$  \hspace{1cm} (224)

Equation (224) is the most convenient means to express the second boundary condition for a free edge normal to the $y_0$-direction.

As a recap and in equation form, the following are the boundary conditions for each of the four edges:

- on edge $x_0 = -a_0/2$, \( w = 0 \); \( w_{x_0 x_0} = 0 \)
- on edge $x_0 = a_0/2$, \( w = 0 \); \( w_{x_0 x_0} = 0 \)
- on edge $y_0 = 0$, \( w = 0 \); \( w_{y_0 y_0} = 0 \) \hspace{1cm} (225)
- on edge $y_0 = b_0$, \( w = 0 \); \( w_{y_0 y_0} = 0 \);

Note that the origin of coordinates in the affine space is taken to be at the center of the clamped edge normal to the $y_0$-direction. This choice of origin location, in general, allows maximum simplicity in manipulations for a lack of symmetry is present in the boundary conditions in the $y_0$-direction.

Just as before, a displacement function $w$ which satisfies the first two stipulations of equations (225) is given by equation (77). Furthermore, substitution of this relation for $w$ into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable $r$ is defined in
Note that this boundary condition (217) is expressed in real space coordinates. No conversion to affine space values has yet been made. $Q_y$ can be expressed in terms of the moments $M_y$ and $M_{xy}$ defined in equation (2) (3:326-327).

$$Q_y = -M_{y,yy} - M_{xy,xy} \quad (218)$$

Appropriate partial differentiation of equation (2) gives the following identities (for symmetric, specially orthotropic laminates):

$$M_{y,yy} = -D_{12} w_{r,xy} - D_{22} w_{r,yyy} \quad (219)$$

$$M_{xy,xy} = -2 D_{66} w_{r,xy} \quad (220)$$

Replacement of terms in equation (217) by those identities given in equations (218), (219), and (220) yields:

$$D_{12} w_{r,xy} + D_{22} w_{r,yyy} + 2 D_{66} w_{r,xy} - N_y w_{r,xy} = 0 \quad (221)$$

Now, transformation of these real space coordinates into affine space coordinates by the rules of equations (8) and utilization of the definition given by equation (13) reshapes equation (221) in the following way:

$$\left[ \frac{D_{12}}{(A^2 B)} \right] \ w_{r,x_0 y_0} + \left[ \frac{D_{22}}{(B^3)} \right] \ w_{r,y_0,y_0}$$

$$+ \left[ 2 \ D_{66} / (A^2 B) \right] \ w_{r,x_0 y_0} + \left[ D_{22} / B \right] \ K_y \ (n/a_0)^2 \ w_{r,y_0} = 0 \quad (222)$$

The constants $A$ and $B$ were defined in the first section in terms of $D_{11}$ and $D_{22}$ such that $A = D_{11}^{1/4}$ and $B = D_{22}^{1/4}$. Application of these bits of information and division of both sides of equation (222) by $D_{22}^{1/2}$ gives:

$$\left[ 2 (D_{12} + 2 D_{66}) / (D_{11} D_{22})^{1/2} \right] \ w_{r,x_0 y_0}$$

$$+ 2 w_{r,y_0,y_0} + 2 K_y \ (n/a_0)^2 \ w_{r,y_0} = 0 \quad (223)$$

Recognize that the coefficient of the first term on the left-hand side of equation (223) is merely $D^*$ (defined in
VI. **Flat Rectangular Composite Laminate Simply Supported in the** $x_o$-**Direction and Clamped and Free on the Two Edges Normal to the** $y_o$-**Direction**

The boundary conditions for a laminate simply supported in the $x_o$-direction and clamped and free on the two edges perpendicular to the $y_o$-direction again display no symmetry in the $y_o$-direction. For the two edges which have normals parallel to the $x_o$-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. For the clamped edge normal to the $y_o$-direction, the vertical displacement and the slope of the vertical displacement with respect to $y_o$ must each equal zero. For the free edge normal to the $y_o$-direction, a moment's reflection allows one to formulate the appropriate boundary conditions.

First, like the simply supported edge, the normal component of the moment to the free edge must be zero. Second, the free edge can support no force on its boundary, so the summation of the shear force and the $y$-buckling load in the direction of the shear must equal zero.

$$ Q_y - N_y w_{y} = 0 $$

(217)

where

- $Q_y$ = resultant shear force per unit length
- $N_y$ = normal force per unit length in the $y$-direction
  positive in tension
Figure 20
X₀-buckling coefficient versus Y₀-buckling coefficient at a constant affine aspect ratio of 4.0 for an S-C-S-S laminate. FOR THE RANGE KYO = -5.0 TO 5.0
is:
\[ Y'''(y_o) = - A \left[ a_m \cos(a_m y_o) - a_m y_m \cos(v_m y_o) \right] \]
\[ + B \left[ a_m \sin(a_m y_o) - v_m \sin(v_m y_o) \right] \]  \hspace{1cm} (258)

Substitution of equations (256), (257), and (258) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(a_m b_o) \{-a_m^2\} & \cos(a_m b_o) \{-a_m^2\} \\
+ \sin(v_m b_o) \{a_m v_m\} & + \cos(v_m b_o) \{v_m^2\}
\end{vmatrix}
\begin{vmatrix}
\cos(a_m b_o) \{-a_m^3\} & \sin(a_m b_o) \{a_m^3\} \\
+ K_{y_o} (n/a_o)^2 a_m \} & - K_{y_o} (n/a_o)^2 a_m} \\
+ \cos(v_m b_o) \{a_m v_m^2\} & + \sin(v_m b_o) \{-v_m^3\}
\end{vmatrix}
= 0
\]  \hspace{1cm} (259)

Expansion of the determinant, division by the common multiple \( a_m \), and multiplication by the quantity \( b_o^4 \) gives:
\[
\begin{align*}
(a_m b_o) (v_m b_o) \sin(a_m b_o) \sin(v_m b_o) \{ (v_m b_o)^2 + (a_m b_o)^2 \\
- 2K_{y_o} (n b_o/a_o)^2 \} & + \cos(a_m b_o) \cos(v_m b_o) \{ 2(a_m b_o)^2 (v_m b_o)^2 \\
- K_{y_o} (n b_o/a_o)^2 [(a_m b_o)^2 + (v_m b_o)^2] & - (a_m b_o)^4 - (v_m b_o)^4 \\
+ K_{y_o} (n b_o/a_o)^2 [(a_m b_o)^2 + (v_m b_o)^2] & = 0
\end{align*}
\]  \hspace{1cm} (260)

Attempts at solution of equation (260) for any combination of \( a_o/b_o \) and \( K_{y_o} \) do not yield the smallest values of \( K_{x_o} \). As a result, further considerations of this equation and this subcase are abandoned.
\( K_{y_0} \) **Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.**

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, \( K_{y_0} \) can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, \( K_{y_0} \) can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that \( K_{y_0} \) range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) reveal that \( Y(y_o) \) must take the form shown in equation (162). As usual, equations (228) through (231) make up the set of boundary conditions for the \( Y(y_o) \) function. First, apply equation (228) to equation (162). This relation fixes \( D_m \) in terms of \( B_m \) such that \( D_m = -B_m \). Utilization of equation (229), on the other hand, yields a relation between \( C_m \) and \( A_m \) such that

\[
C_m = -(\alpha_m / \beta_m)A_m
\]

So equation (162) takes on the following form:

\[
Y(y_o) = A_m \{ \sin(\alpha_m y_o) - (\alpha_m / \beta_m) \sinh(\beta_m y_o) \} \\
+ B_m \{ \cos(\alpha_m y_o) - \cosh(\beta_m y_o) \}
\]

(261)

The first derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[
Y'(y_o) = A_m [\alpha_m \cos(\alpha_m y_o) - \alpha_m \cosh(\beta_m y_o)] \\
- B_m [\alpha_m \sin(\alpha_m y_o) + \beta_m \sinh(\beta_m y_o)]
\]

(262)
The second derivative of $Y(y_0)$ with respect to $y_0$ is:

$$Y''(y_0) = -A_m\{a_m^2 \sin(a_m y_0) + a_m b_m \sinh(b_m y_0)\}$$
$$- B_m\{a_m^2 \cos(a_m y_0) + b_m^2 \cosh(b_m y_0)\} \quad (263)$$

Finally, the third derivative of $Y(y_0)$ with respect to $y_0$ is:

$$Y'''(y_0) = -A_m\{a_m^3 \cos(a_m y_0) + a_m b_m^2 \cosh(b_m y_0)\}$$
$$+ B_m\{a_m^3 \sin(a_m y_0) - b_m^3 \sinh(b_m y_0)\} \quad (264)$$

Substitution of equations (262), (263), and (264) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients $A_m$ and $B_m$. For non-trivial $A_m$ and $B_m$, the following determinental equation must hold:

$$\begin{vmatrix}
\sin(a_m b_0) & \cos(a_m b_0) \\
+ \sinh(b_m b_0) & + \cosh(b_m b_0)
\end{vmatrix}
\begin{vmatrix}
a_m^2 \\
+ b_m^2
\end{vmatrix}
= 0$$

Expansion of the determinant, division by the common multiple $a_m$, and multiplication by the quantity $b_0^4$ gives:

$$(a_m b_0)(b_m b_0) \sin(a_m b_0) \sinh(b_m b_0) \{ (a_m b_0)^2 - (b_m b_0)^2 \}$$
$$- 2 K_{y_0} (n/a_0)^2 a_m$$
$$+ \cos(a_m b_0) \cosh(b_m b_0) \{ 2 (a_m b_0)^2 (b_m b_0)^2 \}$$
$$+ K_{y_0} (n/b_0/a_0)^2 \{ (a_m b_0)^2 - (b_m b_0)^2 \} = 0 \quad (266)$$
Equation (266) represents the governing equation for the combination of any plate aspect ratio and $K_{y_0}$ less than or equal to zero. In other words, the roots of equation (266) yield the smallest values of $K_{x_0}$ for any $a_o/b_o$ and tensile or zero $K_{y_0}$. Consideration of results generated by equation (266) is postponed until the last of the three subcases is presented.

$K_{y_0}$ Ranges from a Relatively Large Negative Number to Negative Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, $K_{y_0}$ can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited by the two inequalities expressed in equations (170) and (171). Furthermore, equations (172), (173), and (174) sequentially illustrate that $Y(y_o)$ must take the form shown in equation (175). Note also that equations (228) through (231) again comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (228) to equation (175). This combination fixes $D_m$ in terms of $B_m$ such that $D_m = -B_m$. Utilization of equation (229), on the other hand, yields a relation between $C_m$ and $A_m$ such that $C_m = -(e_m/\beta_m)A_m$. So equation (175) takes on the following form:
\[ Y(y_0) = A_m \{ \sinh(\epsilon_\text{m}y_0) - (\epsilon_\text{m}/\beta_\text{m}) \sinh(\beta_\text{m}y_0) \} + B_m \{ \cosh(\epsilon_\text{m}y_0) - \cosh(\beta_\text{m}y_0) \} \]  

(267)

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[ Y'(y_0) = A_m \{ \epsilon_\text{m} \cosh(\epsilon_\text{m}y_0) - \epsilon_\text{m} \cosh(\beta_\text{m}y_0) \} + B_m \{ \epsilon_\text{m} \sinh(\epsilon_\text{m}y_0) - \beta_\text{m} \sinh(\beta_\text{m}y_0) \} \]  

(268)

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[ Y''(y_0) = A_m \{ \epsilon_\text{m}^2 \sinh(\epsilon_\text{m}y_0) - \epsilon_\text{m} \beta_\text{m} \sinh(\beta_\text{m}y_0) \} + B_m \{ \epsilon_\text{m}^2 \cosh(\epsilon_\text{m}y_0) - \beta_\text{m}^2 \cosh(\beta_\text{m}y_0) \} \]  

(269)

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[ Y'''(y_0) = A_m \{ \epsilon_\text{m}^3 \cosh(\epsilon_\text{m}y_0) - \epsilon_\text{m} \beta_\text{m}^2 \cosh(\beta_\text{m}y_0) \} + B_m \{ \epsilon_\text{m}^3 \sinh(\epsilon_\text{m}y_0) - \beta_\text{m}^3 \sinh(\beta_\text{m}y_0) \} \]  

(270)

Substitution of equations (268), (269), and (270) into equations (230) and (231), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sinh(\epsilon_\text{m}b_o) \{ \epsilon_\text{m}^2 \} & \cosh(\epsilon_\text{m}b_o) \{ \epsilon_\text{m}^2 \} \\
+ \sinh(\beta_\text{m}b_o) \{ -\epsilon_\text{m}\beta_\text{m} \} & + \cosh(\beta_\text{m}b_o) \{ -\beta_\text{m}^2 \}
\end{vmatrix}
\begin{vmatrix}
\cosh(\epsilon_\text{m}b_o) \{ \epsilon_\text{m}^3 \} & \sinh(\epsilon_\text{m}b_o) \{ \epsilon_\text{m}^3 \} \\
+ K_{y_0} (n/a_o)^2 \epsilon_\text{m} & + K_{y_0} (n/a_o)^2 \epsilon_\text{m}
\end{vmatrix}
\begin{vmatrix}
\cosh(\beta_\text{m}b_o) \{ -\epsilon_\text{m}\beta_\text{m}^2 \} & \sinh(\beta_\text{m}b_o) \{ -\beta_\text{m}^3 \} \\
- K_{y_0} (n/a_o)^2 \epsilon_\text{m} & - K_{y_0} (n/a_o)^2 \beta_\text{m}
\end{vmatrix} = 0
\]

(271)

Expansion of the determinant, division by the common multiple \( \epsilon_\text{m} \), and multiplication by the quantity \( (-b_0^4) \) gives:
\[ (e_{m,b_0})(e_{m,b_0}) \sinh(e_{m,b_0}) \sinh(e_{m,b_0}) \{ (e_{m,b_0})^2 + (e_{m,b_0})^2 \\
+ 2 K_{yo} (nb_o/a_0)^2 \} + (e_{m,b_0})^4 + (e_{m,b_0})^4 + K_{yo} (nb_o/a_0)^2 [(e_{m,b_0})^2 \\
+ (e_{m,b_0})^2] - \cosh(e_{m,b_0}) \cosh(e_{m,b_0}) \{ 2 (e_{m,b_0})^2 (e_{m,b_0})^2 \\
+ K_{yo} (nb_o/a_0)^2 [(e_{m,b_0})^2 + (e_{m,b_0})^2] \} = 0 \quad (272) \]

Attempts at solution of equation (272) for any combination of \( a_o/b_o \) and \( K_{yo} \) do not yield the smallest values of \( K_{xo} \). As a result, further considerations of this equation and this subcase as a whole are abandoned.

**Discussion of Results**

Table X gives selected \( a_o/b_o \), \( K_{yo} \), and \( K_{xo} \) ordered triplets as determined by equation (240). Note that the results generated by equation (240) are given—and indeed are only valid—for compressive or positive \( K_{yo} \). These numbers and plots based upon this data reveal two very important aspects of the buckling characteristics of a laminate supported by this relatively weak set of boundary conditions. First, all values of minimum \( K_{xo} \) for any combination of \( a_o/b_o \) and compressive \( K_{yo} \) are achieved when \( m = 1 \) is utilized in the terms which compose equation (240). More broadly, \( K_{xo} \) plotted versus \( a_o/b_o \) for any constant compressive \( K_{yo} \) results in just one continuous curve. There are no transition points to another curve.

Second, curves of \( K_{xo} \) versus \( a_o/b_o \) for constant compressive \( K_{yo} \) lie well below the zero ordinate for small and intermediate aspect ratios. However, these curves continuously trend upward as \( a_o/b_o \) increases. Indeed, as
ao/bo reaches a certain level, the proper Kxo value lies just slightly and almost indistinguishably below the value characterizing the boundary curve

\[ \{(K_yo/2m^2)^2 + Kxo (a_o/mb_o)^2 - 1\} = 0 \]

Since m = 1 in all cases for positive Ky o, this boundary value of Kxo is given by:

\[ Kxo = (b_o/a_o)^2 \left\{ 1 - (Kyo/2) \right\}^2 \] (273)

For example, virtually the last ao/bo and corresponding Kxo coordinate which can be determined for Ky o = 3.00 are ao/bo = 4.011 and Kxo = -0.0777436. The boundary value of Kxo for the noted yo buckling coefficient and plate aspect ratio is, by equation (273), -0.0776971. Figure 21 is a plot of just this asymptotic approach of Kxo toward the boundary value of Kxo given by equation (273) for constant Ky o = 3.00.

Note that as Ky o increases, the value of ao/bo at which the Kxo versus ao/bo plot becomes almost one with the curve defined by equation (273) is delayed. However, this merging is only postponed; it is never averted. As a result, as ao/bo becomes very large the limiting value of Kxo for any positive Ky o approaches zero (the limiting value of equation (273)).

Table XI gives selected ao/bo, Ky o, and Kxo ordered triplets as determined by equation (266). Note that the results generated by equation (266) are given and are only valid for negative or zero Ky o. Also included is the integer value of m which produces this minimum Kxo. Furthermore, each entry point which corresponds to a transition point from the m curve to the (m+1) curve is superscripted in the ao/bo.
column with a star (*). Especially be aware that discontinuous curves as opposed to the continuous curves as discussed above depict $K_{x_o}$ versus $a_o/b_o$ plots for tensile $K_{y_o}$. The statistics presented in Table XI expose two key characteristics of laminates under tension in the $y_o$-direction.

First, the transition values of $a_o/b_o$ increase as $K_{y_o}$ becomes algebraically larger (or less tensile). For this weak set of boundary conditions, however, transitions occur after longer intervals of aspect ratio than for those stronger groups discussed in prior sections.

Second, irrespective of the negative (or zero) magnitude of $K_{y_o}$, $K_{x_o}$ attains a limiting value of 0.7125 as $a_o/b_o$ approaches infinity. This is a different asymptotic value than that rendered by positive $y_o$-buckling coefficients.

Figure 22 represents a plot of $K_{x_o}$ versus $a_o/b_o$ for twelve distinct values of $K_{y_o}$. The lowest curve characterizes $K_{y_o}=3.0$; whereas, the highest depicts $K_{y_o}=-5.0$ In ascending order the magnitudes of the $y_o$-buckling coefficients which correspond to these remaining curves are: 2.5, 2.0, 1.5, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0, and -4.0 This graph reinforces the concepts that $K_{x_o}$ for a compressive $K_{y_o}$ is determined by one continuous curve and that $K_{x_o}$ for a tensile or zero $K_{y_o}$ is rendered by the lowest ordinates of an infinite number of intersecting curves. In addition, the merging of the family of curves to two separate
asymptotes is readily apparent.

Figure 23 plots in three dimensions the same information as Figure 22. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 22; however, the quantitative aspect of Figure 23 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table XII gives selected coordinates of \( K_y \) and \( K_x \) for three distinct values of \( a_o/b_o \)--1.0, 2.2, and 5.4 Figures 24, 25, and 26 represent two-dimensional plots at these constant \( a_o/b_o \) slices of 1.0, 2.2, and 5.4, respectively. Because the data for each of the curves are derived from two completely different equations ( (240) for \( K_y \) positive; (266) for \( K_y \) zero or negative), a mild variance in slope in the vicinity of \( K_y = 0.0 \) is observed for \( a_o/b_o = 1.0 \) and increasingly larger variances for \( a_o/b_o = 2.2 \) and \( a_o/b_o = 5.4 \). These more distinct breaks typify the wide variance at relatively large aspect ratios between the \( x_o \)-buckling coefficient corresponding to negative \( K_y \) and the \( K_x \) due to positive \( K_y \).
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<th>( K_x )</th>
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### TABLE XI

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the $x_o$-Direction and Clamped and Free on the Two Edges Normal to the $y_o$-Direction
(for $K_{y_0}$ less than or equal to zero)

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<td>-1.0</td>
<td>0.7717</td>
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<td>0.0257</td>
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<td>0.0000</td>
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<tr>
<td>1</td>
<td>5.4000</td>
<td>3.0</td>
<td>-0.0429</td>
</tr>
</tbody>
</table>
Expansion of the determinant and multiplication by the quantity $b_0^4$ results in the following system of equations (the second equation a simplification of the first):

$$\sin^2(U_{bo}) \left\{ -(U_{bo})^5 + K_{yo} \left( n_{bo}/a_o \right)^2 (U_{bo})^3 \right\}$$

$$+ \sin(U_{bo}) \cos(U_{bo}) \left\{ (U_{bo})^4 + K_{yo} \left( n_{bo}/a_o \right)^2 (U_{bo})^2 \right\}$$

$$+ \cos(U_{bo}) \left\{ -(U_{bo})^5 + K_{yo} \left( n_{bo}/a_o \right)^2 (U_{bo})^3 \right\} = 0 \quad (304)$$

$$\sin(U_{bo}) \cos(U_{bo}) \left\{ (U_{bo})^4 + K_{yo} \left( n_{bo}/a_o \right)^2 (U_{bo})^2 \right\}$$

$$- (U_{bo})^5 + K_{yo} \left( n_{bo}/a_o \right)^2 (U_{bo})^3 = 0 \quad (305)$$

No value of $(U_{bo})$ greater than zero can satisfy equation (305). Therefore, no possible solutions exist for the present boundary conditions for

$$\left\{ \frac{(K_{yo}/2m^2)^2 + K_{xo} \left( a_o/m_{bo} \right)^2 - 1}{(K_{yo}/2m^2)^2 + K_{xo} \left( a_o/m_{bo} \right)^2 - 1} \right\} = 0 \quad \text{and} \quad K_{yo} > 0$$

$$\left\{ \frac{(K_{yo}/2m^2)^2 + K_{xo} \left( a_o/m_{bo} \right)^2 - 1}{(K_{yo}/2m^2)^2 + K_{xo} \left( a_o/m_{bo} \right)^2 - 1} \right\} > 0$$

For the quantity $\left\{ \frac{(K_{yo}/2m^2)^2 + K_{xo} \left( a_o/m_{bo} \right)^2 - 1}{(K_{yo}/2m^2)^2 + K_{xo} \left( a_o/m_{bo} \right)^2 - 1} \right\} > 0$ equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown $y(y_o)$ function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of $K_{yo}$ for valid solutions. Three subcases must be considered so that a solution for $K_{xo}$ may be determined for any range of $K_{yo}$. 

-134-
function. The enforcement of equation (277) on equation (132) dictates that \( B_m \) must vanish. In addition, application of equation (278) leads one to the conclusion that \( C_m \) is zero. So equation (132) takes on the following form:

\[
Y(y_0) = A_m \sin(Uy_0) + D_m y_0 \cos(Uy_0) \tag{299}
\]

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'(y_0) = A_m U \cos(Uy_0) + D_m \{ \cos(Uy_0) - Uy_0 \sin(Uy_0) \} \tag{300}
\]

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = -A_m U^2 \sin(Uy_0) + D_m \{ -2U \sin(Uy_0) - U^2 y_0 \cos(Uy_0) \} \tag{301}
\]

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = -A_m U^3 \cos(Uy_0) + D_m \{ -3U^2 \cos(Uy_0) - U^3 y_0 \sin(Uy_0) \} \tag{302}
\]

Substitution of equations (300), (301), and (302) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( D_m \). For non-trivial \( A_m \) and \( D_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(Ub_0) \{-U^2\} & \sin(Ub_0) \{-2U\} & \sin(Ub_0) \{-U\} & \sin(Ub_0) \{-U^2b_0\} \\
\cos(Ub_0) \{-U^3\} & \sin(Ub_0) \{U^3b_0\} & -K_{y_0} (n/a_o)^2 U & -K_{y_0} (n/a_o)^2 Ub_0 \\
+K_{y_0} (n/a_o)^2 U \} & -K_{y_0} (n/a_o)^2 \cos(Ub_0) \{-3U^2\} & +K_{y_0} (n/a_o)^2 \cos(Ub_0) \{-3U^2\} & +K_{y_0} (n/a_o)^2 \cos(Ub_0) \{-3U^2\} \\
\end{vmatrix} = 0
\]

(303)
\( K_{y_0} = 0 \).

For \( K_{y_0} = 0 \) equation (128) constitutes the required shape of the unknown \( Y(y_0) \) function. In addition, equations (277) through (280) are the sets of constraints for this \( Y(y_0) \) function. Moreover, notice that equation (280) reduces to \( Y'''(b_o) = 0 \) in this instance since \( K_{y_0} = 0 \). Equations (277) and (278) imply that \( A_m = C_m = 0 \). Equation (280) similarly necessitates that \( D_m \) vanish. After the imposition of these three conditions, equation (128) reduces to:

\[
Y(y_0) = B_m Y_0
\]

Equation (279) is identically satisfied for this \( Y(y_0) \) given by equation (297). All boundary conditions, equations (277) through (280), are therefore upheld by this \( Y(y_0) \) expressed in equation (297). As a result, if \( K_{y_0} = 0 \),

\[
\{ (K_{y_0}/2m^2)^2 + K_{x_0} (a_0/m b_o)^2 - 1 \} = 0
\]

is a valid solution. Rearrangement of this relation gives:

\[
K_{x_0} = \left[ m / (a_0/b_o) \right]^2
\]

When \( m \) is set equal to unity, equation (298) constitutes the minimum \( K_{x_0} \) for any plate aspect ratio and \( K_{y_0} = 0 \). Consideration of results generated by equation (298) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

\( K_{y_0} > 0 \).

For \( K_{y_0} > 0 \) equations (130) and (131) show that the unknown function \( Y(y_0) \) must fit the relation given by equation (132). Equations (277) through (280) once more comprise the group of boundary conditions for the \( Y(y_0) \).
equations (279) and (280), along with basic algebraic manipulation, yield two homogeneous linear equations in coefficients $A_m$ and $D_m$. For non-trivial $A_m$ and $D_m$, the following determinental equation must hold:

$$\begin{vmatrix}
\sinh(Tb_o) \{ T^2 \} & \sinh(Tb_o) \{ 2T \} \\
& + \cosh(Tb_o) \{ T^2 b_o \} \\
\cosh(Tb_o) \{ T^3 \} & \sinh(Tb_o) \{ T^3 b_o \} \\
& + K_{y_o} (n/a_o)^2 T \} \\
& + K_{y_o} (n/a_o)^2 T b_o \\
& + \cosh(Tb_o) \{ 3T^2 \} \\
& + K_{y_o} (n/a_o)^2 \}
\end{vmatrix} = 0$$

(294)

Expansion of the determinant and multiplication by the quantity $b_o^4$ results in the following equations (the second equation a simplification of the first):

$$\sinh^2(Tb_o) \{ (Tb_o)^5 + K_{y_o} (nb_o/a_o)^2 (Tb_o)^3 \}$$
$$+ \sinh(Tb_o) \cosh(Tb_o) \{ (Tb_o)^4 - K_{y_o} (nb_o/a_o)^2 (Tb_o)^2 \}$$
$$- \cosh^2(Tb_o) \{ (Tb_o)^5 + K_{y_o} (nb_o/a_o)^2 (Tb_o)^3 \} = 0$$

(295)

$$\sinh(Tb_o) \cosh(Tb_o) \{ (Tb_o)^4 - K_{y_o} (nb_o/a_o)^2 (Tb_o)^2 \}$$
$$- (Tb_o)^5 - K_{y_o} (nb_o/a_o)^2 (Tb_o)^3 = 0$$

(296)

No value of $Tb_o$ greater than zero can satisfy equation (296). Therefore, no possible solutions exist for the present boundary conditions for

$$\{(K_{y_o}/2m^2)^2 + K_{x_o} (a_o/m_b)^2 - 1 \} = 0 \text{ and } K_{y_o} < 0$$
\[(K_{y_0}/2m^2)^2 + K_{x_0} (a_{o}/mb_{o})^2 - 1 \} = 0\]

For the quantity \[(K_{y_0}/2m^2)^2 + K_{x_0} (a_{o}/mb_{o})^2 - 1 \} = 0\] equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically for the choice of algebraic sign of \(K_{y_0}\), each possible range of \(K_{y_0}\)--negative, zero, and positive--will be analyzed as different subcases.

\(K_{y_0} < 0\).

For \(K_{y_0} < 0\) equations (121) and (122) lead to the conclusion that the unknown function \(Y(y_0)\) must take the form shown in equation (123). Equations (277) through (280) again comprise the group of boundary conditions for the \(Y(y_0)\) function. The enforcement of equation (277) on equation (123) dictates that \(B_m\) must vanish. In addition, application of equation (278) leads one to the conclusion that \(C_m\) is zero. So equation (123) takes on the following form:

\[Y(y_0) = A_m \sinh(Ty_0) + D_m y_0 \cosh(Ty_0)\]  \hspace{1cm} (290)

The first derivative of \(Y(y_0)\) with respect to \(y_0\) is:

\[Y'(y_0) = A_m T \cosh(Ty_0) + D_m \{ \cosh(Ty_0) + Ty_0 \sinh(Ty_0) \}\]  \hspace{1cm} (291)

The second derivative of \(Y(y_0)\) with respect to \(y_0\) is:

\[Y''(y_0) = A_m T^2 \sinh(Ty_0) + D_m \{ 2T \sinh(Ty_0) \]
\[+ T^2 y_0 \cosh(Ty_0) \}\]  \hspace{1cm} (292)

Finally, the third derivative of \(Y(y_0)\) with respect to \(y_0\) is:

\[Y'''(y_0) = A_m T^3 \cosh(Ty_0) + D_m \{ 3T^2 \cosh(Ty_0) \]
\[+ T^3 y_0 \sinh(Ty_0) \}\]  \hspace{1cm} (293)

Substitution of equations (291), (292), and (293) into
Expansion of the determinant and multiplication by the quantity $b_o$ results in the following equations (the second equation a simplification of the first):

$$
\sin^2(sbo) \left[ e^{2cbo} \{- (cb_o)^4 (sb_o) - 2 (cb_o)^2 (sb_o)^3 - (sb_o)^5
+ K_y (nb_o/a_o)^2 \left( (cb_o)^2 (sb_o) + (sb_o)^3 \right) \right]
+ e^{-2cbo} \left\{ (cb_o)^4 (sb_o) + 2 (cb_o)^2 (sb_o)^3 + (sb_o)^5
- K_y (nb_o/a_o)^2 \left( (cb_o)^2 (sb_o) + (sb_o)^3 \right) \right\}
+ \cos^2(sbo) \left[ e^{2cbo} \{- (cb_o)^4 (sb_o) - 2 (cb_o)^2 (sb_o)^3 - (sb_o)^5
+ K_y (nb_o/a_o)^2 \left( (cb_o)^2 (sb_o) + (sb_o)^3 \right) \right]
+ e^{-2cbo} \left\{ (cb_o)^4 (sb_o) + 2 (cb_o)^2 (sb_o)^3 + (sb_o)^5
- K_y (nb_o/a_o)^2 \left( (cb_o)^2 (sb_o) + (sb_o)^3 \right) \right\}
+ \sin(sbo) \cos(sbo) \left[ 4 (cb_o)^5 + 4 (cb_o) (sb_o)^4 + 8 (cb_o)^3 (sb_o)^2
+ K_y (nb_o/a_o)^2 \left\{ 4 (cb_o)^3 + 4 (cb_o) (sb_o)^2 \right\} \right] = 0 \quad (288)
$$

$$
(e^{2cbo} - e^{-2cbo}) \{- (cb_o)^4 (sb_o) - 2 (cb_o)^2 (sb_o)^3 - (sb_o)^5
+ K_y (nb_o/a_o)^2 \left( (cb_o)^2 (sb_o) + (sb_o)^3 \right) \}
+ 4 \sin(sbo) \cos(sbo) \left\{ (cb_o)^5 + (cb_o) (sb_o)^4
+ 2 (cb_o)^3 (sb_o)^2 + K_y (nb_o/a_o)^2 \left\{ (cb_o)^3
+ (cb_o) (sb_o)^2 \right\} \right\} = 0 \quad (289)
$$

Equation (289) constitutes the governing equation for any plate aspect ratio and $K_y > 0$. In other words, the roots of equation (289) yield the smallest values of $K_y$ for any $a_o/b_o$ and compressive $K_y$. Consideration of results generated by equation (289) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.
\[ Y'''(y_o) = A_m\left( (c^2 - s^2) \sin(sy_o) \ [ e^{cy_o} + e^{-cy_o} ] \right. \\
+ (2cs) \cos(sy_o) \ [ e^{cy_o} - e^{-cy_o} ] \nonumber \\
+ B_m\left( (c^2 - s^2) \cos(sy_o) \ [ e^{cy_o} - e^{-cy_o} ] \right. \\
- (2cs) \sin(sy_o) \ [ e^{cy_o} + e^{-cy_o} ] \nonumber \] (285)

The third derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[ Y''''(y_o) = A_m\left( (3c^2 s - s^3) \cos(sy_o) \ [ e^{cy_o} + e^{-cy_o} ] \right. \\
+ (c^3 - 3cs^2) \sin(sy_o) \ [ e^{cy_o} - e^{-cy_o} ] \nonumber \\
+ B_m\left( (s^3 - 3c^2 s) \sin(sy_o) \ [ e^{cy_o} - e^{-cy_o} ] \right. \\
+ (c^3 - 3cs^2) \cos(sy_o) \ [ e^{cy_o} + e^{-cy_o} ] \nonumber \] (286)

Substitution of equations (284), (285), and (286) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(sb_o)\left[ (c^2 - s^2) (e^{cb_o} + e^{-cb_o}) \right] & \sin(sb_o)\left[ (-2cs) (e^{cb_o} + e^{-cb_o}) \right] \\
\cos(sb_o)\left[ (2cs) (e^{cb_o} - e^{-cb_o}) \right] & \cos(sb_o)\left[ (c^2 - s^2) (e^{cb_o} - e^{-cb_o}) \right]
\end{vmatrix}
= 0
\]

(287)
For equations (275) and (276) to have meaning in the general case, the following conditions must hold:

\[
Y(0) = 0 \quad (277)
\]
\[
Y''(0) = 0 \quad (278)
\]
\[
Y''(b_0) = 0 \quad (279)
\]
\[
Y''(b_0) + K_{y_0} (n/a_0)^2 Y'(b_0) = 0 \quad (280)
\]

First, apply equation (277) to equation (106). This stipulation fixes \(D_m\) in terms of \(B_m\) such that \(D_m = -B_m\). Utilization of equation (278) on the \(Y(y_0)\) equation similarly determines a value \(C_m\) in terms of \(A_m\):

\[
A_m(2cs) + B_m(c^2 - s^2) - C_m(2cs) + D_m(c^2 - s^2) = 0 \quad (281)
\]

But since \(D_m = -B_m\),

\[
C_m = A_m(2cs)/(2cs) = A_m \quad (282)
\]

For these values of \(C_m\) and \(D_m\), equation (106) takes on the following form:

\[
Y(y_0) = A_m\{ \sin(sy_0) \ (e^{cy_0} + e^{-cy_0}) \} \\
+ B_m\{ \cos(sy_0) \ (e^{cy_0} - e^{-cy_0}) \} \quad (283)
\]

The first derivative of \(Y(y_0)\) with respect to \(y_0\) is therefore:

\[
Y'(y_0) = A_m\{ \cos(sy_0) \ (se^{cy_0} + se^{-cy_0}) \} \\
+ \sin(sy_0) \ (ce^{cy_0} - ce^{-cy_0}) \} \}
+ B_m\{ \sin(sy_0) \ (-se^{cy_0} + se^{-cy_0}) \} \\
+ \cos(sy_0) \ (ce^{cy_0} + ce^{-cy_0}) \} \quad (284)
\]

The second derivative of \(Y(y_0)\) with respect to \(y_0\) is:
A lack of symmetry is present in the boundary conditions in the \( y_o \)-direction.

Just as before, a displacement function \( w \) which satisfies the first two stipulations of equation (274) is given by equation (77). Furthermore, substitution of this relation for \( w \) into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable \( r \) is defined in equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states—negative, zero, and positive—of the quantity in the square brackets of equation (81). As explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

\[
\{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0
\]

For the quantity \( \{(K_{y_o}/2m^2)^2 + K_{x_o}(a_o/mb_o)^2 - 1\} < 0 \) equations (82) through (105) illustrate that the unknown function \( Y(y_o) \) must take the form shown in equation (106).

Consider now the boundary conditions, equations (274), for this case of a laminate simply supported on three sides and free on the fourth. When the chosen form of \( w \), equation (77), is substituted into the final three lines of equations (274), the following must hold:

\[
Y(0) \sin(mn/a_o) = 0 ; \quad Y''(0) \sin(mn/a_o) = 0 \quad (275)
\]

\[
Y''(b_o) \sin(mn/a_o) = 0 ; \quad (276)
\]

\[
[Y'''(b_o) + K_{y_o} (n/a_o)^2 Y'(b_o)] \sin(mn/a_o) = 0
\]
VII. Flat Rectangular Composite Laminate Simply Supported in the \( x_o \)-Direction and Simply Supported and Free on the Two Edges Normal to the \( y_o \)-Direction

The boundary conditions for a laminate simply supported in the \( x_o \)-direction and simply supported and free on the two edges perpendicular to the \( y_o \)-direction display no symmetry in the \( y_o \)-direction. For the two edges which have normals parallel to the \( x_o \)-axis, the vertical displacement along each edge and the normal component of the moment to each edge must vanish in the affine space. Similarly, for that edge normal to the \( y_o \)-direction which is simply supported, these same edge conditions hold. However, for the remaining edge which is oriented perpendicular to the \( y_o \)-direction, the vanishing of the normal component of the moment to this free edge and the satisfaction of equation (224) constitute the two requisites. In equation form, the following must hold:

\[
\begin{align*}
\text{on edge } x_o &= -a_o/2, \quad w = 0; \quad w_{x_0x_o} = 0 \\
\text{on edge } x_o &= a_o/2, \quad w = 0; \quad w_{x_0x_o} = 0 \\
\text{on edge } y_o &= 0, \quad w = 0; \quad w_{y_0y_o} = 0 \\
\text{on edge } y_o &= b_o, \quad w_{y_0y_o} = 0; \\
& \quad w_{y_0y_o} + K_{y_o}(n/a_o)^2w_{y_0} = 0
\end{align*}
\]

Note that the origin of coordinates in the affine space is taken to be at the center of the simply supported edge normal to the \( y_o \)-direction. This choice of origin location, in general, allows maximum simplicity in manipulations since
Figure 26
X0-buckling coefficient versus Y0-buckling coefficient at a constant affine aspect ratio of 5.4 for an S-C-S-F laminate

For the range
Kyo = -5.0 to 3.0
Figure 25: Axial buckling coefficient versus axial strain for an S-3-F laminate ratio of 2.2 for an S-3-F laminate.
FIGURE 24
KO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 1.0 FOR AN S-C-S-F LAMINATE

FOR THE RANGE
KYO = -5.0 TO 3.0
FIGURE 23
SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-C-S-F LAMINATE
The quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, $K_{Y_o}$ can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, $K_{Y_o}$ can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that $Y(y_o)$ must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144). Equations (277) through (280) once more comprise the group of boundary conditions for the $Y(y_o)$ function. First, apply equation (277) to equation (144). This stipulation fixes $D_m$ in terms of $B_m$ such that $D_m = -B_m$. Utilization of equation (278) on the $Y(y_o)$ equation, on the other hand, forces $B_m$ and hence $D_m$ to vanish. So equation (144) takes on the following form:

$$Y(y_o) = A_m \sin(a_m y_o) + C_m \sin(v_m y_o) \quad (306)$$

The first derivative of $Y(y_o)$ with respect to $y_o$ is:

$$Y'(y_o) = A_m a_m \cos(a_m y_o) + C_m v_m \cos(v_m y_o) \quad (307)$$

The second derivative of $Y(y_o)$ with respect to $y_o$ is:

$$Y''(y_o) = -A_m a_m^2 \sin(a_m y_o) - C_m v_m^2 \sin(v_m y_o) \quad (308)$$
Finally, the third derivative of $Y(y_0)$ with respect to $y_0$ is:

$$Y'''(y_0) = -A_m a_m^3 \cos(a_m y_0) - C_m v_m^3 \cos(v_m y_0)$$  \quad (309)

Substitution of equations (307), (308), and (309) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients $A_m$ and $C_m$. For non-trivial $A_m$ and $C_m$, the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(a_m b_0) \{-a_m^2\} & \sin(v_m b_0) \{-v_m^2\} \\
\cos(a_m b_0) \{-a_m^3\} & \cos(v_m b_0) \{-v_m^3\} \\
+ K_{y_0} (n/a_0)^2 a_m & + K_{y_0} (n/a_0)^2 v_m
\end{vmatrix} = 0 \quad (310)
\]

Expansion of the determinant, division by the common multiple $(a_m v_m)$, and multiplication by the quantity $b_0^3$ gives:

$$\sin(a_m b_0) \cos(v_m b_0) \{(a_m b_0)(v_m b_0)^2 - K_{y_0} (n b_0 / a_0)^2 (a_m b_0)\}$$

$$+ \cos(a_m b_0) \sin(v_m b_0) \{-(a_m b_0)^2 (v_m b_0)$$

$$+ K_{y_0} (n b_0 / a_0) (v_m b_0) \} = 0 \quad (311)$$

Attempts at solution of equation (311) for any combination of $a_0/b_0$ and $K_{y_0}$ do not yield the smallest values of $K_{x_0}$. As a result, further considerations of this equation and this subcase are abandoned.

$K_{y_0}$ Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, $K_{y_0}$ can take on any value from a comparatively large negative
number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, \( K_{y_0} \) can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that \( K_{y_0} \) range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) reveal that \( Y(y_0) \) must take the form shown in equation (162). As usual, equations (277) through (280) make up the set of boundary conditions for the \( Y(y_0) \) function. First, apply equation (277) to equation (162). This stipulation fixes \( D_m \) in terms of \( B_m \) such that \( D_m = -B_m \). Utilization of equation (278) on the \( Y(y_0) \) equation, on the other hand, forces \( B_m \) and hence \( D_m \) to vanish. So equation (162) takes on the following form:

\[
Y(y_0) = A_m \sin(a_m y_0) + C_m \sinh(\beta_m y_0)
\]  
(312)

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'(y_0) = A_m a_m \cos(a_m y_0) + C_m \beta_m \cosh(\beta_m y_0)
\]  
(313)

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = -A_m a_m^2 \sin(a_m y_0) + C_m \beta_m^2 \sinh(\beta_m y_0)
\]  
(314)

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = -A_m a_m^3 \cos(a_m y_0) + C_m \beta_m^3 \cosh(\beta_m y_0)
\]  
(315)

Substitution of equations (313), (314), and (315) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in
coefficients $A_m$ and $C_m$. For non-trivial $A_m$ and $C_m$, the following determinental equation must hold:

$$\begin{vmatrix}
\sin(a_m b_o) & \sinh(b_m b_o) \\
\cos(a_m b_o) & \cosh(b_m b_o)
\end{vmatrix} a_m = 0 \quad (316)$$

Expansion of the determinant, division by the common multiple $(a_m b_m)$, and multiplication by the quantity $b_o^3$ gives:

$$\sin(a_m b_o) \cosh(b_m b_o) \{-(a_m b_o) (b_m b_o)^2 - K_{y_o} (n b_o / a_o)^2 (a_m b_o) \}$$

$$+ \cos(a_m b_o) \sinh(b_m b_o) \{(a_m b_o)^2 (b_m b_o)$$

$$- K_{y_o} (n b_o / a_o)^2 (b_m b_o) \} = 0 \quad (317)$$

Equation (317) represents the governing equation for the combination of any plate aspect ratio and $K_{y_o}$ less than zero. In other words, the roots of equation (317) yield the smallest values of $K_{x_o}$ for any $a_o / b_o$ and tensile $K_{y_o}$. Consideration of results generated by equation (317) is postponed until the last of the three subcases is presented.

$K_{y_o}$ Ranges from a Relatively Large Negative Number to Negative Infinity.

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, $K_{y_o}$ can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited
by the two inequalities expressed in equations (170) and (171). Furthermore, equations (172), (173), and (174) sequentially illustrate that \( Y(y_o) \) must take the form shown in equation (175). Note also that equations (277) through (280) again comprise the group of boundary conditions for the \( Y(y_o) \) function. First, apply equation (277) to equation (175). This combination fixes \( D_m \) in terms of \( B_m \) such that 
\[ D_m = -B_m \]
Utilization of equation (278) on the \( Y(y_o) \) equation, in contrast, forces \( B_m \) and hence \( D_m \) to vanish. So equation (175) takes on the following form:
\[ Y(y_o) = A_m \sinh(e_m y_o) + C_m \sinh(p_m y_o) \]  
(318)
The first derivative of \( Y(y_o) \) with respect to \( y_o \) is:
\[ Y'(y_o) = A_m e_m \cosh(e_m y_o) + C_m p_m \cosh(p_m y_o) \]  
(319)
The second derivative of \( Y(y_o) \) with respect to \( y_o \) is:
\[ Y''(y_o) = A_m e_m^2 \sinh(e_m y_o) + C_m p_m^2 \sinh(p_m y_o) \]  
(320)
Finally, the third derivative of \( Y(y_o) \) with respect to \( y_o \) is:
\[ Y'''(y_o) = A_m e_m^3 \cosh(e_m y_o) + C_m p_m^3 \cosh(p_m y_o) \]  
(321)
Substitution of equations (319), (320), and (321) into equations (279) and (280), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( C_m \). For non-trivial \( A_m \) and \( C_m \), the following determinental equation must hold:
\[
\begin{align*}
\sinh(e_m b_0) \{ e_m^2 \} & \quad \sinh(p_m b_0) \{ p_m^2 \} \\
+ K_{yo} (n/a_o)^2 e_m & \quad + K_{yo} (n/a_o)^2 p_m \\
\end{align*}
\]

= 0 \hspace{1cm} (322)

Expansion of the determinant, division by the common multiple \((e_m p_m)\), and multiplication by the quantity \(b_o^3\) gives:

\[
\begin{align*}
\sinh(e_m b_0) \cosh(p_m b_0) \{(e_m b_0) (p_m b_0)^2 \\
+ K_{yo} (n b_o/a_o)^2 (e_m b_o)\} \\
- \sinh(p_m b_0) \cosh(e_m b_0) \{(e_m b_0)^2 (p_m b_0) \\
+ K_{yo} (n b_o/a_o)^2 (p_m b_o)\} = 0 \hspace{1cm} (323)
\end{align*}
\]

Attempts at solution of equation (323) for any combination of \(a_o/b_o\) and \(K_{yo}\) do not yield the smallest values of \(K_{xo}\). As a result, further considerations of this equation and this subcase as a whole are abandoned.

Discussion of Results

Table XIII gives selected \(a_o/b_o\), \(K_{yo}\), and \(K_{xo}\) ordered triplets as determined by equation (289). Note that the results generated by equation (289) are given—and indeed are only valid—for compressive or positive \(K_{yo}\). These numbers and plots based upon this data reveal two very important aspects of the buckling characteristics of a laminate supported by this relatively weak set of boundary conditions. First, all values of minimum \(K_{xo}\) for any combination of \(a_o/b_o\) and compressive \(K_{yo}\) are achieved when \(m = 1\) is utilized in the terms which compose equation (289). More broadly, \(K_{xo}\) plotted versus \(a_o/b_o\) for any constant compressive \(K_{yo}\)
results in just one continuous curve. There are no transition points to another curve.

Second, curves of $K_{x_0}$ versus $a_0/b_0$ for constant compressive $K_{y_0}$ lie well below the zero ordinate for small and intermediate affine aspect ratios. However, these curves continuously trend upward as $a_0/b_0$ increases. For any constant $K_{y_0}$, the $K_{x_0}$ versus $a_0/b_0$ plot asymptotically approaches a value of roughly $-0.3 \ K_{y_0}$. For example, the following data reflects $x_0$-buckling coefficients (and corresponding $y_0$-buckling coefficients) for an affine aspect ratio of 60.0:

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<th>$K_{y_0}$</th>
<th>$K_{x_0}$</th>
</tr>
</thead>
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</tr>
<tr>
<td>60.0</td>
<td>3.0</td>
<td>-0.9128</td>
</tr>
</tbody>
</table>

Table XIV gives selected $a_0/b_0, K_{x_0}$ ordered pairs for zero $K_{y_0}$ only. This data is based exclusively upon equation (298). These numbers again bring to light two key details of the uniaxial ($K_{y_0} = 0$) buckling phenomenon. First, identical to the descriptions presented above, all values of minimum $K_{x_0}$ for any affine aspect ratio and null $K_{y_0}$ occur when $m = 1$ is inserted into equation (298). As a result, a
sketch of $K_{x_0}$ versus $a_o/b_o$ for $K_{y_0} = 0$ plots as one continuous curve; transition points are absent.

Second, the $K_{x_0}$ versus $a_o/b_o$ curve for $K_{y_0} = 0$ lies above the zero ordinate for all aspect ratios. As $a_o/b_o$ becomes large, however, the $K_{x_0} = 0$ value is rapidly approached. Indeed, the limiting value of $K_{x_0}$ for this uniaxial buckling case is in fact zero (as can be easily seen by examination of equation (298)).

Table XV gives selected $a_o/b_o$, $K_{y_0}$, and $K_{x_0}$ ordered triplets as determined by equation (317). Note that the results generated by equation (317) are given and are only valid for negative $K_{y_0}$. Also included is the integer value of $m$ which produces this minimum $K_{x_0}$. Furthermore, each entry point which corresponds to a transition point from the $m$ curve to the $(m+1)$ curve is superscripted with a star (*). Especially be aware that discontinuous curves, as opposed to the continuous curves discussed above, depict $K_{x_0}$ versus $a_o/b_o$ plots for tensile $K_{y_0}$. The statistics presented in Table XV expose two key characteristics of laminates under tension in the $y_o$-direction. First, the transition values of $a_o/b_o$ increase as $K_{y_0}$ becomes algebraically larger (or less tensile). For this weak set of boundary conditions, however, transitions occur after longer intervals of aspect ratio than for those stronger groups discussed in prior sections.

Second, irrespective of the negative magnitude of $K_{y_0}$, $K_{x_0}$ attains a limiting value of zero as $a_o/b_o$ approaches infinity. For this set of boundary conditions the rate of
approach toward this zero asymptote is relatively slow. For example, for $K_{yo} = -5.0$, $K_{xo} = 0.0493$ at $a_o/b_o = 50.0$ and $K_{xo} = 0.0247$ at $a_o/b_o = 100.0$. Note that this asymptotic value of zero differs from those asymptotes presented for compressive $K_{yo}$.

Figure 27 represents a plot of $K_{xo}$ versus $a_o/b_o$ for twelve distinct values of $K_{yo}$. The lowest curve characterizes $K_{yo} = 3.0$; whereas, the highest depicts $K_{yo} = -5.0$. In ascending order the magnitudes of the $y_o$-buckling coefficients which correspond to the remaining ten curves are: 2.5, 2.0, 1.5, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0, and -4.0. This graph reinforces the concepts that $K_{xo}$ for a compressive or zero $K_{yo}$ is determined by one continuous curve and that $K_{xo}$ for a tensile $K_{yo}$ is rendered by the lowest values of an infinite number of intersecting curves. In addition, the merging of the family of curves to the zero asymptote for tensile or zero $K_{yo}$ and to distinct, relatively evenly spaced asymptotes for compressive $K_{yo}$ is readily apparent.

Figure 28 plots in three dimensions the same information as Figure 27. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 27; however, the quantitative aspect of Figure 28 is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.
Table XVI gives selected coordinates of $K_{y_0}$ and $K_{x_0}$ for three distinct values of $a_0/b_0=1.4$, 3.0, and 5.4. Figures 29, 30, and 31 represent two-dimensional plots at these constant $a_0/b_0$ slices of 1.4, 3.0, and 5.4, respectively. Because the data for each of the curves are derived from three completely different equations (1289) for $K_{y_0}$ positive; (298) for $K_{y_0}$ zero; (317) for $K_{y_0}$ negative, a mild variance of slope in the vicinity of $K_{y_0} = 0.0$ is observed for $a_0/b_0 = 1.4$ and slightly larger variances for $a_0/b_0 = 3.0$ and $a_0/b_0 = 5.4$. These breaks are decidedly less severe than those observed in corresponding plots for a laminate simply supported on opposite sides and clamped and free on the remaining edges.
TABLE XIII

Buckling Coefficients Versus Plate Aspect Ratio for a
Laminate Simply Supported in the $x_o$-Direction and
Simply Supported and Free on the Two Edges Normal
to the $y_o$-Direction
(for $K_{y_o}$ less than zero)

<table>
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<th>$m$</th>
<th>$a_o/b_o$</th>
<th>$K_{y_o}$</th>
<th>$K_{x_o}$</th>
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TABLE XIV

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the $x_o$-Direction and Simply Supported and Free on the Two Edges Normal to the $y_o$-Direction (for $K_{y_0}$ equal to zero only)

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TABLE XV
Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the \( x_0 \)-Direction and Simply Supported and Free on the Two Edges Normal to the \( y_0 \)-Direction
(for \( K_{y_0} \) greater than zero only)

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## TABLE XVI

**K<sub>x_0</sub> Versus K<sub>y_0</sub> for Various Plate Aspect Ratios for a Laminate Simply Supported in the x<sub>o</sub>-Direction and Simply Supported and Free on the Two Edges Normal to the y<sub>o</sub>-Direction**

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<tr>
<th>m</th>
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\[
\begin{align*}
[ \text{Ze}^{2c_b} - \text{Ve}^{-2c_b} ] & \{ (s_b)^5 + 2(c_b)^2 (s_b)^3 + (c_b)^4 (s_b) \\
& - K_{y_0} (n_b/a_0)^2 \{ (c_b)^2 (s_b) + (s_b)^3 \} \} \\
& + [ 10(S - R)(c_b)^2 (s_b)^3 + 5(R - S)(c_b)^4 (s_b) \\
& + (R - S)(s_b)^5 + 8(RS - VZ + 1)(c_b)^3 (s_b)^2 \\
& + 4(VZ - RS - 1)(c_b)^4 (s_b) \\
& + K_{y_0} (n_b/a_0)^2 \{ 2(RS - VZ + 1)(c_b)(s_b)^2 \\
& + 3(R - S)(c_b)^2 (s_b) + (S - R)(s_b)^3 \} ] \\
& + 2\sin^2(s_b) \{ (VZ - RS)(c_b) (s_b)^4 + (VZ - RS)(c_b)^5 \\
& + 2(c_b)^3 (s_b)^2 + K_{y_0} (n_b/a_0)^2 \{ (VZ - RS)(c_b)^3 \\
& + (c_b) (s_b)^2 \} \} \} \\
& + 2\cos^2(s_b) \{ 2(RS - VZ)(c_b)^3 (s_b)^2 - (c_b) (s_b)^4 \\
& - (c_b)^5 + K_{y_0} (n_b/a_0)^2 \{ (RS - VZ)(c_b) (s_b)^2 \\
& - (c_b)^3 \} \} \} \\
& - 2(R + S) \sin(s_b) \cos(s_b) \{ (c_b) (s_b)^4 \\
& + 2(c_b)^3 (s_b)^2 + (c_b)^5 \\
& + K_{y_0} (n_b/a_0)^2 \{ (c_b) (s_b)^2 + (c_b)^3 \} \} = 0 \quad (350)
\end{align*}
\]

Equation (350) constitutes the governing equation for any plate aspect ratio and \( K_{y_0} \) greater than or equal to zero. In other words, the roots of equation (350) yield the smallest values of \( K_{x_0} \) for any \( a_0/b_0 \) and compressive (or zero) \( K_{y_0} \). Consideration of results generated by equation (350) is postponed until all cases and subcases have been presented for the chosen set of boundary conditions.

\[
\{(K_{y_0}/2m^2)^2 + K_{x_0} (a_0/mb_0)^2 - 1 \} = 0
\]

For the quantity \( \{(K_{y_0}/2m^2)^2 + K_{x_0} (a_0/mb_0)^2 - 1 \} = 0 \)
\[ \sin^2(s_{bo}) \left[ e^{2c_{bo}} \left\{ 2Z(c_{bo})^2 (s_{bo})^3 + Z(c_{bo})^4 (s_{bo}) \right\} \ight.
\[ + \left. Z(s_{bo})^5 - K_{y_0} (n_{bo}/a_0)^2 (Z(c_{bo})^2 (s_{bo}) + Z(s_{bo})^3) \right\} \ight.
\[ - e^{-2c_{bo}} \left\{ V(s_{bo})^5 + 2V(c_{bo})^2 (s_{bo})^3 + V(c_{bo})^4 (s_{bo}) \right\} \ight.
\[ - K_{y_0} (n_{bo}/a_0)^2 (V(c_{bo})^2 (s_{bo}) + V(s_{bo})^3) \right\} \ight.
\[ + \left\{ 8(RS - VZ) (c_{bo})^3 (s_{bo})^2 + 10(S - R) (c_{bo})^2 (s_{bo})^3 \right\} \ight.
\[ + (R - S) (s_{bo})^5 + 6(VZ - RS) (c_{bo}) (s_{bo})^4 \right\} \ight.
\[ + 2(VZ - RS) (c_{bo})^5 + 5(R - S) (c_{bo})^4 (s_{bo}) \right\} \ight.
\[ - 4(c_{bo}) (s_{bo})^4 + 12(c_{bo})^3 (s_{bo})^2 \right\} \right] \ight.
\[ + \cos^2(s_{bo}) \left[ e^{2c_{bo}} \left\{ Z(s_{bo})^5 + 2Z(c_{bo})^2 (s_{bo})^3 \right\} \ight.
\[ + Z(c_{bo})^4 (s_{bo}) - K_{y_0} (n_{bo}/a_0)^2 (Z(s_{bo})^3 \right\} \ight.
\[ + Z(c_{bo})^2 (s_{bo}) \} \right]
\[ - e^{-2c_{bo}} \left\{ V(s_{bo})^5 + 2V(c_{bo})^2 (s_{bo})^3 + V(c_{bo})^4 (s_{bo}) \right\} \ight.
\[ - K_{y_0} (n_{bo}/a_0)^2 (V(c_{bo})^2 (s_{bo}) + V(s_{bo})^3) \right\} \ight.
\[ + \left\{ 12(RS - VZ) (c_{bo})^3 (s_{bo})^2 + 10(S - R) (c_{bo})^2 (s_{bo})^3 \right\} \ight.
\[ + (R - S) (s_{bo})^5 + 4(VZ - RS) (c_{bo}) (s_{bo})^4 \right\} \ight.
\[ + 5(R - S) (c_{bo})^4 (s_{bo}) - 6(c_{bo}) (s_{bo})^4 + 8(c_{bo})^3 (s_{bo})^2 \right\}
\[ - 2(c_{bo})^5 + K_{y_0} (n_{bo}/a_0)^2 (4(RS - VZ) (c_{bo}) (s_{bo})^2 \right\} \ight.
\[ + (S - R) (s_{bo})^3 + 3(R - S) (c_{bo})^2 (s_{bo}) \right\} \ight.
\[ + 2(c_{bo}) (s_{bo})^2 - 2(c_{bo})^3 \} \right] \ight.
\[ - 2\sin(s_{bo}) \cos(s_{bo}) \left[ (R + S) (c_{bo}) (s_{bo})^4 \right. \ight.
\[ + 2(R + S) (c_{bo})^3 (s_{bo})^2 + (R + S) (c_{bo})^5 \right\} \ight.
\[ + K_{y_0} (n_{bo}/a_0)^2 \left[ (R + S) (c_{bo}) (s_{bo})^2 \right. \ight.
\[ + (R + S) (c_{bo})^3 \} \right] = 0 \] (349)
\[
\sin(sb_0) \left[ (Sc^2 - Ss^2 - 2cs)e^{cbo} + (Vc^2 - Vs^2)e^{-cbo} \right] + \cos(sb_0) \left[ (2Scs + c^2 - s^2)e^{cbo} - 2Vcse^{-cbo} \right] = 0
\]

\[
\sin(sb_0) \left[ (Sc^2 - 3Scs + s^2 - 3c^2s + K_y^0(n/a_o)^2 [Sc - s] )e^{cbo} + (3Vc^2 - Vc^3 - K_y^0(n/a_o)^2 Vc)e^{-cbo} \right] + \cos(sb_0) \left[ (3Sc^2s - Ss^3 + c^3 - 3cs^2 + K_y^0(n/a_o)^2 [Ss + c] )e^{cbo} + (3Vc^2s - Vs^3 + K_y^0(n/a_o)^2 Vs)e^{-cbo} \right] = 0
\]

Expansion of the determinant and multiplication by the quantity \(b_0^5\) results in the following equations (the second equation a simplification of the first):
\[ Y'(y_o) = B_m \{ \left[ (Sc - s)e^{cy_o} - Vce^{cy_o} \right] \sin(sy_o) \]
\[ + \left[ (Ss + c)e^{cy_o} + Vse^{cy_o} \right] \cos(sy_o) \} \]
\[ + D_m \{ [Zcse^{cy_o} - (Rc + s)e^{cy_o}] \sin(sy_o) \]
\[ + [Zse^{cy_o} + (Rs - c)e^{cy_o}] \cos(sy_o) \} \]  

(345)

The second derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[ Y''(y_o) = B_m \{ \left[ (Sc^2 - Ss^2 - 2cs)e^{cy_o} + (Vc^2 - Vs^2)e^{-cy_o} \right] \sin(sy_o) \]
\[ + [(2Scs + c^2 - s^2)e^{cy_o} - 2Vcse^{-cy_o}] \cos(sy_o) \} + D_m \{ [(Zc^2 - Zs^2)e^{cy_o} + (Rc^2 - Rs^2 - 2cs)e^{-cy_o}] \sin(sy_o) \]
\[ + [(2Zcse^{cy_o} + (c^2 - s^2 - 2Rcs)e^{cy_o} \cos(sy_o) \} \]  

(346)

Finally, the third derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[ Y'''(y_o) = B_m \{ \left[ (Sc^3 - 3Scs^2 + s^3 - 3cs^2)e^{cy_o} + (3Vcs^2 - Vc^3)e^{-cy_o} \right] \sin(sy_o) \]
\[ + [(3Sc^2s - Ss^3 + c^3 - 3cs^2)e^{cy_o} + (3Vc^2s - Vs^3)e^{-cy_o}] \cos(sy_o) \} \]
\[ + D_m \{ [(Zc^3 - 3Zcs^2)e^{cy_o} + (3Rcs^2 - Rc^3 + s^3 - 3c^2s)e^{-cy_o}] \sin(sy_o) \]
\[ + [(3Zc^2s - Zs^3)e^{cy_o} + (3Rc^2s - Rs^3 + 3cs^2 - c^3)e^{-cy_o}] \cos(sy_o) \} \]  

(347)

Substitution of equations (345), (346), and (347) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( B_m \) and \( D_m \). For non-trivial \( B_m \) and \( D_m \), the following determinental equation must hold:
\[
V = \left\{ -0.5 \frac{s^4}{c} - cs^2 - 0.5 c^3 + K_{y_0} \left(\frac{n}{a_o}\right)^2 \left[0.5 c + 0.5 \frac{s^2}{c}\right] \right\} / \left\{ 2s^3 - 6c^2s - 2 K_{y_0} \left(\frac{n}{a_o}\right)^2 s \right\} \tag{338}
\]

\[
R = \left\{ -0.5 \frac{s^4}{c} + 5cs^2 - 2.5 c^3 + K_{y_0} \left(\frac{n}{a_o}\right)^2 \left[-1.5 c + 0.5 \frac{s^2}{c}\right] \right\} / \left\{ 2s^3 - 6c^2s - 2 K_{y_0} \left(\frac{n}{a_o}\right)^2 s \right\} \tag{339}
\]

Therefore, equation (337) can be written more efficiently in terms of the variables \(V\) and \(R\).

\[
C_m = V B_m + R D_m \tag{340}
\]

In a similar fashion, define two additional variables of convenience:

\[
S = V - (c^2 - s^2)/(2cs) \tag{341}
\]

\[
Z = R - (c^2 - s^2)/(2cs) \tag{342}
\]

With knowledge of the variables \(V\) and \(R\), \(S\) and \(Z\) allow equation (335) to be rewritten in a concise form.

\[
A_m = S B_m + Z D_m \tag{343}
\]

Finally, the \(Y(y_0)\) function, equation (106), can be written in terms of the constants \(B_m\) and \(D_m\) only when equations (340) and (343) are substituted into equation (106).

\[
Y(y_0) = B_m \{ (Se^{cy_0} + Ve^{-cy_0}) \sin(sy_0) + e^{cy_0} \cos(sy_0) \} + D_m \{ (Ze^{cy_0} + Re^{-cy_0}) \sin(sy_0) + e^{-cy_0} \cos(sy_0) \} \tag{344}
\]

It is necessary to differentiate this \(Y(y_0)\) function again. Only two constants, as opposed to four, are now present in this function. The first derivative of \(Y(y_0)\) with respect to \(y_0\) is:
\[ y^{111}(y_0) = A_m e^{c y_0} \left\{ (c^3 - 3c^2 s) \sin(sy_0) \\
+ (3c^2 s - s^3) \cos(sy_0) \right\} \\
+ B_m e^{c y_0} \left\{ (s^3 - 3c^2 s) \sin(sy_0) \\
+ (c^3 - 3c^2 s) \cos(sy_0) \right\} \\
+ C_m e^{-c y_0} \left\{ (3c^2 - c^3) \sin(sy_0) \\
+ (3c^2 s - s^3) \cos(sy_0) \right\} \\
+ D_m e^{-c y_0} \left\{ (s^3 - 3c^2 s) \sin(sy_0) \\
+ (3c^2 s - c^3) \cos(sy_0) \right\} \]  \hspace{1cm} (333)

Apply equation (327) to equation (332). This stipulation fixes \( A_m \) in terms of \( B_m, C_m, \) and \( D_m \).
\[ A_m \{2cs\} + B_m \{ c^2 - s^2\} - C_m \{2cs\} + D_m \{ c^2 - s^2\} = 0 \hspace{1cm} (334) \]
\[ A_m = C_m - \left[ \left( \frac{c^2 - s^2}{2cs} \right) \right] \{ B_m + D_m \} \hspace{1cm} (335) \]

Next, apply equation (328) to the combination of equations (331) and (333). This condition will fix \( C_m \) (and hence \( A_m \)) by equation (335) in terms of \( B_m \) and \( D_m \).
\[ A_m \{3c^2 s - s^3 + K_{y_0} (n/ao)^2 s \} + B_m \{c^3 - 3c^2 s \}
+ K_{y_0} (n/ao)^2 c \} + C_m \{3c^2 s - s^3 + K_{y_0} (n/ao)^2 s \}
+ D_m \{3c^2 s - c^3 - K_{y_0} (n/ao)^2 c \} = 0 \hspace{1cm} (336) \]

When the relation for \( A_m \) expressed in equation (335) is inserted into equation (336), the following expression is returned:
\[ C_m = \left[ B_m \left\{ -0.5 \frac{s^4}{c} - cs^2 - 0.5 c^3 + K_{y_0} (n/ao)^2 (0.5 c \\
+ 0.5 \frac{s^2}{c}) \right\} \right] + D_m \left[ -0.5 \frac{s^4}{c} + 5cs^2 - 2.5 c^3 \\
+ K_{y_0} (n/ao)^2 (-1.5 c + 0.5 \frac{s^2}{c}) \right] \left[ \frac{2s^3}{2s^3}
- 6c^2 s - 2 K_{y_0} (n/ao)^2 s \right] \hspace{1cm} (337) \]

Define the following variables:
\[ Y'(b_0) \sin(mn x_0/a_0) = 0 ; \]

\[ [Y'''(b_0) + K_y (n/a_o)^2 Y'(b_0)] \sin(mn x_0/a_0) = 0 \]

For equations (325) and (326) to have meaning in the general case, the following conditions must hold:

\[ Y''(0) = 0 \] (327)

\[ Y'''(0) + K_y (n/a_o)^2 Y'(0) = 0 \] (328)

\[ Y''(b_0) = 0 \] (329)

\[ Y'''(b_0) + K_y (n/a_o)^2 Y'(b_0) = 0 \] (330)

Again, the function \( Y(y_o) \) is expressed in equation (106). The first derivative of this function with respect to \( y_o \) is:

\[ Y'(y_o) = A_m e^{c y_o} \{ s \cos(s y_o) + c \sin(s y_o) \} \]

\[ - B_m e^{c y_o} \{ s \sin(s y_o) - c \cos(s y_o) \} \]

\[ + C_m e^{-c y_o} \{ s \cos(s y_o) - c \sin(s y_o) \} \]

\[ - D_m e^{-c y_o} \{ s \sin(s y_o) + c \cos(s y_o) \} \] (331)

The second derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[ Y''(y_o) = A_m e^{c y_o} \{(c^2 - s^2) \sin(s y_o) + 2cs \cos(s y_o)\} \]

\[ + B_m e^{c y_o} \{(c^2 - s^2) \cos(s y_o) - 2cs \sin(s y_o)\} \]

\[ + C_m e^{-c y_o} \{(c^2 - s^2) \sin(s y_o) - 2cs \cos(s y_o)\} \]

\[ + D_m e^{-c y_o} \{(c^2 - s^2) \cos(s y_o) + 2cs \sin(s y_o)\} \] (332)

The third derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[ -156- \]
In the same pattern as in prior sections, a displacement function \( w \) which satisfies the first two stipulations of equation (324) is given by equation (77). Furthermore, substitution of this relation for \( w \) into the general buckling equation (14) produces equation (81) by arguments identical to those presented in section IV. The variable \( r \) is defined in equation (81) by equation (39). Equation (81) is now analyzed for the three possible algebraic states—negative, zero, and positive—for the quantity in the square brackets of equation (81). As explained in section IV, it is of the utmost importance to discuss possible solutions in precisely this order.

\[
\left( \frac{K_{y_o}}{2m^2} \right)^2 + K_{x_o} \left( \frac{a_o}{mb_o} \right)^2 - 1 < 0
\]

For the quantity \( \left( \frac{K_{y_o}}{2m^2} \right)^2 + K_{x_o} \left( \frac{a_o}{mb_o} \right)^2 - 1 < 0 \) equations (82) through (105) illustrate that the unknown function \( Y(y_o) \) must take the form shown in equation (106).

Consider now the boundary conditions, equations (324), for this case of a laminate simply supported on one set of opposite sides and free on the other. When the chosen form of \( w \), equation (77), is substituted into the final four lines of equations (324), the following must hold:

\[
Y''(0) \sin(mn_x/a_o) = 0 ;
\]

\[
\left[ Y'''(0) + K_{y_o} \left( \frac{a_o}{n/a_o} \right)^2 Y'(0) \right] \sin(mn_x/a_o) = 0
\]
VIII. **Flat Rectangular Composite Laminate Simply Supported in the \( x_0 \)-Direction and Free in the \( y_0 \)-Direction**

In contrast to the work presented in the last three sections, the boundary conditions for a laminate simply supported in the \( x_0 \)-direction and free in the \( y_0 \)-direction are symmetric with respect to each planar axis. For the edges which have normals parallel to the \( x_0 \)-axis, the vertical displacement along each edge and the normal component of the moment to each edge must be zero in the affine space. On the other hand, for the two edges which have normals parallel to the \( y_0 \)-axis, the vanishing of the normal component of the moment to each edge and the satisfaction of equation (224) constitute the two requisites. In equation form, the following must hold:

\[
\begin{align*}
\text{on edge } x_0 &= -a_0/2, \quad w = 0 \ ; \ w_{rx_0x_0} = 0 \\
\text{on edge } x_0 &= a_0/2, \quad w = 0 \ ; \ w_{rx_0x_0} = 0 \\
\text{on edge } y_0 &= 0, \quad w_{ry_0y_0} = 0 \\
\quad w_{ry_0y_0} + K_{y_0} (n/a_0)^2 w_{ry_0} &= 0 \\
\text{on edge } y_0 &= b_0, \quad w_{ry_0y_0} = 0 \\
\quad w_{ry_0y_0} + K_{y_0} (n/a_0)^2 w_{ry_0} &= 0
\end{align*}
\] (324)

Note that the origin of coordinates in the affine space is taken to be at the center of one of the free edges normal to the \( y_0 \)-direction. This choice of origin location, despite the symmetric boundary conditions, affords one maximum simplicity in numerical manipulations.
FIGURE 31
X0-BUCKLING COEFFICIENT VERSUS Y0-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 5.4 FOR AN S-S-S-F LAMINATE

FOR THE RANGE
KYO = -5.0 TO 3.0
Figure 30

Xo-buckling coefficient versus K10 for an S-S-F laminate with a constant aspect ratio of 3.0 for the range K10 = -5.0 to 3.0.
FIGURE 2A
SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-S-S-F LAMINATE
equation (81) simplifies to equation (120). Since the character of equation (120) differs drastically for the choice of algebraic sign of \( K_{y_0} \), each possible range of \( K_{y_0} \) --negative, zero, and positive--will be analyzed as different subcases.

\[ K_{y_0} < 0. \]

For \( K_{y_0} < 0 \) equations (121) and (122) lead to the conclusion that the unknown \( Y(y_0) \) function must take the form shown in equation (123). The first derivative of this function with respect to \( y_0 \) is:

\[
Y'(y_0) = A_m T \cosh(Ty_0) + B_m T \sinh(Ty_0) \\
+ C_m \{ \sinh(Ty_0) + Ty_0 \cosh(Ty_0) \} \\
+ D_m \{ \cosh(Ty_0) + Ty_0 \sinh(Ty_0) \} \quad (351)
\]

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = A_m T^2 \sinh(Ty_0) + B_m T^2 \cosh(Ty_0) \\
+ C_m \{ 2T \cosh(Ty_0) + T^2 y_0 \sinh(Ty_0) \} \\
+ D_m \{ 2T \sinh(Ty_0) + T^2 y_0 \cosh(Ty_0) \} \quad (352)
\]

The third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = A_m T^3 \cosh(Ty_0) + B_m T^3 \sinh(Ty_0) \\
+ C_m \{ 3T^2 \sinh(Ty_0) + T^3 y_0 \cosh(Ty_0) \} \\
+ D_m \{ 3T^2 \cosh(Ty_0) + T^3 y_0 \sinh(Ty_0) \} \quad (353)
\]

Apply equation (327) to equation (352). This combination fixes \( C_m \) in terms of \( B_m \) such that \( C_m = -(T/2)B_m \). Next, couple equations (351) and (353) so that they adhere to the boundary condition expressed in equation (328). \( D_m \) is therefore related to \( A_m \) in the following manner:

\[
D_m = -A_m \{ T^3 + K_{y_0} (n/a_0)^2 T \} / \{ 3T^2 + K_{y_0} (n/a_0)^2 \} \quad (354)
\]
Define the following variable:

\[
H = - \left\{ \frac{T^3 + K_y (\pi/a_o)^2 T}{3T^2 + K_y (\pi/a_o)^2} \right\} \quad (355)
\]

With the variable \(H\) so defined, the \(Y(y_o)\) function, equation (123), can be written in terms of the constants \(A_m\) and \(B_m\) only.

\[
Y(y_o) = A_m \{ \sinh(Ty_o) + H y_o \cosh(Ty_o) \} + B_m \{ \cosh(Ty_o) - (T/2) y_o \sinh(Ty_o) \} \quad (356)
\]

As explained in the previous paragraphs, it is now necessary to differentiate this slimmed down expression three times. The first derivative of \(Y(y_o)\) with respect to \(y_o\) is:

\[
Y'(y_o) = A_m \{ (T + H) \cosh(Ty_o) + HT y_o \sinh(Ty_o) \} + B_m \{ (T/2) \sinh(Ty_o) - (T^2/2) y_o \cosh(Ty_o) \} \quad (357)
\]

The second derivative of \(Y(y_o)\) with respect to \(y_o\) is:

\[
Y''(y_o) = A_m \{ (T^2 + 2HT) \sinh(Ty_o) + T^2 H y_o \cosh(Ty_o) \} - B_m \{ (T^3/2) y_o \sinh(Ty_o) \} \quad (358)
\]

Finally, the third derivative of \(Y(y_o)\) with respect to \(y_o\) is:

\[
Y'''(y_o) = A_m \{ (T^3 + 3T^2 H) \cosh(Ty_o) + T^3 H y_o \sinh(Ty_o) \} - B_m \{ (T^3/2) \sinh(Ty_o) + (T^4/2) y_o \cosh(Ty_o) \} \quad (359)
\]

Substitution of equations (357), (358), and (359) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \(A_m\) and \(B_m\). For non-trivial \(A_m\) and \(B_m\), the following determinental equation must hold:
\[
\begin{aligned}
\sinh(Tb) \{T^2 + 2TH\} &+ \cosh(Tb) \{-T^3b_0/2\} \\
\sinh(Tb) \{T^3Hb_0\} &+ K_{y_0}(n/a_0)^2HTb \\
+ \cosh(Tb) \{T^3 + 3T^2H\} &+ \cosh(Tb) \{-T^4b_0/2\} \\
+ K_{y_0}(n/a_0)^2(T + H)} &- K_{y_0}(n/a_0)^2(T^2b_0/2) \\
&= 0
\end{aligned}
\]

Expansion of the determinant and multiplication by the quantity \(b_0^5\) results in the following equations (the second a simplification of the first):

\[
\sinh^2(Tb) \left[ 0.5 (Hb_0)(Tb_0)^6 - 0.5 (Tb_0)^5 - (Hb_0)(Tb_0)^4 \\
+ K_{y_0}(n/a_0)^2 \{ 0.5 (Tb_0)^3 + (Hb_0)(Tb_0)^2 \\
+ 0.5 (Hb_0)(Tb_0)^4 \} \right] \\
- \cosh^2(Tb) \left[ 0.5 (Hb_0)(Tb_0)^6 \\
+ K_{y_0}(n/a_0)^2 \{ 0.5 (Hb_0)(Tb_0)^4 \} \right] = 0
\] (361)

\[
\sinh^2(Tb) \left[ - (Tb_0)^5 - 2(Hb_0)(Tb_0)^4 \\
+ K_{y_0}(n/a_0)^2 \{ (Tb_0)^3 + 2(Hb_0)(Tb_0)^2 \} \right] \\
- (Hb_0)(Tb_0)^6 - K_{y_0}(n/a_0)^2 \{ (Hb_0)(Tb_0)^4 \} = 0
\] (362)

No value of \((Tb_0)\) greater than zero can satisfy equation (362). Therefore, no possible solutions exist for the present boundary conditions for

\[\{(K_{y_0}/2m^2)^2 + K_x(a_0/mb_0)^2 - 1 \} = 0 \text{ and } K_{y_0} < 0 \]

\[K_{y_0} = 0.\]

For \(K_{y_0} = 0\) equation (128) constitutes the required shape of the unknown \(Y(y_0)\) function. In addition, equations (327) through (330) are the sets of constraints for this
Y(\(y_0\)) function. Moreover, notice that equations (328) and (330) reduce to \(Y'''(0) = 0\) and \(Y'''(b_0) = 0\) respectively because \(K_{y_0} = 0\) Equations (327) and (328) imply that \(C_m = D_m = 0\) After the imposition of these two conditions, equation (128) reduces to:

\[
Y(\(y_0\)) = A_m + B_m y_0
\]

Equations (329) and (330) are automatically satisfied for this \(Y(\(y_0\))\) given by equation (363). All boundary conditions, equations (327) through (330), are therefore upheld by this \(Y(\(y_0\)).\) As a result, if \(K_{y_0} = 0\) equation (298) is a valid solution for \(K_{y_0}\). However, equation (298) does not yield the minimum \(K_{y_0}\) for any combination of plate aspect ratio and \(K_{y_0} = 0\) Consequently, further consideration of this subcase is dropped.

\(K_{y_0} > 0.\)

For \(K_{y_0} > 0\) equations (130) and (131) show that the unknown function \(Y(\(y_0\))\) must fit the relation given by equation (132). The first derivative of this function with respect to \(y_0\) is:

\[
Y'(\(y_0\)) = A_m U \cos(Uy_0) - B_m U \sin(Uy_0)
+ C_m \{ \sin(Uy_0) + Uy_0 \cos(Uy_0) \} \\
+ D_m \{ \cos(Uy_0) - Uy_0 \sin(Uy_0) \}
\]

(364)

The second derivative of \(Y(\(y_0\))\) with respect to \(y_0\) is:

\[
Y''(\(y_0\)) = -A_m U^2 \sin(Uy_0) - B_m U^2 \cos(Uy_0)
+ C_m \{ 2U \cos(Uy_0) - U^2 y_0 \sin(Uy_0) \} \\
- D_m \{ 2U \sin(Uy_0) - U^2 y_0 \cos(Uy_0) \}
\]

(365)
The third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = - A_m U^3 \cos(Uy_0) + B_m U^3 \sin(Uy_0) \\
- C_m \{ 3U^2 \sin(Uy_0) + U^3 y_0 \cos(Uy_0) \} \\
- D_m \{ 3U^2 \cos(Uy_0) - U^3 y_0 \sin(Uy_0) \} \tag{366}
\]

Apply equation (327) to equation (365). This combination fixes \( C_m \) in terms of \( B_m \) such that \( C_m = (U/2) B_m \)

Next, couple equations (364) and (366) so that they adhere to the boundary condition expressed in equation (328). \( D_m \) is thus related to \( A_m \) in the following manner:

\[
D_m = A_m \{ U^3 - K_{y_0} (n/a_o)^2 U \} / \{-3U^2 + K_{y_0} (n/a_o)^2 \} \tag{367}
\]

Define the following variable:

\[
G = \{ U^3 - K_{y_0} (n/a_o)^2 U \} / \{-3U^2 + K_{y_0} (n/a_o)^2 \} \tag{368}
\]

With the variable \( G \) so defined, the \( Y(y_0) \) function, equation (132), can be written in terms of the constants \( A_m \) and \( B_m \) only.

\[
Y(y_0) = A_m \{ \sin(Uy_0) + G y_0 \cos(Uy_0) \} \\
+ B_m \{ \cos(Uy_0) + (U/2) y_0 \sin(Uy_0) \} \tag{369}
\]

Again, this function must be differentiated three times.

The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'(y_0) = A_m \{ (U + G) \cos(Uy_0) - UG y_0 \sin(Uy_0) \} \\
- B_m \{ (U/2) \sin(Uy_0) - (U^2 y_0/2) \cos(Uy_0) \} \tag{370}
\]

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = - A_m \{ (U^2 + 2UG) \sin(Uy_0) + U^2 G y_0 \cos(Uy_0) \} \\
- B_m \{ (U^3 y_0/2) \sin(Uy_0) \} \tag{371}
\]

Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:
\[ y'''(y_o) = A_m\left\{-\left(U^3 + 3Gy_o^2\right) \cos(Uy_o) + U^3 Gy_o \sin(Uy_o) \right\} \]
\[ - B_m\left\{ \left(U^3/2\right) \sin(Uy_o) + \left(U^4/2\right) y_o \cos(Uy_o) \right\} \quad (372) \]

Substitution of equations (370), (371), and (372) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinantal equation must hold:

\[
\begin{vmatrix}
\sin(Ub_o) \{-U^2 - 2UG\} & \sin(Ub_o) \{-U^3b_o/2\} \\
+ \cos(Ub_o) \{-U^2Gb_o\} & \sin(Ub_o) \{-U^3/2\} \\
- \sin(Ub_o) \{U^3Gb_o\} & - K_yo (n/a_o)^2 UGb_o \\
- K_yo (n/a_o)^2 UGb_o & - K_yo (n/a_o)^2 U/2 \\
+ \cos(Ub_o) \{-U^3 - 3U^2G\} & + \cos(Ub_o) \{-U^4b_o/2\} \\
+ K_yo (n/a_o)^2 (U + G) & + K_yo (n/a_o)^2 \{U^2b_o/2\}
\end{vmatrix}
= 0
\]

(373)

Expansion of the determinant and multiplication by the quantity \( b_o^5 \) results in the following system of equations (the second equation a simplification of the first):

\[
sin^2(Ub_o) \left\{ 0.5 \left(Gb_o\right)(Ub_o)^6 + 0.5 \left(Ub_o\right)^5 + \left(Gb_o\right)(Ub_o)^4 + K_yo (nbo/a_o)^2 \left[ 0.5 \left(Ub_o\right)^3 + \left(Gb_o\right)(Ub_o)^2 - 0.5 \left(Gb_o\right)(Ub_o)^4 \right] \right\} = 0 \quad (374)
\]

\[
sin^2(Ub_o) \left\{ (Ub_o)^5 + 2\left(Gb_o\right)(Ub_o)^4 + K_yo (nbo/a_o)^2 \left[ (Ub_o)^3 + 2\left(Gb_o\right)(Ub_o)^2 \right] \right\} + \left(Gb_o\right)(Ub_o)^6 - K_yo (nbo/a_o)^2 \left[ (Gb_o)(Ub_o)^4 \right] = 0 \quad (375)
\]
No value of \((Ubo)\) greater than zero can satisfy equation (375). Therefore, no possible solutions exist for the present boundary conditions for

\[\{(K_y/2m^2)^2 + K_x (a_o/mbo)^2 - 1 \} = 0 \quad \text{and} \quad K_y > 0\]

\[\{(K_y/2m^2)^2 + K_x (a_o/mbo)^2 - 1 \} > 0\]

For the quantity \(\{(K_y/2m^2)^2 + K_x (a_o/mbo)^2 - 1 \} > 0\) equation (81) reduces to equations (137) and (138). Each of these two equations determines two roots for the unknown \(Y(y_o)\) function. Peak interest centers on the positive or negative characters of those quantities contained in the curly brackets of equations (137) and (138), for these aspects imply not only different solution forms but different domains of \(K_y\) for valid solutions. Three subcases must be considered so that a solution for \(K_x\) may be determined for any range of \(K_y\).

\(K_y\) **Ranges from a Relatively Large Positive Number to Positive Infinity.**

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, \(K_y\) can take on any value from a comparatively large negative number to positive infinity. Similarly, if the quantity likewise bracketed in equation (138) cannot be positive, \(K_y\) can validly range from a relatively large positive number to positive infinity. The intersection of these two domains is then merely the last quoted domain. In equation form, the search for a solution is limited by the two inequalities
expressed in equations (139) and (140). Furthermore, equations (141) through (143) explicitly demonstrate that \( Y(y_0) \) must take the form shown in equation (144). Note also that equations (145) and (146) define the variables in equation (144).

The first derivative with respect to \( y_0 \) of this function \( Y(y_0) \) of equation (144) is:

\[
Y'(y_0) = A_m a_m \cos(a_m y_0) - B_m a_m \sin(a_m y_0) \\
+ C_m v_m \cos(v_m y_0) - D_m v_m \sin(v_m y_0)
\]  

(376)

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = -A_m a_m^2 \sin(a_m y_0) - B_m a_m^2 \cos(a_m y_0) \\
- C_m v_m^2 \sin(v_m y_0) - D_m v_m^2 \cos(v_m y_0)
\]  

(377)

The third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = -A_m a_m^3 \cos(a_m y_0) + B_m a_m^3 \sin(a_m y_0) \\
- C_m v_m^3 \cos(v_m y_0) + D_m v_m^3 \cos(v_m y_0)
\]  

(378)

Apply equation (327) to equation (377). This combination fixes \( D_m \) in terms of \( B_m \) such that \( D_m = -(a_m/v_m)^2 B_m \). Next, couple equations (376) and (378) so that they mirror the boundary condition expressed in equation (328). \( C_m \) is thus related to \( A_m \) in the following manner:

\[
C_m = A_m \{ a_m^3 - K_{y_0} (n/a_o)^2 a_m \} / \{ -v_m^3 + K_{y_0} (n/a_o)^2 v_m \}
\]  

(379)

Define the following variable:

\[
L = \{ a_m^3 - K_{y_0} (n/a_o)^2 a_m \} / \{ -v_m^3 + K_{y_0} (n/a_o)^2 v_m \}
\]  

(380)

With the variable \( L \) so defined, the \( Y(y_0) \) function, equation (144), can be written in terms of the constants \( A_m \) and \( B_m \) only.
Y(y₀) = Aₘ{ sin(αₘy₀) + L sin(γₘy₀) }
+ Bₘ{ cos(αₘy₀) - (αₘ/γₘ)² cos(γₘy₀) }  \hspace{1cm} (381)

For reasons detailed in prior work, equation (381) must be differentiated three times. The first derivative of Y(y₀) with respect to y₀ is:

Y'(y₀) = Aₘ{αₘ cos(αₘy₀) + Lγₘ cos(γₘy₀) }
- Bₘ{αₘ sin(αₘy₀) - (αₘ²/γₘ) sin(γₘy₀) }  \hspace{1cm} (382)

The second derivative of Y(y₀) with respect to y₀ is:

Y''(y₀) = -Aₘ{αₘ² sin(αₘy₀) + Lγₘ² sin(γₘy₀) }
- Bₘ{αₘ² cos(αₘy₀) - αₘγₘ sin(γₘy₀) }  \hspace{1cm} (383)

Finally, the third derivative of Y(y₀) with respect to y₀ is:

Y'''(y₀) = -Aₘ{αₘ³ cos(αₘy₀) + Lγₘ³ cos(γₘy₀) }
+ Bₘ{αₘ³ sin(αₘy₀) - αₘγₘ² sin(γₘy₀) }  \hspace{1cm} (384)

Substitution of equations (382), (383), and (384) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients Aₘ and Bₘ. For non-trivial Aₘ and Bₘ, the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(αₘb₀) \{-aₘ²\} & \cos(αₘb₀) \{-aₘ²\} \\
+ \sin(γₘb₀) \{-Lγₘ²\} & + \cos(γₘb₀) \{aₘ²\}
\end{vmatrix}
+ Kₙ \left(\pi/α₀\right)^{2} aₘ 
+ \cos(γₘb₀) \{-Lγₘ³\}
+ Kₙ \left(\pi/α₀\right)^{2} Lγₘ
- Kₙ \left(\pi/α₀\right)^{2} aₘ
\]

= 0

Expansion of the determinant, division by the common
multiple $a_m$, and multiplication by the quantity $v_m b_o$ gives:
\[
\sin(a_m b_o) \sin(v_m b_o) \left[ (a_m b_o)^3 (v_m b_o)^2 - L (a_m b_o)^2 (v_m b_o)^3 \right.
+ K_{y_o} (n b_o / a_o)^2 \{ L (v_m b_o)^3 - (a_m b_o)^3 \} \]
+ \cos(a_m b_o) \cos(v_m b_o) \left[ (a_m b_o)^4 (v_m b_o) - L (a_m b_o) (v_m b_o)^4 \right.
+ K_{y_o} (n b_o / a_o)^2 \{ L (a_m b_o) (v_m b_o)^2 - (a_m b_o)^2 (v_m b_o) \} \]
- (a_m b_o)^4 (v_m b_o) + L (a_m b_o) (v_m b_o)^4
+ K_{y_o} (n b_o / a_o)^2 \{ (a_m b_o)^2 (v_m b_o) - L (a_m b_o) (v_m b_o)^2 \} = 0 \quad (386)

Attempts at solution of equation (386) for any combination of $a_o / b_o$ and $K_{y_o}$ do not yield the smallest values of $K_{y_o}$. As a result, further considerations of this equation and this subcase are dropped.

$K_{y_o}$ Ranges from a Relatively Large Negative Number to a Relatively Large Positive Number.

If the quantity contained in the curly brackets of equation (137) is constrained to remain less than zero, $K_{y_o}$ can take on any value from a comparatively large negative number to positive infinity. On the other hand, if the quantity bracketed in equation (138) must be positive, $K_{y_o}$ can validly range from a relatively large positive number to negative infinity. The intersection of these two domains dictates that $K_{y_o}$ range from a relatively large negative number to a relatively large positive value. In equation form, the search for a solution is limited by the three inequalities expressed in equations (156), (157), and (158). Furthermore, equations (159), (160), and (161) reveal that $Y(y_o)$ must take the form shown in equation (162).

The first derivative with respect to $y_o$ of this function
\( Y(y_o) \) in equation (162) is:

\[
Y'(y_o) = A_m a_m \cos(a_m y_o) - B_m a_m \sin(a_m y_o) \\
+ C_m b_m \cosh(b_m y_o) + D_m b_m \sinh(b_m y_o) 
\]  
(387)

The second derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[
Y''(y_o) = -A_m a_m^2 \sin(a_m y_o) - B_m a_m^2 \cos(a_m y_o) \\
+ C_m b_m^2 \sinh(b_m y_o) + D_m b_m^2 \cosh(b_m y_o) 
\]  
(388)

The third derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[
Y'''(y_o) = -A_m a_m^3 \cos(a_m y_o) + B_m a_m^3 \sin(a_m y_o) \\
+ C_m b_m^3 \cosh(b_m y_o) + D_m b_m^3 \sinh(b_m y_o) 
\]  
(389)

Apply equation (327) to equation (388). This combination fixes \( D_m \) in terms of \( B_m \) such that \( D_m = \left(\frac{a_m}{b_m}\right)^2 B_m \)

Next, couple equations (387) and (389) so that they conform to the boundary condition expressed in equation (328). \( C_m \) is thus related to \( A_m \) in the following manner:

\[
C_m = A_m \left\{ \frac{a_m^3 - Ky_o (n/a_o)^2 a_m}{b_m^3 + Ky_o (n/a_o)^2 b_m} \right\}  
\]  
(390)

Define the following variable:

\[
J = \left\{ \frac{a_m^3 - Ky_0 (n/a_o)^2 a_m}{b_m^3 + Ky_0 (n/a_o)^2 b_m} \right\}  
\]  
(391)

With the variable \( J \) so defined, the \( Y(y_o) \) function, equation (162), can be written in terms of the constants \( A_m \) and \( B_m \) only.

\[
Y(y_o) = A_m \left\{ \sin(a_m y_o) + J \sinh(b_m y_o) \right\} \\
+ B_m \left\{ \cos(a_m y_o) + \left(a_m/b_m\right)^2 \cosh(b_m y_o) \right\} 
\]  
(392)

As before, equation (392) must be differentiated three times. The first derivative of \( Y(y_o) \) with respect to \( y_o \) is:

\[
Y'(y_o) = A_m \left\{ a_m \cos(a_m y_o) + J b_m \cosh(b_m y_o) \right\} \\
- B_m \left\{ a_m \sin(a_m y_o) - \left(a_m/b_m\right) \sinh(b_m y_o) \right\} 
\]  
(393)

The second derivative of \( Y(y_o) \) with respect to \( y_o \) is:
Finally, the third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = -A_m \left[ a_m^2 \sin(a_m y_0) - J \beta_m \sinh(\beta_m y_0) \right] - B_m \left[ a_m^2 \cos(a_m y_0) - a_m \beta_m \cosh(\beta_m y_0) \right]
\]

(394)

Substitution of equations (393), (394), and (395) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients \( A_m \) and \( B_m \). For non-trivial \( A_m \) and \( B_m \), the following determinental equation must hold:

\[
\begin{vmatrix}
\sin(a_m b_0) \left\{ -a_m^2 \right\} & \cos(a_m b_0) \left\{ -a_m^2 \right\} \\
+ \sinh(\beta_m b_0) \left\{ J \beta_m^2 \right\} & + \cosh(\beta_m b_0) \left\{ a_m^2 \beta_m \right\}
\end{vmatrix}
= 0
\]

(396)

Expansion of the determinant, division by the common multiple \( a_m \), and multiplication by the quantity \( b_0 \) gives:
\[
\sin(a_m b_o) \sinh(\beta_m b_o) \left\{ -(a_m b_o)^3 (\beta_m b_o)^2 + J(a_m b_o)^2 (\beta_m b_o)^3 \right\} \\
- K_{yo} \left( nb_o/a_o \right)^2 \left[ (a_m b_o)^3 + J(\beta_m b_o)^3 \right] \\
+ \cos(a_m b_o) \cosh(\beta_m b_o) \left\{ (a_m b_o)^4 (\beta_m b_o) + J(a_m b_o) (\beta_m b_o)^4 \right\} \\
+ K_{yo} \left( nb_o/a_o \right)^2 \left[ J(\beta_m b_o) (a_m b_o)^2 - (a_m b_o)^2 (\beta_m b_o) \right] \\
+ \left\{ -(a_m b_o)^4 (\beta_m b_o) - J(a_m b_o) (\beta_m b_o)^4 \right\} \\
+ K_{yo} \left( nb_o/a_o \right)^2 \left[ (a_m b_o)^2 (\beta_m b_o) \right. \\
\left. - J(a_m b_o) (\beta_m b_o)^2 \right] \right\} = 0 \quad (397)
\]

Equation (397) represents the governing equation for the combination of any plate aspect ratio and \( K_{yo} \) less than zero. In other words, the roots of equation (397) yield the smallest values of \( K_x \) for any \( a_o/b_o \) and tensile \( K_{yo} \). Consideration of results generated by equation (397) is postponed until the last of the three subcases is presented.

\( K_{yo} \) **Ranges from a Relatively Large Negative Number to Negative Infinity.**

If the quantity contained in the curly brackets of equation (137) is constrained to be greater than zero, \( K_{yo} \) can take on any value from a comparatively large negative number to negative infinity. Furthermore, this stipulation of positivism in equation (137) ensures that the bracketed quantity in equation (138) will be similarly greater than zero. In equation form, the search for a solution is limited by the two inequalities expressed in equations (170) and (171). Moreover, equations (172), (173), and (174) sequentially illustrate that \( Y(y_o) \) must take the form shown in equation (175). Note also that equations (163) and (176) define the variables in equation (175).
The first derivative with respect to \( y_0 \) of this function \( Y(y_0) \) in equation (175) is:

\[
Y'(y_0) = A_m \theta_m \cosh(\theta_m y_0) + B_m \theta_m \sinh(\theta_m y_0) \\
+ C_m \beta_m \cosh(\beta_m y_0) + D_m \beta_m \sinh(\beta_m y_0)
\]  
(398)

The second derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y''(y_0) = A_m \theta_m^2 \sinh(\theta_m y_0) + B_m \theta_m^2 \cosh(\theta_m y_0) \\
+ C_m \beta_m^2 \sinh(\beta_m y_0) + D_m \beta_m^2 \cosh(\beta_m y_0)
\]  
(399)

The third derivative of \( Y(y_0) \) with respect to \( y_0 \) is:

\[
Y'''(y_0) = A_m \theta_m^3 \cosh(\theta_m y_0) + B_m \theta_m^3 \sinh(\theta_m y_0) \\
+ C_m \beta_m^3 \cosh(\beta_m y_0) + D_m \beta_m^3 \sinh(\beta_m y_0)
\]  
(400)

Apply equation (327) to equation (399). This combination fixes \( D_m \) in terms of \( B_m \) such that

\[
D_m = -\left(\frac{\theta_m}{\beta_m}\right)^2 B_m
\]

Next, couple equations (398) and (400) in such a fashion so that they adhere to the boundary condition expressed in equation (328). \( C_m \) is thus related to \( A_m \) in the following manner:

\[
C_m = -A_m \left\{ \theta_m^3 + K_{y_0} (n/a_0)^2 \theta_m \right\} / \left\{ \beta_m^3 + K_{y_0} (n/a_0)^2 \beta_m \right\}
\]
(401)

Define the following variable:

\[
W = -\left\{ \theta_m^3 + K_{y_0} (n/a_0)^2 \theta_m \right\} / \left\{ \beta_m^3 + K_{y_0} (n/a_0)^2 \beta_m \right\}
\]
(402)

With the variable \( W \) so defined, the \( Y(y_0) \) function, equation (175), can be written in terms of the constants \( A_m \) and \( B_m \) only.

\[
Y(y_0) = A_m \left\{ \sinh(\theta_m y_0) + W \sinh(\beta_m y_0) \right\} \\
+ B_m \left\{ \cosh(\theta_m y_0) - \left(\frac{\theta_m}{\beta_m}\right)^2 \cosh(\beta_m y_0) \right\}
\]
(403)

Three derivatives must again be taken of this \( Y(y_0) \) function. The first derivative of \( Y(y_0) \) with respect to \( y_0 \) is:
The second derivative of $Y(y_0)$ with respect to $y_0$ is:

$$Y''(y_0) = A_m\{ e_m \cosh(e_m y_0) + W \beta_m \cosh(\beta_m y_0) \}$$

$$+ B_m\{ e_m \sinh(e_m y_0) - \frac{e_m^2}{\beta_m} \sinh(\beta_m y_0) \} \quad (404)$$

Finally, the third derivative of $Y(y_0)$ with respect to $y_0$ is:

$$Y'''(y_0) = A_m\{ e_m^3 \cosh(e_m y_0) + W \beta_m^3 \cosh(\beta_m y_0) \}$$

$$+ B_m\{ e_m^3 \sinh(e_m y_0) - e_m^2 \beta_m \sinh(\beta_m y_0) \} \quad (405)$$

Substitution of equations (404), (405), and (406) into equations (329) and (330), along with basic algebraic manipulation, yields two homogeneous linear equations in coefficients $A_m$ and $B_m$. For non-trivial $A_m$ and $B_m$, the following determinental equation must hold:

$$\begin{vmatrix}
\sinh(e_m b_0) \{e_m^2 \} & \cosh(e_m b_0) \{e_m^2 \} \\
+ \sinh(\beta_m b_0) \{W \beta_m^2 \} & + \cosh(\beta_m b_0) \{-e_m^2 \} \\
\end{vmatrix}
= 0$$

$$\begin{vmatrix}
\cosh(e_m b_0) \{e_m^3 \} & \sinh(e_m b_0) \{e_m^3 \} \\
+ K_{y_0} (n/a_0)^2 e_m \} & + K_{y_0} (n/a_0)^2 \beta_m \} \\
+ \cosh(\beta_m b_0) \{W \beta_m^3 \} & - \sinh(\beta_m b_0) \{e_m^2 \beta_m \} \\
+ K_{y_0} (n/a_0)^2 W \beta_m \} & + K_{y_0} (n/a_0)^2 \beta_m \beta_m \} \\
\end{vmatrix}
= 0 \quad (407)$$

Expansion of the determinant, division by the common multiple $e_m$, and multiplication by the quantity $\beta_m b_0^5$ gives:
\[
\sinh(e_m b_0) \sinh(e_m b_0) \left\{ -(e_m b_0)^3 (e_m b_0)^2 + W(e_m b_0)^2 (e_m b_0)^3 
+ K_{y_0} (n b_0/a_0)^2 \left[ W(e_m b_0)^3 - (e_m b_0)^3 \right] \right\} 
+ \cosh(e_m b_0) \cosh(e_m b_0) \left\{ -W(e_m b_0)^4 (e_m b_0) + (e_m b_0)^4 (e_m b_0) 
+ K_{y_0} (n b_0/a_0)^2 \left[ (e_m b_0)^2 (e_m b_0) - W(e_m b_0) (e_m b_0)^2 \right] \right\} 
+ \left\{ -(e_m b_0)^4 (e_m b_0) + W(e_m b_0) (e_m b_0)^4 
+ K_{y_0} (n b_0/a_0)^2 \left[ W(e_m b_0) (e_m b_0)^2 
- (e_m b_0)^2 (e_m b_0) \right] \right\} = 0 \quad (408)
\]

Attempts at solution of equation (408) for any combination of \(a_0/b_0\) and \(K_{y_0}\) do not yield the smallest values of \(K_{x_0}\). As a result, further considerations of this equation and this subcase as a whole are dropped.

**Discussion of Results**

Table XVII gives selected \(a_0/b_0\), \(K_{y_0}\), and \(K_{x_0}\) ordered triplets as determined by equation (350). Note that the results generated by equation (350) are given—and indeed are only valid—for positive, or compressive, and zero \(K_{y_0}\). These numbers and plots based upon this data reveal three very important aspects of the buckling characteristics of a laminate supported by this relatively weak set of boundary conditions. First, all values of minimum \(K_{x_0}\) for any combination of \(a_0/b_0\) and compressive or zero \(K_{y_0}\) are achieved when \(m = 1\) is utilized in the terms which compose equation (350). More broadly, \(K_{x_0}\) plotted versus \(a_0/b_0\) for any constant \(K_{y_0}\) greater than or equal to zero results in just one continuous curve. There exist no transition points to another curve.
Second, for \( K_{y_0} = 0 \) data points for a \( K_{x_0} \) versus \( a_o/b_o \) sketch lie just and virtually indistinguishably below the boundary curve \( \{(K_{y_0}/2m^2)^2 + K_{x_0} (a_o/m_b)^2 - 1 \} = 0 \). Since \( m = 1 \) in all cases for this uniaxial buckling (\( K_{y_0} = 0 \)), this boundary value of \( x_o \)-buckling coefficient is given by equation (298). Moreover, as just stated the boundary value quoted in equation (298) is a virtually exact approximation for \( K_{x_0} \) for the null value of \( y_o \)-buckling coefficient.

Third, curves of \( K_{x_0} \) versus \( a_o/b_o \) for constant compressive \( K_{y_0} \) lie well below the zero ordinate for small and intermediate affine aspect ratios. However, these curves continuously trend upward as \( a_o/b_o \) increases. For any constant \( K_{y_0} \), the \( K_{x_0} \) versus \( a_o/b_o \) plot asymptotically approaches a value of approximately \(-1.2 K_{y_0} \). For example, the following data reflects \( x_o \)-buckling coefficients (and corresponding \( y_o \)-buckling coefficients) for an affine aspect ratio of 100.0:

\[
\begin{array}{ccc}
   a_o/b_o & K_{y_0} & K_{x_0} \\
   100.0 & 0.5 & -0.6078 \\
   100.0 & 1.0 & -1.2158 \\
   100.0 & 1.5 & -1.8238 \\
   100.0 & 2.0 & -2.4318 \\
   100.0 & 2.5 & -3.0398 \\
   100.0 & 3.0 & -3.6479 \\
\end{array}
\]

Table XVIII gives selected \( a_o/b_o \), \( K_{y_0} \), and \( K_{x_0} \) ordered...
triplets as determined by equation (397). Note that the results generated by equation (397) are given and are only valid for negative \( K_{y_o} \). Also included is the integer value of \( m \) which produces this minimum \( K_{x_o} \). Furthermore, each entry point which corresponds to a transition point from the \( m \) curve to the \((m +1)\) curve is superscripted with a star (*). Especially be aware that discontinuous curves as opposed to the continuous curves discussed above highlight \( K_{x_o} \) versus \( a_o/b_o \) plots for tensile \( K_{y_o} \). The statistics presented in Table XVIII expose two key characteristics of laminates under tension in the \( y_o \)-direction. First, the transition values of \( a_o/b_o \) increase as \( K_{y_o} \) becomes algebraically larger (or less tensile). For this weak set of boundary conditions, however, transitions occur after longer intervals of aspect ratio than for those stronger groups discussed in prior sections.

Second, irrespective of the negative magnitude of \( K_{y_o} \), \( K_{x_o} \) attains a limiting value of zero as \( a_o/b_o \) approaches infinity. For this set of boundary conditions, the rate of approach toward this zero asymptote is decidedly slow. For example, for \( K_{y_o} = -3.0 \), \( K_{x_o} = 0.0382 \) at \( a_o/b_o = 100.0 \) and \( K_{x_o} = 0.0348 \) at \( a_o/b_o = 110.0 \) Note that this asymptotic value of zero differs from those asymptotes presented for compressive \( K_{y_o} \).

Figure 32 represents a plot of \( K_{x_o} \) versus \( a_o/b_o \) for twelve distinct values of \( K_{y_o} \). The lowest curve characterizes \( K_{y_o} = 3.0 \); whereas, the highest depicts
In ascending order the magnitudes of the $y_o$-buckling coefficients which correspond to the remaining ten curves are: 2.5, 2.0, 1.5, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0, and -4.0. This graph reinforces the concepts that $K_{xo}$ for a compressive or zero $K_{yo}$ is determined by one continuous curve and that $K_{xo}$ for a tensile $K_{yo}$ is rendered by the lowest values of an infinite number of intersecting curves. In addition, the merging of the family of curves to the zero asymptote for tensile or zero $K_{yo}$ and to distinct, relatively evenly spaced asymptotes for compressive $K_{yo}$ is readily apparent.

Figure 33 plots in three dimensions the same information as Figure 32. Qualitatively, this sketch expresses the nature of the buckling surface better than does Figure 32; however, the quantitative aspect is not as appealing. Computer-generated plots are skewed by the angle at which the "artist" draws the sketch. Consequently, extraction of accurate data from the three-dimensional plot is virtually impossible.

Table XIX gives selected coordinates of $K_{yo}$ and $K_{xo}$ for three distinct values of $a_o/b_o$ -- 1.2, 2.6, and 5.4. Figures 34, 35, and 36 represent two-dimensional plots at these constant $a_o/b_o$ slices of 1.2, 2.6, and 5.4, respectively. Because the data for each of the curves are derived for two completely different equations ( (350) for $K_{yo}$ positive or zero; (397) for $K_{yo}$ negative), a mild variance of slope in the vicinity of $K_{yo} = 0.0$ is observed for $a_o/b_o = 1.2$ and
slightly larger variances for $a_o/b_o = 2.6$ and $a_o/b_o = 5.4$. These breaks are certainly less severe than those observed in corresponding plots for a laminate simply supported on opposite sides and clamped and free on the remaining edges.
TABLE XVII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the \( x_0 \)-Direction and Free in the \( y_0 \)-Direction (for \( K_{y_0} \) greater than or equal to zero)

<table>
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<th>( a_0/b_0 )</th>
<th>( K_{y_0} )</th>
<th>( K_{x_0} )</th>
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TABLE XVIII

Buckling Coefficients Versus Plate Aspect Ratio for a Laminate Simply Supported in the $x_0$-Direction and Free in the $y_0$-Direction
(for $K_{y_0}$ less than zero)

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TABLE XIX

\( K_{x_0} \) Versus \( K_{y_0} \) for Various Plate Aspect Ratios for a Laminate Simply Supported in the \( x_0 \)-Direction and Free in the \( y_0 \)-Direction

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FIGURE 33
SURFACE REPRESENTING RELATION BETWEEN BUCKLING COEFFICIENTS AND AFFINE ASPECT RATIO
FOR AN S-F-S-F LAMINATE
Figure 34
XO-buckling coefficient versus YO-buckling coefficient at a constant affine aspect ratio of 1.2 for an S-F-S-F laminate

For the range
KYD = -5.0 to 3.0
FIGURE 35
XO-BUCKLING COEFFICIENT VERSUS YO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT RATIO OF 2.6 FOR AN S-F-S-F LAMINATE

FOR THE RANGE
KYO = -5.0 TO 3.0
FOR THE RANGE
$K_{10} = 0.0$ TO $3.0$

XO-BUCKLING COEFFICIENT VS.
XO-BUCKLING COEFFICIENT AT A CONSTANT AFFINE ASPECT
RATIO OF 5.4 FOR AN S-F-S-F LAMINATE

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IX. Conclusion

The plate buckling problem for a specially orthotropic, symmetric composite laminate depends rather weakly on the starred bending stiffness ratio $D^*$. This ratio of course lies within the range zero to one, and in fact proves to be quite invariant of the actual magnitude within these limits. As a result, the selection of the null value of $D^*$ both facilitates the solution process and returns excellent approximations of plate buckling behavior.

The seven sets of edge constraints examined blanket a wide range of support condition combinations. Any configuration for which at least one set of opposite edges is simply supported has been rigorously examined. Moreover, the clamped on all sides case (probably the most important) is similarly and rather thoroughly presented. It is hoped that this group of boundary conditions constitutes a set large enough so that they may be meaningfully employed in a design process.

Finally, the fact that buckling coefficients and aspect ratio are expressed in affine space allows sweeping generality. The buckling surfaces for each of the seven sets of boundary conditions can be utilized for any composite material. The coordinates need be scaled only by the first
two diagonal elements of the bending stiffness array. From this flexibility one can further infer that these buckling results hold for specially orthotropic, symmetric laminates of today and even for those yet to be devised.
Bibliography


VITA

James P. McFadden was born on 13 September 1960 in East Chicago, Indiana. He graduated from high school in Whiting, Indiana in 1978 and attended the University of Notre Dame from which he received a Bachelor of Science in Civil Engineering in May 1982. Upon graduation, he worked for Aeronautical Systems Division at Wright-Patterson AFB. He entered the School of Engineering, Air Force Institute of Technology, as a part-time student in July 1982.

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Title: BIAXIAL BUCKLING OF SPECIALLY ORTHOTROPIC, SYMMETRIC RECTANGULAR PLATES

Thesis Chairman: Dr. E. J. Brunelle
The biaxial plate buckling problem for specially orthotropic, symmetric laminates is transformed from Cartesian to doubly affine space. Setting the starred bending stiffness ratio $D^*$ (which ranges from zero to one) to the null value enables ready and very accurate solution of the buckling problem. Seven sets of boundary restraint configurations are examined, and corresponding buckling surfaces are presented. The character of these results vary widely between the strongest and weakest sets of support conditions. In order to prevent buckling for the weakest conditions, considerable tension must be provided on parallel edges for just small amounts of compression applied on the opposite set of edges.
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