Introduction

This interim report constitutes a summary of research performed under Grant AFOSR-81-0047 during the year beginning October 1, 1982. First we present a list of the personnel involved in the research effort. In the next section we present a summary of the research results that have been achieved. Then in the following section we briefly comment upon the research in progress. This is followed by a list of publications supported during this grant year.
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Summary of Results

In this section we will present a brief summary of our research results published after October 1, 1982. The reference numbers in this section are keyed to the publication list in the last section. Most of our research results are in the areas of quantization theory and detection theory. We will begin our summary by discussing our results in quantization theory; this will be followed by a presentation of our results in detection theory; and finally, we will mention our results in other areas.

Quantization is the process by which data is reduced to a simpler, more compatible with digital processing. Loosely speaking, quantization is at the heart of analog to digital conversion. It is an area which has increased in importance in the last few years due to the burgeoning advances in digital technology. The typical goal of quantization is to reduce data to a simpler representation without causing much distortion; that is, the output of a quantizer should be close to the input, with some appropriate measure of distance. An N-level k-dimensional vector quantizer is a mapping $Q: \mathbb{R}^k \rightarrow \mathbb{R}^k$ which assigns the input vector $x$ to an output vector $Q(x)$ chosen from a set of $N$ vectors $\{y_i: y_i \in \mathbb{R}^k, i=1,...,N\}$.

Generally, the quantizer input is modeled as a random vector $X$ described by a $k$-variate distribution $F$. A measure of quantizer performance is the distortion function

$$D(Q,F) = \int d(x,Q(x))dF(x). \quad (1)$$

where $d: \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ is an appropriately chosen cost function. An optimal $N$-level quantizer $Q$ for the random vector $X$ is one that minimizes (1) over the class of all $N$-level quantizers.

There had apparently been a long-standing belief among researchers in quantization theory that optimal quantizers always exist. This existence is important from the viewpoint of numerical design algorithms and in studying convergence properties of sequences of quantizers; also, several results in quantization theory are hypothesized upon the existence of optimal quantizers. Recently we have been investigating this existence belief and we have produced several results which validate it in many cases. In [3] we showed that if $d(x,y) = C(\|x-y\|)$, where $\|\cdot\|$ represents the Euclidean norm and $C: [0,\infty) \rightarrow [0,\infty)$ is non-decreasing and lower semi-continuous, then an optimal $N$-level quantizer exists for any positive integer $N$ and for any distribution.
F. Then in [1] and [16] we considered non-difference based cost functions, e.g., \( d(x,y) \), and in a fairly general context we established existence of optimal quantizers. Finally, in [6] we established conditions for optimal quantizers to exist in a metric space. Our work on quantization of metric spaces is the most general treatment of quantizers in metric spaces of which we are aware.

We have also been active in establishing convergence properties of sequences of quantizers. These convergence results are important from the viewpoint of numerical design algorithms, and they yield considerable insight into the limiting behavior of sequences of quantizers. Suppose that a sequence of probability measures \( \mu_n \) converges weakly to a probability measure \( \mu \). Let \( Q_n \) be an optimal \( N \)-level quantizer for \( \mu_n \). Does the distortion associated with the quantizer \( Q_n \) and the measure \( \mu_n \) converge to the optimal distortion for quantizing \( \mu \) with \( N \)-levels? Does the sequence of optimal quantizers for the \( \mu_n \)'s converge to a quantizer \( Q \); and if so, is \( Q \) optimal for \( \mu \)? Several of our convergence results have been focused about these two questions. In [12] we considered difference based cost functions, e.g. \( d(x-y) \), and we established results sufficient for affirmative answers to the above questions. Then in [1] and [16] we established conditions sufficient for affirmative answers for non-difference based cost functions, e.g. \( d(x,y) \). In [6] we extended these results to metric spaces. In each of the above works (i.e. [12] for difference based cost functions, [1] and [16] for non-difference based cost functions, and [6] for metric spaces), we also considered the above two questions where \( \mu_n \) represented the empirical measure based on \( n \) iid samples drawn from the measure \( \mu \), and we established conditions sufficient for almost sure convergence in the above two questions. In all of the above convergence results we chose to put conditions on the cost function rather than the distribution; the cost function is easier to control than the (frequently not exactly known) underlying distribution.

These convergence results for sequences of quantizers are fairly general and they form powerful tools for the study of quantization. For example, one of the more practical problems associated with quantizers is the problem of how to construct them. Most of the algorithms for quantizer design involve successively improving a suboptimal quantizer, with the procedure hopefully converging to an optimal quantizer. One of the currently most popular design algorithms for vector quantization is the so-called "Linde-Buzo-Gray algorithm"
As a by-product of one of our results in [12], we established convergence of this algorithm for r-th power distortion measures, i.e. \( d(x,y) = \|x-y\|^r \). This is the first rigorous convergence result for this algorithm in a reasonably general context.

We have also investigated some practical aspects of scalar quantization. A popular way of realizing a scalar quantizer is via a method known as companding. A companding system consists of an invertible function \( G: \mathbb{R} \to [0,1] \) followed by a uniform N-level quantizer on \([0,1]\), followed by the inverse function \( G^{-1}(\cdot) \). Any arbitrary N level scalar quantizer can be realized via a companding system. (For \( k > 1 \), this statement is false in \( \mathbb{R}^k \).) In [4] we considered the situation where a noisy channel was inserted between the uniform quantizer and the inverse function \( G^{-1}(\cdot) \). Thus the input \( X \) goes through the "compressor" \( G \), the output is then uniformly quantized and transmitted over a noisy channel, and the result is put through the "expander" \( G^{-1} \). Ideally, the output should be close to the input. In [4] we assumed that the number of quantization levels \( N \) was very large and we gave an approximation for the compressor \( G \) that took into account not only the error due to the quantization but also the error due to the channel noise.

In [15] we presented a simple and straightforward technique for constructing minimum mean squared error symmetric uniform scalar quantizers for some common distributions on the data.

In the context of scalar minimum mean squared error quantization, one of the most popular design techniques is the Lloyd-Max algorithm (IRE Transactions on Information Theory, March 1960 and IEEE Transactions on Information Theory, March 1982). Unfortunately, two potential problems arise with the Lloyd-Max scheme. The first problem is how to get a good initial guess for starting the iterative scheme, and the second problem is how to intelligently update the algorithm. Both of these problems were addressed in [10] for some common distributions on the data. Our modifications of the Lloyd-Max algorithm result in a very fast design algorithm for scalar minimum mean squared error quantization. For example, we can design a 64-level quantizer for a Gaussian distribution with a high degree of accuracy (the terminating condition for the Lloyd-Max algorithm was set at \( 10^{-8} \)) in 0.136 seconds of computer time on a CDC Cyber 170/750.

Another research area in which we have recently obtained results is the area of signal detection. The detection problem is modeled as a test between
two statistical hypotheses; we assume that under the null hypothesis noise alone is being observed, and under the alternate hypothesis a signal plus noise is being observed. In discrete time detection we assume that the observation is indexed by a subset of the integers, e.g. $x_1, x_2, \ldots, x_n$, and in continuous time detection we assume that the observation is indexed by an interval, e.g. $x(t)$, $0 \leq t \leq T$. In [9] we considered using a continuous time filter for a discrete time problem. That is, we supposed that the signal $s(\cdot)$ was known at certain discrete instants of time $\{t_k\}$, and we showed that under certain regularity conditions (including bandlimitedness of $s(t)$) we could derive a signal $\hat{s}(\cdot)$ such that $\hat{s}(t_k) = s(t_k)$ in such a way that if we used a continuous time filter based on the signal $\hat{s}(t)$, $0 \leq t \leq T$, then the performance of the detector was robust to the inexact knowledge of the actual signal $s(t)$, $0 \leq t \leq T$. We showed that the performance of the continuous time detector based on the signal $\hat{s}(t)$, $0 \leq t \leq T$, could be considerably better than the performance of the discrete time detector based on the signal $\{s(t_k)\}$, even though the signal $s(t)$ is not known for $t$ outside the set $\{t_k\}$.

In the case of discrete time detection where the noise and the signal are stationary and the samples are independent, it is well known that the Neyman-Pearson test has a test statistic which can be expressed as

$$\sum_{i=1}^{n} g(X_i)$$

where $X_i$, $i=1,\ldots,n$, represent the observations, and $g(\cdot)$ is an appropriately chosen function. In earlier work we had considered the problem of constraining the test statistic to be of the above form and letting the noise samples be "slightly" dependent. We then tried to choose the function $g(\cdot)$ to best account for the dependency structure of the noise, in the sense of the asymptotic relative efficiency (or Pitman efficiency) with respect to any other choice for $g(\cdot)$. In [14] we investigated the problem of how to choose $g(\cdot)$ when both the signal and the noise were modelled as $\phi$-mixing random processes, where we also allowed the noise to be dependent on the signal over a finite window, such as signal dependent noise induced through reverberation effects. In [13] we considered the problem of approximating an optimal $g(\cdot)$ by a sequence of Borel measurable functions $\{g_i(\cdot)\}$. We compared the performance resulting from the approximate nonlinearities to the optimal performance, and we showed that the loss in performance can be made arbitrarily
small by making $g_i(*)$ appropriately close to $g(*)$. We allowed a strong-mixing dependency structure for the (random) signal and the noise, and we considered as examples specific forms, e.g. quantizers, polynomials, for the $g_i(*)$.

In [11] we considered the discrete time detection of time varying deterministic signals in weakly dependent noise. We considered a finite set of observations, and we were concerned with the situation where the dependency in the noise was ignored; that is, the detector was designed assuming that the noise samples were mutually independent. We investigated quantitative conditions which allowed determining when the dependency can be ignored, and we presented a result which allowed bounding the variation in false alarm rate and detection probability induced by ignoring the dependency.

There has recently been a great deal of interest in robust hypothesis testing, and engineers have attempted to apply results in this area to problems in signal detection. This may be attractive when one realizes that the statistical characterizations adopted by engineers are frequently only reasonable approximations. In [5] we studied the actual performance associated with a robust test for a constant signal in iid Laplace noise, and we showed that in many cases the degradation in performance (i.e. the performance loss due to the robust test as compared to the performance of the Neyman-Pearson optimal test) was indeed very small.

The relative efficiency between two detectors is a ratio of the amount of data required by one detector, relative to another, to attain a prescribed level of performance. Although this concept is of fundamental importance in the theory of signal detection, it has been successfully investigated in only very few special cases. As an approximation to the relative efficiency, engineers have frequently employed the asymptotic relative efficiency (ARE), the limiting value of the relative efficiency (under suitable regularity conditions) as the sample sizes required by the detectors approach infinity. The ARE was introduced in the statistical literature, where it is known as the Pitman efficiency. Usually it can be determined in a fairly straightforward fashion, and this is due principally to an appeal to the central limit theorem. The ARE is a limiting result; and in any practical engineering situation, only a finite number of samples can be taken in the context of discrete time detection. Thus it might not always be appropriate to approximate the relative efficiency with the ARE. In [7] we considered the discrete time detection of a known time varying signal in additive noise.
where the noise sequence is assumed to be a sequence of iid random variables; and we studied the relative efficiency of the sign detector, a popular nonparametric detector, and the correlation detector, which is Neyman-Pearson optimal in the case when the noise is Gaussian. In this work [7] we presented results illustrating the convergence of relative efficiencies for both Gaussian noise and Laplace noise. Some examples were given where the relative efficiencies did not quickly converge to the ARE.

In another area of research, we have investigated some effects of a form of nonlinear distortion on spectral properties of random processes. Let 

\[ \{X(t), t \in \mathbb{R}\} \]

denote a second order random process that is second order stationary (i.e. the bivariate distribution is invariant to a translation of the time axis), and mean square continuous. Then it is well known that \(X(t)\) possesses a spectral distribution function \(F\). (We take all spectral distribution functions to be zero at minus infinity.) Let \(g: \mathbb{R} \rightarrow \mathbb{R}\) be a Borel measurable function such that \(E\{(g[X(t)])^2\} < \infty\). Then \(g[X(t)]\) is mean square continuous and thus possesses a spectral distribution function \(G\). For \(p \geq 2\), let

\[ A_p = \frac{\int |u|^p dF(u)}{F(\infty)} \]

and let

\[ \overline{A}_p = \frac{\int |u|^p dG(u)}{G(\infty)} \]

where we assume that \(F(\infty) > 0\) and \(G(\infty) > 0\) (i.e. neither \(X(t)\) nor \(g(\cdot)\) is "degenerate"). The \(A_p\) and \(\overline{A}_p\) represent the \(p\)-th absolute moments of the input and output normalized spectral distributions, respectively. These quantities are important since they relate to measures of spectral dispersion, and thus bandwidth. Bandwidth is a fundamental property of signal transmission; and nonlinearities, such as \(g(\cdot)\), are encountered in numerous situations. In [2] we compared \(A_p\) to \(\overline{A}_p\) for various conditions placed on \(g(\cdot)\) and \(X(t)\), and with certain conditions on \(X(t)\) we gave both upper and lower bounds to \(\overline{A}_p\) in terms of \(A_p\) and \(g(\cdot)\). For example, it follows from [2] that if \(X(t)\) is a zero mean Gaussian process with a nonconstant autocorrelation function and if \(g(\cdot)\) is odd and not almost everywhere constant, then \(A_2 \leq \overline{A}_2\), with equality if and only if \(g\) is almost everywhere linear.
In [17] we investigated an existing method (Delp and Mitchell, IEEE Transactions on Communications, September 1979) for image compression known as block truncation coding. The basic block truncation coding approach employs a two level quantizer whose output levels are obtained through matching the first two sample moments of the data before and after quantization. We generalized this basic block truncation coding approach by using two level quantizers which preserve higher order moments. This generalization offered the potential for improved performance. Some examples were given to illustrate the improvement in image quality.

To conclude the survey of our recent work, in [8] we summarized several mistakes that are frequently made in communication theory. For example, if $X(t)$ is a zero mean stationary Gaussian process and $R(\tau) = E\{X(t)X(t+\tau)\}$, then it is possible for $X(t)$ to be ergodic but $R(\tau)$ to not converge to zero as $\tau \to \infty$. We point out that ergodicity does not imply "asymptotic independence". As another example, we point out that the conditional expectation does not always yield a minimum mean squared error estimate. That is, we constructed two bounded random variables $X$ and $Y$ and a function $f: \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$ pointwise on the underlying probability space but $E[|Y - E(Y|X)|^2] > 10^4$. 
Research in Progress

Our research is progressing very well in several directions. Most of our efforts are concentrated upon a rigorous analysis of problems in signal processing that are dictated by practical concerns, and most of our current research is directed toward non-Gaussian phenomena. In the remainder of this section we will briefly describe the problems we are currently investigating.

We are presently studying algorithms for quantizer design, and we are considering both the case where the distribution of the data to be quantized is known and the case where it is unknown. For the case of a known distribution, we are currently approaching the Lloyd-Max algorithm from the viewpoint of a problem in control theory. That is, one can view a "run" through the algorithm as a plant, and one can view the updating of the algorithm as a feedback control. We can thus view the fast convergence of the algorithm as a matter of using a good feedback in the control system. We have recently successfully modified the Lloyd-Max algorithm in the context of scalar minimum mean squared error quantization, and we are presently extending this analysis to minimum mean absolute error quantization.

In practice the distribution of data to be quantized is often unknown and one must use an estimate of the true distribution. Consider an algorithm for vector quantization which assumes a known distribution for the input. If an estimate of this distribution were used instead, what kind of performance can we expect? Under what conditions can we get strongly consistent estimates of the optimal output levels? We are currently attempting to answer these questions in a reasonably general sense.

Recently a group of researchers (Gray, Kieffer, and Linde, Information and Control, May 1980) used a sample empirical distribution as an estimator in the so-called Linde-Buzo-Gray algorithm. However, their analysis holds only for this algorithm. Will a similar result hold for other algorithms? We are currently investigating this general idea in an "algorithm-free" context; that is, we are attempting to derive results which are independent of the particular algorithm used.

Another area we are currently investigating is the relative efficiencies of detectors. Consider two discrete time detectors, say $D_1$ and $D_2$. Assume that for detector $D_1$, the minimum number of samples needed to achieve a false
alarm probability no greater than \( \alpha \) and a detection probability of at least \( \beta \) is \( n_1 \). Then the relative efficiency of \( D_2 \) with respect to \( D_1 \) is \( n_1/n_2 \).

In the context of discrete time detection, the relative efficiency, as a function of \( n_1 \), has rarely yielded to successful analysis. Our preliminary results in [7] are notable in the sense that they presently represent the most successful approach to the study of the relative efficiency in the literature. We are currently extending these results to the case where the discrete time detection is in the context of dependent noise. We plan to also consider the situation where the time varying signal \( \{s_i\} \) is not precisely known.

The above summary describes our on-going research. In the near future we plan to also consider some aspects of non-Gaussian random processes, and we also plan to investigate the performance of detectors and how this depends on errors or inaccuracies in the assumptions governing the design of the detection system.
List of Publications


The research topics investigated were primarily in the areas of quantization and detection in signal processing. The existence of optimal quantizers was established under very general conditions and convergence properties of sequences of quantizers to optimal quantizers were investigated. A simple and straightforward technique was presented for constructing minimum mean-squared error symmetric uniform scalar quantizers for some common input distributions. The Lloyd-Max algorithm was modified to give a very fast design algorithm for scalar minimum mean-squared error quantization. In the area of detection, a continuous time filter was designed for a discrete time problem and outperformed the corresponding discrete time detector. Detection problems were investigated when both signal and noise were modelled as phi-tighting random processes. Several situations were investigated in which asymptotic relative efficiency is an inappropriate measure of efficiency. In a different area of research, the effects of a form of nonlinear distortion on spectral properties of random processes were investigated.
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