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INFLUENCE OF SURFACE TENSION AND SURFACE SHEAR ON FINAL COAT THICKNESS IN JET-STRIPPED CONTINUOUS COATING OF SHEET MATERIALS

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ABSTRACT

Air jets are used as a wiping device to reduce the final thickness in some industrial coating applications. Existing theories allow only a given internal normal stress, i.e. impose a pressure distribution in the coating liquid, due to the jet. In this note, surface tension and tangential stresses are also included. In particular, numerical results are obtained which quantify the extent to which surface tension reduces the wiping effect of the jet.

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There are many situations in which a liquid film needs to be wiped off a solid surface, and one of the most common technological stripping mechanisms used for that purpose is a high-speed fluid jet. The industrial importance of this method of film wiping has prompted scientific studies of the process, but so far, these have accounted only for the pressure exerted by the jet on the film.

The following gives consideration also to surface tension and tangential stresses, and analyzes under what circumstances they have an important influence on what can be achieved stripping a film with a jet. The equations governing the process in those circumstances are established and analyzed. Numerical results, moreover, are obtained which quantify the extent to which surface tension reduces the wiping effect of the jet.

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INFLUENCE OF SURFACE TENSION AND SURFACE SHEAR ON FINAL COAT THICKNESS IN JET-STRIPPED CONTINUOUS COATING OF SHEET MATERIALS

E. O. Tuck* and J.-M. Vanden-Broeck

1. Introduction

The influence of a stripping jet on continuous coating processes has been the subject of investigations by Thornton & Graff (1976) and by Tuck (1982). In these studies, the jet is represented as a given distribution of pressure within the coating material. In applications such as to the steel-galvanising industry, this pressure distribution is created by blowing a thin high-speed air jet onto the coated steel sheet, just after it emerges from the bath of molten coating material.

It is the purpose of the present note to consider two phenomena not incorporated in previous studies, namely tangential stresses and surface tension effects. That is, the air jet not only exerts a normal stress or external pressure on the surface of the coating, but also causes a shear or tangential stress. Furthermore, the external pressure must communicate itself to the coating layer across its free surface, and if that free surface possesses curvature and non-zero surface tension, the internal pressure will differ from that in the jet.

In the galvanising application, both of these phenomena are expected to have only minor effects. However, quantitative estimates of their magnitude have not here-to-fore been available, and such estimates are provided here.

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We first set up the equations describing the process in a suitable non-dimensional framework, and use lubrication-type simplifications to reduce the problem to that of solving a partial differential equation for the layer thickness \( h(y,t) \) as a function of height \( y \) and time \( t \). This derivation reveals that surface-tension effects are negligible if the capillary number (i.e. non-dimensional inverse surface tension) far exceeds the cube of the slope or thickness/height scale of the layer. Similarly, shear forces have an effect on the coating process only if their magnitude is greater than a multiple of the pressure forces which is of the order of the slope of the coating.

The influence of surface tension is explored numerically by solving the ordinary differential equation of the 4th-order that describes the steady-state thickness \( h(y) \). Results are obtained for the final coating thickness as a function of the jet's maximum pressure gradient, for various values of a suitably-scaled surface-tension coefficient. It is a remarkable feature of the results of Thornton and Graff (1976) at zero surface tension that the final coating thickness depends only on this maximum pressure gradient, and is insensitive to the variation in pressure away from the point of maximum gradient. Such a conclusion no longer holds when surface tension is present, and in principle the complete shape of the pressure curve is significant.

The numerical results show that surface tension has the expected effect of inhibiting the stripping tendency of the jet. That is, the reason for use of the jet is to obtain thinner coatings than would apply from pure draining under gravity, and this is achieved via a decrease in
thickness from below to above the jet. Since this introduces curvature into the free surface of the coating layer, surface tension will resist it, and the influence of the jet will be less than if there was no surface tension. Two different shapes for the pressure distribution were used, but there was little difference in the coated thickness for jets with the same maximum pressure and pressure gradient, even when surface tension was significant.
2. Derivation of Equations

Assuming two-dimensional flow of an incompressible Newtonian fluid, as in Figure 1, we have to solve the Navier-Stokes equations

\[ u_t + uu_x + vu_y = -\frac{1}{\rho} p_x + v\gamma^2 u \]  
\[ v_t + uv_x + vv_y = -g - \frac{1}{\rho} p_y + v\gamma^2 v \]

and

\[ u_x + v_y = 0. \]

The boundary conditions on the moving sheet are

\[ u=0, \quad v=v \quad \text{on} \quad x=0. \]

If the free boundary has equation \( x = h(y,t) \), the kinematic boundary condition is

\[ u = h_t + vh_y \quad \text{on} \quad x=h. \]

The normal stress is

\[ \tau_1 = -p + \frac{2\mu}{1+h_y^2} [u_x - h_y (u_y + v_x) + h^2 v_y] \]

and the tangential stress is

\[ \tau_y = \frac{\mu}{1+h_y^2} [2h_y (u_x - v_y) + (1-h_y^2)(u_y + v_x)]. \]

These stresses act on the fluid side \( x = h-0 \) of the interface.

We suppose that on the other side \( x = h+0 \) of this interface, there is a known pressure \( P(y,t) \) and tangential stress \( T(y,t) \), and that the interface possesses surface tension \( \sigma \). Thus, the dynamic boundary conditions are

\[ \tau_1 = -P + \sigma h_{yy} (1+h_y^2)^{-3/2} \]
Figure 1. Sketch of flow and coordinate system.
and

$$\tau_x = T$$  \hspace{2cm} (2.9)$$

where $\tau_x$, $\tau_y$ are given by (2.6), (2.7) respectively. We envisage $P$ and $T$ as due to the action of an externally-applied gas jet, although this is not a necessary assumption.

The above is an exact formulation for any boundary $h$. We now suppose that this boundary has a small $O(\epsilon)$ slope to the vertical. That is, if $L$ denotes a length scale for changes in the $y$-direction, prescribed for example by the $y$-wise scale of the pressure $P$, we assume that any layer of interest has a thickness much less than $L$, with $h/L = O(\epsilon), \epsilon << 1$.

This is the situation that applies in lubrication theory (Cameron, 1966), and we now derive asymptotic simplifications of the system (2.1)-(2.9), consistent with lubrication theory, in the limit as $\epsilon \to 0$. For example, we take $L,V$ as the length and velocity scales in the $y$-direction, but $cL$ and $cV$ as corresponding scales in the $x$-direction, and we take $L/V$ as the time scale, if the problem is unsteady. Lubrication pressures are high, of order $\mu VL/h^2$. Thus, in order that gravity and the jet both have an effect, we must demand that both the hydrostatic pressure $\rho g L$ and the jet pressure $P$ also have this scale, i.e.

$$\rho g L, P, P = O(\mu VL^{-1} \epsilon^{-2}) \hspace{2cm} (2.10)$$

If we now examine all equations for order of magnitude, we find from (2.1) and (2.2) that

$$p_x [1+O(\epsilon^2)+O(\epsilon^2 VL/V)] = 0 \hspace{2cm} (2.11)$$

and
Equations (2.3)-(2.5) are unsimplified, while (2.6)-(2.9) yield

\[(p-P+\sigma h_{yy})[1+O(\varepsilon^2)] = 0\]  \hspace{1cm} (2.13)

and

\[(\mu v_x-T)[1+O(\varepsilon^2)] = 0.\]  \hspace{1cm} (2.14)

The simplified equations are obtained by letting \(\varepsilon \to 0\), replacing the square bracket in (2.11)-(2.14) by unity in each case. In (2.11) and (2.12), this involves not only the assumption that \(\varepsilon\) is small, but also the assumption that \(\varepsilon^2 VL/\nu\) is small, i.e. that the Reynolds' number \(VL/\nu\) is not too large, specifically not as large as \(O(\varepsilon^{-2})\). The terms corresponding to \(O(\varepsilon^2 VL/\nu)\) are simply the inertia terms on the left of (2.1) and (2.2), and we assume these are negligible relative to the (very large) viscous terms.

Thus we have

\[p_x = 0\]  \hspace{1cm} (2.15)

\[v_{xx} = \frac{\rho g + p_y}{\mu}\]  \hspace{1cm} (2.16)

and (2.3), to be satisfied in \(0 < x < h\), subject to (2.4) on \(x=0\) and (2.5),

\[p = P - \sigma h_{yy}\]  \hspace{1cm} (2.17)

and

\[v_x = T/\mu\]  \hspace{1cm} (2.18)

on \(x=h\).

Note that (2.17) is consistent in order of magnitude only if \(\sigma = O(\mu \nu c^{-3})\). If \(Ca\) denotes the capillary number

\[Ca = \mu \nu/\sigma\]  \hspace{1cm} (2.19)
then we need

$$Ca = O(\varepsilon^3)$$ \hspace{1cm} (2.20)

If $Ca >> O(\varepsilon^3)$, surface tension is negligible. Similarly, (2.18) is consistent in order of magnitude only if $T = O(\mu VL^{-1} \varepsilon^{-1})$ i.e.

$$T/P = O(\varepsilon)$$ \hspace{1cm} (2.21)

If $T/P \ll O(\varepsilon)$, shear forces due to the jet are negligible.
3. Solution of Flow Equations

In view of (2.15), \( p = p(y,t) \), and hence by (2.17),

\[
p(y,t) = P(y,t) - \partial h_y(y,t)
\]

(3.1)
is known through the flow field, once \( h(y,t) \) is known. Now (2.16) indicates that the vertical velocity profile is parabolic in \( x \), i.e. the solution of (2.16) subject to (2.4) and (2.18) is

\[
v = V + \frac{\rho g + P - \partial h_y(y)}{\mu} \left( \frac{1}{2} x^2 - h x \right) + \frac{T x}{\mu} .
\]

(3.2)
The corresponding \( x \)-wise velocity is obtained by solving (2.3) subject to (2.4) i.e.

\[
u = h_y \frac{\rho g + P - \partial h_{yy}}{2\mu} x^2 - \frac{\partial h_{yy}}{\mu} \left( \frac{1}{6} x^3 - \frac{1}{2} h x^2 \right) - \frac{T x^2}{2\mu} .
\]

(3.3)
Finally, the kinematic boundary condition (2.5) gives

\[
h_t + \partial h_y = f - \frac{\sigma}{3\mu} \left( h^3 h_{yy} \right)_y
\]

(3.4)
where

\[
c = V + \frac{T h}{\mu} - \frac{\rho g + P}{2\mu} h^2
\]

(3.5)
and

\[
f = \frac{h^3}{3\mu} p_{yy} - \frac{h^2}{2\mu} T_y
\]

(3.6)

In general, equation (3.4) is a partial differential equation to determine \( h = h(y,t) \). One may hope to solve it, for any given initial profile \( h(y,0) \). If \( \sigma = 0 \), i.e. if there is no surface tension, or more accurately in view of (2.20), if \( Ca \gg \varepsilon \), (3.4) is a first-order equation and can be solved by the method of characteristics, (Tuck, 1982).

In the special case of steady flow in which no flow variable depends upon time \( t \), (3.4) integrates once to give (with primes for \( \frac{d}{dy} \))
\[ Q = V h + \frac{T}{2\mu} h^2 - \frac{\rho g \cdot P' \cdot h'''}{3\mu} h^3 \]  

(3.7)

where \( Q \) is a constant, physically identifiable as the net flux

\[ Q = \int_0^h v \, dx . \]  

(3.8)

Thus, constancy of (3.7) simply reflects conservation of mass for steady flow.
4. Effect of Surface Tension on Pressure Stripping

We shall be concerned principally with the case when the jet's action is dominated by its pressure, i.e. shall set \( T=0 \), or more accurately in view of (2.21), \( T/P \ll O(\varepsilon) \). Now (3.7) becomes

\[
Q = Vh - \frac{2g + P'}{3\mu} h^3.
\]

(4.1)

If \( \sigma=0 \) i.e. \( Ca >> O(\varepsilon^3) \), then (4.1) reduces again to

\[
Q = Vh - \frac{2g + P'}{3\mu} h^3.
\]

(4.2)

and if, further, there is no jet, i.e. \( P=0 \), to

\[
Q = Vh - \frac{2g}{3\mu} h^3.
\]

(4.3)

Equations (4.1)-(4.3) can all be interpreted as equations to determine the coating thickness \( h \), given \( Q \). But (4.1) is a 3rd-order ordinary differential equation, whereas (4.2) and (4.3) are cubic algebraic equations. Further, (4.3) is independent of the co-ordinate \( y \); thus solutions of (4.3) represent uniform coating thickness \( h = \text{constant} \).

As a function of \( h \), the right-hand side of (4.3) possesses a maximum, of value

\[
Q_m = \frac{2}{3} Vh_m;
\]

(4.4)

at \( h = h_m \), where

\[
h_m = \left(\frac{V\sqrt{g}}{h}\right)^{\frac{1}{3}}.
\]

(4.5)

So long as \( 0 < Q < Q_m \), there exist two positive solutions \( h = h_1 \) and \( h = h_2 \) of the cubic equation (4.3), satisfying

\[
0 < h_1 < h_m < h_2.
\]

(4.6)
The hypothesis of Deryaguin (1945) and Hrbek (1961) is that, in the absence of the jet, the actual thickness of the coating is \( h = h_0 \) as given by (4.5), namely that value that maximises the flux. The purpose of the jet is to reduce the actual coated thickness below this value \( h = h_0 \).

Any such jet will have a finite range of effect, and in particular, far above or below the position where it acts, we expect that \( P \to 0 \). Hence solutions of (4.2) will approach solutions of (4.3), for some \( Q \). Since \( h \) becomes asymptotically constant, then \( h'' \) vanishes, and hence solutions of the problem (4.1) with non-zero surface tension will also approach solutions of (4.3) far above or below the jet.

The above argument suggests that appropriate boundary conditions for the differential equation (4.1) are that \( h \) approaches constant values satisfying (4.3), when \( y \to \pm \infty \). There are several possibilities. We are not interested in \( Q > Q_m \) (for which there is no positive solution for \( h \)) or \( Q < 0 \) (which can never provide a coating). If \( 0 < Q < Q_m \), there are two positive solutions \( h_1 \) and \( h_2 \) of (4.3), as discussed above, and we could have (i) \( h_1 \) at both ends (ii) \( h_2 \) at both ends (iii) \( h_1 \) at \( y = -\infty \) and \( h_2 \) at \( y = +\infty \), or (iv) \( h_2 \) at \( y = -\infty \) and \( h_1 \) at \( y = +\infty \).

The "normal" situation is one of (i) or (ii). That is, if \( h = h(y) \) varies continuously as \( P(y) \) varies, it is to be expected that, when \( P(y) \) returns to its original (zero) value, \( h \) will also return to its original value, be it \( h_1 \) or \( h_2 \). Situations (iii) and (iv) are "extra-ordinary" (Tuck, 1982) in that co-incidence between peaks of \( P'(y) \) and of \( Q = Q(h(y)) \) allows a transition between \( h_1 \) and \( h_2 \).
However, situation (iv) is precisely that of present interest. That is, we are interested only in cases where the jet has the effect of reducing the coating thickness from the thick value $h_2$ below it to the thin value $h_1$ above it. In order to achieve this special result for a given $P(y)$, the value of $Q$ must be allowed to vary, until its value coincides with what is needed for the peak coincidence described above.

The above argument was used by Tuck (1982) to discuss appropriate solutions of the cubic algebraic equation (4.2). In the present generalisation to the 3rd-order differential equation (4.1), we need appropriate boundary conditions on $h(y)$ as $y \rightarrow \pm \infty$. We choose to use simply vanishing slope and curvature conditions, i.e.

$$h', h'' \rightarrow 0 \quad \text{as} \quad y \rightarrow \pm \infty. \quad (4.7)$$

At first sight, the system (4.1), (4.7) appears over-determined, since it involves 4 boundary conditions for a 3rd-order ordinary differential equation. However, in view of the above discussion, we must allow the parameter $Q$ to be determined by the solution, so that the order of the system is in effect 4 rather than 3. Indeed, if (4.1) is differentiated once to eliminate $Q$, this 4th-order character is made explicit. However, we prefer to work with (4.1) as it stands.

The boundary conditions (4.7) do not in themselves prescribe which of the situations (i)-(iv) will be achieved. However, (4.1) is a non-linear equation that will be solved by iteration. We do not expect that the system (4.1), (4.7) has a unique solution, and the solution that one actually achieves in any successfully completed
iterative procedure will depend upon the nature of the starting guess. We therefore "seed" our solution, by starting with a guessed profile $h(y)$ having $h > h_m$ for $y < 0$, and $h < h_m$ for $y > 0$. This causes convergence toward a solution of type (iv).

Equation (4.1) can be written in non-dimensional form as follows. If $h = h_m h^*$, $Q = Q_m Q^*$, $P = pgP^*$ and $y = Ly^*$, where $h_m$ and $Q_m$ are as defined by (4.5) and (4.4), and $L$ is the length scale for variations in $P$, then (omitting stars),

$$1 + P'(y) - \tau h'''(y) = \frac{3h - 2Q}{h^*}$$

where

$$\tau = \frac{\sigma h_m}{(pgL^3)}$$

i.e.

$$\tau^2 = Ca \frac{\sigma^2 \nu \rho}{g L^6}$$

where $Ca$ is the capillary number as defined earlier, and

$$Bo = \sigma/(pgL^2)$$

is a Bond number. In fact, it is the parameter $\tau$ that truly measures the influence of surface tension in the present problem; if $\tau$ is small, surface tension has a negligible effect.

We use the following direct numerical method to solve (4.8) subject to (4.7). Let $\Delta y$ be a suitably small interval, and write $h_j = h(y_0+j\Delta y)$, $j=1,2,...,N-1$, where $y_0 << 0$ and $y_0+N\Delta y >> 0$. We satisfy (4.7) by introducing 4 extra points $h_{-1}$, $h_0$, $h_N$, $h_{N+1}$ such that $h_{-1} = h_0 = h_1$ and $h_{N+1} = h_N = h_{N-1}$. Now, on this extended mesh, we use the 2nd-order accurate estimate

$$h'''(y + (j-\frac{1}{2})\Delta y) = \frac{h_{j+1} - 2h_j + 3h_{j-1}}{\Delta y^3}$$
for the 3rd derivative in (4.8) and force (4.8) to hold at \( y = y_0 + (j - \frac{1}{2}) \Delta y \), \( j=1,2,3,...,N \), by linearly interpolating the right-hand side. This gives \( N \) non-linear algebraic equations, that we solve by Newton iteration. There are a total of \( N \) unknowns, namely the \( N-1 \) values of \( h = h_j, \) \( j=1,2,...,N-1 \), and the unknown flux \( Q \).

In fact, it is \( Q \) itself that is the primary output of interest, since this quantity measures the final (solidified) coating thickness. Figure 2 shows \( Q \) versus \( P_1 \text{max} \ P'(y) \), for various values of \( \tau \). The curve for \( \tau=0 \) is that first given by Thornton & Graff (1976), namely

\[
Q = (1+P_1)^{-\frac{1}{2}}
\]

Notably, this zero-surface-tension result depends only on the non-dimensional maximum pressure gradient \( P_1 \), and is quite independent of the shape of the \( P'(y) \) curve.

If \( \tau > 0 \), the shape of the \( P'(y) \) curve does influence the value of \( Q \), and Figure 2 includes two different examples. The solid curves are for a piecewise-quadratic pressure gradient, i.e.

\[
P'(y)/P_1 = \begin{cases} 
0 & , \ y < -1 \\
-4y - 4y^2 & , \ -1 < y < 0 \\
-4y + 4y^2 & , \ 0 < y < 1 \\
0 & , \ y > 1 
\end{cases}
\]

with peaks at \( y = \pm 1/2 \). The dashed curves are for a Gaussian-type pressure distribution, whose gradient

\[
P'(y)/P_1 = -\frac{4}{3} y \exp(1-9ey^2/8)
\]

has peaks at \( y = \pm \frac{1}{3} \sqrt{8} = \pm 0.4 \). The normalization is such that in both cases the maximum non-dimensional pressure gradient is \( P_1 \), and the maximum non-dimensional pressure is \( \frac{1}{3} P_1 \). Thus, the length scale \( L \) used in the definition (4.9) of \( \tau \) has been taken as \( 1\frac{1}{2} \) times the...
Figure 2. Scaled coating flux $Q$ versus maximum pressure gradient $P_1$, for various values of the surface-tension parameter $\tau$. The solid curves are for a piecewise quadratic pressure gradient, and the dashed curves for an exponential pressure distribution.
ratio between the actual maximum pressure and the actual maximum pressure gradient.

As \( T \) increases above the zero value where (4.13) applies, a stronger jet (larger \( P_1 \)) is needed to produce the same stripping effect, since surface tension tends to keep the free surface plane. At the same time, different shapes of jet pressure distribution are in principle capable of yielding different stripping properties. However, at least when the length scale \( L \) is defined as above in such a way as to keep constant both the maximum pressure gradient and the maximum pressure, there is only a very small difference between the results using (4.14) and (4.15), over the complete range of values of \( T \) used.

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References


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Coating, surface tension

Air jets are used as a wiping device to reduce the final thickness in some industrial coating applications. Existing theories allow only a given internal normal stress, i.e. impose a pressure distribution in the coating liquid, due to the jet. In this note, surface tension and tangential stresses are also included. In particular, numerical results are obtained which quantify the extent to which surface tension reduces the wiping effect of the jet.