THEORIES OF KINEMATIC ANALYSIS AND SYNTHESIS 
OF SPATIAL MECHANISMS CONTAINING LOWER AND HIGHER PAIRS

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Theories of Kinematic Analysis and Synthesis of Spatial Mechanisms Containing Lower and Higher Pairs.

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Accomplishments to date include vector-theories for the analysis of spatial function, path, and motion generators containing higher-pair joints. Also completed are design theories which assure that a synthesized mechanism is free from branching defects. Additional theories have been developed for synthesizing several types of single-input spatial motion generator mechanism with complete input crank rotation, optimal transmission characteristics, and correct order of output positions. Methods have also been developed for efficiently formulating and solving systems of non-linear.
BLOCK 20. Cont..

equations which commonly arise in the synthesis of spatial mechanisms. The theories developed under the sponsorship of this grant have expanded the utility of spatial mechanisms. It has led to simplified analysis and design theories for spatial mechanisms containing higher pairs and it has produced a new "wholeistic" approach to spatial.
4. List of Material Contained in the Appendix

The appendix contains copies of the following papers which have been published. The titles and authors have been listed below, all other publication material is contained in section 5c. of this report.


2) "Kinematic Analysis of Four-Link Space Mechanisms Containing Sphere-Groove and Sphere-Slotted-Cylinder Higher Pair," A. Ghosal, D. Kohli, and G. N. Sandor.

The abstracts of the following Masters Thesis and Doctoral Dissertations have also been included in the appendix.


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5.a. State of the Problem Studied

Design and analysis theories for planar mechanisms are well developed and such devices are in common use. However, many automation tasks require mechanisms which can generate spatial motion. One solution is to employ multi-degree-of-freedom, multiple-input robotic manipulators. However, these devices are limited in speed and accuracy, and require sophisticated electronic control systems. On the other hand, single-input spatial mechanisms, the topic of this research, are purely mechanical, and are better suited for performing highly repetitive automation tasks of limited complexity more efficiently, reliably and economically than robotic manipulators.

Single-input spatial mechanisms are much more difficult to design and analyze than planar mechanisms. As a result, their use to date has been quite limited. This is especially true of spatial mechanisms containing higher pairs (joints which develop only point or line contact and allow several degrees of freedom of relative motion).

The research being conducted under this grant attempted to develop simplified theories for designing and analyzing single-input spatial mechanisms.

5.b. Summary of Most Important Results

Accomplishments to date include vector-theories for the analysis of spatial function, path and motion generators, containing higher-pair joints which allow minimizing the number of mechanical parts. For example, a newly analyzed class of spatial function generators has only two moving links: the input and the output. Also completed are design theories which assure that a synthesized mechanism is free from the "branching defect" (i.e. satisfies the physical motion requirements as well as the mathematical criteria. Additional theories have been developed for synthesizing several types of single-input spatial motion generator mechanisms to have complete input crank rotation, to have optimal transmission characteristics and to have the correct order of output positions.
Methods have been developed for efficiently formulating and solving systems of non-linear equations which commonly arise in the synthesis of spatial mechanisms.

It is believed that the theories developed under the sponsorship of this grant have greatly expanded the utility of spatial mechanisms in two important ways. First, it has led to simplified design and analysis theories for spatial mechanisms containing higher pairs. Second, it has produced a new "wholeistic" approach to spatial mechanism design, wherein many of the "real-world" constraint conditions are considered in the design process.

5.c. List of Publications


5.d. Participating Scientific Personnel

Personnel Drawing Support from this Project:
1) Dr. George N. Sandor, P.I.
2) Dr. Dilip Kohli, Consultant
3) Mr. Ashitava Ghosal, earned Ph.D., August, 1983.
4) Dr. Charles Reinholtz, earned Ph.D., August, 1983.
5) Mr. Partha De, Master's Degree Candidate

Personnel Contributing to the Research but not drawing support from this Project:
1) Mr. Xirong Zhuang, Visiting Engineer from the People's Republic of China.
2) Dr. Manuel V. Hernandez, earned Ph.D., May, 1983.
6. BIBLIOGRAPHY OF REFERENCES
Sources of Information Used in this Research


Generators with Spheric Joints Part I - Theory", presentation at the 1984 ASME Mechanisms conference and accepted for publication in the Transactions of the ASME.


7. APPENDICES
KINEMATIC ANALYSIS OF THREE-LINK SPATIAL MECHANISMS CONTAINING SPHERE-PLANE AND SPHERE-GROOVE PAIRS

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Abstract—Kinematic pairs in a spatial mechanism are viewed either as allowing relative screw motion between links or as constraining the motion of the two chains of the mechanism connected to the two elements of the pair. Using pair geometry constraints of the sphere-plane and sphere-groove kinematic pairs, the displacement, velocity, and acceleration equations are derived for, $R-SP-R$, $R-SP-P$, $P-SP-P$, $R-SP-R$, and $R-SP-C$ three-link mechanisms. For known values of the input variable, other variables are computed in closed form. The analysis procedures are illustrated using numerical examples.

1. INTRODUCTION

The mechanisms containing higher pairs such as cam, sphere-plane, sphere-groove, or cylinder-plane provide the designer with the capabilities of designing machines and mechanisms to satisfy more complex and exact functional requirements than feasible with only lower pair mechanisms. These mechanisms in general are compact and contain fewer links than those with lower pairs.

In recent years, there has been considerable development in the tools for kinematic analysis of spatial mechanisms containing lower pairs.

Kinematic analysis of space mechanisms was initiated by the significant contribution of Dimentiong[1]. Dimentberg[2, 3] demonstrated the use of dual numbers and screw calculus to obtain closed-form displacement relationships of an $RCCC'$ and other four-, five-, six- and seven-link spatial mechanisms containing revolute, cylinder, prismatic and helical pairs. Denavit[4] derived closed-form displacement relationships for a spatial $RCCC$ mechanism using dual Euler angles. Yang[5] also derived such relationships for $RCCC$ mechanisms using dual quaternions.

Vectors were first used by Chace[6] to derive closed-form displacement relations of $RCCC$ mechanisms. Wallace and Freudenstein[7] also used vectors to obtain closed-form displacement relations of $RRSRR$ and $RRP_{RR}$ mechanisms.

Yang[8] proposed a general formulation using dual numbers to conduct displacement analysis of $RCCR$ spatial five-link mechanisms. Soni and Pamidi[9] extended this application of $(3 \times 3)$ matrices with dual elements to obtain closed-form displacement relations of $RCCRR$ mechanisms.

Yuan[10] employed screw coordinates to obtain closed-form displacement relations for $RRCCR$ and other spatial mechanisms.


Bagai[19] used a $(3 \times 3)$ screw matrix for displacement analysis of a mechanism containing two revo-
lute pairs, one cylinder pair and one spheric pair. Dobrovolski[20] used the method of spherical images to analyze space mechanisms containing revolute and cylinder pairs. Duffy[21, 22], Duffy and Habib-Olahi[23] used the method of spherical triangles to derive displacement relations for five and six link mechanisms containing revolute and cylinder pairs. Keller[25] and Gupta[26] also analyzed space mechanisms containing revolute, prismatic, cylinder, helical and spheric pairs. Recently Kohli and Soni[26] and Singh and Kohli[27] used the method of pair constraint geometry and successive screw displacements to conduct analyses of single and multi-loop mechanisms.

In the present paper, screw displacements expressed in vector form and the pair geometry constraints, also expressed in vector form, are used to derive the displacement, velocity and acceleration equations for R-Sp-R, R-Sp-P, P-Sp-R, P-Sp-P and R-Sp-C three link mechanisms.

Since Revolute (R) and Prismatic (P) pairs are special cases of the cylinder pair (in prismatic pairs, the rotation is zero; for revolute pairs sliding is zero), we derive the analysis equation for C-Sp-C and C-Sg-C mechanisms, and then force rotations or translations at one or more pairs to zero, to obtain the equations for the above described three-link one degree of freedom mechanisms.

Briefly, the procedure for obtaining the analysis equations is as follows.

Step 1. Consider the C-Sp-C mechanism and the C-Sg-C mechanism.

Step 2. Separate the two moving links (Bodies 1 & 2) at the sphere-plane pair for the C-Sp-C case and at the sphere-grove pair for the C-Sg-C.

Step 3. Use the screw displacements in vector form to describe the new (ith) position of the sphere-plane (Sp) or sphere-grove (Sg) pairs from two sides of the pair.

Step 4. Use the pair geometry constraints on the position of the pair obtained from two sides.

Step 5. Force the cylindrical (C) joints as revolute (R) or prismatic (P) joints by setting the sliding or the rotation equal to zero at cylindrical pairs.

2. THE THREE-LINK MECHANISM AND ASSOCIATED VECTORS

Figure 1 shows the initial position of two rigid bodies grounded via cylindrical pairs and connected together by a sphere-plane pair. Also shown are the following vectors and scalar quantities:

- \( \mathbf{u}_s \) unit vector defining the direction of the axis of cylindrical pair A.
- \( \mathbf{u}_e \) unit vector defining the direction of the axis of cylindrical pair B.
- \( \mathbf{P} \) vector locating the axis of cylindrical pair at A in the fixed coordinate system.
- \( \mathbf{Q} \) vector locating the axis of cylindrical pair at B in the fixed coordinate system.
- \( \mathbf{A} \) unit vector perpendicular to the plane of the Sp pair embedded in body 1.
- \( \mathbf{A}' \) vector embedded in body 2, congruent with A in the starting position, as shown in Fig. 1.
- \( \mathbf{R} \) vector locating point R, the sphere center in the fixed coordinate system.
- \( \theta_1 \) rotation of link 1 about axis \( \mathbf{u}_s \).
- \( \theta_2 \) rotation of link 2 about axis \( \mathbf{u}_e \).
- \( \mathbf{S}_1 \) translation of link 1 along axis \( \mathbf{u}_s \).
- \( \mathbf{S}_2 \) translation of link 2 along axis \( \mathbf{u}_e \).

Figure 2 shows the C-Sg-C mechanism with all associated vectors and scalars. Description of all parameters are the same as for the C-Sp-C mechanism except for the direction of the vector \( \mathbf{A} \), which is now along the direction of the groove and also the addition of \( \mathbf{S}_p \), which is the translation of the sphere along the direction of \( \mathbf{A} \).

3. PAIR GEOMETRY CONSTRAINT EQUATIONS

Figures 3 and 4 show a sphere-plane (Sp) pair and a sphere-grove (Sg) pair with the vector \( \mathbf{R} \) locating \( \mathbf{R} \), the sphere center. The vector \( \mathbf{A} \), in the Sp pair is defined as a vector perpendicular to the plane in which the sphere moves. In the Sg pair, the vector \( \mathbf{A} \) defines the direction of the groove.
We can now define the vectors $R$, $A_1$, $R'$ and $A_1'$. These new vectors will define the displaced position and direction of initially coincident point $R$ and vector $A$ in bodies 1 and 2 respectively after some relative motion between bodies 1 and 2. The prime notation here is used for new position expressed from the motion of body 2, whereas the unprimed notations are used for new positions expressed from the motion of body 1.

The pair geometry constraint equation for the $Sp$ pair is

$$\frac{d}{dt}[(R_j - R'_j) \cdot A_j] = 0, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (1)

which expresses that any relative motion between the sphere and the plane must be perpendicular to the vector $A_j$ (Fig. 1).

The pair geometry constraint equation for the $Sg$ pair is

$$\frac{d}{dt}R_j \cdot \frac{d}{dt}R'_j = \frac{d}{dt}(A_j' S_g) \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (2)

where $S_g$ is the translation of the sphere along the groove in the direction of $A_j'$. The constraint equation for the $Sg$ pair expresses that any relative motion between the sphere and the groove must be along the groove which is in the direction of $A_j'$ (Fig. 2).

4. WORKING EQUATIONS

Referring to Fig. 1, let $A$ be a vector in body 1 $A'$ a momentarily congruent vector in body 2 in the first position, perpendicular to the plane of the $Sp$ pair.

\[
A_j = \cos \theta A_j (A - (A' \cdot u_j) u_j) + \sin \theta (u_j \times A) \\
+ (A' \cdot u_j) u_j
\]

\[
A_j' = \cos \theta A_j (A' - (A' \cdot u_j) u_j) + \sin \theta (u_j + A) \\
+ (A' \cdot u_j) u_j
\]

Also, the displaced position of the point $R$ in rigid bodies 1 and 2 are given by:

\[
R_j = \cos \theta (R - P) - ((R - P) \cdot u_j) + \sin \theta (u_j \times (R - P)) + [(R - P) \cdot u_j] u_j \\
+ u_j S_g + P
\]

\[
R'_j = \cos \theta (R - Q) - ((R - Q) \cdot u_j) u_j + \sin \theta (u_j \times (R - Q)) + [(R - Q) \cdot u_j] u_j \\
+ u_j S_g + Q
\]

Using the identity $[A - (A' \cdot u_j) u_j] = (u_j \times A) \times u_j$, introducing the vectors

\[
K = R - P
\]

\[
L = R - Q
\]

and the following notation for any two vectors $u_j$ and $D$.

\[
U_{c0} = (u_c \times D) \times u_c
\]

we can substitute eqns (7) and (7a) into eqns (5) and (6) to get

\[
R_j = R + u_j S_g + (\cos \theta - 1) U_{jK} + \sin \theta (u_j \times K)
\]

(5a)

and

\[
R'_j = R + u_j S_g + (\cos \theta - 1) U_{jL} + \sin \theta (u_j \times L)
\]

(6a)

We now take the time-derivatives of equations for $R_j$ and $R'_j$ and using the notation of dots above the
variables to indicate time derivatives, we obtain the following equations

\[ \dot{R}_j = u_x \dot{S}_y + \cos \theta_s(u_x \times K) - \sin \theta_s u_{x \times x} \dot{S}_y \]  
\[ \dot{R}_k = u_x \dot{S}_y + \cos \theta_s(u_x \times L) - \sin \theta_s u_{x \times L} \dot{S}_y \]  
\[ \dot{R}_r = u_x \dot{S}_y - \cos \theta_s u_{x \times x} + \sin \theta_s u_{x \times K} \dot{S}_y \]  
\[ + \cos \theta_s(u_x \times K) - \sin \theta_s u_{x \times x} \dot{S}_y \]  
\[ + \cos \theta_s(u_x \times L) - \sin \theta_s u_{x \times L} \dot{S}_y \]  

Substituting eqn (7a) into eqn (4), using eqns (5a) and (6a), and by making the following substitutions

\[ M_{x} = \cos \theta_s(u_x \times K) - \sin \theta_s u_{x \times x} \]  
\[ M_{y} = \cos \theta_s(u_x \times L) - \sin \theta_s u_{x \times L} \]  
\[ N_{x} = \cos \theta_s(u_x \times x) + \sin \theta_s u_{x \times K} \]  
\[ N_{y} = \cos \theta_s(u_x \times L) + \sin \theta_s u_{x \times L} \]  

we can derive the following working equations

\[ A' = A + (\cos \theta_s - 1)U_{x \times x} + \sin \theta_s u_{x \times A} \]  
\[ A' = \cos \theta_s(u_x \times A) - \sin \theta_s u_{x \times A} \]  
\[ A' = [\cos \theta_s(u_x \times A) - \sin \theta_s u_{x \times A}] \dot{S}_y \]  
\[ + [\cos \theta_s(u_x \times x) + \sin \theta_s u_{x \times K}] \dot{S}_y \]  
\[ = V_{y} \dot{S}_y - W_{y} \dot{S}_y \]  

where

\[ V_{y} = \cos \theta_s(u_x \times A) - \sin \theta_s u_{x \times A} \]  
and
\[ W_{y} = \cos \theta_s(u_x \times x) + \sin \theta_s u_{x \times K} \]  

For the \( Sp \) pair,

\[ (R_j - R_j) \cdot A_j = 0. \]  

For the \( Sg \) pair,

\[ (R_j - R_j) = A_j S_{g}. \]  

Observe that eqns (20) and (21) are eqns (1) and (2) with \( n = 0 \).

The cylindrical pairs used in the derivation may be forced to work as prismatic \( (P) \) pairs by letting \( \theta = 0 \) or may be forced to work as revolute \( (R) \) pairs by letting \( S = 0 \).

5.1 The \( P\)-\( Sp\)-\( P \) case

For this mechanism, we use \( \theta_s = \theta_s = 0 \) and eqns (14) and (17) are simplified to

\[ R_j - R_j = u_x S_j - u_x S_j \]  
and
\[ A_j = A. \]  

Substituting in eqn (20), we get

\[ (u_x S_j - u_x S_j) \cdot A = 0 \]  
which simplifies to the input/output equation

\[ S_j = \frac{u_x \cdot A}{u_x \cdot A}. \]  

5.2 The \( R\)-\( Sp\)-\( P \) case

\( \theta_s \) is the input; \( S_p \) is the output and \( \theta_s = \theta_s = 0 \). Equations (14) and (17) with \( \theta_s = \theta_s = 0 \) substituted in eqn (20) provide,

\[ \left[ -u_x S_p + (\cos \theta_s - 1)U_{x \times x} + \sin \theta_s u_{x \times K} \right] \cdot A = 0. \]  

After simplification we obtain

\[ S_p = \frac{[(\cos \theta_s - 1)U_{x \times x} + \sin \theta_s u_{x \times K}] \cdot A}{u_x \cdot A}. \]  

5.3 The \( R\)-\( Sp\)-\( R \) case

We have for this case \( S_p = S_p = 0 \), and eqn (14) and (17) are simplified to obtain

\[ R_j - R_j = (\cos \theta_s - 1)U_{x \times x} + \sin \theta_s u_{x \times K} \]  
and
\[ A_j = A + (\cos \theta_s - 1)U_{x \times x} + \sin \theta_s u_{x \times A}. \]  

Substituting the above equations into eqn (20), and simplifying the resulting equation, we obtain

\[ S_j + (\cos \theta_s - 1)U_{x \times x} + \sin \theta_s u_{x \times A} = 0. \]
where $S_j$ is the known vector

$$S_j = (\cos \theta_j - 1)U_{ax} + \sin \theta_j(u_x \times K). \tag{26}$$

Equation (25) can be solved for $\theta_j$ by using the following identities

$$1 - \tan^2 \frac{\theta_j}{2} = \frac{2 \tan \frac{\theta_j}{2}}{2 \tan \frac{\theta_j}{2}}$$

and simplifying the resulting quadratic equation to yield

$$\tan \frac{\theta_j}{2} = \frac{-b + \sqrt{b^2 - 4ac}}{2c} \tag{28}$$

where:

$$a = -U_{ax} \cdot A - S_j \cdot U_{ax}$$

$$b = (u_x \times L) \cdot A - S_j \cdot (u_x \times A)$$

$$c = -S_j \cdot A.$$

5.4 The P–Sp–R case

Here, $\theta_j = S_j = 0$ and we have

$$R_j = R'_j = u_x S_j - (\cos \theta_j - 1)U_{ax} - \sin \theta_j(u_x \times L)$$

and

$$A'_j = A + (\cos \theta_j - 1)U_{ax} + \sin \theta_j(u_x \times L).$$

Substituting the above equation into eqn (20) and simplifying, we get

$$\cos \theta_j - 1 - u_x S_j \cdot A - S_j(u_x \cdot U_{ax}) + \sin \theta_j$$

$$\sin \theta_j(u_x \times A) - S_j \mu_x \cdot A = 0. \tag{29}$$

Substituting eqns (27) in eqn (29) and simplifying the resulting quadratic gives us

$$\tan \frac{\theta_j}{2} = \frac{-b + \sqrt{b^2 - 4ac}}{2c} \tag{30}$$

where this time

$$a = -U_{ax} \cdot A - S_j(u_x \cdot U_{ax})$$

$$S = (u_x \times L) \cdot A - S_j \mu_x \cdot (u_x \times A)$$

$$c = -S_j \mu_x \cdot A.$$

5.5 The R–Sp–C case

Only $S_j$ in eqn (17) is identically zero, so we get

$$R_j - R'_j = -u_x S_j + S_j - (\cos \theta_j - 1)U_{ax}$$

$$- \sin \theta_j(u_x \times L)$$

where $S_j$ is given by eqn (26). Also,

$$A'_j = A + (\cos \theta_j - 1)U_{ax} + \sin \theta_j(u_x \times A).$$

Substituting in eqn (21), we have

$$u_x S_j - S_j + (\cos \theta_j - 1)U_{ax}$$

$$+ \sin \theta_j(u_x \times L) + A'_j S_j = 0.$$
For the $S_g$ pair

\begin{equation}
\mathbf{\dot{R}}_g - \mathbf{\dot{R}}'_g = A'_g S_{\theta_4} + A'_g S_{\theta_4}
\end{equation}

and

\begin{equation}
\mathbf{\ddot{R}}_g - \mathbf{\ddot{R}}'_g = \dot{\lambda}_g S_{\theta_4} + 2\dot{A}'_g S_{\theta_4} + A'_g S_{\theta_4}
\end{equation}

6.1 The P-Sp-P case

Here we can use the time derivatives of the displacement equation to get

\begin{equation}
S_{\theta_4} = \frac{u_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

\begin{equation}
\dot{S}_{\theta_4} = \frac{u_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

6.2 The R-Sp-P case

Equations (18) and (19) become

\begin{equation}
\mathbf{\dot{R}}_g - \mathbf{\dot{R}}'_g = M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4}
\end{equation}

and

\begin{equation}
\mathbf{\ddot{R}}_g - \mathbf{\ddot{R}}'_g = -N_{\theta_4} \dot{\beta}_{\theta_4} + M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4}
\end{equation}

Also

\begin{equation}
\dot{\lambda}_g = \lambda_{\theta_4} \dot{\lambda}_g = 0.
\end{equation}

Substituting in eqns (36) and (37), we get

\begin{equation}
\dot{S}_{\theta_4} = \frac{M_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

and

\begin{equation}
\dot{S}_{\theta_4} = -\frac{N_{\theta_4}}{u_{\theta_4}} A_{\theta_4} + \frac{M_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

6.3 The R-Sp-R case

\begin{equation}
S_{\theta_4} = S_{\theta_4} = S_{\theta_4} = S_{\theta_4} = S_{\theta_4} = 0.
\end{equation}

Equations (18) and (9) become

\begin{equation}
\mathbf{\dot{R}}_g - \mathbf{\dot{R}}'_g = M_{\theta_4} \beta_{\theta_4} - M_{\theta_4} \theta_{\theta_4}
\end{equation}

and

\begin{equation}
\mathbf{\ddot{R}}_g - \mathbf{\ddot{R}}'_g = -N_{\theta_4} \dot{\beta}_{\theta_4} + M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4}
\end{equation}

Also,

\begin{equation}
\dot{\lambda}_g = \lambda_{\theta_4} \dot{\lambda}_g = \lambda_{\theta_4} \dot{\lambda}_g = 0.
\end{equation}

Substituting in eqns (36) and (37), we get

\begin{equation}
\dot{S}_{\theta_4} = \frac{M_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

and

\begin{equation}
\dot{S}_{\theta_4} = -\frac{N_{\theta_4}}{u_{\theta_4}} A_{\theta_4} + \frac{M_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

6.4 The P-Sp-R case

Equations (18) and (19) are

\begin{equation}
\mathbf{\dot{R}}_g - \mathbf{\dot{R}}'_g = u_s S_{\theta_4} - M_{\theta_4} \theta_{\theta_4}
\end{equation}

and

\begin{equation}
\mathbf{\ddot{R}}_g - \mathbf{\ddot{R}}'_g = u_s S_{\theta_4} + N_{\theta_4} \dot{\beta}_{\theta_4} - M_{\theta_4} \theta_{\theta_4}
\end{equation}

Also,

\begin{equation}
\dot{\lambda}_g = V_{\theta_4} \theta_{\theta_4} \text{ and } \lambda_{\theta_4} = V_{\theta_4} \theta_{\theta_4} - W_s \theta_{\theta_4}.
\end{equation}

Substituting in eqns (36) and (37) we get

\begin{equation}
\dot{S}_{\theta_4} = \frac{u_{\theta_4}}{u_{\theta_4}} A_{\theta_4}
\end{equation}

and

\begin{equation}
\dot{S}_{\theta_4} = \frac{1}{D} \left( u_{\theta_4} S_{\theta_4} + 2 \left( u_{\theta_4} V_{\theta_4} S_{\theta_4} \right) \right)
\end{equation}

where $D$ is given by eqn (46).

6.5 The R-Sp-C case

Only $S_{\theta_4}$, $S_{\theta_4}$ and $S_{\theta_4}$ are zero and eqns (18) and (19) become

\begin{equation}
\mathbf{\dot{R}}_g - \mathbf{\dot{R}}'_g = M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4} - M_{\theta_4} \theta_{\theta_4}
\end{equation}

and

\begin{equation}
\mathbf{\ddot{R}}_g - \mathbf{\ddot{R}}'_g = -N_{\theta_4} \dot{\beta}_{\theta_4} + M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4}
\end{equation}

Also,

\begin{equation}
\dot{\lambda}_g = V_{\theta_4} \theta_{\theta_4} \text{ and } \lambda_{\theta_4} = V_{\theta_4} \theta_{\theta_4} - W_s \theta_{\theta_4}.
\end{equation}

Substituting the expression for $(\dot{R}_g - \dot{R}_g)$ just obtained into eqn (38) we get

\begin{equation}
M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4} - M_{\theta_4} \theta_{\theta_4} = \lambda_{\theta_4} S_{\theta_4} + V_{\theta_4} \theta_{\theta_4} S_{\theta_4} (49)
\end{equation}

$\dot{S}_{\theta_4}$, $S_{\theta_4}$ and $\dot{S}_{\theta_4}$ are unknowns in eqn (49).

Taking the dot product of eqn (49) with $(A'_\theta \times u_\theta)$, we get

\begin{equation}
(M_{\theta_4} \beta_{\theta_4} - M_{\theta_4} \theta_{\theta_4}) \cdot (A'_\theta \times u_\theta) = V_{\theta_4} \cdot (A'_\theta \times u_\theta) S_{\theta_4} C_{\theta_4} + V_{\theta_4} \theta_{\theta_4} S_{\theta_4}
\end{equation}

or

\begin{equation}
\dot{\theta}_{\theta_4} = \frac{M_{\theta_4} \cdot A'_\theta \cdot u_\theta}{(S_{\theta_4} V_{\theta_4} + M_{\theta_4}) \cdot A'_\theta \cdot u_\theta}
\end{equation}

Now, taking the dot product of eqn (49) with $[A'_\theta \times (S_{\theta_4} V_{\theta_4} + M_{\theta_4})]$, we have

\begin{equation}
(M_{\theta_4} \beta_{\theta_4} - u_s S_{\theta_4}) \cdot A'_\theta \cdot (S_{\theta_4} V_{\theta_4} + M_{\theta_4}) = 0
\end{equation}
Kinematic analysis of three-link spatial mechanisms

or

\[ S_b = \frac{M_d \cdot A'_b \times (S_q V_q + M_b)}{u_p \cdot A'_b \times (S_q V_q + M_b)} \theta_p \tag{51} \]

Letting \( X \) be equal to the r.h.s. of eqn (53) and by using the same technique of taking the dot product of eqn (53) with the proper cross-products, we will obtain the following:

\[ \theta_a = \frac{X \cdot A'_b \times u_p}{(S_q V_q + M_b) \cdot A'_b \times u_p} \tag{54} \]

\[ S_b = \frac{X \cdot A'_b \times (S_q V_q + M_b)}{u_p \cdot A'_b \times (S_q V_q + M_b)} \tag{55} \]

\[ S_q = \frac{X \cdot u_b \times (S_q V_q + M_b)}{A'_b \cdot u_b \times (S_q V_q + M_b)} \tag{56} \]

7. NUMERICAL EXAMPLES

1. Analysis of a \( R-S_p-R \) mechanism.

The vectors describing the mechanism are

\[ u_a = 0i + j + 0.6 \]

\[ u_b = (3i + j + 0.6)/\sqrt{10} \]

\[ p = 0i + 0.6 \]

\[ Q = 0i + 0.75j \]

\[ R = 1i + 1.5j + 2 \]

\[ A = 0i + 0.6j + 1 \]

The plot of the output displacement (\( \theta_a \)), velocity (\( \phi_a \)) and acceleration (\( \phi_a \)) are given in Fig. 5.

2. Displacement, velocity and acceleration analysis of a \( R-S_p-C \) mechanism.

![Graph showing \( \theta_a \), \( \phi_a \), and \( \phi_a \) for the \( R-S_p-R \) mechanism.](attachment:image.png)
The mechanism parameters are:

\[ u_a = (I + J + K l) \sqrt{6} \]
\[ u_b = (I + J + K l) \sqrt{6} \]
\[ P = 0 + 0 + 0 \]
\[ Q = 0 + 0 + 0 \]
\[ R = 3l + 3l + 3l \]
\[ A = (I + J + K l) \sqrt{6} \]

The motion parameters are: \( \theta_n \) is one unit of angular velocity and \( \theta_n \) is zero, both constant for \( j = 0, 1, 2, \ldots \). The results of the analysis for the \( R-Sg-C \) mechanism are shown in a table on the next page.

The direction of the rotations and linear motions are established using the right hand rule. Rotations are positive counterclockwise looking at the head of the unit vectors \( u_a \) and \( u_b \). Linear motions are positive when they are in the direction of the vectors they are associated with.

It is to be mentioned here also that although the quadratic equations gave two sets of solutions, only one set will define the motion of the mechanism. The other set of solutions are for those positions in which the mechanism has to be disassembled into the other possible configuration.

## CONCLUSIONS

Displacements, velocities and accelerations have been derived for several three-link spatial mechanisms containing sphere-plane and sphere-groove pairs. The groove of the sphere-groove pair was assumed to be a cylindrical groove, resulting in straight line axis of the groove. However, a more generalized groove may be one whose axis is a spatial curve. The authors are working on developing analysis procedures for mechanisms containing such a generalized sphere-groove pair. The expected results of their work will be the subject of a forthcoming paper. Similarly, the authors also have the generalization of the sphere-plane pair in progress, in which the parallel of the pair are generalized to form equidistant curved surfaces.

Acknowledgements—The authors wish to acknowledge the support of this research under Grant No. DAAG29-81-K-0125 at the University of Florida sponsored by the U.S. Army Research Office. The second author also wishes to acknowledge the support of the College of Engineering at the University of Wisconsin, Milwaukee.

## REFERENCES

Kinematic analysis of three-link spatial mechanisms


APPENDIX

1. Sphere–plane constraint equation

The complete displacement constraint equations of the Sp-pair are

\[ R_1 - R_2 = S_n \omega_n + S_p \omega_p \]

(a)

and

\[ \omega_p \cdot A' = 0 \]

(b)

where \( \omega_p \) is a unit vector in the plane of the Sp pair, perpendicular to \( A' \) and is in the direction of the relative motion of point \( R \) of body 1 with respect to the initially coincident point \( R' \) of body 2.

Derivatives of equations (a) and (b) with respect to time are taken to give the following velocity and acceleration constraint equations:

Velocity

\[ \dot{R}_1 - \dot{R}_2 = S_n \omega_n + S_p \omega_p \]

(c)

and

\[ \dot{\omega}_n \cdot A' + \omega_n \cdot \dot{A'} = 0 \]

(d)

Acceleration:

\[ \ddot{R}_1 - \ddot{R}_2 = S_n \ddot{\omega}_n + 2S_p \omega_p + S_p \ddot{\omega}_p \]

(e)

and

\[ \ddot{\omega}_p \cdot A' + 2\omega_p \cdot \ddot{A'} + \omega_p \cdot \dot{\omega}_p = 0 \]

(f)

The constraint eqns (a)–(f) are complete in the sense that all of the important variables in the motion of the joint elements are included. Also, the Coriolis component in the acceleration constraint eqn (f) is evident since \( A' \) is a function of \( \dot{\omega}_n \).

2. Proof that \( (d^2 \omega_n^a)(\dot{R}_1 - \dot{R}_2) \cdot A' = 0 \) for \( n = 1, 2 \) satisfies the complete Sp pair constraint equation

Without loss of generality, we can let \( S_n = S_n \omega_n \) and write the complete constraint equation as

\[ \frac{d}{dt}(R_1 - R_2) = \frac{d}{dt}(S_n) \]

(a)

and

\[ \frac{d}{dt}(S_n \cdot A') = 0 \]

(b)

Displacement: For \( n = 0 \), eqn (a) and (b) are

\[ (R_1 - R_2) = S_n \]

(c)

and

\[ S_n \cdot A' = 0 \]

(d)

Taking the dot product of eqn (c) with \( A' \) gives us the displacement constraint equation for the Sp pair.

\[ (R_1 - R_2) \cdot A' = 0 \]

(e)

Velocity: With \( n = 1 \), eqns (a) and (b) will become

\[ \dot{R}_1 - \dot{R}_2 = S_n \]

(f)

and

\[ S_n \cdot A' = - S_n \cdot A' \]

(g)

Taking the dot product of eqn (f) with \( A' \) gives us

\[ (R_1 - R_2) \cdot A' = S_n \cdot A' \]

(h)

Substituting eqn (g) into (b), we have

\[ (R_1 - R_2) \cdot A' = - S_n \cdot A' \]

(i)

Equation (c) can now be substituted in eqn (i) to get

\[ (R_1 - R_2) \cdot A' = - (R_1 - R_2) \cdot A' \]

or

\[ (R_1 - R_2) \cdot A' = (R_1 - R_2) \cdot A' \]

(j)

which is really

\[ \frac{d}{dt}[(R_1 - R_2) \cdot A'] = 0 \]

(k)

Acceleration: For \( n = 2 \), eqns (a) and (b) will be

\[ R_1 - R_2 = S_n \]

(l)
Taking the dot product of eqn (1) with \( A_i \) and substituting
\[
S_{n_i} \cdot A_i' + 2S_{n_i} \cdot A_i' + S_{n_i} \cdot A_i' = 0 \tag{m}
\]
Substituting eqns (c) and (f) into eqn (n) gives us
\[
(R_j - R_j) \cdot A_i' + 2(R_j - R_j) \cdot A_i' + (R_j - R_j) \cdot A_i' = 0
\]
which is
\[
\frac{d^2}{dt^2} ([R_j - R_j] \cdot A_j] = 0. \tag{o}
\]

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**ANALYSE CINÉMATIQUE DES MÉCANISMES SPATIAUX À TROIS BARRES CONTENANT LES PAIRES SPHERE-PLAN ET SPHERE-RAINURE**

G.N. Sandor, D. Kohli, M. Hernandez, Jr., A. Chosal

Résumé - On considère généralement qu'une paire dans un mécanisme spatial permet un mouvement relatif de vis entre les membres, où qu'elle reste le mouvement des éléments qui lui sont reliés.

En employant les contraintes géométriques des paires de sphère-plan et de sphère-rainure cinématiques, les équations pour le déplacement, la vitesse et l'accélération sont dérivées pour les mécanismes avec trois membres R-Sp-R, R-Sp-P, P-Sp-P, P-Sp-R et R-Sp-C (R: révolue; P: prismatique; C: cylindrique; S: sphérique; Sp: sphère-plan; Sr: sphère-rainure). Pour les valeurs connues de la variable d'entrée, les autres variables sont calculées par des formules non-itératives. Le procédé d'analyse est illustré par des exemples numériques.
The geometric constraints of two higher pairs, namely sphere-groove and sphere-slotted-cylinder, are derived. Using these pair geometry constraints, input-output relationships are derived for several mechanisms containing sphere-groove and sphere-slotted-cylinder pairs. The input-output equation for the R-Sg-R-R linkage is obtained as a fourth degree polynomial in the half-tangent of the output crank angle. For other cases of mechanisms containing a sphere-groove pair (such as R-Sg-R-P, R-Sg-P-R) the input-output equation is quadratic. The input-output equations for the R-Sc-C-R and R-Sc-R-C are obtained as eighth degree polynomials in the half-tangent of their output angles. For mechanisms with prismatic output containing a sphere-slotted-cylinder pair, the input-output equation is a second degree polynomial in the output translation.

Vectors were first used by Chace [7,8] to obtain vector equations for position, velocity and acceleration analysis.

Yang [9] used dual numbers to analyze RCRCR five link spatial mechanisms. Soni and Pamidi [10] extended this application of (3x3) matrices with dual elements to obtain closed-form displacement relationships for RCRCR spatial mechanisms. Soni, Dukkipati and Huang [11] also used (3x3) matrices with dual elements to analyze 5 link single loop and two loop spatial mechanisms containing revolute, prismatic and cylindrical pairs. Yuan [12] developed the use of screw co-ordinates by way of which they developed closed-form displacement relationships for all 3R-2C type spatial mechanisms. Duffy [14] has demonstrated the use of spherical trigonometry and dual numbers to obtain closed form input-output relation for four-, five-, and six-link spatial mechanisms. Duffy and Crane [15] also use the same method for the displacement analysis of a general spatial 7-link, 7R mechanism.

Iterative techniques for analysis of spatial mechanisms were developed by Hartenberg and Denavit [16]. Uicker [17] explored in further detail the matrix approach of Hartenberg and Denavit. Soni and Harrissberger [18] used (3x3) matrices with dual elements for an iterative approach to analyze spatial mechanisms.

Finite screws were used by Kohli and Soni [19,20] to conduct displacement analysis of single-loop and two-loop space mechanisms involving R,C,P,H and S pairs. Recently Kohli and Soni [21], and Kohli and Singh [22] used the method of pair geometric constraints and successive screw displacements to conduct analysis of spatial mechanisms containing lower and higher pairs. Sandor, Kohli etc. [23] used the above method to conduct displacement, velocity and acceleration analysis of three link spatial mechanisms containing sphere-plane and sphere-groove pairs.

In the present paper, finite screw displacement analysis,
expressed in vector form, and pair geometry constraints, also expressed in vector form, are used to derive the displacement equations for four-link spatial mechanisms containing sphere-groove and sphere-in-slotted-cylinder pairs. Although, the analysis of spatial mechanisms containing sphere-groove and sphere-in-slotted-cylinder pairs can be done by modeling these higher pairs as SP and RRSP, the procedure is made unnecessarily complicated by introducing these hypothetical joints. The use of finite screws and pair geometry constraints avoids this.

![Diagram of the C-Sg-C-C mechanism](image1)

**Figure 1.** The C-Sg-C-C Mechanism.

![Diagram of the C-Sc-f-C mechanisms](image2)

**Figure 2.** The C-Sc-f-C Mechanisms.

3. THE FOUR-LINK MECHANISMS AND ASSOCIATED VECTORS

Figure 1 shows the C-Sg-C-C mechanism, figure 2 shows the C-Sc-f-C mechanism. Also shown in the figures are the following vectors and scalar quantities. The vectors are denoted by wavy underscores.

- \( u_1 \) unit vector defining the direction of the first joint axis - grounded cylinder pair 1.
- \( u_3 \) unit vector defining the direction of the third joint axis in the initial position - cylinder pair 3.
- \( u_4 \) unit vector for defining the direction of the fourth joint axis - grounded cylinder pair 4.
- \( P_1 \) locates the first joint axis \( u_1 \).
- \( P_3 \) locates the third joint axis \( u_3 \) in its initial position.
- \( P_4 \) locates the fourth joint axis \( u_4 \).
- \( A \) unit vector along the direction of the groove embedded in the groove element.
- \( b \) unit vector perpendicular to the axial centerline of the groove defining the orientation of the slot in the initial position, embedded in the groove element.
- \( c \) - unit vector along the direction of the groove embedded in the groove element.
- \( b^* \) unit vector, coincident with \( b \) in the initial position, embedded in the sphere element.
- \( c^* \) unit vector initially coincident with \( c \), embedded in the sphere element.
- \( A^* \) unit vector embedded in the sphere element, initially coincident with \( A \).
- \( R \) vector locating the sphere center in the initial position.
- \( R^* \) vector locating a point on the groove axis, but initially coincident with \( R \).
- \( \theta_1 \) rotation of the groove element pivoted at the first C joint.
- \( \theta_3 \) relative rotation of the sphere element pivoted at the third C joint, with respect to link 3 (Fig. 2).
- \( \theta_4 \) rotation of link 3 pivoted at the fourth C joint.
- \( S_1 \) scalar translation along \( u_1 \) at joint 1.
- \( S_3 \) relative scalar translation of link 2 with respect to link 3 at joint 3.
- \( S_4 \) scalar translation at joint 4.
- \( S_g \) relative scalar translation of the sphere element along the groove for Sg pair.
- \( R \) relative scalar translation of the sphere element along the groove for Sc pair.
4. Finite Screw Displacements

Fig. 3 and 4 show a rigid body \( I \) connected to ground by means of one cylinder pair and by a chain containing two cylinder pairs respectively. By giving successive screw displacements and using a shorthand notation, the displaced position of \( A \) and \( R \) are obtained as shown in Table 1.

### Table 1 (continued)

where \( s_1 = A - (\cos \theta \sin \beta) R - \sin \theta \sin \beta \omega e \),

\[ s_2 = b - (\cos \theta \sin \beta) R - \sin \theta \sin \beta \omega e \]

where \( \theta \) and \( \beta \) are angles of rotation. \( s_1 \) and \( s_2 \) are the translations at the joints. \( P_1 \) and \( P_2 \) locate the axis of rotation \( s_1 \) and \( s_2 \) in the starting position.

Equations (c), (d), (e) and (f) can be written in a different form:

\[ s_1 = \cos \theta \sin \beta \omega e + \sin \theta \sin \beta \omega e \]

\[ s_2 = \cos \theta \sin \beta \omega e + \sin \theta \sin \beta \omega e \]

where \( P_1 \) and \( P_2 \) are functions of \( s_1 \) and \( s_2 \) only and can be expanded like equation (c) and (f)

\[ P_1 = \cos \theta \sin \beta \omega e + \sin \theta \sin \beta \omega e \]

\[ P_2 = \cos \theta \sin \beta \omega e + \sin \theta \sin \beta \omega e \]

In the case of a \( R \) or \( P \) joint, the translation or rotation respectively vanishes and the resulting equations can be considerably simplified.
The constraint equations for velocity and acceleration can be obtained by taking time derivatives of equations (1), (2), and (3). The general constraint equations for Sg pair can be written as:
\[
\frac{d^n}{dt^n}(R^1_j - R^3_j) = \frac{d^n}{dt^n}(S^1_j - S^3_j), \quad n = 0, 1, 2
\]
and for the Sc pair
\[
\frac{d^n}{dt^n}(R^1_j - R^3_j) = \frac{d^n}{dt^n}(C^1_j - C^3_j), \quad n = 0, 1, 2
\]

6. ANALYSIS OF MECHANISMS CONTAINING Sg PAIR

To analyze four link mechanisms containing an Sg pair, we need to consider equation (1). The terms can be expanded as in equations (b), (1) and (j) by using proper subscripts and superscripts. In order to get an input-output relation between \(q_1\) and \(q_2\) or \(S_1\) (if the fourth joint is prismatic) it is necessary to eliminate \(c_3\) or \(S_3\) (depending whether the third joint is a revolute or prismatic) and \(S_4\). The Sg pair has four degrees of freedom, so the other joints have to be either P or R joints.

6a. The R-Sg-R-R Case

In this case \(q_1\) is the input, \(q_2\) is the output.

All the cylinder joints have to be forced to have zero translation to make them revolute joints. Therefore we have the following expressions:
\[
R^1_j = R + (\cos q_1 - 1)(u_1)(R - P) + \sin q_1 (u_1)(R - P)
\]
(7)
\[
A^1_j = A + (\cos q_1 - 1)(u_1)(A) + \sin q_1 (u_1)(A)
\]
(8)
\[
R^3_j = S_3 + (\cos \theta_3^1 - 1)(C_3^1) + \sin \theta_3^1 (D_3^1)
\]
(9)

\(R^1_j, C_3^1, D_3^1\), and \(S_3\) can be expanded like equation (1) with proper change of subscripts and superscripts. Taking the cross product of equation (1) with \(A_j^1\) we have
\[
R^1_j \times A_j^1 = S_3^1 \times A_j^1
\]
(10)

The left hand side is known since \(\theta_3^1\) is the input. Expanding and then simplifying the right hand side of equation (10) we have,
\[
A_j^1 \times R_j^1 = A_j^1 \times R_j^3 + (\cos \theta_3^1 - 1)(A_j^1 \times C_3^1) + \sin \theta_3^1 (A_j^1 \times D_3^1)
\]
(11)

Taking the cross product of both sides with \(A_j^1 \times D_3^1\) we have
\[
(\cos \theta_3^1 - 1)(A_j^1 \times C_3^1) \times (A_j^1 \times D_3^1) = [A_j^1 \times (R_j^1 - R_j^3)] \times (A_j^1 \times D_3^1)
\]
(12)

Simplifying and noting that equation (12) is actually a scalar equation as both vectors are along \(A_j^1\), we
have,

\[(\cos^2 \theta - 1)A^2 - (\cos^2 \theta - 1)B^2 + 2A \sin \theta \cos \theta - 2B \sin \theta \cos \theta + 1 = 0\]  

(13)

Similarly taking cross product with \(A \times C\) using

\[a \times (b \times c) = (a \cdot c)b - (a \cdot b)c\]

and simplifying we have

\[\sin^2 \theta \left[A^2 + B^2 + 2A \cos \theta \sin \theta - 2B \cos \theta \sin \theta + 1\right] = 0\]

(14)

Equations (13) and (14) can be written as

\[A \cos^2 \theta = B + A\]

(15)

\[-A \sin^2 \theta = C\]

(16)

where

\[A = A^2 + B^2 + 2A \cos \theta \sin \theta - 2B \cos \theta \sin \theta + 1\]

\[B = A \cos \theta - B \sin \theta\]

\[C = A \sin \theta + B \cos \theta\]

(17)

Squaring and adding we have the input-output relationships as,

\[b^2 + 2ab + c^2 = 0\]

(18)

Simplifying expressions in (17) it can be shown that all terms A, B, C are linear in \(\sin^2 \theta\) and \(\cos^2 \theta\) (Table II).

Table II

<table>
<thead>
<tr>
<th>Simplifications of some Vector Expressions:</th>
</tr>
</thead>
<tbody>
<tr>
<td>[c_1 \cdot c_2 = c_1 \cdot (c_2 + c_3) + c_1 \cdot (c_2 - c_3)]</td>
</tr>
<tr>
<td>[= \sin \theta \cos \theta (c_1 \cdot c_2) + \cos \theta \sin \theta (c_1 \cdot c_3)]</td>
</tr>
<tr>
<td>[= \sin \theta \cos \theta (c_1 \cdot c_2) + \sin \theta \cos \theta (c_1 \cdot c_3)]</td>
</tr>
<tr>
<td>[= \sin \theta \cos \theta (c_1 \cdot c_2) + \sin \theta \cos \theta (c_1 \cdot c_3)]</td>
</tr>
<tr>
<td>[= \sin \theta \cos \theta (c_1 \cdot c_2) + \sin \theta \cos \theta (c_1 \cdot c_3)]</td>
</tr>
</tbody>
</table>

Simplifications of these expressions:

\[A \cos^2 \theta = B + A\]

(15)

\[-A \sin^2 \theta = C\]

(16)

\[A^2 = A^2 + B^2 + 2A \cos \theta \sin \theta - 2B \cos \theta \sin \theta + 1\]

\[B^2 = A \cos \theta - B \sin \theta\]

\[C^2 = A \sin \theta + B \cos \theta\]

(17)

Squaring and adding we have the input-output relationships as,

\[b^2 + 2ab + c^2 = 0\]

(18)

Simplifying expressions in (17) it can be shown that all terms A, B, C are linear in \(\sin^2 \theta\) and \(\cos^2 \theta\) (Table II).

So the input-output relation is of fourth degree in the tangent of the output half-angle. The input-output relationship is of the form,

\[a_4 (\cos^2 \theta - 1)^2 + a_2 \sin^2 \theta + a_4 \sin^2 \theta (\cos^2 \theta - 1) + a_6 (\cos^2 \theta - 1) + a_8 \sin^2 \theta + x_6 = 0\]

(19)

\[\text{using } \cos^2 \theta = \frac{1 - x_4}{1 + x_4} \quad \text{and } \sin^2 \theta = \frac{2x_4}{1 + x_4^2} \quad \text{we get,}\]

\[x_4^2 (2a_4 - 1 - 2a_6) + x_4^4 (2a_4 - 2a_6 + 2a_8) + x_4 (2a_4 - 1) + x_6 = 0\]

(20)

where \(a_i\), \(i = 1, 2, 3, 4, 5\) are defined in Table III.

Table III

<table>
<thead>
<tr>
<th>(a_1), (a_2), (a_3), (a_4), (a_5), (a_6), (a_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1 = \frac{1}{2} \left(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_8\right))</td>
</tr>
<tr>
<td>(a_2 = \frac{1}{2} \left(a_2 + a_3 + a_4 + a_5 + a_6 + a_8\right))</td>
</tr>
<tr>
<td>(a_3 = \frac{1}{2} \left(a_3 + a_4 + a_5 + a_6 + a_8\right))</td>
</tr>
<tr>
<td>(a_4 = \frac{1}{2} \left(a_4 + a_5 + a_6 + a_8\right))</td>
</tr>
<tr>
<td>(a_5 = \frac{1}{2} \left(a_5 + a_6 + a_8\right))</td>
</tr>
<tr>
<td>(a_6 = \frac{1}{2} \left(a_6 + a_8\right))</td>
</tr>
<tr>
<td>(a_8 = \frac{1}{2} \left(a_8\right))</td>
</tr>
</tbody>
</table>

Again using \(\cos^2 \theta = \frac{1 - x_3}{2} \quad \text{and } \sin^2 \theta = \frac{2x_3}{1 + x_3^2} \quad \text{we get,}\]

\[x_3 = \tan \frac{\theta}{2}\]

and using trigonometric identities:

\[\sin^2 \theta \cdot x_3 + \cos^2 \theta = 1\]

(21)

\[\sin^2 \theta - \sin \theta \cos \theta = x_3\]

(22)

we have, \(x_3 = B/C = C/B + 2A\)

(23)

using equation (1) and taking the scalar product with \(A_i\) we get,

\[S = \left(\frac{1 - a_1}{a_1 a_2 a_3 a_4 a_5 a_6 a_8}\right)^{\frac{1}{3}}\]

(24)

Taking the derivatives of equation (20), (23) and (24) we can obtain expressions for velocities and accelerations.

6b. The R-Sg-R-P, R-Sg-Z-P and R-Sg-P-P Cases.

For mechanisms containing \(S\) pair, like the R-Sg-R-P mechanism, the R-Sg-P-P mechanism and the R-Sg-Z-P mechanism, the input-output relationships can be found after simplifying the expressions for finite screw displacements and the pair geometry constraints (see reference [24] for details).

"See also Ref. [24]."
For the R-Sg-R-P mechanism the input-output relationship is quadratic in output displacement. For the R-Sg-P-R mechanism, the input-output relationship is quadratic in the output tangent half angle. For the R-Sg-P-P mechanism the input-output relationship is linear.

7. ANALYSIS OF FOUR LINK MACHINERYS CONTAINING AN Sc PAIR

The Sc pair has three degrees of freedom, so there can be only one cylindrical joint and the two other joints are revolute or prismatic. To analyze four link mechanisms containing an Sc pair we need to consider equations (2) and (3). In order to get an expression between input $\theta_3$ and output(s) $\theta_3$ and/or $\theta_4$, we need to eliminate one joint rotation and/or translation.

7a. The R-Sc-C-R Case

In this case, $S_1 = S_2 = 0$. Equation (2) can be written as,

$$R^3 + (\cos \theta_3 - 1)C^3j + \sin \theta_3 D^3j + u_3j S_3 + c_4^1 T = R_3^3 \tag{25}$$

By writing the dot product of equation (25) with $c_j^1$ and $u_3j$ we get after simplification,

$$(1 - \cos \theta_3) a_1 + \sin \theta_3 b_1 + c_1 = 0 \tag{26}$$

where $a_1 = \frac{1}{3} j^3 D_3$, $b_1 = \frac{1}{3} j^3 C_3$, $c_1 = \frac{1}{3} (\delta_3 u_3j) - u_3j (c_j^3)$.

Simplifying equation (3) we obtain

$$a_2 (1 - \cos \theta_3) + b_2 \sin \theta_3 + c_2 = 0 \tag{27}$$

where $a_2 = -(c_j^3 b_j^3 - X_4 b_j^3)$,

$$b_2 = (c_j^3 b_j^3 - X_4 b_j^3) \tag{28}$$

and $c_2 = (c_j^3 b_j^3 - X_4 b_j^3).$

In Table II the expanded terms for $X_4 b_j^3, Y_4 b_j^3$ are given. We can eliminate $\theta_3$ from (26) and (27) after using $\cos \theta_3 = \frac{1 - x_3^2}{2}$ and $\sin \theta_3 = \frac{2x_3}{1 + x_3^2}$

$$x_3 = \tan \theta_3 / 2$$

The solution is given by $|abc| |bc| - |ac|^2 = 0$, (30) and the common root is given by $x_3 = \pm \frac{|ac|}{|bc|} = \pm \frac{X_3}{|bc|}$

$$|ab| = (2a_1 c_1)(2b_2) - (2b_1)(2a_2 c_2) \tag{31}$$

$$|bc| = 2b_1 c_2 - 2b_2 c_1$$

$$|ac| = (2a_1 c_1) c_2 - (2a_2 c_2) c_1 = 2a_1 c_2 - 2a_2 c_1$$

Equation (29) is the input-output relationship. All the terms $a_1, b_4, c_2, i = 1,2$ are linear in $\sin \theta_3$ and $\cos \theta_3$. Equation (29) can be simplified to give

$$\delta_3 x_3^4 = 0$$

where $\delta_3$ and link parameters (refer to reference [24] and Table IV for details).

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<td>$a_1 = (c_1^3 b_1^3 - X_4 b_1^3)$</td>
</tr>
<tr>
<td>$b_2 = (c_1^3 b_1^3 - X_4 b_1^3)$</td>
</tr>
<tr>
<td>$c_2 = (c_1^3 b_1^3 - X_4 b_1^3) + (c_0^1 \theta_3 + \theta_3)$</td>
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<tr>
<td>$c_1 = (c_1^3 b_1^3 - X_4 b_1^3)$</td>
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<tr>
<td>$c_3 = (c_1^3 b_1^3 - X_4 b_1^3)$</td>
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Some other terms can be...
Table IV (continued)

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<td>Reference</td>
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<tr>
<td>1.5</td>
<td>Reference</td>
</tr>
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The input-output relationship for the Sc-P-C case is given by:

\[ T = \frac{(R_3N)^4 - (\cos S_3 - 1)^2}{(c_3 - c_4)^4} \cdot \left( \frac{V_3}{x_3} \right) \]

The input-output relationship for the Sc-P-C case is given by:

\[ T = \frac{(R_3N)^4 - (\cos S_3 - 1)^2}{(c_3 - c_4)^4} \cdot \left( \frac{V_3}{x_3} \right) \]

For the Sc-P-C mechanism, the input-output relationship is quadratic. For the Sc-P-C mechanism, the input-output equation is an eighth degree polynomial in the input tangent half-angle. For the Sc-P-C case, the input-output relationship is quadratic.

V. NUMERICAL EXAMPLE

1. R-Sc-R-C Mechanism

Given:

\[ \theta_1 = 2i + 3j + 0k \]

For the R-Sc-R-C mechanism, the unknowns are \( \theta_1, \theta_2, S_4 \), and \( \theta_3 \).

The plots of the \( \theta_3 \) and \( S_4 \) in terms of \( \theta_1 \) are shown in Figure 7 and 8.

2. R-Sc-C-R Mechanism

Given:

\[ \theta_1 = 2i + 3j + 0k \]

For the R-Sc-C-R mechanism, the unknowns are \( \theta_1, \theta_2, S_4 \), and \( \theta_3 \).

The plot of \( \theta_3 \) in terms of \( \theta_1 \) is given in Figure 9.
9. CONCLUSION

Displacement equations have been derived for several four-link spatial mechanisms containing sphere-groove and sphere-slotted-cylinder pairs. Velocity and acceleration relationships can be obtained by differentiating the displacement equations. The grooves of these pairs are assumed to have straight axial centerlines. However, a more generalized groove may be one where the centerline is a spatial curve. The authors are working on the analysis of these and also of other three, four, five and six link mechanisms containing other higher pairs. The expected result of the work will be reported in forthcoming papers.

ACKNOWLEDGMENT

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10. REFERENCES

ANALYSIS OF SPATIAL MECHANISMS
CONTAINING HIGHER PAIRS

BY

ASHITAVA GHOSAL

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ANALYSIS OF SPATIAL MECHANISMS CONTAINING HIGHER PAIRS

By

Ashitava Ghosal

August 1982

Chairman: George N. Sandor
Major Department: Mechanical Engineering

In this work, results of the investigation dealing with analysis of spatial mechanisms containing higher pairs are presented. Complete analytical expressions for the position, velocity and acceleration analysis for several three-, four-, and five-link spatial mechanisms containing higher pairs are presented. Also presented are computer programs for the analysis of some of these mechanisms.

A higher pair as distinct from a lower pair allows more degrees-of-freedom between its elements. The kinematic analysis of spatial mechanisms containing higher pairs is based on the concept of finite screws and pair geometry constraints. Expressions for finite screws and their derivatives have been developed and expressed in a shorthand notation. The pair geometry constraints for sphere-plane, sphere-groove, sphere-slotted-cylinder
and cylinder-plane higher pairs are presented. Using these pair
geometry constraints, and the finite screws and its derivatives,
several mechanisms containing the above mentioned higher pairs
have been analyzed for position, velocity and acceleration.

Computer programs are given for the analysis of some of the
typical three-, four-, and five-link mechanisms. The use of
the programs are demonstrated in the examples in Chapter VII.

George F. Sanders
Chairman
OPTIMIZATION OF
SPATIAL MECHANISMS

By

CHARLES FREDERICK REINHOLTZ

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
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OPTIMIZATION OF SPATIAL MECHANISMS

By

Charles Frederick Reinholtz

August, 1983

Chairman: George N. Sandor
Cochairman: Joseph Duffy
Major Department: Mechanical Engineering

The material in this dissertation can be effectively divided into two subtopics: philosophy of optimal mechanism design, and optimization of dyad-based spatial mechanisms.

The first subtopic, philosophy of optimal mechanism design, is intended to be general in nature, applying to all types of mechanisms, both higher and lower pair, and both planar and spatial. This is covered in Chapters One through Three. Chapter One examines past approaches to mechanism optimization. Chapter Two is a brief review of optimization theory, particularly as it applies to mechanism optimization. Chapter Three draws upon the insights gained in the first two chapters to formulate a general approach to the mechanism optimization problem.

The second subtopic of this dissertation, optimization of dyad-based spatial mechanisms, is covered in Chapters
Four through Seven. This is actually a rather limited example of applying the philosophy developed in the first three chapters. Nevertheless, the mechanisms treated in this section are believed to represent some of the most useful motion generating spatial mechanisms, and, therefore, those for which improved design theories are most urgently needed. In Chapter Four, closed-form synthesis equations are derived for dyads containing revolute (R), spheric (S) and cylindric (C) pairs. Chapters Five and Six present detailed examples of the optimization of the four-link RCCC and five-link RSSR-SC and RSSR-SS mechanisms. Finally, Chapter Seven outlines procedures for the optimization of other dyad-based spatial mechanisms, and offers suggestions for further research.
KINEMATIC SYNTHESIS AND ANALYSIS OF THREE-LINK SPATIAL FUNCTION GENERATORS WITH HIGHER PAIRS

BY

MANUEL VENADAS HERNANDEZ JR.

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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Abstract of Dissertation Presented to the Graduate Council of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

KINEMATIC SYNTHESIS AND ANALYSIS
OF
THREE-LINK SPATIAL FUNCTION GENERATORS WITH HIGHER PAIRS

By
Manuel Venadas Hernandez Jr.

April 1983

Chairman: Dr. George N. Sandor
Major Department: Mechanical Engineering

Function generation synthesis of spatial mechanisms with only three links is achieved by employing higher pairs (sphere-plane (Sp), cylinder-plane (Cp) and sphere-groove (Sg) pairs) to constrain the motion of two links.

This dissertation shows the methods and procedures for obtaining the equations for multiply-separated-precision point (MSP) synthesis for four spatial function generators - R-Sp-R, R-Sp-P, R-Cp-C and R-Sg-C.

Higher pair constraint equations in vector form are utilized to obtain closed-form solutions for the different synthesis cases of various numbers of positions and specified and unknown parameters. The method of elimination is used extensively to solve the resulting non-linear systems of equations.
Kinematic synthesis and analysis of the four spatial function generators is performed in vector notations and with screw displacements in vector form. Explicit equations are also obtained from the investigation of the transmission characteristics of these mechanisms.

The synthesis procedures just completed were then augmented by developing design criteria for the pair elements to assure range of mobility and avoidance of interference.