ALTERNATIVE MODELS FOR CALCULATION OF ELEVATION ANGLES AND RAY TRANSIT TI... (U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA C D MAIN SEP 84
THESIS

ALTERNATIVE MODELS FOR CALCULATION OF ELEVATION ANGLES AND RAY TRANSIT TIMES FOR RAY TRACING OF HYDROPHONIC TRACKING DATA

by

Carl D. Mein

September 1994

Thesis Advisor: P. P. Post

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<td>This thesis treats the problem of calculating the elevation angle needed to initiate an acoustic signal. The algorithm used to locate the source of a signal is presented. At regularly spaced time intervals, a synchronized sound signal is transmitted. A hydrophone array containing four hydrophones is placed at the four corners of a large, flat area. The data obtained from these hydrophones are input into the algorithm.</td>
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are determined from the arrival times at the four hydrophones. Current methods for using such time data to produce an apparent position suitable for ray tracing are reviewed. Then four new methods are developed and documented mathematically. All methods are compared under a simulated environment of a sound speed profile which is linear with depth. One of the new methods is judged to be an improvement over current methods in this idealized environment. Finally the improved method is used to estimate the variability in the time data from a real hydrophonic tracking problem.
Alternative Models for Calculation of Elevation Angles and Bay Transit Times for Ray Tracing of Hydrophonic Tracking Data

by

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ABSTRACT

This thesis treats the problem of determining the elevation angle needed to initialize an underwater sound ray tracing algorithm used to locate the position of a target vehicle. At regularly spaced time intervals the vehicle pings a synchronized sound signal which is received by a (short base line) sonar array containing four hydrophones positioned at four of the corners of a cube. The wavefront direction angles are determined from the arrival times at the four hydrophones.

Current methods for using such time data to produce an apparent position suitable for ray tracing are reviewed. Then four new methods are developed and documented mathematically. All methods are compared under a simulated environment of a sound speed profile which is linear with depth. One of the new methods is judged to be an improvement over current methods in this idealized environment. Finally the improved method is used to estimate the variability in the time data from a real hydrophonic tracking problem.
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I. INTRODUCTION AND BACKGROUND

A. BACKGROUND

Suppose that an underwater acoustical source emits a signal at a known time. Suppose that the times of arrival of that signal at the hydrophones on a three-dimensional sensing array are also known. Finally suppose that the speed of sound in water is modeled as being homogeneous over time and horizontal displacement, varying only with depth. Then the angle of elevation \( A \), and the time \( T \) of the arrival of the signal at the acoustic center of the hydrophone array can be determined. If the relationship between the speed of sound and the depth underwater is known exactly, then the angle \( A \) and the time \( T \) can be used to trace the signal trajectory back over its ray path (called ray tracing) to determine the original position of the acoustical source [Ref. 1]. However, full realization of this method in actual hydrophonic tracking is prevented by two primary sources of inaccuracies in the process. The first and probably greatest problem is that the speed of sound profile can be approximated at only a few locations, usually at great cost to achieve even moderate accuracy. In addition the profile is certain to fluctuate over time and location. The second problem, confounding the first, is that there may be inaccuracies, of unknown size, in the time data values recorded by the sensing array.

B. PURPOSE

As noted in [Ref. 1] the procedure of determining a sound source position by ray tracing is very sensitive to even small errors in the angle of elevation or speed of
sound at the sensing array. Of course ray tracing is also highly dependent on the accurate determination of the transit time from source to array. The purpose of this study is to develop an appropriate model of the timing data observed in the hydrophonic tracking problem. The objective of the desired model is to estimate the error in the time values, and produce improved estimates of the initial angle and ray transit time, so as to reduce the effect of these errors on the ray tracing procedure.

In pursuit of these objectives this thesis first reviews the currently used models, which are called the NAVY and NAVY_A methods. Then four alternative models are developed, called the L.S., L.S.C., M.L.P. and M.L.S. methods. Finally the performance of all six models are evaluated and compared using simulation studies. The comparisons are made under the idealized conditions of a known linear speed of sound versus depth relationship.

C. TRACKING RANGE CONFIGURATION

The tracking range which supplied data for this study consists of several separate three dimensional hydrophone arrays sitting on the sea bottom. They are laid out in a rough grid, each array being approximately 750 feet from the next, so that the sound source being tracked is never more than about 5000 feet from the nearest array. The arrays are at depths of roughly 1000 to 1300 feet.

Each array (see figure 1.1) has four independent hydrophones defining an orthogonal coordinate system. The phones are referred to as the x, y, z, and c hydrophones, and are on four adjacent vertices of a cube (see figure 1.2 with sides of length D (usually 30 feet). The arrays are linked to shore based computers by electronic cable. The origin of the local coordinate system of each array is at the acoustic
center of the array, which is the center of the cube defined by the four hydrophones. Therefore the hydrophones are in the positions

\[
\begin{align*}
x & : \quad (D, -D, -D) / 2 \\
y & : \quad (-D, D, -D) / 2 \\
z & : \quad (-D, -D, D) / 2 \\
c & : \quad (-D, -D, -D) / 2
\end{align*}
\]

in terms of the array's local coordinate system.
Figure 1.2  Geometry of Hydrophonic Arrays.

The sound source is equipped with a clock synchronized with the shore based computers and emits a signal at specified intervals. The times of receipt of those signals at the four hydrophones are recorded, and the corresponding travel times are calculated by subtraction of the signal emission time.

D. APPARENT POSITIONS

The first step in estimating the position of a sound source is to do so under the assumption that the sound wave travelled its entire trajectory through water which had a constant speed of sound. The result, called the apparent position, is obviously erroneous, but is then corrected by
the ray tracing procedure (a description of which follows later). The constant velocity value used is usually the estimated speed of sound \( V \) at the depth of the sensing array.

The constant velocity assumption implies that the wavefront is an expanding sphere. Therefore the apparent position is calculated using simple spherical equations involving squared distance calculations. Specifically, if \( T_x \) is the travel time recorded for the \( x \)-phone, then \( V \cdot T_x \) is the distance between the apparent position and the \( x \)-phone. This distance is also equal to the usual geometric distance between the two positions

\[
( X , Y , Z ) \quad \text{(apparent position)}
\]

and \( ( D , -D , -D ) / 2 \) (\( x \) hydrophone position).

Equating these two squared distances, and equating their counterparts for the other three hydrophones, the equations (1.1) are obtained.

\[
\begin{align*}
( X - D/2 )^2 + ( Y + D/2 )^2 + ( Z + D/2 )^2 &= V_x^2 T_x^2 \\
( X + D/2 )^2 + ( Y - D/2 )^2 + ( Z + D/2 )^2 &= V_y^2 T_y^2 \\
( X + D/2 )^2 + ( Y + D/2 )^2 + ( Z - D/2 )^2 &= V_z^2 T_z^2 \\
( X + D/2 )^2 + ( Y + D/2 )^2 + ( Z + D/2 )^2 &= V_c^2 T_c^2
\end{align*}
\]

Assuming that the times \( T_x, T_y, T_z, \) and \( T_c \) are known, (1.1) is a system of four equations in three unknowns \( X, Y, \) and \( Z \). This overdetermined system will, in general, have no exact solution. In fact, even if the time values were exactly correct, the system would still have no exact solution. This is because the equations correspond to the straight line ray paths due to the constant velocity assumption, whereas the time values correspond to the true ray paths which are not straight due to the actual variation of velocity of sound along the ray path. This is a subtle, but very important point.
To throw out one of the equations arbitrarily, so as to reduce the system to three equations in three unknowns, is to throw away information from one of the hydrophones. The pseudo solution currently utilized is to subtract the fourth equation from each of the first three, yielding a system of three equations in three unknowns which allows an exact solution involving information from all four hydrophones. However that solution will not, in general, satisfy any of the original four equations, and is only one of many reasonable ways to choose an approximate solution.

E. INITIAL ELEVATION ANGLE AND RAY TRANSIT TIME

Assuming that a solution \((X_a, Y_a, Z_a)\) has been determined for the apparent position, then the initial angle of elevation is just the angle of elevation of that solution, given by (1.2).

\[ \alpha = \arcsin \left( \frac{Z_a}{\sqrt{X_a^2 + Y_a^2 + Z_a^2}} \right) \quad (1.2) \]

The objective is to find an apparent position \((X_a, Y_a, Z_a)\) such that (1.2) computes an angle which approximates the physically correct elevation angle as closely as possible. The solution \((X_a, Y_a, Z_a)\) and the times \(T_x, T_y, T_z\) and \(T_c\) can then be used to determine an appropriate value for \(T\), which is the 'ray transit time', or time of arrival of the sound wave at the acoustic center of the sensing array. The currently employed method uses the proportional relationship of equation (1.3), where \(R\) and \(R_c\) are the ranges from the apparent position to the acoustic center and to the c hydrophone respectively, as in (1.4).
\[ \frac{R_s}{T_c} = V_1 = \frac{R}{T} \left( \begin{array}{l}
\text{or } T = \frac{T_c R}{R_c}
\end{array} \right) \quad (1.3) \]

\[ R = \sqrt{V_a^2 + V_a^2 + Z_a^2} \quad (1.4) \]

\[ P_c = \sqrt{(X_a + D/2)^2 + (Y_a + D/2)^2 + (Z_a + D/2)^2} \]

**F. RAY TRACING**

Whichever method is selected to produce the apparent position \((X_a, Y_a, Z_a)\), it is transformed into the estimate of the true sound source position \((X, Y, Z)\) by the procedure of ray tracing. When there is velocity layering in the water, the ray path is no longer a straight line. This is treated using repeated applications of Snell's Law [Ref. 2 p. 131], starting with the layer of water in which the array sits, and backtracking upwards through successive velocity layers, until the estimated ray transit time \(T\) is consumed.

The layering effect is artificially induced by the limitation that the speed of sound can be estimated at only a finite number of depths, the result of which is commonly called the water column. For example, at the tracking range studied the speed of sound is measured every 25 feet, starting at the depth of 12.5 feet. Hence, for example, the third layer from 50 to 75 feet deep is assumed to have a constant speed of sound equal to that measured at 62.5 feet.

The first layer processed [Ref. 3 p. 4] is the partial layer lying between the array and the deepest layer boundary that is shallower than the array, with thickness \(Z_1\) (see figure 1.3).

The incremental slant range in the first layer is \(S_1\) given by \((1.5)\), where \(A1\) is the initial elevation angle estimate.
The incremental travel time in the first layer is $T_1$ given by (1.6), where $V_1$ is the velocity estimated for the layer in which the array is situated.

$$S_1 = \frac{Z_1}{\sin (A_1)} \quad (1.5)$$
The incremental horizontal distance travelled by the ray in the first layer is \( H_1 \) given by (1.7).

\[
H_1 = S_1 \cos(A_1) \quad (1.7)
\]

To determine the angle of elevation in the next layer, Snell's law (1.8) is applied, where \( V_2 \) is the speed of sound estimated for that layer.

\[
\frac{\cos(A_1)}{V_1} = \frac{\cos(A_2)}{V_2} \quad (1.8)
\]

When (1.8) is solved for the cosine of the angle of entry into the next layer, (1.9) is obtained.

\[
\cos(A_2) = \frac{V_2 \cos(A_1)}{V_1} \quad (1.9)
\]

The procedure of computing the incremental values of slant range, time and horizontal distance are now repeated for the second layer. The overall procedure is repeated upwards through layers 2, ..., \( n \), where \( n \) is the first layer in which the sum of the incremental travel times exceeds the total ray transit time, as in (1.10).

\[
(T_1 + T_2 + \cdots + T_n) > T \quad (1.10)
\]

In the last, uppermost layer the values \( T_n, S_n, H_n, \) and \( Z_n \) must be adjusted to compensate for overshooting the total time \( T \). The values \( H_i \) and \( Z_i \) (\( i=1, \ldots, n \)) are then accumulated to get (1.11).

\[
H = \sum_{i=1}^{n} H_i \quad Z = \sum_{i=1}^{n} Z_i \quad (1.11)
\]

Now the raytraced position estimate is given by (1.12).
The sensing array is usually not aligned with the coordinate system of the overall tracking range. Therefore the apparent position, which is in terms of the local array coordinates, must be changed by a suitable geometric transformation prior to ray tracing so as to account for the angle of tilt at which the array sits on the sea bottom. After ray tracing, the position estimate must be again transformed to account for rotation of the array about its Z axis away from a position which is aligned with the range coordinate axes. Finally a simple translation must be applied to reference the position estimate to the range coordinate system origin vice the acoustic center of the array. The end result is a position in terms of the overall range coordinate system. These transformations are not given here because they are used after the estimation of the initial angle and time, and hence do not affect the accuracy of those estimates. See [Ref. 3] for further details.

G. DISCUSSION

If the velocity versus depth relationship is smooth and estimated accurately, then the ray tracing procedure is surprisingly robust with respect to the thickness of the layers. For example, if velocity is linear versus depth, and is known exactly, then the exact hydrophone times, ray transit time and initial angles can be computed [Ref. 4] for any given sound source position. Then the ray tracing procedure, with layers as thick as 25 feet and targets as far away as 3000 feet, still estimates positions to within inches of each of the true coordinate values. This seems to indicate that errors resulting from position estimation are
not due to the approximation by layers, except possibly when there are radical changes in the velocity pattern within single layers such as frequently occur in layers near the water surface. Rather such errors apparently are due mostly to inaccuracies either in the estimation of the speed of sound profile itself, or in the initial angle and transit time estimates. This study will focus on those errors which are involved in the time values observed at the hydrophones, and attempt to produce methods of initial angle and transit time estimation which reduce the effects of those errors on the overall position estimation procedure.
II. CURRENTLY EMPLOYED METHODS

A. BASIC CONCEPTS

The method currently used for estimation of an initial angle and ray path transit time focuses on the four spherical equations (1.1) given in Chapter I. As previously discussed, these equations have no exact solution because:

1. they form an overdetermined system of four equations in three unknowns;
2. the time values recorded at the hydrophones may be inaccurate due to unknown sources of error in the observation process; and
3. even if the time values were exact, they correspond to a nonconstant velocity profile, and so will not be correct for the constant velocity geometry (spherical wavefront) assumed by the equations.

As previously mentioned, the pseudo solution chosen by the current method is to subtract the fourth spherical equation from each of the first three, and solve the resulting system of three equations in three unknowns. This method has the beneficial quality that information is retained from all four hydrophones, whereas to just drop the fourth equation (or any one of the equations) without the initial subtraction would cause complete loss of the information from the data recorded by one of the hydrophones. However, it is important to realize that the solution thus obtained does not actually satisfy any of the original four equations.

It should be noted that the development of this solution in [Ref. 3] is done entirely from a geometrical point of view, and does not mention the system of four spherical
equations. The text of [Ref. 3] does not draw attention to the fact that the solution developed is just one of many plausible choices, none of which will satisfy all four spherical constraints. Therefore the solution chosen is treated as though it were the exact solution, only subject to errors in the observed hydrophone time values. However, even with exactly correct time values, this currently employed method will not yield the true elevation angle and ray transit time. This is due to the assumption of a constant velocity vice the true nonconstant velocity profile. This conflict introduces an automatic bias in the initial angle and time estimates currently used for ray tracing.

B. COMPUTATIONS

To simplify notation, let \((X, Y, Z)\) be the coordinates of the apparent position which was formerly denoted \((X_a, Y_a, Z_a)\). Then when the current solution is applied, the first step is to subtract the fourth equation from each of the other three, which produces the equations (2.1).

\[
\begin{align*}
(X - D/2)^2 - (X + D/2)^2 &= v^2 (\tau_c^2 - \tau_x^2) \\
(Y - D/2)^2 - (Y + D/2)^2 &= v^2 (\tau_c^2 - \tau_y^2) \\
(Z - D/2)^2 - (Z + D/2)^2 &= v^2 (\tau_c^2 - \tau_z^2)
\end{align*}
\]

The solution to these are easily obtained, as in (2.2).

\[
\begin{align*}
x &= v^2 (\tau_c^2 - \tau_x^2) / 2 \, D \\
y &= v^2 (\tau_c^2 - \tau_y^2) / 2 \, D \\
z &= v^2 (\tau_c^2 - \tau_z^2) / 2 \, D
\end{align*}
\]

Then the initial elevation angle estimate is (2.3).

\[
A = \arcsin \left( \sqrt{x^2 + y^2 + z^2} \right)
\]
The ray path transit time to the acoustic center is (2.4), where \( R \) and \( R_c \) are as defined in Chapter I by (1.4).

\[
T = T_c R / R_c
\]  

(2.4)

This method of determining the apparent position shall hereafter be referred to as the 'Navy Method', or 'NAVY' for short.

C. ADJUSTMENTS TO THE ORIGINAL SOLUTION

Experience has shown that the NAVY method produces an apparent position estimate which usually can be improved by an adjustment which is described in this section.

The cosine of the angle between the \( i \)-th axis and the straight line from the origin out to the apparent position is called the \( i \)-th direction cosine \( C_i \). It is a fact of geometry that the sum of the squares of the three direction cosines must equal unity. Therefore the method is now adjusted to reflect that constraint.

The direction cosines used are the angles made by the ray path at the \( c \) hydrophone. Therefore the \((X,Y,Z)\) coordinates calculated by the original method are first translated to coordinates referenced to the \( c \)-phone as the temporary origin, as in (2.5).

\[
X_c = X + D/2 \quad Y_c = Y + D/2 \quad Z_c = Z + D/2
\]  

(2.5)

Therefore the three direction cosines are given by (2.6).

\[
C_x = X_c / V T_c \quad C_y = Y_c / V T_c \quad C_z = Z_c / V T_c
\]  

(2.6)

The denominators in (2.6) are all \( V \cdot T_c \) because that is the range from the apparent position to the \( c \) hydrophone, as estimated by the time from the \( c \) hydrophone. Ideally these
cosines should add to unity when squared. Therefore if DCC is the 'direction cosines correction' factor defined by (2.7), then DCC should be close to one.

\[
DCC = \sqrt{C_x^2 + C_y^2 + C_z^2}
\]  

(2.7)

Deviation of DCC from unity is interpreted as an indication of receiver timing errors, array malformation or invalid data at one or more of the hydrophones [Ref. 3 p.C-3]. Currently if DCC lies outside the interval (0.98, 1.02), the data is discarded as being excessively full of error. The direction cosines of the remaining acceptable data points are rescaled using (2.8) to assure satisfaction of the direction cosines constraint.

\[
C'_x = \frac{C_x}{DCC} \quad C'_y = \frac{C_y}{DCC} \quad C'_z = \frac{C_z}{DCC}
\]  

(2.8)

A corrected set of new coordinates are computed by (2.9), still being referenced to the c hydrophone.

\[
X_c = V_T c_x C'_x \quad Y_c = V_T c_y C'_y \quad Z_c = V_T c_z C'_z
\]  

(2.9)

These are then translated by (2.10) to coordinates referenced to the acoustic center.

\[
x = X_c - D/2 \quad y = Y_c - D/2 \quad z = Z_c - D/2
\]  

(2.10)

This adjusted method of determining the apparent position shall hereafter be referred to as the 'Navy Adjusted Method', or 'NAVY_A' for short.

D. DISCUSSION

When a sound source is within the detection range of more than one sensing array, each array produces timing data. The data from each array can be processed to produce
a position estimate. Ideally these multiple estimates of position will be in reasonably close agreement. However experience with actual tracking data has shown that this is not the case in many of the multiple detection opportunities. This tendency toward disagreement between multiple estimates of the same position is commonly called the crossover, or crosstalk, problem. This problem often occurs when the sound source is moving away from the tracking domain of one array into the tracking domain of another. This study focuses on improvement of the initial angle and time estimates, which hopefully will help alleviate the crossover problem.

The current choice of a 'best' compromise solution appears to be based on reasons of simplifying geometry and calculations. These are worthwhile objectives, but do not in themselves reflect the need to estimate accurately the initial elevation angle and ray transit time. Since there exist physically correct values for both the angle and the time, those values will produce the exact position after ray tracing, provided that the velocity profile is known exactly. The desire then is to estimate these true values as accurately as possible.

The question at this point is whether or not the direction cosines adjustment causes the estimated apparent position to be closer to the true apparent position. Experience has indicated that it does [Ref. 3 p.C-7]. However the effect of the adjustment can be interpreted in terms of the original four spherical equations (1.1). The rescaling of the direction cosines given by (2.6), so as to assure that their squares add to unity, is equivalent to rescaling the quantities in (2.5) so as to assure that their squares add to \((V \cdot Tc)^2\). That is exactly the constraint stated by the fourth spherical equation of (1.1). So the effect of the adjustment is to require that the fourth equation
concerning the data at hydrophone c, is exactly satisfied. This requirement will, in general, assure that the other three equations are not satisfied. Since experience shows that the adjustment often improves the solution, this seems to imply that the fourth equation is somehow more important than the other three. Or it may just be that exact satisfaction of one of the equations usually assures a reasonably good compromise solution.

To summarize, the NAVY method provides a useful apparent position suitable as input for ray tracing. But the direction cosines adjustment used in the NAVY_A method, for reasons not understood at this time, appears to improve that position as indicated by test results. Those results are supported by the results of this thesis (see Chapter V). However, as will be demonstrated by the example considered in the next section, the DSC correction factor of the NAVY_A method has an effect which must be something more than just the smoothing of timing errors.

E. A COMPUTATIONAL EXAMPLE

For the purposes of illustration and comparison, suppose that a 30 foot sensing array is at a depth of 1300 feet, that the coordinate system origin is at the array acoustic center, and that the array arms are parallel to the coordinate axes. This implies that the four hydrophones are in the positions:

\[ x : (15, -15, -15) \]
\[ y : (-15, 15, -15) \]
\[ z : (-15, -15, 15) \]
\[ c : (-15, -15, -15) \]

If there is a sound source known to be in position

\( (1000, 3000, 900) \)

then the depth of that source is \(1300 - 900 = 40\) feet.
Figure 2.1 Sample Depth Versus Velocity Profile.
Figure 2.1 shows the estimated sound velocity profile which was estimated for the data used in the course of this thesis. As can be seen, the profile is primarily linear at depths greater than 100 feet. The profile at depths greater than 100 feet is reasonably approximated by the linear relationship

\[ V = 4840.7 + 0.03314 \cdot \text{DEPTH}. \]

Therefore suppose that in this example problem the velocity profile is known exactly, and is given by the above linear relationship.

Under these circumstances, with known linear velocity profile, and known sound source location, the exact times of arrival of the sound wave at the four hydrophones can be computed using the methods set forth in Appendix A. Those exact times (in seconds) are:

\[
\begin{align*}
T_x : & \quad 0.6779686893 \\
T_y : & \quad 0.6742257788 \\
T_z : & \quad 0.6782243197 \\
T_c : & \quad 0.6798324156.
\end{align*}
\]

The corresponding exact values for the initial elevation angle, ray transit time and resulting apparent position are also directly computable. Those computations will hereafter be known as the EXACT method, and will produce the correct true position after ray tracing. The results of the EXACT method are given in Table I, along with the corresponding apparent position estimates produced when the two methods, NAVY and NAVYA, are applied to the (errorless) time values.

At first glance the differences in Table I might seem rather small. However it is important to recall that these are produced under the ideal conditions of a very smooth and exactly known velocity profile. These idealizations are far from the realities of a nonlinear velocity profile which is estimated by a procedure involving errors which are unknown and probably significant. Such realities might well cause the differences in Table I to be significantly larger. The
TABLE I
Single Example Comparison of NAVY and NAVY_A Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Transit Time (secs)</th>
<th>Elev. Angle A (degs)</th>
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<tr>
<td>NAVY_A</td>
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<td>15.26461</td>
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<tr>
<td>EXACT</td>
<td>0.67527043</td>
<td>15.27002</td>
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<table>
<thead>
<tr>
<th>Apparent Position Estimate</th>
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</thead>
<tbody>
<tr>
<td>NAVY</td>
</tr>
<tr>
<td>NAVY_A</td>
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<tr>
<td>EXACT</td>
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</table>

nature and size of those differences remain difficult to determine until more is known about the the velocity profile estimation errors and their effect on the position estimating process. In any event even the small differences in Table I might be magnified during the ray tracing process under the conditions of a realistic velocity profile.

The differences illustrate the very important point that the direction cosines adjustment causes changes in the estimates even when the time data is free of all error. Hence the deviation from 1.0 of the correction factor DCC is not just due to array malfunction, receiver timing error or other sources of non-valid data as previously assumed.

In the example above, the NAVY_A method produced a slightly better time and angle value than the NAVY method. However, in this same example the NAVY_A method produced an apparent position estimate which is slightly farther away from the EXACT answer than the position estimated by the
NAVI method. This is only one example, and its results should not be generalized. However, it illustrates the point that apparently the true effect of the adjustment may not be well understood.

Furthermore, the adjustment seems to place heavy emphasis on the time value recorded at microphone c. Therefore the effect of the adjustment may well depend largely on the accuracy of that one particular data value, which is a relatively unbalanced dependence in the presence of data errors.
III. PLANAR WAVEFRONT MODELS

A. THE PLANAR WAVEFRONT ASSUMPTION

When a sound wave travels in constant velocity water, it expands in the shape of a sphere. If the velocity profile is variable instead, but is reasonably well behaved versus depth, then the expanding wave is a smooth distortion of a spherical surface. In either case, if the wave has travelled a long distance when it arrives at a hydrophonic array, then that small piece of the wavefront which passes through the 30 foot cube spanned by the array may be approximated reasonably by a flat planar surface. This approximation is the basis of the planar wavefront models developed in this chapter.

B. PLANE EQUATIONS

A plane in space is fully defined by identifying any point \((X_0, Y_0, Z_0)\) on the plane, and also a vector \((C_1, C_2, C_3)\) of unit length which is perpendicular to that plane. The vector is called the unit normal vector for that plane. Then any point \((X, Y, Z)\) on that plane must satisfy the equation of the plane, namely (3.1).

\[
C_1 (X - X_0) + C_2 (Y - Y_0) + C_3 (Z - Z_0) = 0 \quad (3.1)
\]

The perpendicular distance from the plane to any point \((X_1, Y_1, Z_1)\) not on the plane is the absolute value of (3.2).

\[
C_1 (X_1 - X_0) + C_2 (Y_1 - Y_0) + C_3 (Z_1 - Z_0) \quad (3.2)
\]
C. BASIC MODEL EQUATIONS

Consider a coordinate system whose origin is at hydrophone \( C \), and whose axes are aligned with the three array arms. Let \( C_1, C_2 \) and \( C_3 \) be the direction cosines on the \( X, Y \) and \( Z \) axes respectively for the vector from the origin to the apparent sound source position. Then \((C_1,C_2,C_3)\) is itself a vector, of unit length, which is perpendicular to the planar wavefront emanating from the sound source. Therefore \((C_1,C_2,C_3)\) can be used as the normal vector for the wavefront plane.

In the coordinate system referenced to the hydrophone as the origin, the acoustic center has coordinates \((D/2, D, D/2)\), where \( D \) is the length of an array arm. When the soundwave plane arrives at the acoustic center, it will have the equation (3.3).

\[ C_1 X + C_2 Y + C_3 Z = D (C_1 + C_2 + C_3) / 2 \]  

(3.3)

The x-phone has coordinates \((D, D, 0)\), and the distance between it and the soundwave plane at the acoustic center is (3.4), which then simplifies to (3.5).

\[ C_1 (D/2 - D) + C_2 (D/2 - 0) + C_3 (D/2 - 0) \]  

(3.4)

\[ D (-C_1 + C_2 + C_3) / 2 \]  

(3.5)

This distance is measured in a direction perpendicular to the wavefront plane, and so is measured in the direction of travel of the soundwave. Therefore the same distance is also equal to (3.6).

\[ V (\tau_x - \tau) \]  

(3.6)
In (3.6) \( V \) is the velocity of sound at the array, \( T_x \) is the time of arrival of the sound wave at the x-phone, and \( T \) is the time of arrival of the sound wave at the acoustic center. The term "distance" is used loosely in (3.5) and (3.6), because these quantities may be negative. The true distances are the absolute values of these quantities. Since the next step is to equate these two distances, it is only necessary to show that these two quantities always have the same sign. There are two cases to consider, depending on whether the first component of the apparent position is positive \((X>0)\), or negative \((X<0)\). Let \((X',0,0)\) be the intersection of the X axis with the wavefront plane as it passes through the acoustic center. Then (3.3) can be used to solve for \( X' \), namely

\[
X' = \frac{D (C_1 + C_2 + C_3)}{2 C_1}.
\]

Now consider the case where \( X>0 \). Then \( C_1>0 \) also, and if (3.6) is positive, then

\[ T_x > T \]

which implies that the wave arrives at the acoustic center before it arrives at the x hydrophone. Therefore, since \( X>0 \), the plane at the acoustic center will intersect the X axis at a point beyond the x hydrophone, or \( X'>D \), and hence

\[
X' > D \Rightarrow \frac{(C_1 + C_2 + C_3)}{2 C_1} > 1 \\
\Rightarrow C_1 + C_2 + C_3 > 2 C_1 \quad \text{(since } C_1>0) \\
\Rightarrow -C_1 + C_2 + C_3 > 0 \\
\Rightarrow (3.5) \text{ is positive.}
\]

A parallel argument applies for the case of \( X>0 \), thus establishing that (3.5) and (3.6) always have the same sign. Equation (3.7) is the result of equating these two quantities.

\[
-C_1 + C_2 + C_3 + \left( \frac{2 V T}{D} \right) - \left( \frac{2 V T_1}{D} \right) = 0 \quad (3.7)
\]
For convenience, let $K = 2V/D$, and then apply the same logic to the distances of the $y$, $z$, and $c$ hydrophones from the wavefront plane as it passes through the acoustic center, to obtain the equations of (3.8).

\[
\begin{align*}
C_1 - C_2 - C_3 - KT + KT_x &= 0 \\
-C_1 + C_2 - C_3 - KT + KT_y &= 0 \\
-C_1 - C_2 + C_3 - KT + KT_z &= 0 \\
C_1 + C_2 + C_3 + KT - KT_c &= 0
\end{align*}
\] (3.3)

The system (3.3) is four equations in four unknowns, and is the planar model's version of the equations (1.1). The unknowns in (3.8) are $C_1$, $C_2$, $C_3$ and $T$. However there is the additional constraint that $C_1$, $C_2$, and $C_3$ are direction cosines, and therefore the direction cosines constraint (3.9) is a fifth equation, creating a system of five equations in four unknowns.

\[
C_1^2 + C_2^2 + C_3^2 = 1
\] (3.9)

Generally there is no set of values $(C_1, C_2, C_3, T)$ which will satisfy all five equations at once. This is because of the realities of a nonplanar wavefront and the presence of timing errors. The next section develops a method that produces set of values for the unknowns which is intended to satisfy those equations reasonably well.

D. MINIMIZATION OF SUM OF SQUARED ERRORS

Let $E_i$, $(i=1,2,3,4)$ be the value of the left hand side of the $i$-th equation of (3.8). Then $E_i$ measures the amount of error in the $i$-th equation caused by the chosen solution. It is impossible to have $E_i=0$ for all $i=1,2,3,4$. However some compromise may be made. Specifically the compromise chosen here is the classic minimization of (3.10), the sum of squared errors.
Minimization is to be done subject to the direction cosine constraint (3.9). Application of the Lagrange multiplier technique [Ref. 5 p.55] calls for the minimization of (3.11) over all possible choices of \( C_1, C_2, C_3, T \) and \( \lambda \).

\[
L = \sum_{i=1}^{4} E_i^2 - \lambda \left( \sum_{i=1}^{3} C_i^2 - 1 \right) \quad (3.11)
\]

Taking the partial derivative of \( L \) with respect to \( T \) yields (3.12).

\[
\frac{\partial L}{\partial T} = \sum_{i=1}^{4} (-2K E_i) = -2K \left( E_1 + E_2 + E_3 + E_4 \right)
= 2K \left[ -2K T + K \left( T_1 + T_2 + T_3 - T_4 \right) \right] \quad (3.12)
\]

Equating (3.12) to zero and solving for \( T \) yields immediately the appropriate estimate (3.13) of \( T \), the ray transit time.

\[
T = \frac{1}{2} \left( T_1 + T_2 + T_3 - T_4 \right) \quad (3.13)
\]

The partial derivative of \( L \) with respect to \( C_1 \) is (3.14).

\[
\frac{\partial L}{\partial C_1} = 2 \left( E_1 - E_2 + E_3 + E_4 \right) - 2 \lambda C_1
= 2 \left[ 4C_1 + 2KT + K \left( T_1 + T_2 + T_3 - T_4 \right) - \lambda C_1 \right] \quad (3.14)
\]

If (3.13) is used for \( T \) in (3.14), then (3.15) results.

\[
\frac{\partial L}{\partial C_1} = 2 \left[ \left( 4 - \lambda \right) C_1 + 2K \left( T_1 - T_4 \right) \right] \quad (3.15)
\]
If the same procedure is used for the partial derivatives of $L$ with respect to each of $C_1$, $C_2$ and $C_3$, and all are equated to zero, then the results are (3.16).

\[(4 - \lambda) C_i = 2K(T_4 - T_i) \quad i = 1, 2, 3 \quad (3.16)\]

In order to solve for the Lagrange multiplier $\lambda$, square both sides of the three equations of (3.16), and add the resulting equations together. Then use the sum of squares constraint (3.9) and solve for $\lambda$ to yield (3.17).

\[\lambda = 4 + 2K \left[ \sum_{i=1}^{3} (T_4 - T_i)^2 \right] \quad (3.17)\]

Substitute (3.17) in (3.16) and simplify to obtain the appropriate estimate of $C_i$, namely (3.18).

\[C_i = \frac{T_4 - T_i}{\sqrt{\sum_{j=1}^{3} (T_4 - T_j)^2}} \quad i = 1, 2, 3 \quad (3.18)\]

The choice of sign in (3.18) is determined by the fact that $C_i$ is positive if and only if the sound wave arrives at the $i$-th phone before it arrives at the $c$-phone, which in turn implies that $(T_4 - T_i) > 0$.

E. THE LEAST SQUARES METHOD

In summary, the first alternative method for determining an apparent position starts with estimation of the ray transit time to the acoustic center by (3.19). Then the apparent position estimates are computed using (3.20).

\[T = \left( T_1 + T_2 + T_3 - T_4 \right) / 2 \quad (3.19)\]
\[ x = \frac{v \cdot \sum_{j=1}^{2} (T_4 - T_j)}{\sqrt{\sum_{j=1}^{2} (T_4 - T_j)^2}} \]
\[ y = \frac{v \cdot \sum_{j=1}^{2} (T_4 - T_j)}{\sqrt{\sum_{j=1}^{2} (T_4 - T_j)^2}} \]
\[ z = \frac{v \cdot \sum_{j=1}^{2} (T_4 - T_j)}{\sqrt{\sum_{j=1}^{2} (T_4 - T_j)^2}} \]

This method shall be hereafter referred to as the 'Least Squares Method', or 'L.S.' for short. The apparent advantages of the L.S. method are that:

1. all four hydrophone times have equal weight in a simple expression for the ray transit time \( T \), rather than using an expression so heavily dependent on \( T_c \) as in the NAVY and NAVYA methods;
2. the differences of squared time values which appear in the solutions of the NAVY methods are avoided in the L.S. method, thereby lessening the tendency toward computational roundoff problems;
3. the direction cosines already add to unity when squared, requiring no arbitrary adjustments; and
4. computation of the initial angle allows cancellation of several terms, resulting in the simple expression (3.21).

\[ A = \arcsin \left( \frac{T_4 - T_3}{\sqrt{\sum_{j=1}^{2} (T_4 - T_j)^2}} \right) \] (3.21)

**F. BIAS IN THE LEAST SQUARES METHOD**

Unfortunately a potentially serious problem exists with the L.S. method. That concerns the consequences of the assumption of a planar wavefront. The effect is difficult
to derive explicitly because it is difficult to determine
the true shape of a wavefront corresponding to a nonconstant
velocity profile. However, if it is assumed that a spherical
assumption is more accurate than the planar assumption, then
the bias can be estimated roughly, and then subtracted from the
original L.S. position estimate. This is still a difficult
problem because, as previously discussed, the four basic
spherical equations themselves have no exact solution.

Nevertheless, as a rough estimate of the bias, the
following procedure is used. First estimate the apparent
position by the L.S. method. Then calculate the straight
line distances from that position to each of the four array
hydrophones. Divide those distances by the velocity of
sound at the array to obtain the corresponding times. Use
these times to recalculate the apparent position using the
L.S. method again. The difference between the original and
recalculated L.S. positions roughly measures the error that
would be made by the L.S. method when it is applied to the
time values which correspond to a spherically spreading
sound wave whose source is in the vicinity of the original
apparent position. Therefore this difference can be used as
a bias vector which can be subtracted from the original L.S.
solution. This bias correction is the basis of the method
set forth in the next section.

G. THE LEAST SQUARES CORRECTED METHOD

The second alternative method for estimating an apparent
position is as follows:

1. calculate the apparent position \( P_1 \) by using the L.S.
   method;

2. calculate the distances from that position to the
   four hydrophones, and convert them to times by
   dividing by the speed of sound at the array;
3. Use the new times to calculate a new apparent position $P_2$, using the L.S. method;

4. Calculate the difference vector $P_2 - P_1$ (see Figure 3.1), and subtract it from the original position $P_1$ to obtain the corrected position $P$:

$$P = P_1 - (P_2 - P_1) = 2P_1 - P_2$$

5. Finally, adjust the ray transit time $T$ calculated for the original position $P_1$, by using the proportional transformation:

$$T' = T \cdot R / R_1$$

where $R$ is the range to the new position $P$, and $R_1$ is the range to the original position $P_1$.

---

**Figure 3.1 Bias Adjustment for the L.S. Method.**

This method shall hereafter be referred to as the 'Least Squares Corrected Method', or L.S.C. for short. The properties of the L.S.C. method, like those of the NAVY A method, are not well understood at this time. It is offered only as
an alternative which may combine the beneficial properties of the L.S. and NAVY methods, namely that it will:

1. estimate a ray transit time value equally dependent on all four data times, thereby smoothing out excessive error in any one of the time values; and
2. reflect the spherical wave assumption, believed to provide a more accurate description than the planar assumption.

For targets at a range of 3000 feet, the length of the error vector ε varies from 0 to as much as 10 or 11 feet (see Table II). The error vector lengths seem to be dependent on both azimuth and elevation angles of the target from the array. These patterns indicate a potential for further investigation to relate estimation errors to such variables.

H. MAXIMUM LIKELIHOOD CALCULATIONS

One shortcoming of all methods described thus far is that none of them can be used to produce an estimate of the underlying error in the time data values. The method set forth in this section is a first attempt to estimate that noise, and is again based on the assumption of a planar wavefront.

Let $T_i$ be the time recorded from by $i$-th hydrophone. Let $T_i$ be the true time which, under absolutely error free conditions, would have been recorded at the $i$-th hydrophone. An assumption of Gaussian noise is now made, namely that

$T_i = U_i + E_i$

where the $E_i$ are independent identically distributed normal random variables with mean zero and variance $\sigma^2$. Therefore the $T_i (i=1,2,3,4)$ are also normally distributed with the same variance, but with means $U_i (i=1,2,3,4)$. 

40
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**TABLE II**

Error Vector Lengths for the L.S.C. Method

Tabled entries are lengths in feet of bias vectors.

Angles are in multiples of $\pi$. 
Now let \( C_j \) be the \( j \)-th direction cosine \((j=1, 2, 3)\). Then geometrically, using the assumption of a planar wavefront, \( C_j \) is given by \((3.22)\), where \( V \) is the speed of sound at the array, and \( D \) is the array dimension (30 feet).

\[
C_j = \frac{V (U_j - U_j)}{D}
\]

\((3.22)\)

Letting \( U = U_4 \), then \((3.22)\) can be solved for each \( U_j \) in terms of \( U \) and \( C_j \), as in \((3.23)\).

\[
U_j = U - DC_j / V
\]

\((3.23)\)

The probability density of each time value \( T_i \) is given by \((3.24)\).

\[
f_1(T_i) = \frac{1}{\sqrt{2\pi s}} \exp \left[ -\frac{(T_i - U + DC_i/V)^2}{2s^2} \right]
\]

\((3.24)\)

Using \((3.23)\) in \((3.24)\), and multiplying the four densities together, the likelihood function \((3.25)\) is formed.

\[
L_0 = \left( \frac{1}{\sqrt{2\pi s}} \right)^4 \exp \left[ -\frac{1}{2s^2} \sum_{i=1}^{4} (T_i - U + DC_i/V)^2 \right]
\]

\((3.25)\)

In \((3.25)\) \( C_4 \) has been used for notational convenience, and is defined to be zero. Then the log-likelihood function is formed by taking natural logs of \((3.25)\), yielding \((3.26)\).

\[
L_1 = -2 \ln(2\pi) - 4 \ln(s) - \frac{1}{2s^2} \sum_{i=1}^{4} (T_i - U + DC_i/V)^2
\]

\((3.26)\)

Since the values of the \( C_i \) are to be direction cosines, the usual direction cosines constraint must be added to \( L_1 \) to form the Lagrangian function \( L_2 \) given in \((3.27)\).
The objective is to maximize $L_2$ over the possible choices of $C_1$, $C_2$, $C_3$, $U$, $S$ and lambda. Ignoring $S$ for the moment, maximization of $L_2$ can be achieved by minimization of $L$ given in (3.28).

$$L = \frac{1}{2} \sum_{i=1}^{4} (T_i - U + DC_i/V)^2 + \lambda \left[ \sum_{i=1}^{3} (C_i^2) - 1 \right]$$ (3.28)

Take partial derivatives of $L$ to get (3.29) and (3.30).

$$\frac{\partial L}{\partial C_i} = \frac{2D}{V} (T_i - U + DC_i/V) - 2 \lambda C_i$$ (3.29)

$$\frac{\partial L}{\partial U} = 2 \left[ \sum_{i=1}^{4} T_i - 4U + \frac{D}{V} \sum_{i=1}^{3} C_i \right]$$ (3.30)

Equate (3.30) to zero and solve for $U$ to obtain (3.31), the maximum likelihood estimate for the time at the c hydrophone.

$$U = \left( \frac{\sum_{j=1}^{4} T_j + \frac{D}{V} \sum_{j=1}^{3} C_j}{4} \right)$$ (3.31)

Equate the three equations of (3.29) to zero, and multiply each equation by $C_i$ respectively, to obtain (3.32).

$$\lambda C_i^2 = \frac{C}{V} \left( T_i C_i - U C_i + \frac{D}{V} C_i^2 \right) \quad i = 1,2,3$$ (3.32)

Add the three equations of (3.32) together, and use the direction cosine constraint to obtain (3.33).

$$\lambda = \frac{D}{V} \left[ \sum_{i=1}^{3} T_i C_i - U \sum_{i=1}^{3} C_i + \frac{D}{V} \right]$$ (3.33)

Equation (3.34) results when (3.29) is equated to zero.
\[ C_i = \frac{\frac{T_i - U}{V} \lambda - \frac{D}{V}}{\sum_{j=1}^{3} T_j C_j - U \sum_{j=1}^{3} C_j} \] (3.34)

Substitute (3.33) into (3.34) to obtain (3.35), the maximum likelihood estimate of the direction cosines, where \( U \) is the estimate calculated by (3.31).

\[ C_i = \frac{T_i - U}{\sum_{j=1}^{3} T_j C_j - U \sum_{j=1}^{3} C_j} \] (3.35)

As the reader will perhaps have noticed, the equations of (3.35) define each of the unknowns \( C_i \) in terms of all three unknowns. Such a structure suggests that (3.35) can be used as an iteration function. That is, if reasonable initial values are used for the three unknowns in the right hand side of (3.35) then new values are produced. Repeat the process using the new values until the answer stabilizes within acceptable tolerances. Although convergence to the correct solution is not guaranteed, the method has never failed for the equations of (3.35). Unlike the L.S. method, (3.35) does not have any known closed form solution.

Returning to the standard deviation \( S \), take the partial derivative of (3.27) with respect to \( S \) to get (3.36).

\[ \frac{\partial L_2}{\partial S} = \frac{-4}{S} + \frac{1}{S^3} \sum_{i=1}^{4} (T_i - U + \frac{D}{V} C_i)^2 \] (3.36)

Multiply (3.36) by \( S^3 \) and solve for \( S^2 \) to get the maximum likelihood estimate of the variance, given by (3.37).

\[ S^2 = \frac{1}{4} \left\{ \sum_{i=1}^{4} (T_i - U) + \frac{D}{V} \sum_{i=1}^{3} C_i \right\} \] (3.37)
I. THE MAXIMUM LIKELIHOOD PLANAR METHOD

In summary, the third method for estimation of an apparent position is as follows:
1. let $U = T4$ initially;
2. use the L.S. method to calculate the initial values for $C_i$, ($i=1,2,3$), using (3.31);
3. use $U$ and $C_i$ ($i=1,2,3$) in the right hand side of (3.35) to obtain new estimates for $C_i$ ($i=1,2,3$);
4. recalculate $U$, using (3.31);
5. reiterate steps 3 and 4 until the values $C_i$ ($i=1,2,3$) and $U$ converge within acceptable tolerances;
6. calculate $S^2$, using (3.37);
7. calculate the estimated apparent position relative to the c-phone, using (3.38);

$$X_c = VUC_1 \quad Y_c = VUC_2 \quad Z_c = VUC_3 \quad (3.38)$$

8. lastly translate this solution and its corresponding time estimate to a solution and time relative to the acoustic center, using (3.39), where $R$ and $R_c$ are as defined by (1.4) in Chapter I.

$$X = X_c + D/2 \quad Y = Y_c + D/2 \quad Z = Z_c + D/2 \quad (3.39)$$

$$T = U/R/R_c$$

This method shall be hereafter referred to as the 'Maximum Likelihood Planar Method', or M.L.P. for short. Originally the hopes for this method were rather high, especially since it was the first method to produce a variance estimate. However subsequent experience with the method indicates that it probably suffers significantly from at least two factors:

1. the planar wavefront assumption probably builds in a position bias as in the case of the L.S. method; and
2. the variance estimate is inflated since part of the noise being measured is due to the inadequacy of the planar assumption.

J. COMPUTATIONAL EXAMPLE

For a quick comparison, the three methods developed in this chapter are now applied to the example which was used in Chapter II. The EXACT results calculated previously are included in Table III for comparison.

<table>
<thead>
<tr>
<th>Method</th>
<th>Transit Time $\Delta T$ (Secs)</th>
<th>Tilt Angle $\Delta \theta$ (Deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L.S.</td>
<td>0.67529319</td>
<td>15.22798</td>
</tr>
<tr>
<td>L.S.C.</td>
<td>0.67527149</td>
<td>15.26372</td>
</tr>
<tr>
<td>M.L.P.</td>
<td>0.67529007</td>
<td>15.10437</td>
</tr>
<tr>
<td>EXACT</td>
<td>0.67527043</td>
<td>15.27002</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Apparent Position Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>L.S. (1003.809, 3019.751, 866.16)</td>
</tr>
<tr>
<td>L.S.C. (1006.089, 3018.266, 868.235)</td>
</tr>
<tr>
<td>M.L.P. (997.722, 3023.685, 859.355)</td>
</tr>
<tr>
<td>EXACT (1005.966, 3017.899, 868.555)</td>
</tr>
</tbody>
</table>

As can be seen easily, the M.L.P. method fares rather poorly in all regards, even worse than the L.S. method. This will be confirmed by the evaluations made in Chapter V. Also worthy of note is the apparent tendency of the L.S.C.
method to correct the L.S. method back toward the exact solution. The evaluations of Chapter V will confirm that the L.S.C. method almost always yields a better solution than the original L.S. solution.
IV. A SPHERICAL WAVEFRONT MODEL

A. THE SPHERICAL WAVEFRONT ASSUMPTION

All the models developed in Chapter III are limited primarily by the assumption that the wavefront is planar upon arrival at the hydrophone array. In this chapter a model is developed under the assumption that the wavefront is spherical upon arrival at the array. If the sound velocity profile were constant with depth then the spherical model would be exact. This is of course not the case, but it is suspected that the wavefront is better modelled as a sphere than as a plane because that small piece of the wavefront which passes through the 30 foot cube spanned by the array is locally spherical. That is because every part of that piece travelled through approximately the same regions of water, experiencing the same general raytending patterns.

The spherical assumption is accurate if and only if the speed of sound is constant over the ray path, and consequently the original four spherical equations apply once again. They were:

$$\begin{align*}
(X - D/2)^2 + (Y + D/2)^2 + (Z + D/2)^2 &= v_x^2 \\
(X + D/2)^2 + (Y - D/2)^2 + (Z + D/2)^2 &= v_y^2 \\
(X + D/2)^2 + (Y + D/2)^2 + (Z - D/2)^2 &= v_z^2 \\
(X + D/2)^2 + (Y + D/2)^2 + (Z + D/2)^2 &= v_c^2
\end{align*}$$

(4.1)

It has been stressed previously that there is no exact solution \((X,Y,Z)\) satisfying all four equations (4.1). That is because the time values on the right hand side correspond to the reality of a variable velocity profile. However, if the spherical wavefront assumption is to be accurate, then a
constant velocity is the assumed case, and any inaccuracies in the time values are regarded as due to timing errors only. Therefore, under the spherical assumption, if the true time values \( t_i \) \((i=1,2,3,4)\) were known and substituted into (4.1), an exact solution to the overdetermined system would be realized. In that case a solution to any three of the equations would also be equal to that unique exact solution. In particular, the NAVY solution of Chapter II would be the true solution. Therefore in terms of the coordinate system referenced to the \( c \) hydrophone, \( x \) would be given by (4.2).

\[
x = \frac{D}{c} + \frac{v^2}{cD} \left( \frac{t_4^2 - t_i^2}{4} \right)
\]

However, \( x \) is also given by (4.3), where \( C_1 \) is the direction cosine along the \( x \) axis of the vector from the \( c \) hydrophone to the sound source.

\[
x = v^2 U_i C_1
\]

The time value of interest at the moment is \( t_4 \), the time at the \( c \) hydrophone. Therefore let \( U = U_4 \) for clarity of presentation, and then equate the expressions in (4.2) and (4.3) in order to solve for \( U \). If this same logic is also applied to the similar expressions for the distances to the \( y \) and \( z \) hydrophones, the results are (4.4).

\[
U_i = \sqrt{U^2 - \left(2D U_i / V \right) + \left(D / V \right)^2} \quad i = 1, 2, 3
\]

The expression (4.4) will be useful in the development of the model of this chapter.

B. LEAST SQUARES MODELS

A direct approach might be to apply the least squares error technique to the spherical equations, in a manner
paralleling that used on the four planar equations of the L.S. model in Chapter III. However the formulae and equations that result are exceedingly complex, involve fourth degree powers of the data values $T_i$, and have thus far defied all solution attempts. Therefore this idea was abandoned in favor of the maximum likelihood approach which follows.

C. MAXIMUM LIKELIHOOD COMPUTATIONS

As in the M.L.P. model, Gaussian noise is assumed for the time data values. Therefore

$T_i = U_i + E_i \quad i = 1, 2, 3$

where the $E_i$ are independent identically distributed normal random variables with mean zero and variance $S^2$. The density of each $T_i$ is therefore (4.5).

$$ f_i(T_i) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \frac{1}{S^2} (T_i - U_i)^2 \right] \quad i = 1, 2, 3, 4 \quad (4.5) $$

To form the likelihood function, multiply the four densities together, to obtain (4.6).

$$ L_0 = \left( \frac{1}{\sqrt{2\pi}} \right)^4 \exp \left[ -\frac{1}{2} \frac{1}{S^2} \sum_{i=1}^{4} (T_i - U_i)^2 \right] \quad (4.6) $$

Form the log likelihood function by taking natural logarithms of (4.6) to obtain $L_1$ of (4.7).

$$ L_1 = -2 \ln(2\pi) - 4 \ln(S) - \frac{1}{2 S^2} \left[ \sum_{i=1}^{4} (T_i - U_i)^2 \right] \quad (4.7) $$

In order to maximize $L_1$ with respect to $C1, C2, C3$ and $U$, it is sufficient to minimize $L_2$ of (4.8), where a substitution for $U_i$ has been made using (4.4).
$L_2 = (T_i - U)^2 + \sum_{i=1}^{3} \left[ T_i - \sqrt{U^2 - \left( \frac{2DU}{V} \right)^2 + \frac{U}{V}} \right]^2 \quad (4.3)$

Now add the usual direction cosines constraint and form the Lagrangian function $L$ of (4.3):

$$L = L_2 - \lambda \left[ \sum_{i=1}^{3} c_i^2 - 1 \right] \quad (4.9)$$

For notational convenience, let $K_i$ be given by (4.10).

$$K_i = U^2 - (2 DU c_i / V) + (D/V)^2 \quad i = 1, 2, 3 \quad (4.10)$$

Take the partial derivative of $L$ with respect to $c_i$ to get (4.11).

$$\frac{\partial L}{\partial c_i} = \left( \frac{2 (T_i - K_i)}{-2 K_i} \right) \left( \frac{-2 DU}{V} \right) - 2 \lambda \quad i = 1, 2, 3 \quad (4.11)$$

Simplify (4.11) and equate to zero to yield (4.12).

$$c_i = \frac{DU}{\lambda V} \left( \frac{T_i - \sqrt{K_i}}{\sqrt{K_i}} \right) \quad i = 1, 2, 3 \quad (4.12)$$

Multiply the three equation in (4.12) by $c_i$ (i=1,2,3) respectively, and add them together. Then use the constraint on the sum of squares of the direction cosines to obtain (4.13).

$$\lambda = \frac{DU}{V} \sum_{i=1}^{3} c_i \left( \frac{T_i - \sqrt{K_i}}{\sqrt{K_i}} \right) \quad (4.13)$$

Substitute (4.13) into (4.12) to get the maximum likelihood estimate for the direction cosines, as in (4.14).
Now take the partial derivative of $L$ with respect to $U$, as in (4.15).

$$\frac{\partial L}{\partial U} = 2 \sum_{i=1}^{3} \left[ \frac{I_i - \sqrt{K_i}}{\sqrt{K_i}} \right] (2U - 2DC_i V) + 2 (U - T_4) \quad (4.15)$$

Equate (4.15) to zero and solve for $U$ to obtain the maximum likelihood estimate of the time value (4.16) at the $c$ hydrophone.

$$U = \frac{T_4 - \frac{D}{V} \sum_{j=1}^{3} C_j \left( \frac{T_j - \sqrt{R_j}}{\sqrt{R_j}} \right)}{1 - \sum_{j=1}^{3} \frac{T_j - \sqrt{R_j}}{\sqrt{R_j}}} \quad (4.16)$$

Finally take the partial derivative of $L$ in (4.17) with respect to $S$. Equate it to zero and solve to get (4.17), the maximum likelihood estimate of the variance.

$$\varepsilon^2 = \frac{1}{4} \left[ \sum_{j=1}^{3} \left( \frac{T_j - \sqrt{R_j}}{\sqrt{R_j}} \right)^2 + (T_4 - U)^2 \right] \quad (4.17)$$

D. SOLUTION BY A MODIFICATION OF NEWTON'S METHOD

The solutions given by equations (4.14), (4.16), and (4.17) once again form a set of equations which would seem to be solvable by natural iteration as in the M.P. method. Unfortunately this time the technique fails to converge. A mathematical tool is needed which is stronger than natural iteration. What is used is a modified four dimensional
version of Newton's Method [Ref. 5 p.47] to search for the roots of a set of four equations.

The objective is to determine the values $C_1, C_2, C_3$ and $U$ which satisfy (4.14) and (4.16). For further notational convenience define the values $M$ and $N$ as in (4.18) and (4.19).

$$M = \sum_{j=1}^{3} \left( \frac{T_j - \sqrt{R_j}}{\sqrt{K_j}} \right)$$  \hspace{1cm} (4.13)

$$N = \sum_{j=1}^{3} \left( \frac{T_j}{\sqrt{K_j}} \right)$$  \hspace{1cm} (4.19)

Given those definitions of $M$ and $N$, then the equations whose roots are desired can be simplified to (4.20).

$$C_i = h_i(C_1, C_2, C_3, U) = \frac{T_i - \sqrt{R_i}}{\sqrt{K_i} M} \hspace{1cm} i = 1, 2, 3$$  \hspace{1cm} (4.20)

$$U = h_4(C_1, C_2, C_3, U) = \frac{T_4 - B \cdot M}{N}$$

Now define error functions as in (4.21). These evaluate the amount of error in each of the equations (4.20) for any set of values for $(C_1, C_2, C_3, U)$.

$$e_i = C_i - h_i \hspace{1cm} i = 1, 2, 3$$  \hspace{1cm} (4.21)

Let $G$ be the four dimensional column vector $(g_1, g_2, g_3, g_4)'$. Finally let $G_0$ be the matrix of partial derivatives of $G$, as in (4.22).

Newton's Method in four dimensions says that if $\vec{X}(n)$ is a four dimensional vector holding the current approximate roots $C_1, C_2, C_3$ and $U$, then $\vec{X}(n+1)$ will be an improved answer, where $\vec{X}(n+1)$ is given by (4.23).
Of course it is necessary to calculate the derivatives held in the matrix \( GP \) in order to use (4.23). Those derivatives are given in Appendix B.

Unfortunately when multidimensional versions of Newton's Method are applied, there is often a tendency for the method to converge slowly, or even diverge. This is because it tends to overshoot the best answer for each iteration. To alleviate this problem, a modification is made to the method. At the end of each Newton iteration, prior to proceeding with the next iteration, a Golden Section Search [Ref. 6] is performed to find the best possible answer in the direction of the new iterative solution. Specifically the line in four dimensional space from \( X_{n-1} \) of the previous iteration to \( X_n \) of the present iteration is searched for the best answer. The definition of the 'best' answer is that point along the search line which minimizes the sum of squared error functions, namely (4.24).

\[
\sum_{i=1}^{4} g_i^2
\]  

(4.24)
The current iterative solution $\mathbf{x}(n)$ is then given the value of the minimizing point resulting from the Golden Section Search. Then the next iteration of Newton's Method is performed, along with another Golden Section Search to find the next iterative solution $\mathbf{x}(n+1)$.

E. THE MAXIMUM LIKELIHOOD METHOD

In summary, the fourth and final alternative method to estimate apparent positions is as follows:

1. let $U = T_4$ initially;
2. use the L.S. method (6e321) to initially estimate the values of $C_j$ $(j=1,2,3)$;
3. set $\mathbf{x}(1) = (C_1,C_2,C_3,U)$;
4. initialize the Newton Iteration counter : $I = 1$;
5. calculate the values $K_i$, $M$ and $N$ in accordance with equations (4.10), (4.18) and (4.19);
6. calculate the error function vector $\mathbf{j}(\mathbf{x}(I))$, using (4.20) and (4.21);
7. calculate the derivative matrix $\mathbf{G}\mathbf{p}(\mathbf{x}(I))$, using the results in Appendix B;
8. invert $\mathbf{G}\mathbf{p}(\mathbf{x}(I))$;
9. calculate the new Newton estimate $\mathbf{x}(I+1)$ from (4.23);
10. perform a Golden Section Search along the line between $\mathbf{x}(I)$ and $\mathbf{x}(I+1)$ to find the point which minimizes (4.24);
11. let $\mathbf{x}(I+1)$ be equal to the minimizing point found in the previous step;
12. increase the Newton iteration counter : $I = I+1$;
13. reiterate steps 5 through 12 until the values $\mathbf{x}(I) = (C_1,C_2,C_3,U)$ converge within acceptable tolerances;
14. calculate the estimate of $S^2$ using (4.17).
This method shall hereafter be referred to as the 'Maximum Likelihood Spherical Method', or M.L.S. for short. As will be seen from the evaluations made in Chapter 7, the M.L.S. method is apparently the only alternative to consistently rival the performance of the currently used NAVY_A method. It has the additional advantage that it estimates the error present in the time data values.

F. A COMPUTATIONAL EXAMPLE

This latest method, M.L.S., is now applied to the same example considered in Chapters II and III. For a quick comparison, Table IV lists the results of using all the methods. The error vector lengths are the distances of each position estimate from the EXACT apparent position.

Since this is only one example, this table is not presented for the purpose of any broad conclusions. However it is of interest to note that in this example

1. the M.L.S. method outperforms all others, including the NAVY_A method; and

2. in many ways the original NAVY method outperforms the adjusted NAVY_A method.

G. VARIANCE ESTIMATION

In the computational example, the time data values used were exact since the methods of Appendix A could be used with the known linear velocity profile. Therefore the appropriate value for variance in the time values would be zero. For this error free example, the estimates shown in Table V were obtained for the standard deviations of timing noise, using the two maximum likelihood methods.

In the case of this one example, the spherical assumption is apparently an improvement over the planar, since the M.L.S. estimate of error is only 4% of the M.L.P. estimate.
<table>
<thead>
<tr>
<th>Method</th>
<th>Transit Time T (sec)</th>
<th>Elev. Angle A (deg)</th>
<th>Length of Error Vector E (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>0.67526969</td>
<td>15.26459</td>
<td>0.4128</td>
</tr>
<tr>
<td>NAVY_A</td>
<td>0.67527027</td>
<td>15.26461</td>
<td>0.4840</td>
</tr>
<tr>
<td>L.S.</td>
<td>0.67529319</td>
<td>15.22798</td>
<td>3.733</td>
</tr>
<tr>
<td>L.S.C.</td>
<td>0.67527149</td>
<td>15.26372</td>
<td>0.522</td>
</tr>
<tr>
<td>M.L.P.</td>
<td>0.67529007</td>
<td>15.10437</td>
<td>13.641</td>
</tr>
<tr>
<td>M.LS.</td>
<td>0.67527049</td>
<td>15.26677</td>
<td>0.4117</td>
</tr>
<tr>
<td>EXACT</td>
<td>0.67527043</td>
<td>15.27002</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Apparent Position Estimate</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.LS.</th>
<th>EXACT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1005.957, 3017.874, 868.113)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1006.087, 3018.259, 868.235)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1003.809, 3019.751, 866.126)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1006.089, 3018.266, 868.235)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(997.722, 3023.685, 859.355)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1006.195, 3018.190, 868.375)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1005.966, 3017.899, 868.535)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

That is regarded as an improvement because the correct answer is zero. The higher M.L.P. estimate is indicative of the inflation due to the planar assumption.

The tracking range which provided data for this study records time values to seven decimal places. Therefore the standard deviation estimated by the M.L.S. method is of particular interest, since it indicates errors in the seventh decimal place even when there is no error present. Since the data was actually error free in this example, the
estimate is a measure of the variation induced by the spherical wavefront assumption. A broader discussion of error estimation will be undertaken in Chapter VI.
V. EVALUATIONS OF MODELS

A. GENERAL

There are many sources of errors in the overall hydrophonic tracking problem. These include, among others:

1. errors in estimation of the sound velocity profile;
2. inhomogeneity of the velocity profile over time and horizontal displacement; and
3. possible errors in measuring the positions, and the angles of tilt and rotation for the hydrophonic arrays.

This study focuses on those errors which occur during the computations preceding the ray tracing procedure. To evaluate the performance of the methods developed in Chapters II, III and IV, it is necessary to control strictly the ray tracing procedure. Only in that way can the differences found between methods be attributed to the differences between models, and not to any source of error outside those methods.

It was originally hoped that the various methods might be compared by applying them to real tracking data. However it was found that the overall tracking problem had too many large sources of error to allow the methods to demonstrate any differences. Therefore the methods were compared under a more tightly controlled simulated environment.

B. SIMULATION SCENARIO

Two different simulations are used to compare the six different methods. Both use a basic scenario similar to that of the computational example explored in Chapters II, III and IV. That example assumed that:
1. the velocity versus depth relationship is linear, and given exactly by:

\[ V = 4840.7 + 0.03314 \cdot \text{DEPTH} \] ;

2. the acoustic center of each array is at a depth of 1300 feet;

3. the hydrophone arrays are all level, and their X, Y and Z arms are parallel to the respective coordinate axes of the tracking range.

Under these circumstances the methods set forth in Appendix A can be used to compute the exact values for the hydrophone times, ray transit time and elevation angle for a sound wave emanating from a source at any specified location. Therefore when the methods are applied to those exact times, the resulting estimated positions can be compared to the known true position.

C. SIMULATED ERROR FOR TIMING DATA

The models developed in this study were designed to improve position estimation, especially in the presence of errors in the timing data. Lacking any better model at this time for timing errors, the simulated environment includes an assumption of Gaussian errors for the hydrophone times. Therefore realistic timing data can be simulated by adding to each exact time value a random quantity of normally distributed error. The mean of the error is assumed to be zero. The variance was estimated from real tracking data, using the variance estimating property of the M.L.S. model. The data from one tracking run was used, involving six hydrophone arrays and 733 position estimates (see Chapter VI for data selection details). Each position from the tracking run produces one estimate of the variance. The variance value chosen for use in the simulations was the median of the 733 variance estimates produced by the M.L.S. method. That value was
\[ S^2 = 9.1204 \text{ E-12 secs}^2. \]

That is the same as a standard deviation of
\[ S = 3.02 \text{ E-6 secs}. \]

Each of the two types of simulations was run four separate times. Each run was done with a different specified variation. Those four distinct error conditions are delineated in Table VI.

### TABLE VI
Simulation Error Levels

<table>
<thead>
<tr>
<th>RUN</th>
<th>LEVEL</th>
<th>VARIANCE</th>
<th>STD. DEVIATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>zero</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>low</td>
<td>9.12 E-14</td>
<td>3.02 E-7</td>
</tr>
<tr>
<td>3</td>
<td>medium</td>
<td>9.12 E-12</td>
<td>3.02 E-6</td>
</tr>
<tr>
<td>4</td>
<td>high</td>
<td>9.12 E-10</td>
<td>3.02 E-5</td>
</tr>
</tbody>
</table>

D. SINGLE ARRAY SIMULATION

In the first simulation, the intent was to compare the methods pairwise, so as to determine which method is more likely to produce the more accurate estimate of a given sound source position. One thousand positions were chosen at random. Each position was 3000 feet from the array. The positions were uniformly spread over the surface of a sphere of radius 3000 feet, centered at the array, truncated above by the water surface (depth 0) and below by the depth of the array (1300 feet). The methods set forth in Appendix C were used to assure that the random positions selected were
uniformly distributed over the surface area of the truncated hemisphere.

Each of the 1000 randomly chosen source positions was then processed as follows:

1. calculate the exact hydrophone times, using the methods of Appendix A;

2. add to each of the four exact times a random value of error at the specified level (zero, low, medium or high);

3. apply each of the six methods to the hydrophone times, generating six different apparent positions;

4. apply the ray tracing procedure to each of the six positions, using layers that are 25 feet thick, and utilizing the known linear velocity profile, thereby producing six different estimates of the sound source location;

5. compare the six different estimates pairwise to see which method in each pair produced the estimate closest to the true sound source location.

The comparison being made is that one method is considered preferable to the other if it more frequently produces the more accurate estimate.

The layer thickness of 25 feet was selected order to simulate the actual procedure at the tracking range which provided data for this study. However, as discussed in Chapter I, when the velocity profile is smooth and known exactly, the process is very robust with respect to the thickness used. The thickness values 1, 10, and 25 were all tried, with virtually no changes in any of the comparisons between methods.
E. SINGLE ARRAY SIMULATION RESULTS

Tables VII and VIII contain the results of the single array simulation. Each tabled entry represents the fraction of time that the method of that row produced a better estimate than the method of that column. For example, in Table VII, the L.S. method outperformed the M.L.S. method in only 5.3% of the 1000 trials with low error values.

In a one tailed test that one method is better than another, these binomial proportions are significant at the 0.05 level if they exceed 0.526. Symmetrically, one method is significantly worse than another at the 0.05 level if the proportion is less than 0.474. For the 0.01 level the corresponding critical values are 0.537 and 0.463 respectively.

The results indicate that:

1. as previously claimed, the NAVY_A method usually outperforms the unadjusted NAVY method; of particular interest is the case of zero error which actually compares the relative ability of each method to produce the exact answer when given the exact times; in those cases the NAVY_A method does extremely well against the NAVY method;

2. under all error conditions the most successful performer is the M.L.S. method, since it always has a favorable (greater than 0.5) comparison fraction against all other methods; the M.L.S. fractions vary little over the four error levels;

3. under all error conditions, the worst performer is always the M.I.P. method;

4. spherical methods consistently outperform planar methods; and

5. increased error levels tend to lessen the distinction between methods; in the zero error case comparisons
TABLE VII

Single Array Simulation Results for Lower Error Levels

<table>
<thead>
<tr>
<th>ERROR LEVEL : ZERO</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.L.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>.011</td>
<td>.950</td>
<td>.676</td>
<td>.987</td>
<td>.420</td>
<td></td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.989</td>
<td>.950</td>
<td>.728</td>
<td>.987</td>
<td>.434</td>
<td></td>
</tr>
<tr>
<td>L.S.</td>
<td>.050</td>
<td>.050</td>
<td>.987</td>
<td>.057</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.324</td>
<td>.272</td>
<td>.950</td>
<td>.346</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.013</td>
<td>.013</td>
<td>.013</td>
<td>.013</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.560</td>
<td>.566</td>
<td>.943</td>
<td>.987</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ERROR LEVEL : LOW</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.L.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>.165</td>
<td>.952</td>
<td>.648</td>
<td>.987</td>
<td>.396</td>
<td></td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.835</td>
<td>.952</td>
<td>.693</td>
<td>.987</td>
<td>.405</td>
<td></td>
</tr>
<tr>
<td>L.S.</td>
<td>.048</td>
<td>.048</td>
<td>.987</td>
<td>.058</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.352</td>
<td>.307</td>
<td>.952</td>
<td>.369</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.013</td>
<td>.013</td>
<td>.013</td>
<td>.013</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.604</td>
<td>.595</td>
<td>.942</td>
<td>.987</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

are in the interval (0.01,0.99), while in the high error case that interval is narrowed considerably to (0.37,0.63).

F. DOUBLE ARRAY SIMULATION

In the second simulation, the intent again was to compare the methods pairwise, this time determining which method is more likely to produce positions which agree more closely in the two array crossover problem. This is not the
TABLE VIII
Single Array Simulation Results for Higher Error Levels

<table>
<thead>
<tr>
<th>ERROR LEVEL : MEDIUM</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.L.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>.464</td>
<td>.819</td>
<td>.559</td>
<td>.987</td>
<td>.435</td>
<td></td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.546</td>
<td>.821</td>
<td>.566</td>
<td>.987</td>
<td>.437</td>
<td></td>
</tr>
<tr>
<td>L.S.</td>
<td>.191</td>
<td>.179</td>
<td>.181</td>
<td>.971</td>
<td>.177</td>
<td></td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.441</td>
<td>.434</td>
<td>.819</td>
<td>.987</td>
<td>.450</td>
<td></td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.013</td>
<td>.013</td>
<td>.029</td>
<td>.013</td>
<td>.013</td>
<td></td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.565</td>
<td>.563</td>
<td>.823</td>
<td>.550</td>
<td>.937</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ERROR LEVEL : HIGH</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.L.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>.466</td>
<td>.544</td>
<td>.439</td>
<td>.562</td>
<td>.431</td>
<td></td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.514</td>
<td>.546</td>
<td>.516</td>
<td>.559</td>
<td>.431</td>
<td></td>
</tr>
<tr>
<td>L.S.</td>
<td>.456</td>
<td>.454</td>
<td>.457</td>
<td>.557</td>
<td>.400</td>
<td></td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.561</td>
<td>.484</td>
<td>.543</td>
<td>.562</td>
<td>.427</td>
<td></td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.438</td>
<td>.441</td>
<td>.443</td>
<td>.438</td>
<td>.373</td>
<td></td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.569</td>
<td>.569</td>
<td>.600</td>
<td>.573</td>
<td>.627</td>
<td></td>
</tr>
</tbody>
</table>

The same question as that addressed by the single array simulation. The two estimates produced by any one method may be very close to each other, and yet be far away from the true position.

For the double array scenario two arrays are used, separated by 7500 feet, both at depths of 1300 feet. The arms of each array are parallel to the corresponding coordinate system axes. Once again 1000 positions were randomly generated in a uniform manner, this time over a 3 dimensional box.
Figure 5.1 Double Array Simulation Configuration.

Running crossways between the two arrays. The box is 1300 feet deep, 5000 feet long and 1000 feet across (see figure 5.1).

Each of the 1000 randomly chosen sound source positions in the two array simulation were processed as follows:

1. calculate the exact hydrophone times for the first array using the methods set forth in Appendix A;
2. add to the exact times some random error at the specified level;
3. Repeat steps 1 and 2 for the second array, using a new set of random error values;
4. apply each of the six methods to the time values from both arrays, producing six different pairs of apparent positions;
5. apply the ray tracing procedure to both apparent positions in each of the six pairs, utilizing the known linear velocity profile, producing six pairs of estimated sound source positions;
6. for each of the six pairs of positions, calculate the distance between the two positions in the pair;
7. make pairwise comparisons of the six different distances, to see which method in each pair exhibits the closest agreement between its two position estimates.

This time the comparison being made is that one method is considered better than another if the positions it produces agree more closely more often than those of the other method.

G. DOUBLE ARRAY SIMULATION RESULTS

Tables IX and X contain the results of the double array simulation. Each tabled entry represents the fraction of time that the method of that row produced a pair of estimates which were in closer agreement than the estimates produced by the method of that column. Significance criteria for these proportions are the same as for the single array simulation.

These results are similar to those of the single array simulation, because they indicate that:
TABLE IX
Double Array Simulation Results for Lower Error Levels

<table>
<thead>
<tr>
<th></th>
<th>EPROF LEVEL : ZERO</th>
<th>ERROR LEVEL : LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NAVY</td>
<td>NAVY_A</td>
</tr>
<tr>
<td>NAVY</td>
<td>.366</td>
<td>.989</td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.634</td>
<td>.989</td>
</tr>
<tr>
<td>L.S.</td>
<td>.011</td>
<td>.011</td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.274</td>
<td>.266</td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.032</td>
<td>.032</td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.788</td>
<td>.788</td>
</tr>
<tr>
<td></td>
<td>NAVY</td>
<td>NAVY_A</td>
</tr>
<tr>
<td>NAVY</td>
<td>.448</td>
<td>.989</td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.552</td>
<td>.989</td>
</tr>
<tr>
<td>L.S.</td>
<td>.011</td>
<td>.011</td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.339</td>
<td>.316</td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.048</td>
<td>.048</td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.690</td>
<td>.688</td>
</tr>
</tbody>
</table>

1. the NAVY_A method outperforms the original unadjusted NAVY method, although the difference is not significant in the higher error level cases;
2. the most successful performer is consistently the M.L.S. method;
3. spherical methods almost always outperform planar methods;
4. increased error levels tend to lessen the distinction between methods.
### TABLE X

**Double Array Simulation Results for Higher Error Levels**

<table>
<thead>
<tr>
<th>ERROR LEVEL: MEDIUM</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.L.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>.498</td>
<td>.841</td>
<td>.513</td>
<td>.808</td>
<td>.443</td>
<td></td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.532</td>
<td>.842</td>
<td>.542</td>
<td>.809</td>
<td>.440</td>
<td></td>
</tr>
<tr>
<td>L.S.</td>
<td>.159</td>
<td>.158</td>
<td>.159</td>
<td>.525</td>
<td>.160</td>
<td></td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.487</td>
<td>.458</td>
<td>.841</td>
<td>.808</td>
<td>.433</td>
<td></td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.192</td>
<td>.191</td>
<td>.475</td>
<td>.192</td>
<td>.144</td>
<td></td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.557</td>
<td>.560</td>
<td>.840</td>
<td>.567</td>
<td>.856</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ERROR LEVEL: HIGH</th>
<th>NAVY</th>
<th>NAVY_A</th>
<th>L.S.</th>
<th>L.S.C.</th>
<th>M.L.P.</th>
<th>M.L.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAVY</td>
<td>.487</td>
<td>.556</td>
<td>.535</td>
<td>.479</td>
<td>.442</td>
<td></td>
</tr>
<tr>
<td>NAVY_A</td>
<td>.513</td>
<td>.558</td>
<td>.499</td>
<td>.481</td>
<td>.440</td>
<td></td>
</tr>
<tr>
<td>L.S.</td>
<td>.444</td>
<td>.442</td>
<td>.445</td>
<td>.451</td>
<td>.422</td>
<td></td>
</tr>
<tr>
<td>L.S.C.</td>
<td>.465</td>
<td>.501</td>
<td>.555</td>
<td>.480</td>
<td>.438</td>
<td></td>
</tr>
<tr>
<td>M.L.P.</td>
<td>.521</td>
<td>.519</td>
<td>.549</td>
<td>.520</td>
<td>.428</td>
<td></td>
</tr>
<tr>
<td>M.L.S.</td>
<td>.558</td>
<td>.560</td>
<td>.578</td>
<td>.562</td>
<td>.572</td>
<td></td>
</tr>
</tbody>
</table>

There are however some indications from these results which are different from those of the single array simulation, such as:

1. the worst performer was consistently the L.S. method, rather than the M.L.P. method;
2. in the high error case the M.L.P. method is equal to, or even marginally better than, every other method except the M.L.S. method;
3. The performance of the M.L.S. method is noticeably better under error free conditions.

The last two inferences are perhaps the most interesting. The maximum likelihood approach was intended to estimate and account for Gaussian errors in the timing data values. Hence it is really not very surprising that the M.L.S. method does well with error prone data. But it is interesting to note that the M.L.P. method also does well under high error levels, even though it probably suffers from a bias due to the planar assumption.

Of even greater interest is that the M.L.S. method seems to be at its best when compared to other methods under error free conditions. This result was unexpected, and indicates that the M.L.S. method not only handles timing errors well, as was intended, but apparently also does an even better job of approximating the elusive exact solution to the original four spherical equations of (4.1) and (2.1) when the exact time values are available.

H. LIMITATIONS ON INTERPRETATION OF RESULTS

The results of both simulations seem to imply that the M.L.S. method outperforms the currently used NAVY_A method on any randomly chosen sound source position, with or without timing errors. These are encouraging results. Nevertheless the reader is cautioned that these tests are just simulations, and like all simulations, must make assumptions which cannot fully reflect the reality of actual hydrophonic tracking conditions. The most important assumptions made for these simulations are:

1. The sound velocity profile is known exactly, and is linear;
2. The errors in the timing values from any hydrophone are normally distributed with mean zero, and are independent of the noise in any other hydrophone; and
3. The parameters of the simulation were fixed; for example, the single array simulation used a fixed range of 3000 feet, and both simulations used arrays at 1300 feet of depth, with fixed orientations to the range coordinate system.

The first assumption is probably of little consequence. It not only greatly facilitates computations, but also helps to isolate the initial angle and time estimation problem from the unrelated errors involved in the velocity profile estimation procedure.

The second assumption is somewhat more troublesome. Errors may not be Gaussian at all, or if they are, the mean may not be zero. Unfortunately each position estimation involved only four equations, and therefore did not allow for estimation of more than four parameters. Therefore the error mean, being a fifth parameter, could not be estimated. Also the errors of any one hydrophone may very well not be independent of the errors of the other three phones on that array. Fortunately these concerns are offset somewhat by the results of the double array simulation, wherein it was found that the M.L.S. method was at its best when there was no noise at all.

Also the error type and level may depend on other factors, such as the target's range, elevation and azimuth angles from the array. This highlights the concerns of the third assumption. There is considerable room here for future work concerning the dependency of results on such complicating factors.

Lastly it should be pointed out that the simulations make comparisons only on the binomial basis of better versus worse in 1000 trials. The magnitudes of the actual differences are ignored. It is possible, though perhaps unlikely, that while one method marginally outperforms a second method in most trials, in all the remaining trials the first method
is much worse than the second. There is also room here for further work.
VI. CONCLUSIONS AND RECOMMENDATIONS

A. ESTIMATION OF TIMING DATA VARIATION

One of the primary purposes of this study was to estimate the amount of variability in the timing data being recorded during actual tracking runs. This problem was addressed by the M.L.P. and M.L.S. maximum likelihood models. The M.L.P. model, as previously discussed, suffered from a bias due to the planar assumption, was the poorest estimator of positions among all the models, and produced an inflated variance estimate. Therefore the spherical model M.L.S. is used to estimate the data variance.

The variability that is measured by the M.L.S. model is made up of three components. First there is the variance induced by the spherical assumption. Then there is the variance caused by the seven decimal accuracy used when recording the data. Finally there are the errors inherent in the physical process, due to such factors as hydrophone variability or malfunction, local distortions of the sound wave, and inexactness of the water column which estimates the speed of sound profile. The last two sources of error together make up the variability that is involved in the time values which are ultimately used in position estimation, and is therefore the variation that is to be estimated.

If it is assumed that the variability induced by the spherical assumption is independent of the data variability, then the M.L.S. variance estimate is the sum of those two variances, or

\[ \sigma_{mls}^2 = \sigma_{sph}^2 + \sigma_{time}^2 \]
Therefore the data variance can be estimated by first estimating the variability induced by the spherical assumption, and then subtracting it from the M.L.S. estimate of the variation.

The M.L.S. estimate was obtained by applying the M.L.S. method to the data from a tracking run at the Naceose torpedo tracking range on May 6, 1980. That run involved position estimation by several different hydrophone arrays. The run made several thousand position estimates, 733 of which were at depths of 100 feet or more and involved targets not more than 4700 feet from the sensing array. The depth limitation was imposed to avoid the excessive complications caused by the radical changes in the velocity profile above that depth (see figure 2.1). The maximum range limitation imitates the data validation procedure at the tracking range, where positions farther than 4700 feet from the array are discarded.

The tracking data yielded the following range of estimates for the standard deviation of the data noise, using the M.L.S. model.

<table>
<thead>
<tr>
<th>Value</th>
<th>2.89 E-5 secs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAXIMUM VALUE</td>
<td>1.18 E-5 secs.</td>
</tr>
<tr>
<td>0.95 QUANTILE</td>
<td>3.02 E-6 secs.</td>
</tr>
<tr>
<td>MEDIAN VALUE</td>
<td>4.11 E-7 secs.</td>
</tr>
<tr>
<td>0.05 QUANTILE</td>
<td>3.13 E-8 secs.</td>
</tr>
<tr>
<td>MINIMUM VALUE</td>
<td>3.13 E-8 secs.</td>
</tr>
</tbody>
</table>

For an overall estimate of the noise, the median value was used, so that:

$$\sigma^2_{\text{M.L.S.}} = (3.02 \ E^{-6})^2$$

The variability induced by the spherical assumption was estimated by applying the M.L.S. procedure to perfectly noiseless data in the idealized environment of the single array simulation of Chapter V. This was done for targets at ranges of 1500 to 4500 feet, at 500 feet increments, with
1000 randomly chosen targets at each range. The results are collected in Table XI.

**TABLE XI**

<table>
<thead>
<tr>
<th>RANGE</th>
<th>MINIMUM</th>
<th>Q(.05)</th>
<th>MEDIAN</th>
<th>Q(.95)</th>
<th>MAXIMUM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500</td>
<td>2.72E-10</td>
<td>2.84E-8</td>
<td>2.08E-7</td>
<td>3.13E-7</td>
<td>3.29E-7</td>
</tr>
<tr>
<td>2000</td>
<td>2.12E-9</td>
<td>5.88E-8</td>
<td>2.26E-7</td>
<td>3.14E-7</td>
<td>3.30E-7</td>
</tr>
<tr>
<td>2500</td>
<td>3.90E-8</td>
<td>1.09E-7</td>
<td>2.34E-7</td>
<td>3.14E-7</td>
<td>3.31E-7</td>
</tr>
<tr>
<td>3000</td>
<td>8.35E-8</td>
<td>1.50E-7</td>
<td>2.37E-7</td>
<td>3.15E-7</td>
<td>3.24E-7</td>
</tr>
<tr>
<td>3500</td>
<td>1.21E-7</td>
<td>1.59E-7</td>
<td>2.37E-7</td>
<td>3.13E-7</td>
<td>3.24E-7</td>
</tr>
<tr>
<td>4000</td>
<td>1.47E-7</td>
<td>1.65E-7</td>
<td>2.39E-7</td>
<td>3.16E-7</td>
<td>3.25E-7</td>
</tr>
<tr>
<td>4500</td>
<td>1.46E-7</td>
<td>1.69E-7</td>
<td>2.42E-7</td>
<td>3.14E-7</td>
<td>3.25E-7</td>
</tr>
</tbody>
</table>

Table XI shows that the median and maximum inflation values are reasonably independent of target range. The minimum values vary somewhat, but only for range values below 3000 feet. This represents a very stable situation overall. Therefore, the inflation due to the spherical assumption is estimated by the median value at a range of 3000 feet, namely:

\[
\sigma^2 = (2.37 \times 10^{-7})^2
\]

Combining these two estimates, the variance estimated for the timing data is
\[ \sigma_{\text{time}}^2 = \sigma_{\text{mls}}^2 - \sigma_{\text{sph}}^2 = (3.02 \times 10^{-6})^2 - (2.37 \times 10^{-7})^2 = (3.011 \times 10^{-6})^2 . \]

As can be seen, the error induced by the spherical model is less than 10% of the M.L.S. error estimate. Therefore when it is accounted for by subtraction from the M.L.S. variation estimate, the final variance estimate changes little.

The estimated value indicates a standard error in the 6th decimal place. With a typical speed of sound value of 4880 feet per second, this represents a position differential of about
\[ 4880 \times 3.011 \times 10^{-6} = 0.015 \text{ feet} . \]

This estimate is quite low, indicating that the time values being recorded are sufficiently accurate.

There is considerable opportunity for additional work determining the relationship, if any, between the time variance and other factors such as angles of elevation and azimuth of the target from the array.

There is also the problem that the time variation is likely to be array dependent. For example, consider figure 6.1, wherein the standard error estimates from the actual tracking run are plotted versus the range of the target. The plot does not indicate that there is any simple relationship between range and error level. However the plot does show a bunching pattern. When the error estimates are plotted separately for each array, then the bunching pattern becomes clearly associated with the individual arrays. Consider figures 6.2 and 6.3, where the separate plots have been made for four different arrays. It is still not clear from these plots whether the principle effect is due to the individual arrays, or the ranges of the targets. However some level of array dependency seems likely, indicating a need for additional investigation.
Figure 6.1 Error Estimation Versus Range of Target.

B. CHOICE OF METHOD

Clearly all indications are that the planar wavefront models, L.S. and M.L.F. are not candidates for use as position estimators. Furthermore the hybrid model L.S.C. is an interesting improvement, but never really performs well enough compared to the M.L.S. and NAVY models.

The original NAVY model is usually outperformed by the adjusted NAVY_A method. However the differences are not always significant.

The spherical model M.L.S., on the other hand, consistently outperformed all other methods during the simulated evaluations. It would seem that M.L.S. is the model of choice. It does the best job of handling normally distributed errors in the data. But that is not the strongest
Figure 6.2  Error Estimation for Arrays 7 and 14.
Figure 6.3 Error Estimation for Arrays 56 and 57.
A more important, and surprising, argument in its favor is that when the exact, error free times are used, the M.L.S. apparent position estimate will usually produce the most accurate estimate of the true apparent position. This is the desired overall result, so that the sensitive ray tracing procedure will be affected minimally by apparent position estimation errors.

There are nevertheless several notes of caution which should be considered before embracing the M.L.S. method wholeheartedly. The first caution has been stressed before, namely that these conclusions were arrived at under the idealized conditions of the simulations scenarios. The second caution, also previously stressed, is that the actual magnitudes of the differences between position estimates has been ignored. It is conceivable that while one method always produces a better estimate than another, the difference between any two position estimates is acceptably small.

Lastly there is the caution that the M.L.S. model involves a complicated iterative procedure which uses considerable computer time. It is probably too slow a procedure for use with 'real time' analysis during the execution of tracking runs. For real time tracking the NAVY_A method currently in use is probably preferrable due to its simple computations.

However, for post run analysis, and also possibly for calibration of the hydrophone arrays, the M.L.S. method is recommended as being a more exact and more robust position estimator than those methods currently in use.

C. RECOMMENDATIONS FOR FUTURE INQUIRY

Several recommendations have already been made for work needed to estimate the effect of suitable independent variables on both timing errors and the bias in certain methods.
In addition there exist at least two other areas for possibly fruitful investigation.

The first area concerns the interplay between methods. Specifically, the binomial comparisons of Chapter V show that even the worst methods are better than each of the other methods at least part of the time. Hence it is possible that the best method overall would be a suitable combination of methods, wherein each method is used where it is most effective. For example, even though the M.L.S. method has been indicated as the best method for any randomly selected position, it may be consistently outperformed by another method under certain circumstances, such as extremely high or low elevation angles.

The second area for possible work addresses the question of how to next improve upon the existing models. It is herein suggested that the next improvement in modelling would be a method which is based on a linear velocity assumption. As figure 2.1 demonstrates, a linear velocity profile is a reasonable approximation for most depths. This would be the next logical step above the constant velocity assumption which is associated with the spherical models. Most of the mathematical basis for such a model is contained in Appendix A. Possibly a suitable set of equations could be developed involving the hydrophone times and reflecting the linear velocity assumption. If so, the least squares or maximum likelihood techniques might provide useful results.
APPENDIX A
LINEAR VELOCITY PROFILE THEORY

All computations in this study are made under the assumption that the sound velocity is directly related to depth in a linear manner, and is known exactly. Under these circumstances many closed form results, not otherwise available, can be obtained and used in those computations.

Suppose that the velocity profile is given by
\[ V = V_0 + V_1 z \]
where \( V_0 \) and \( V_1 \) are known constants, and \( z \) is the depth variable, measured down from the water's surface. In this case it is known [Ref. 4] that the path of a sound ray from a signal source to a hydrophone will have the shape of an arc of a circle. The center of that circle will be somewhere above the surface of the water. The vertical placement of that center is determined by the value of \( z \) at which the speed of sound equals zero (see figure A.1). Although that depth is negative and is not really a depth at all, it nevertheless has geometric meaning.

Consider the vertical plane containing the circle center, the sound source and the hydrophone. Let \( h \) be the variable which measures the horizontal position in that plane. Let \( (h, z) = (a_1, a_2) \) be the position of the hydrophone, and let \( (h, z) = (p_1, p_2) \) be the position of the sound source. Then \( C_2 \) given by (A.1) is the \( z \) coordinate of the circle center.

\[
C_2 = \frac{-V_0}{V_1} \quad \text{(A.1)}
\]

What must be found is \( C_1 \), the \( x \) coordinate of the circle center, and \( r \), the radius of the circle. To solve for these values, the equation of the circle is used, evaluated at the two known locations, yielding (A.2).
Figure A.1  Circular Ray Paths for a Linear Velocity Profile.

\[(a_1 - C_1)^2 + (a_2 - C_2)^2 = r^2 \]  
\[(p_1 - C_1)^2 + (p_2 - C_2)^2 = r^2 \]  (A.2)

The left hand sides of the two equations of (A.2) can be equated, and solved for $C_1$, leading to (A.3).

\[C_1 = \frac{(p_1 + a_1)}{2} + \frac{(p_2 - a_2)}{2(p_1 - a_1)}(p_2 + a_2 - 2C_2) \]  (A.3)

When this value of $C_1$ is substituted into the first equation of (A.2), then the result is given by (A.4).
\[
\begin{align*}
    r &= \sqrt{(a_1 - c_1)^2 + (a_2 - c_2)^2} \quad (A.4)
\end{align*}
\]

The circular arc length between the hydrophone and the sound source is easily computed. Unfortunately the velocity of sound does not stay constant along that arc. Therefore in order to determine the amount of time \((T)\) required for the sound ray to travel the ray path, the effect of the velocity must be integrated along the arc. This is done by (A.5), where \(S\) is the arc length between the two points, and \(V(s)\) is the speed of sound as a function of position on the arc.

\[
T = \int_0^S \frac{ds}{V(s)} \quad (A.5)
\]

In (A.5) the sound source position corresponds to an arc length of \(s = 0\), and the hydrophone position corresponds to arc length \(s = S\). In [Ref. 4] this integral is shown to be equivalent to (A.6).

\[
T = -\frac{1}{V_1} \int_{A_0}^{A_1} \frac{da}{\cos(a)} \quad (A.6)
\]

In (A.6) \(A_0\) is the angle of elevation of the ray path at the sound source, and \(A_1\) is the elevation angle at the hydrophone. The antiderivative in (A.7) can be used to solve (A.6), leading to the ray path transit time expression in (A.8).

\[
\int \frac{da}{\cos(a)} = \ln \left( \frac{1 + \sin(a)}{\cos(a)} \right) \quad (A.7)
\]

\[
T = -\frac{1}{V_1} \ln \left( \frac{\cos(A_1)}{\cos(A_0)} \cdot \frac{1 + \sin(A_0)}{1 + \sin(A_1)} \right) \quad (A.8)
\]
If the elevation angle at any point along the arc is denoted by \( A \), then (A.9) relates the angle to the derivative of \( z \) with respect to \( h \) along the ray path.

\[
\tan (A) = \frac{dz}{dh} \tag{A.9}
\]

Implicit differentiation of the equation of the circle yields (A.10) as another expression for the same derivative.

\[
\frac{dz}{dn} = \frac{h - C_1}{z - C_2} \tag{A.10}
\]

Equate the two derivatives to give (A.11) which relates the elevation angle and the position \((h, z)\) on the arc.

\[
\tan (A) = \frac{h - C_1}{z - C_2} \tag{A.11}
\]

From (A.11) a simple geometric argument produces the equations in (A.12).

\[
\cos (A) = \frac{z - C_2}{r} \quad \sin (A) = \frac{h - C_1}{r} \tag{A.12}
\]

First let \( A, h \) and \( z \) be equal to \((A_1, a_1, a_2)\) in (A.12), and then let them equal \((A_0, p_1, p_2)\). Then substitute both expressions into (A.8). The result is (A.13), the desired expression for the ray path transit time in terms of the positions of the sound source and the hydrophone.

\[
T = \frac{1}{V_1} \ln \left( \frac{a_2 - C_2}{P_2 - C_2} \left( \frac{r + P_1 - C_1}{r + a_1 - C_1} \right) \right) \tag{A.13}
\]

To summarize, if a ray path is to terminate at the three dimensional position \( (X_1, Y_1, Z_1) \), and the sound source is at \( (X_2, Y_2, Z_2) \), then perform the following steps in order to calculate the exact ray path time and elevation angle that would correspond to a linear velocity profile:
1. Use (A.14) to translate the three dimensional positions to the two dimensional coordinates of the previously described vertical plane, with the origin at the water surface directly above the end of the ray path:

\[ P_1 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad (A.14) \]

\[ P_2 = z_1, \quad a_1 = 0, \quad a_2 = z_2 \]

2. Calculate C2, C1 and r using (A.1), (A.3), and (A.4);

3. Calculate the exact ray path transit time \( T \) using (A.13); and

4. Calculate the exact elevation angle \( \Theta \) at the hydrophone, using (A.15):

\[ \Theta = \arccos \left( \frac{a_2 - C_2}{r} \right) \quad (A.15) \]
APPENDIX B
PARTIAL DERIVATIVE FORMULAE FOR NEWTON'S METHOD

The central tool used by Newton's Method in the development of the M.L.S. model in Chapter IV is the matrix GP. It is the matrix of first order derivatives of the error expressions $g_1$, $g_2$, $g_3$ and $g_4$. Those derivatives are set forth in this appendix.

If $X = (C_1, C_2, C_3, U)$, then the error functions are defined by (B.1),

$$g_i(X) = C_i - \frac{T_i - \sqrt{K_i}}{\sqrt{K_i}} \quad i = 1, 2, 3 \quad (B.1)$$

$$g_4(X) = U - \frac{T_4 - D M / V}{1 - N}$$

where the values $K_i$, $M$ and $N$ are the functions given by (B.2).

$$K_i = U^2 - (2UDC_i / V) + \frac{D^2}{V^2} \quad (B.2)$$

$$N = \sum_{j=1}^{3} C_j \left( \frac{T_j - \sqrt{K_j}}{\sqrt{K_j}} \right)$$

$$M = \sum_{j=1}^{3} \frac{T_j - \sqrt{K_j}}{\sqrt{K_j}}$$

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Recall that \( GP \) is the matrix shown in (B.3).

\[
GP (C_1, C_2, C_3, U) = \begin{bmatrix}
\frac{\partial g_1}{\partial C_1} & \frac{\partial g_1}{\partial C_2} & \frac{\partial g_1}{\partial C_3} & \frac{\partial g_1}{\partial C_4} \\
\frac{\partial g_2}{\partial C_1} & \frac{\partial g_2}{\partial C_2} & \frac{\partial g_2}{\partial C_3} & \frac{\partial g_2}{\partial C_4} \\
\frac{\partial g_3}{\partial C_1} & \frac{\partial g_3}{\partial C_2} & \frac{\partial g_3}{\partial C_3} & \frac{\partial g_3}{\partial C_4} \\
\frac{\partial g_4}{\partial C_1} & \frac{\partial g_4}{\partial C_2} & \frac{\partial g_4}{\partial C_3} & \frac{\partial g_4}{\partial C_4}
\end{bmatrix}
\]  

Then given the definitions above, the following derivatives are the result of straightforward, though tedious, differentiation, and are offered without detailed proof.

**Lemma 1**

\[
\frac{\partial M}{\partial C_i} = \frac{V K_i (T_i - K_i) + C_i T_i U D}{V K_i \sqrt{K_i}} \quad i = 1, 2, 3 (B.4)
\]

**Lemma 2**

\[
\frac{\partial M}{\partial U} = \sum_{j=1}^{3} \frac{C_j T_j (D C_j - V U)}{V K_j \sqrt{K_j}} \quad (B.5)
\]

**Lemma 3**

\[
\frac{\partial N}{\partial C_i} = \frac{T_i U D}{V K_i \sqrt{K_i}} \quad i = 1, 2, 3 (B.6)
\]

**Lemma 4**

\[
\frac{\partial N}{\partial U} = \sum_{j=1}^{3} \frac{T_j (D C_j - V U)}{V K_j \sqrt{K_j}} \quad (B.7)
\]
The first three diagonal elements of \( GP \) are the derivatives of each \( g_i \) with respect to \( C_i \) (\( i=1,2,3 \)) and are given by (B.8).

\[
\frac{\partial g_i}{\partial C_i} = 1 - \left[ \frac{T_i D M - V K_i (T_i - \sqrt{K_i})}{V M^2 K_i \sqrt{K_i}} \right] \quad (B.8)
\]

The off diagonal elements in the first three rows and columns of \( GP \) are the derivatives of each \( g_i \) with respect to \( C_j \), and are given by (B.9).

\[
\frac{\partial g_i}{\partial C_j} = \left( \frac{T_i - \sqrt{K_i}}{M^2 \sqrt{K_i}} \right) \frac{\partial M}{\partial C_j} \quad \text{for } i = 1,2,3 \quad j = 1,2,3 \quad j \neq i
\]

The derivative of each \( g_i \) with respect to \( U \) is given by (B.10).

\[
\frac{\partial g_i}{\partial U} = \frac{T_i M (VU - DC_i) + V K_i (T_i - \sqrt{K_i}) \frac{\partial M}{\partial U}}{V M^2 K_i \sqrt{K_i}} \quad (B.10)
\]

The first three elements of the last row of \( GP \) are the derivatives of \( g_4 \) with respect to each \( C_i \), and are given by (B.11).

\[
\frac{\partial g_4}{\partial C_i} = \frac{\partial^N}{\partial C_i} \left( VT_4 - D M \right) - D \frac{\partial M}{\partial C_i} (1-N) \quad (B.11)
\]

\[
\frac{\partial^N}{\partial (U - N)^2}
\]
FORMULA 5

The last row, last column of $\mathcal{C}P$ is the derivative of $g_u$ with respect to $U$, and is given by (B.12).

$$\frac{\partial g_4}{\partial U} = 1 - \frac{\partial N}{\partial U} (vT_4 - DM) - D \left( \frac{\partial M}{\partial U} \right) (1 - N) \quad (B.12)$$

$$V (1 - N)^2$$
APPENDIX C
UNIFORM SAMPLES ON A TRUNCATED HEMISPHERE

The single array simulation of Chapter V required a random sample of positions in space uniformly distributed over the surface of a truncated hemisphere. The hemisphere is to be of radius \( r \) (3000 feet) about the acoustic center of a hydrophonic array. The truncation of the overall sphere is due to the fact that the upper portion of the hemisphere is above the water surface, and the lower half is below the sea bottom.

\[
w = r_1 \, dH \\
r_1 = r \, \cos(E) \\
w = r \, \cos(E) \, dH
\]

Figure C.1 Hemispherical Geometry.

Let \( E \) be the variable denoting angles of elevation above the horizontal. Let \( H \) be the variable denoting horizontal azimuth angles about the center of the hemisphere. Consider a small piece of hemispherical surface area bounded by the
elevation angles $\theta_1$ and $(\theta_1 + d\theta)$, and by the azimuth angles $\beta_1$ and $(\beta_1 + d\beta)$ (see figure C.1). If $w$ is the horizontal width of that piece, then $w$ is given by (C.1).

$$w = r_1 d\theta = r \cos(\theta) \, d\theta \quad \text{(C.1)}$$

If $A_1$ is the area of that piece, then $A_1$ is approximated by (C.2).

$$A_1 = w \, d\beta = r \cos(\theta) \, d\theta \, d\beta \quad \text{(C.2)}$$

Now suppose that $0 < \theta_1 < \theta_2 < \theta_{\max}$ where $\theta_{\max}$ is the elevation angle of the top of the truncated hemisphere. Then the ratio $A_1/A_2$ of two different areas at elevation angles $\theta_1$ and $\theta_2$ is given by (C.3).

$$\frac{A_1}{A_2} = \frac{r \cos(\theta_1)}{r \cos(\theta_2)} \frac{d\theta \, d\beta}{d\theta \, d\beta} = \frac{\cos(\theta_1)}{\cos(\theta_2)} \quad \text{(C.3)}$$

If $n_1$ and $n_2$ are to be the (relative) sample sizes from the two areas $A_1$ and $A_2$ respectively, then uniformity of the sample requires that (C.4) hold.

$$\frac{A_1}{n_1} = \frac{A_2}{n_2} \quad \text{(C.4)}$$

The combination of (C.4) and (C.3) implies (C.5).

$$n_2 = n_1 \frac{\cos(\theta_2)}{\cos(\theta_1)} \quad \text{(C.5)}$$

Letting $\theta_1 = 0$, then the relative sample size $n_2$ for area $A_2$ is given by (C.6), where $n$ is the relative sample size at the base of the hemisphere.

$$n_2 = n \cos(\theta_2) \quad \text{(C.6)}$$

Now the differential probability of drawing a random position that has elevation angle $\theta$ is given by (C.7), where
N is the total sample size to be drawn from the surface on the sector bounded by the azimuth angles \( H_1 \) and \( (H_1 + dH) \).

\[
f_E(E) = \frac{n}{N} = \frac{n}{N} \cos(E) \quad \text{(C.7)}
\]

Therefore, if \( K \) is defined to be the constant ratio \( n/N \), then the corresponding cumulative distribution is given by (C.8).

\[
F_E(E) = \int_0^E K \cos(e) \, de = K \sin(E) \quad \text{(C.8)}
\]

For the highest elevation angle, \( E_{\text{max}} \), the cumulative distribution function must equal one, so that (C.9) holds.

\[
F_E(E_{\text{max}}) = 1 = K \sin(E_{\text{max}}) \quad \text{(C.9)}
\]

As a result, the constant \( K \) is determined by (C.10).

\[
K = \frac{1}{\sin(E_{\text{max}})} \quad \text{(C.10)}
\]

Therefore the complete cumulative distribution function is given by (C.11).

\[
F_E(E) = \frac{\sin(E)}{\sin(E_{\text{max}})} \quad \text{(C.11)}
\]

Now the inverse probability transform can be used. If \( U \) is a uniform random variable on the interval \((0, 1)\), then let \( E \) be given by (C.12).

\[
E = \arcsin \left( \frac{U}{\sin(E_{\text{max}})} \right) \quad \text{(C.12)}
\]

Now choose an azimuth angle \( H \) randomly and uniformly over the interval \((0, 2\pi)\). Then \((X, Y, Z)\) given by

\[
X = r \cos(H) \cos(E) \\
Y = r \sin(H) \cos(E) \\
Z = r \sin(E)
\]

93
is the corresponding position in spherical coordinates. That position is a random position drawn from a population of positions uniformly distributed across the surface of a truncated hemisphere with maximum elevation angle $E_{\text{max}}$. 
APPENDIX D

SINGLE ARRAY SIMULATION COMPUTER PROGRAM

This FORTRAN program selects a random sample of size M of 3-dimensional positions all at a range of R feet from the acoustic center of a hydrophonic array. The arms of the array are assumed to be aligned with the coordinate system of the range and is in a position given by the vector A.

The speed of sound profile is assumed linearly given by

\[ V = V_1 + V_2 \cdot \text{depth} \]

For each of the positions selected, exact hydrophone times are calculated using the subroutine ICOMP and random errors are added to the exact times. The error distribution is normal, with mean zero and standard deviation SDEV.

The time values are then used to estimate an apparent position by two different methods, using appropriate subroutines. The resulting apparent positions are then processed by the subroutine TRACE to obtain the corresponding actual position estimates. The resulting position estimates are then compared to the original true position to see which method produced the most accurate estimate.

This is done for all possible pairs of the six methods.

```
INTEGER M, J, K, N, METH, METH1, METH2, NYES, NYEST, NNOISE
DIMENSION REL(1606), RA(1000), STUFF(400)
DOUBLE PRECISION P(1000,3), A(3), VV(2), T(1000,4), VV(2,2), VV(2,3)
DOUBLE PRECISION TEST(1000), T(1000,4), DIF(1000)
DOUBLE PRECISION DIFT(1000), AC(3), TC(1000)
DOUBLE PRECISION V, Z, SDEV, PI, T(1000), EL(1000)
DOUBLE PRECISION DIFT(1000)

M = 1000
R = 3000.0DO
SDEV = 3.464d-5
FORMAT(2X, 'NOISE STD. DEV. = ', F15.10)
WRITE(6, 1) SDEV
WRITE(6, 2) SDEV
SEED = 931947.0DO
PI = 2.0DO * DARCOS(0.0DO)

SET INITIAL VALUES OF SAMPLE SIZE, RANGE, AND ERROR STANDARD DEVIATION

M = 1000
R = 3000.0DO
SDEV = 3.464d-5

FORMAT(2X, 'NOISE STD. DEV. = ', F15.10)
WRITE(6, 1) SDEV
WRITE(6, 2) SDEV

SET ARRAY POSITION AND LINEAR PROFILE FOR SPEED OF SOUND.

A(1) = 0.0DO
A(2) = 1300.0DO
AC(1) = A(1) + 15.0
AC(2) = A(2) + 15.0
AC(3) = A(3) - 15.0

```

ALTERNATIVE MODELS FOR CALCULATION OF ELEVATION ANGLES AND RAY TRANSIT

NAVY POSTGRADUATE SCHOOL
MONTEREY, CA  C.D. MAIN SEP 84

END

F/G 17/1
NL
SET UPPER LIMIT ON ELEVATION ANGLE, IF NECESSARY

IF (PHIMAX .GT. A(3)) 30, 10

PHIMAX = 2.566D0

CALL GGUBS (SEED, M, RAZ)

CONVERT ANGLES TO 3-D COORDINATES

DO 11 I = 1, M
   EL(I) = REL(I)
   AZ(I) = RAZ(I)
   PHI(I) = DARSIN(EL(I)/FK)
   P(I,1) = R*DOSIN(PHI) * DCOS(AZ(I)) * PI*2.D0
   P(I,2) = R*DOSIN(PHI) * DCSIN(AZ(I)) * PI*2.D0
   P(I,3) = A(3) - R*DOSIN(PHI)

CONTINUE

COMPUTE EXACT HYDROPHONE TIMES UNDER THE LINEAR VELOCITY PROFILE ASSUMPTION.

CALL TCOMP (P, AC, M, VV, TC)

GENERATE AND ADD ERRORS TO TIMES.

NOISE = 4*M

CALL GGNNML (SEED, NOISE, STUFF)

DO 1111 METH1 = 1, 5
   KMETH = METH1 + 1
   DO 1222 METH2 = KMETH, 6
      IF (METH1 .EQ. METH2) GO TO 1222

WRITE (6, 260) METH = METH1

FOUR CUTER LOOPS RUN THROUGH ALL PAIRS OF THE SIX POSSIBLE POSITION ESTIMATING METHODS.

DO 1111 METH1 = 1, 5
   KMETH = METH1 + 1
   DO 1222 METH2 = KMETH, 6
      IF (METH1 .EQ. METH2) GO TO 1222

WRITE (6, 260) METH = METH1

SAMPLE SIZE = 'I, ' M

96
CALL SUBROUTINES TO IMPLEMENT THE METHODS

130 IF (METH .NE. 1) GO TO 131
CALL POSNAV (TN,V,M,PEST,TEST)
FORMAT(3X,'NAVY, UNCORRECTED')
WRITE(6,201) GO TO 150
131 IF (METH .NE. 2) GO TO 132
CALL POSNAV (TN,V,M,PEST,TEST)
FORMAT(2X,'NAVY, CORRECTED')
WRITE(6,202) GO TO 150
132 IF (METH .NE. 3) GO TO 133
CALL POSLS (TN,V,M,PEST,TEST)
FORMAT(2X,'LEAST SQUARES, UNCORRECTED')
WRITE(6,203) GO TO 150
133 IF (METH .NE. 4) GO TO 134
CALL POSLS (TN,V,M,PEST,TEST)
FORMAT(18X,'LEAST SQUARES, CORRECTED')
WRITE(6,204) GO TO 150
134 IF (METH .NE. 5) GO TO 135
CALL POSMLP (TN,V,M,PEST,TEST)
FORMAT(18X,'MAX. LIKELIHOOD, PLANAR')
WRITE(6,205) GO TO 150
135 IF (METH .NE. 6) GO TO 136
CALL POSMLP (TN,V,M,PEST,TEST)
FORMAT(18X,'MAX. LIKELIHOOD, SPHERICAL')
WRITE(6,206) GO TO 150
136 WRITE(6,137) METH,METH2
137 FORMAT(2T5,14,1) AND/OR 1,14)
GO TO 1500

THE APPARENT POSITION ESTIMATES ARE RELATIVE TO THE ARRAY CENTER, AND MUST BE TRANSLATED TO THE TRACKING RANGE COORDINATE SYSTEM.

150 DO 145 I = 1,M
PEST(I,1) = PEST(I,1) + A(1)
PEST(I,2) = PEST(I,2) + A(2)
PEST(I,3) = A(3) - PEST(I,3)
CONTINUE

CONVERT APPARENT POSITIONS TO ACTUAL ESTIMATES BY USING THE TRACING PROCEDURE.

CALL TRACE (PEST,PT,TEST,A,M,VV)

CALCULATE THE DIFFERENCE BETWEEN THE 1ST POSITION ESTIMATE AND THE TRUE POSITION.

IF (METH .NE. METH1) GO TO 160
DO 155 J = 1,M
DIF(J) = 0.0 DO (DIF(J) = DABS(PEST(J) - TC(J,4))
DO 156 J = 1,M
DIF(J) = DIF(J) + (PT(J) - P(I,J))**2
CONTINUE
CONTINUE
METH = METH2  
GO TO 130

CALCULATE DIFFERENCE BETWEEN THE 2ND POSITION ESTIMATE AND THE TRUE POSITION, AND COMPARE IT TO THE 1ST POSITION DIFFERENCE.

160 NYES = 0
NYEST = 0
DO 166 I = 1,M
    DO 167 J = 1,3
167 CONTINUE
    DIFT(I) = DIFT(I) - DABS (T(I) - TC(I,4))
    IF ( DIFT(I) .LT. 0.0D0 ) GO TO 165
    IF ( DIFT(I) .GT. 0.0D0 ) GO TO 166
    NYEST = NYEST + 1
    CONTINUE
166 CONTINUE
    FRACT = ( DFLOAT(NYES) / DFLOAT(M) )
300 FORMAT (2X, 'FRACTION OF TIME METHOD #1 IS: ' , F7.3 )
301 FORMAT (2X, 'IN POSITION: ', F7.3 )
    WRITE (6,300) FRACT
    WRITE (6,301) FRACT
    WRITE (6,999) FRACT
700 FORMAT (1X, ' ===---------------')
993 FORMAT (2X, 'STOP')
END
APPENDIX E
DOUBLE ARRAY SIMULATION COMPUTER PROGRAM

THIS FORTRAN PROGRAM SELECTS A RANDOM SAMPLE OF SIZE M OF 3-DIMENSIONAL POSITIONS IN A BOX RUNNING CROSSWAYS BETWEEN TWO HYDROPHONIC ARRAYS. THE BOX HAS DIMENSIONS GIVEN BY THE VECTOR BOX. THE ARRAYS ARE SEPARATED BY 7500 FEET, AND HAVE COORDINATES GIVEN BY THE VECTORS A1 AND A2. THE ARMS OF THE ARRAYS ARE ALIGNED WITH THE RANGE COORDINATE SYSTEM.

THE SPEED OF SOUND PROFILE IS ASSUMED TO BE LINEAR, AND IS GIVEN BY
\[ v = v_v(1) + v_v(2) \cdot \text{DEPTH} \].

FOR EACH OF THE POSITIONS SELECTED, EXACT TIME VALUES FOR SOUND WAVE ARRIVAL AT THE HYDROPHONES OF EACH ARRAY ARE COMPUTED BY THE SUBROUTINE TCOMP. THEN RANDOM ERRORS ARE ADDED TO ALL TIMES. THE ERROR DISTRIBUTION IS NORMAL, WITH MEAN ZERO AND STANDARD DEVIATION SDEV.

THE TIME VALUES ARE THEN USED TO ESTIMATE APPARENT POSITIONS BY TWO DIFFERENT METHODS, USING THE APPROPRIATE SUBROUTINES. EACH METHOD PRODUCES TWO DIFFERENT ESTIMATES, ONE FOR EACH ARRAY. THE APPARENT POSITIONS ARE THEN TRANSLATED TO ACTUAL POSITIONS BY THE SUBROUTINE TRACE. THEN THE LENGTH OF THE DIFFERENCE VECTOR FOR EACH METHOD IS COMPUTED. THE TWO DIFFERENCE LENGTHS ARE COMPARED TO SEE WHICH METHOD PRODUCES POSITION PAIRS IN CLOSEST AGREEMENT. THIS COMPARISON IS DONE FOR ALL POSSIBLE PAIRS OF THE SIX METHODS.

INTEGER M, I, J, K, METH, METH1, METH2, NYES, NOISE, NARRAY
DIMENSION RP(1000), STUFF(4000), NAM1(6), NAM2(6)
DOUBLE PRECISION A1(3), A2(3), T1(1000, 4), T2(1000, 4)
DOUBLE PRECISION VAR(1000), NOISE, FRAC, XYZ, DEPTH
DOUBLE PRECISION TEST(1000), T2(1000, 3), SDEV(1000, 4)
DOUBLE PRECISION SDEV, SEED, BOX(3), PEST(1000, 3), A(3)
DOUBLE PRECISION T(1000, 4), V, VV(2), DIF(1000)

DATA NAM2(1), NAM2(2), NAM2(3), NAM2(4), NAM2(5), NAM2(6) /'CORR', 'PLNR', 'PLNR', 'PLNR', 'PLNR', 'PLNR' /

INITIALIZE VALUES FOR SAMPLE SIZE, RANGE, DEPTH AND ERROR STANDARD DEVIATION.

M = 1000
SDEV = 3.464D-5
DEPTH = 1300.0D0
FORMAT(2X, NOISE, SDEV = 'F15.10
WRITE (6, 1) SDEV
WRITE (6, 999) SEED
SEED = 9347.6D0
SET UP ARRAY POSITIONS, AND SOUND VELOCITY PROFILE

A1(1) = -3750. D0
A1(2) = 0. D0
A1(3) = DEPTH
A2(1) = 3750. D0
A2(2) = 0. D0
A2(3) = DEPTH
VV(1) = 4640.7 D0
VV(2) = 3314. D-5
V = VV(1) * VV(2) * DEPTH

FORMAT(2X,' (SAMPLE SIZE = ',I1,' )')
WRITE (6,30) M
WRITE (6,999)

DRAW UNDERWATER BOX FOR RANDOM POSITIONS

BOX(1) = 1000. D0
BOX(2) = 5000. D0
BOX(3) = DEPTH

GENERATE RANDOM POSITIONS IN BOX

DO 11 J = 1,2
   CALL GGUSBS(SEED,M,RP)
   DO 22 I = 1,M
      XYZ = RP(I)
      P1(I,J) = (XYZ - 0.5D0) * BOX(J)
      CONTINUE
   11 CONTINUE
   CALL GGUSBS(SEED,M,RP)
   DO 33 I = 1,M
      XYZ = RP(I)
      P2(I,3) = XYZ*BOX(3)
   33 CONTINUE

COMPUTE EXACT HYDROPHONE ARRIVAL TIMES, AND ADD RANDOM ERRORS TO TIME VALUES.

NOISE = M*4
CALL TCOMP(P1,A1,M,VV,T1)
CALL GGNML(SEED,NOISE,STUFF)
K = 1
DO 44 I = 1,M
   DO 45 J = 1,M
      NOISE = STUFF(K)
      NOISE = NOISE*SDEV
      T1(I,J) = T1(I,J) + NOISE
      K = K+1
   45 CONTINUE
   44 CONTINUE

CALL TCOMP(P1,A2,M,VV,T2)
CALL GGNML(SEED,NOISE,STUFF)
K = 1
DO 55 I = 1,M
   DO 56 J = 1,M
      NOISE = STUFF(K)
      NOISE = NOISE*SDEV
      T2(I,J) = T2(I,J) + NOISE
      K = K+1
   56 CONTINUE
   55 CONTINUE
SET UP LOOPS TO RUN THROUGH ALL PAIRS OF METHODS.

DO 1111 METH1 = 1, 5
     KMETH = METH1 + 1
DO 1222 METH2 = KMETH, 6
     IF (METH1 .EQ. METH2) GO TO 1222

FORMAT (2X, 'METHODS COMPARED ARE 1. ', A4, 2X, A4)
FORMAT (2X, 'AND 2. ', A4, 2X, A4)
WRITE (6, 10) NAM1(METH1), NAM2(METH1)
WRITE (6, 20) NAM1(METH2), NAM2(METH2)
WRITE (6, 999)

METH = METH1
NARRAY = 1

SET UP ARRAY BEING USED

A(I) = A1(I)
A(J) = A1(J)
A(K) = A1(K)
DO 66 I = 1, M
     DO 67 J = 1, 4
       T(I, J) = T1(I, J)
     CONTINUE
66 CONTINUE

CALL SUBROUTINES TO PERFORM ESTIMATION METHODS.

IF (METH .NE. 1) GO TO 131
     CALL POSNAV (T, V, M, PEST, TEST)
     GO TO 150
131 IF (METH .NE. 2) GO TO 132
     CALL POSNV (T, V, M, PEST, TEST)
     GO TO 150
132 IF (METH .NE. 3) GO TO 133
     CALL POSLS (T, V, M, PEST, TEST)
     GO TO 150
133 IF (METH .NE. 4) GO TO 134
     CALL POSLS (T, V, M, PEST, TEST)
     GO TO 150
134 IF (METH .NE. 5) GO TO 135
     CALL POSPLP (T, V, M, PEST, TEST, VAR)
     GO TO 150
135 IF (METH .NE. 6) GO TO 136
     CALL POSMLP (T, V, M, PEST, TEST, VAR)
     GO TO 150
136 WRITE (6, 137) METH1, METH2
137 FORMAT (2X, 'CANT FIND METHODS ', A4, ' AND/OR ', A4)
     GO TO 1000

IF (NARRAY .EQ. 2) GO TO 152

APPARENT POSITION ESTIMATES ARE IN LOCAL ARRAY COORDINATES AND MUST BE TRANSLATED TO TRACKING RANGE SYSTEM COORDINATES.

DO 144 I = 1, M
     PEST(I, 1) = PEST(I, 1) + A1(I)
     PEST(I, 2) = PEST(I, 2) + A1(I)
     PEST(I, 3) = A1(I) - PEST(I, 3)
CONTINUE

CORRECT APPARENT POSITIONS BY RAY TRACING.

CALL TRACE (PEST, P1, TEST, A, M, VV)
GO ON TO SECOND ARRAY

C
NAARRAY = 2
A(1) = A2(1)
A(2) = A2(2)
A(3) = A2(3)
DO 78 I = 1, M
    DO 78 J = 1, 4
        T(I, J) = T2(I, J)
    CONTINUE
78 CONTINUE
GO TO 130

C TRANSLATE 2ND ARRAY APPARENT POSITIONS TO RANGE COORDINATE SYSTEM, AND THEN RAY TRACE.

C
DO 145 I = 1, M
    PEST(I, 1) = PEST(I, 1) + A2(1)
    PEST(I, 3) = PEST(I, 3) - A2(3)
145 CONTINUE

C CALL TRACE (PEST, P2, TEST, A, M, VV)

C COMPUTE DIFFERENCE VECTORS FOR 1ST METHOD

C IF ( METH .NE. METH1 ) GO TO 160

DO 155 I = 1, M
    DIF(I) = 0.0
    DO 156 J = 1, 3
        DIF(I) = DIF(I) + (P1(I, J) - P2(I, J))**2
    CONTINUE
155 CONTINUE

GO ON TO SECOND METHOD

METH = METH2
NAARRAY = 1
GO TO 125

C COMPUTE DIFFERENCE VECTORS FOR 2ND METHOD AND COMPARE TO DIFFERENCES FOR 1ST METHOD.

C NYES = 0
DO 166 I = 1, M
    DO 167 J = 1, 3
        DIF(I) = DIF(I) - (P1(I, J) - P2(I, J))**2
    CONTINUE
    IF ( DIF(I) .GT. 0.0 ) GO TO 166
166 CONTINUE

WRITE (6, 999)
FRACT = (DFLOAT(NYES) / DFLOAT(M))
300 FORMAT(2X,' FRACTION OF TIME METHOD #1 : ', F7.3)
310 FORMAT(2X,' IS BETTER IS ', F7.3)
WRITE (6, 300) FRACT
WRITE (6, 310) FRACT
WRITE (6, 999)
WRITE (6, 999)

CONTINUE
1111 CONTINUE

WRITE (6, 999)
FRACT = (DFLOAT(NYES) / DFLOAT(M))
300 FORMAT(2X,' FRACTION OF TIME METHOD #1 : ', F7.3)
310 FORMAT(2X,' IS BETTER IS ', F7.3)
WRITE (6, 300) FRACT
WRITE (6, 310) FRACT
WRITE (6, 999)
WRITE (6, 999)

CONTINUE
1111 CONTINUE

STOP
COMPUTER SUBROUTINES FOR TIME CALCULATION AND RAYTRACING

SUBROUTINE TCOP (P,A,, VVT)

C THIS FORTRAN SUBROUTINE COMPUTES THE EXACT TIME
C REQUIRED FOR A SOUND WAVE TO TRAVEL FROM ITS
C SOURCE (VECTOR F) TO THE FOUR HYDROPHONES ON AN
C ARRAY WHOSE ACOUSTIC CENTER IS SPECIFIED BY
C VECTOR A.

C THE METHODOLOGY USED IS SET FORTH IN APPENDIX A
C OF THE THESIS. THE BASIC ASSUMPTION IS ONE OF
C A LINEAR VELOCITY PROFILE, WHOSE COEFFICIENTS ARE
C GIVEN BY THE VECTOR VV.

INTEGER M, I, J
DOUBLE PRECISION P(1000, 3), A(3), VV(2), T(1) 00, 4)
DOUBLE PRECISION AA(4, 3), P2, R, C1, C2

C SET UP COORDINATES OF FOUR HYDROPHONES ON THE ARRAY

DO 11 I = 1, 4
   DO 22 J = 1, 3
      AA(I, J) = -15.DO + A(J)
22 CONTINUE
   AA(I, 3) = 15.DO + A(3)
11 CONTINUE

AA(1, 1) = 15.DO + A(1)
AA(2, 2) = 15.DO + A(2)
AA(3, 3) = -15.DO + A(3)

C CALCULATE QUANTITIES C1, C2, R, AND THE TIMES T(*, 4)

C2 = -1.DO*(VV(1)/VV(2))

C LOOP THROUGH THE M SOURCE LOCATIONS

DO 33 I = 1, M
   P2 = P(I, 3)
   DO 44 J = 1, 4
      P1 = DSQRT((P(I, 1)-AA(J, 1))**2 + (P(I, 2)-AA(J, 2))**2)
      C1 = P1**2+P2**2-2*AA(J, 3)**2 +2.DO*C2*(AA(J, 3)-P2)
      C1 = C1/(2.DO*P1)
      R = DSQRT(C1**2+AA(J, 3)-C2)**2)
      T(I, J) = ((AA(J, 3)-C2)/(P2-C2)) *((R-C1+P1)/(R-C1))
44 CONTINUE
33 CONTINUE

RETURN
END
SUBROUTINE TRACE (P, PT, T, A, M, VV)

THIS FORTRAN SUBROUTINE TAKES AN APPARENT POSITION ESTIMATE P AND CONVERTS IT TO AN ACTUAL POSITION ESTIMATE PT BY RAY TRACING. THE RAY TRACING ASSUMES A LINEAR VELOCITY PROFILE, WITH COEFFICIENTS IN THE VECTOR VV. OTHER INPUTS ARE THE ARRAY LOCATION VECTOR A AND THE RAY TRANSIT TIME TO THE ACOUSTIC CENTER. THE RAY TRACING LAYERS ARE DEL FT THICK WITH THE FIRST (BOTTOM) LAYER BEING THE FIRST DEL FT IMMEDIATELY ABOVE THE ARRAY.

INTEGER M, I
DOUBLE PRECISION T(1000), P(1000, 3), PT(1000, 3), A(3)
DOUBLE PRECISION DEL, H, SA, CA, DEPTH, V, V1, VV(2)
DOUBLE PRECISION DS, DT, DZ, DH

DEL = 25.DO

DO 11 I = 1, M

INITIALIZE INCREMENTAL VALUES FOR EACH POSITION

H = DSQRT(P(I, 1) - A(1)) **2 + (P(I, 2) - A(2)) **2
CA = H/DSQRT(H**2 + (P(I, 3) - A(3)) **2)
DT = 0.DO
DZ = 0.DO
DH = 0.DO
DEPTH = A(3) - DEL/2.DO

V = VV(1) * VV(2) * DEPTH

INNER LOOP: PROCESS DATA UPWARDS, LAYER BY LAYER, UNTIL RAY TRANSIT TIME IS EXHAUSTED

10 SA = DSQRT(1.DO - CA**2)
DZ = DZ * DEL
DS = DEL/SA
DT = DT + DS/V
DH = DH + DS * CA

IF (DT. GT. T(I)) GO TO 20

DEPTH = DEPTH - DEL
V1 = VV(1) * VV(2) * DEPTH
CA = CA * (V1/V)

V = V1
GO TO 10

ADJUST FOR OVERSHOOTING IN LAST LAYER

20 DS = V * (DT - T(I))
DH = DH - CA * DS

ADJUST APPARENT POSITION TO GET ACTUAL POSITION

PT(I, 1) = (P(I, 1) - A(1)) * (DH/H) + A(1)
PT(I, 2) = (P(I, 2) - A(2)) * (DH/H) + A(2)
PT(I, 3) = A(3) - (DZ - SA * DS)

CONTINUE
RETURN
END
APPENDIX G
COMPUTER SUBROUTINES FOR POSITION ESTIMATION METHODS

C
C SUBROUTINE POSNAV (T, V, M, P, NT)
C
INTEGER M, I
C
THIS FORTRAN SUBROUTINE IMPLEMENTS THE ORIGINAL
UNADJUSTED NAVY APPARENT POSITION ESTIMATION METHOD.

INPUT ARE THE HYDROPHONE TIMES T FOR M SOURCES
POSITIONS, AND VELOCITY V AT THE ARRAY.

OUTPUT ARE M APPARENT POSITION ESTIMATES P ALONG WITH
THE CORRESPONDING M RAY TRANSIT TIMES NT. ALL THE
POSITIONS ARE REFERENCED TO THE ACOUSTIC CENTER, WITH
THE Z COMPONENT BEING MEASURED UPWARDS FROM THE ARRAY.

DOUBLE PRECISION T(1000,4), P(1000,3), NT(10) 0
DOUBLE PRECISION V, TC, D, LISC, NUMER, DENOM
C
D = 30. DO
DC 11 I = 1, M
DC 15 NUMER = 0. DO
DC 16 DENOM = 0. DO
DC 17 TC = T(I,4)
DC 18 DO 22 J = 1, 3
DC 19 P(I, J) = V*V*(TC*TC-T(I, J)**2)/(D*2. DO)
DC 20 NUMER = NUMER + P(I, J)**2
DC 21 DENOM = DENOM + (P(I, J) + 15. DO)**2
DC 22 CONTINUE
DC 11 CONTINUE
DC 12 NT(I) = TC*DSQRT(NUMER/DENOM)
C
RETURN
END
C
SUBROUTINE POSNVC (T, V, P, NT)

THIS FORTRAN SUBROUTINE IMPLEMENTS THE ADJUSTED NAVY METHOD NAVY-A FOR ESTIMATING APPARENT POSITIONS.

INPUT ARE THE HYDROPHONES TIMES T FOR M SOUND SOURCE LOCATIONS, AND THE VELOCITY V AT THE ARRAY.

OUTPUT ARE THE M APPARENT POSITION ESTIMATES P, AND THE CORRESPONDING ARRAY TRANSIT TIMES NT. ALL POSITIONS ARE REFERENCED TO THE ACOUSTIC CENTER OF THE ARRAY, WITH THE Z COMPONENT MEASURED UPWARDS FROM THE ARRAY.

INTEGER M, I
DOUBLE PRECISION T(1000,4), P(1000,3), NT(1)
DOUBLE PRECISION V, D, NUMER, DENOM, TC, DCC

D = 30.D0
DO 10 I = 1, M
   TC = T(I,4)
   DO 20 J = 1, 3
      P(I, J) = 15.D0 + (V*V*(TC*T(I, J)**2)/(D*2.D0))
      DCC = DCC + P(I, J)**2
20 CONTINUE
NUMER = 0.D0
DENOM = 0.D0
DO 40 J = 1, 3
   P(I, J) = (P(I, J)*V*TC)/DSQRT(DCC)
   DENOM = DENOM + P(I, J)**2
40 CONTINUE
NUMER = NUMER + P(I, J)**2
NT(I) = TC*DSQRT(NUMER/DENOM)
CONTINUE
RETURN
END
SUBROUTINE POSLS (T, V, M, P, LST)

THIS FORTRAN SUBROUTINE IMPLEMENTS THE LEAST SQUARES PLANAR METHOD I.S.

INPUT ARE THE HYDROPHONE TIMES T FOR M SOUND SOURCE POSITIONS, AND THE VELOCITY V AT THE ARRAY.

INPUTS AND OUTPUTS ARE THE SAME AS FOR THE SUBROUTINE POSNAV.

INTEGER M, I, J
DOUBLE PRECISION T(1000,4), P(1000,3), LST(1000)
DOUBLE PRECISION V, DISC, TC

DC 11 I = 1, M
DC 22 J = 1, 3
P(I, J) = T(I, 4) - T(I, J)
DISC = DISC + P(I, J)**2
CONTINUE
LST(I) = (T(I,1) + T(I,2) + T(I,3) - T(I, 4))/2. DO
P(I, J) = (V*LSI(I)*P(I, J))/DSQRT(DISC)
CONTINUE
RETURN
END
SUBROUTINE POSLSC (T, V, M, P, LST)

THIS FORTRAN SUBROUTINE IMPLEMENTS THE BIAS ADJUSTED LEAST SQUARES PLANAR METHOD L.S.C. FOR ESTIMATING APPARENT POSITIONS.

INPUTS AND OUTPUTS ARE THE SAME AS FOR THE SUBROUTINE POSLS.

INTEGER I, M, J, K, L
DOUBLE PRECISION T(1000,4), P(1000,3), LST(1000)
DOUBLE PRECISION V, A(4,3), B(4), DISC, TD, RSQR, R2SQR, E(3)

SET UP HYDROPHONE POSITIONS
DO 44 J = 1, 3
DO 55 I = 1, 4
A(I, J) = '-15.00
CONTINUE
A(I, J) = 15.00
CONTINUE

CALCULATE ORIGINAL L.S. SOLUTION
DO 11 I = 1, M
DO 22 J = 1, 3
E(I, J) = T(I, 4) - T(I, J)
DISC = DISC + E(I, J)**2
CONTINUE
LST(I) = (T(I, 1)*T(I, 2)*T(I, 3) - T(I, 4))/2.
DO 33 J = 1, 3
P(I, J) = (V*LST(I) - P(I, J))/DSQRT(DISC)
CONTINUE
DISC = P(I, 1)**2 + P(I, 2)**2 + P(I, 3)**2
D(4) = 0.00
DO 89 K = 1, 3
D(J) = D(4) + (P(I, K) - A(4, K))**2
CONTINUE

CALCULATE BIAS VECTOR E
DISC = DSQRT(DISC)
DO 66 J = 1, 4
E(I, J) = (TD - D(4))* (D(4) - D(J))/ (DISC*2.00) - P(I, J)
CONTINUE
R2SQR = 0.00
DO 99 J = 1, 3
E(I, J) = P(I, J) - E(I, J)
R2SQR = R2SQR + P(I, J)**2
CONTINUE

ADJUST ORIGINAL RAY TRANSIT TIME
LST(I) = LST(I) * (DSQRT(R2SQR/RSQR))
CONTINUE
RETURN
END

108
SUBROUTINE POSLMP (T, V, M, PMT, VAR)

THIS FORTRAN SUBROUTINE IMPLEMENTS THE MAXIMUM LIKELIHOOD PLANAR METHOD M.L.P. FOR ESTIMATING APPARENT POSITIONS.

INPUTS ARE THE HYDROPHONE TIMES T FOR M SOURCE POSITIONS, AND THE VELOCITY OF SOUND V AT THE ARRAY.

OUTPUTS ARE THE APPARENT POSITION ESTIMATES P AND RAY TRANSIT TIMES MT FOR EACH OF THE M POSITIONS. ALSO OUTPUT IS A VECTOR VAR OF M ESTIMATES OF THE TIMING ERROR VARIANCE. POSITIONS ARE ALL REFERENCED TO THE ACOUSTIC CENTER OF THE ARRAY, WITH THE Z COMPONENT MEASURED UPWARDS FROM THE ARRAY.

INTEGER M, I, J
DOUBLE PRECISION T(1000, 4), P(1000, 3), MT(1000)
DOUBLE PRECISION V, TC, D, DIFF, DE NO, TL, C(3)
DOUBLE PRECISION VAR(1000), C0(3), X(3)

D = 30.0
TC = 1.0

INITIALIZE THE DIRECTION COSINE ESTIMATE BY USING THE LEAST SQUARES SOLUTION.

LOCATE THROUGH THE M POSITIONS

DO 11 I = 1, M
   TC = T(I, 4)
   DENOM = (TC - T(I, 1)) ** 2 + (TC - T(I, 2)) ** 2
   DO 12 J = 1, 3
      C(J) = (4. / (TC - T(I, J))) / DSQRT(DENOM)
   CONTINUE

12 ITC = TC

DO 22 ITER = 0, 10
   DE NO = 0
   ITC = 0.0
   DO 22 ITER = ITC + 1
   DENOM = NO
   DO 22 ITER = ITC + 1

USE ITERATION FORMULAE TO DEVELOPE COSINES AND TIME VALUES.

DO 33 J = 1, 3
   C0(J) = C(J)
   DENOM = DENOM + T(I, J) * C(J)
   TC = TC + T(I, J)
   CONTINUE

33 CONTINUE

TC = (TC + T(I, 4)) * D * T(C(1) + C(2) + C(3)) / V / 4.0
DENOM = DENOM - TC * (C(1) + C(2) + C(3))
DO 66 J = 1, 3
   X(J) = (1.0 - TC) / DENOM
   CONTINUE

66 CONTINUE

CHECK ITERATION TOLERANCES

DIFF = DMAX1(X(1), X(2), X(3))
IF (DIFF > 10.0) GO TO 70

WRITE(6, 60) ITER
WRITE(6, 61) I
GOTO 11

60 FORMAT (2X, 'ITERATIONS EXCEED ', I6)
61 FORMAT (2X, 'IN POSITION NO. ', I6)

109
IF ( TOL .LT. DIFF ) GO TO 22

NUMER = 0.D0
DENOM = 0.D0

CONVERT COSINES TO POSITIONS, AND TRANS.ATE TO
ACOUSTIC CENTER REFERENCED COORDINATES.

DO 44 J = 1, P

DENOM = DENOM + P(I,J)**2
DENOM = DENOM + P(I,J)**2

CONTINUE

MT(I) = TC*DSQRT(NUMER/DENOM)

ESTIMATE VARIANCE

VAR(I) = 0.D0
DO 55 J = 1, P

VAR(I) = VAR(I) + (T(I,J) - TC*D*C(J)/I)**2

CONTINUE

VAR(I) = VAR(I)/4.D0

RETURN

END
SUBROUTINE POSMLS (TIME, V, M, P, MLT, VARML)

THIS FORTRAN SUBROUTINE IMPLEMENTS THE MAXIMUM LIKELIHOOD SPHERICAL METHOD M.L.S. FOR ESTIMATING APPARENT POSITIONS.

INPUTS AND OUTPUTS ARE THE SAME AS FOR THE SUBROUTINE POSMIP.

INTEGER M, I, J, ITER, II, ITER1, ITER2
DOUBLE PRECISION TIME (1000, 4), P (1000, 3), VARML (1000), SDEV (1000), MLT (1000), R (3), DSRC (3)
DOUBLE PRECISION R, TOL1, TOL2, EPS, L, S, DIFF
DOUBLE PRECISION T (3), XC, TAU, TK (3), VK (3), C (3), CI (3)
DOUBLE PRECISION A (4), B (4), I (4), EPS
DOUBLE PRECISION GR, X1 (4), X2 (4), A (4), B (4), T

C SET GLOBAL VALUES
R = (3.0 - DSQRT(5.0)) / 2.0
C = 0.0
TOL1 = 1.0 D - 5
TOL2 = 1.0 D - 5

C START OUTERMOST LOOP (ONE FOR EACH SOURCE POSITION)
DO 5 II = 1, M
DENOM = 0.0
TAU = TIME (II, 4)
EPS = 1.0 D - 3
TC = TAU
DO 11 J = 1, 3
T (J) = TIME (II, J)
C (J) = ((V ** 2) ** (C ** 2 - T (J) ** 2) / (D * 2.0)) + D / 2.0
DENOM = DENOM + C (J) ** 2
11 CONTINUE

C START MIDDLE LOOP (CREATE DERIVATIVE MATRIX GP (, ) )
DO 122 J = 1, 3
C (J) = C (J) / DSQRT (DENOM)
122 CONTINUE

C START MIDDLE LOOP (CREATE DERIVATIVE MATRIX GP (, ) )
DO 11 II = 1, M
CI (I) = C (I)
K (I) = TAU * ((D * V) ** 2 - (2.0 D * TAU * C (I)) / V)
VK (I) = V * K (I) * DSQRT (K (I))
TK (I) = T (I) + DSQRT (K (I))
DLDC (I) = ((V ** 2) ** 2 - T (I) * C (I) * (T (I) D * TAU) / VK (I))
DSDC (I) = (T (I) D * TAU) / VK (I)
DLDT = DLD T * (C (I) T (I) * (D * C (I) - V * TAU) / VK (I))

111
DSDT = DSDT + T(I) * (D*C(I) - V*TAU) / VK(I)
I = L*(C(I)*TR(I)) / DSQR(K(I))
S = S*K(I) / DSQR(K(I))

CONTINUE

DO 22 I = 1, 3
DO 33 J = 1, 3
GP(I,J) = (TK(I) *DLD(T(J))) / (L*L*DSQR(K(I)))
CONTINUE
GP(4, I) = DSCC(I) * (V*TC - D*L)
GP(4, I) = (GP(4, I) - D*LDC(I) * (1.0 - S))
/G(V*(1.0 - S)**2)
GP(I, I) = (I*T(I) *D*TAU) - D*LD(TC(I) *K(I) * V*TK(I))
GP(I, I) = 1.0 - (GP(I, I) / (VK(I) *L*L))
GP(I, 4) = (I*I) *L*(V*TAU - D*C(I)) + V*K(I) *L*K(I) *DLDT
GP(4, 4) = GP(4, 4) / (VK(I) *L*L)

CONTINUE
GP(4, 4) = 1.0 - (DSDT * (V*TC - D*L)) - D*LDC*(1.0 - S))
/(V*(1.5 - S)**2)

INVERT DERIVATIVE MATRIX
CALL GAUSS3 (4, 0, GP, GP, IER, 4)

CALCULATE INITIAL NEWTON SEARCH SOLUTION
DO 44 I = 1, 3
GP(I) = C(I) - TK(I) / (DSQR(K(I)) * L)
CONTINUE
GP(4) = TAU
DO 55 I = 1, 3
B(I) = B(I) - GP(I, 4) * GP(4, I)
CONTINUE
B(4) = B(4) - GP(I, 4) * GP(4, I)

PREPARE FOR GOLDEN SECTION SEARCH
EPS = DMAX1 ((1.0 - 6), (EPS / 1.0))
DO 66 I = 1, 3
A(I) = C(I)
X(I) = A(I) * R*(B(I) - A(I))
CONTINUE
A(4) = TAU
X(4) = A(4) * R*(B(4) - A(4))
TAU = X(4)

START INNERMOST LOOP (COMMENCE GOLDEN SECTION SEARCH TO IMPROVE THE INITIAL NEWTON SOLUTION)

ITR2 = 0
CALL OBJFCN (F1, L, K, TK, S, X1, D, V, T, TC)
IF (ITR2 LT 50) GO TO 400
FORMAT (2X, 'EXCESSIVE G.S.S. ITERS., P'S X', I6)
WRITE (6, 410) II
GO TO 5

DIFF = 0.0
DO 77 I = 1, 3
DIFF = DIFF + (A(I) - B(I))**2
X2(I) = A(I) + B(I) - X1(I)
C(I) = X2(I)
CONTINUE
X2(4) = A(4) + B(4) - X1(4)
TAU = X2(4)
DIFF = DSQRAT(DIFF + (A(4) - B(4)) ** 2)

CHECK TOLERANCES - STOP G.S.S. IF TIGHT ENOUGH

IF ( EPS .GT. DIFF ) GO TO 100

CALL OBJFCN(F2,L,K,TK,S,X2,D,V,T,TC)

CHOOSE IMPROVING DIRECTION

IF ( F1 .LT. F2 ) GO TO 90
DO 88 I = 1,3
  A(I) = X1(I)
  X1(I) = X2(I)
  C(I) = X1(I)
  CONTINUE

88 A(4) = X1(4)
X1(4) = X2(4)
GO TO 70

DO 90 I = 1,3
  B(I) = X2(I)
  X2(I) = A(I) + B(I) - X1(I)
  C(I) = X1(I)
CONTINUE

B(4) = X2(4)
X1(4) = A(4) + B(4) - X1(4)
GO TO 70

CHECK TOLERANCES FOR NEWTON SEARCH ITERATIONS
AND PREP FOR NEXT NEWTON ITERATION IF NECESSARY

100 DO 144 I = 1,3
  X(I) = DABS(C(I) - C1(I))
  CONTINUE
  DIFF = DSQRT(MAX1(X(1),X(2),X(3)))
  IF ( TOL1 .LT. DIFF ) GO TO 10
  DIFF = DABS(TAU - T1)
  IF ( TOL2 .LT. DIFF ) GO TO 10

DONE WITH II-TH SET OF TIMES AND POSITIONS. MAKE
ESTIMATES, AND GO ON TO NEXT POSITION TO BE ESTIMATED

VARML(II) = 0.D0
DENOM = 0.D0
NUMER = 0.D0
DO 155 J = 1,3
  VARML(II) = VARML(II) + TK(J) * TK(J)
  P(II,J) = VTAU*C(J)
  DENOM = DENOM + P(II,J) ** 2
  NUMER = NUMER + P(II,J) ** 2
155 CONTINUE

VARML(II) = (VARML(II) + (TC - TAU) ** 2) / 4.D0
SDEV(II) = DSQRT(VARML(II))
MLT(II) = TAU * DSQRT(NUMER / DENOM)

CONTINUE
WRITE(7,900) (SDEV(I),I = 1,M)
C900 FORMAT(5E15.5)
RETURN
END

113
SUBROUTINE OBJFNC (F, L, K, TK, S, X, D, V, T, TC)

This FORTRAN subroutine is called by the subroutine POSMLS. It calculates new values for several variables as well as a functional value which is the decision factor determining the appropriate improving direction for the golden section search.

INTEGER I

DOUBLE PRECISION F, L, K(3), TK(3), S, X(4), D, V, T, TC, T(3)

S = 0. DO
L = 0. DO
D0 11 I = 1, 3

K(I) = X(4) ** 2 + (D/V)**2 - ((2. DO* D*X(4)*X(I))/V)

TK(I) = A(I) - DSQRT(K(I))

S = S + TK(I)/DSQRT(K(I))

L = L + (X(I)*TK(I))/DSQRT(K(I))

CONTINUE

F = 0. DO
D0 22 T = 1, 3

F = F + (X(I) - TK(I) / (DSQRT(K(I))*L))**2

CONTINUE

F = F + (X(4) - (V*T - D*L) / (V*(1. DO - S)))**2

RETURN

END
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