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A THEORETICAL ANALYSIS OF
NONLINEAR EFFECTS ON THE FLUTTER AND
DIVERGENCE OF A TUBE CONVEYING FLUID

ENOCH CH'NG, '78

AMS Report No. 1343

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A THEORETICAL ANALYSIS OF
NONLINEAR EFFECTS ON THE FLUTTER AND
DIVERGENCE OF A TUBE CONVEYING FLUID

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Department of Aerospace and Mechanical Sciences
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Princeton, NJ 08540

August 2, 1977
ABSTRACT

A theoretical model capable of describing the flutter and divergence of a tube conveying high velocity fluid is developed in this study. The nonlinear effect due to tension induced by bending is considered along with other linear forces. The partial differential equations of motion are reduced to a set of ordinary differential equations by Galerkin's method and the time histories of the tube motion are evaluated numerically. Numerical results for the thresholds of flutter instability and the mode shapes of a cantilevered tube are in good agreement with those obtained theoretically and experimentally by other investigators. (1)

A static or divergence instability occurs at a certain critical fluid velocity if the tube is simply supported at both ends. At higher fluid velocity, the tube becomes dynamically unstable. The limit cycle oscillations of the cantilever tube and the characteristics of the simply supported tube motion predicted by the present nonlinear analysis remain to be verified quantitatively by experiment. Plans have been made to set up an appropriate experiment.
ACKNOWLEDGEMENTS

I would like to thank Professor E. H. Dowell for his unfailing assistance in every aspect of this research. His constant encouragement throughout the course of study, his patience and his painstaking review of my work are deeply appreciated.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>iv</td>
</tr>
<tr>
<td>PART I: INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>PART II: CANTILEVERED TUBE</td>
<td>4</td>
</tr>
<tr>
<td>1. Derivation of Equations of Motion</td>
<td>5</td>
</tr>
<tr>
<td>1.1 General Approach</td>
<td>5</td>
</tr>
<tr>
<td>1.2 The Nonlinear Tension Effect</td>
<td>7</td>
</tr>
<tr>
<td>2. Methods of Solution</td>
<td>10</td>
</tr>
<tr>
<td>2.1 Nondimensional Equations of Motion</td>
<td>10</td>
</tr>
<tr>
<td>2.2 Galerkin's Method</td>
<td>11</td>
</tr>
<tr>
<td>2.3 Numerical Methods</td>
<td>13</td>
</tr>
<tr>
<td>3. Numerical Results</td>
<td>16</td>
</tr>
<tr>
<td>3.1 Comparison of Numerical and Previous Solutions</td>
<td>16</td>
</tr>
<tr>
<td>3.2 Theoretical Limit Cycle Predictions</td>
<td>17</td>
</tr>
<tr>
<td>3.3 Mode Shapes and Stress Distribution</td>
<td>17</td>
</tr>
<tr>
<td>PART III: SIMPLY SUPPORTED TUBE</td>
<td>19</td>
</tr>
<tr>
<td>1. Equations of Motion</td>
<td>20</td>
</tr>
<tr>
<td>2. Predicted Characteristics</td>
<td>22</td>
</tr>
<tr>
<td>PART IV: CONCLUDING REMARKS</td>
<td>24</td>
</tr>
<tr>
<td>References</td>
<td>26</td>
</tr>
<tr>
<td>Tables</td>
<td>27</td>
</tr>
<tr>
<td>Figures</td>
<td>31</td>
</tr>
<tr>
<td>Appendices</td>
<td>95</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$a_{jk}$</td>
<td>mode shape integral given by Eq. 41</td>
</tr>
<tr>
<td>$a$</td>
<td>cross sectional area of tube</td>
</tr>
<tr>
<td>$A$</td>
<td>notation for matrix</td>
</tr>
<tr>
<td>$b_{jk}$</td>
<td>mode shape integral given by Eq. 42</td>
</tr>
<tr>
<td>$c_{hi}$</td>
<td>mode shape integral given by Eq. A-15</td>
</tr>
<tr>
<td>$ds$</td>
<td>differential length</td>
</tr>
<tr>
<td>$D$</td>
<td>damping factor given in Eq. 9</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus</td>
</tr>
<tr>
<td>$f_{damp}$</td>
<td>damping force</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>$I$</td>
<td>moment of inertia of tube section</td>
</tr>
<tr>
<td>$\ell$</td>
<td>length of tube</td>
</tr>
<tr>
<td>$m$</td>
<td>mass per unit length of tube</td>
</tr>
<tr>
<td>$m_f$</td>
<td>mass per unit length of fluid</td>
</tr>
<tr>
<td>$\hat{q}$</td>
<td>column vector of $q_k$</td>
</tr>
<tr>
<td>$q_k$</td>
<td>generalized coordinate</td>
</tr>
</tbody>
</table>
\( q^* \) some constant \( q_k \)

\( r \) radius of gyration

\( R \) end reaction force

\( \overline{R} \) radius of tube

\( \overline{R}_1 \) inside radius of tube

\( \overline{R}_2 \) outside radius of tube

\( t \) time

\( t \) thickness (Appendix III)

\( T \) kinetic energy

\( T_0 \) tension

\( U \) potential energy

\( V \) velocity of fluid in tube

\( W \) work

\( x \) coordinate

\( y \) displacement of tube

\( \beta \) nondimensional velocity = \( \sqrt{\frac{m_f k^2}{EI}} V = 3.52 \sqrt{\frac{\mu}{T-\mu}} \frac{V}{\omega_f} \)

\( \varepsilon_x \) strain (x-direction)

\( \gamma_k \) \( k^{th} \) cantilever beam mode given in Eq. 34
\( \gamma_n \) \text{ \( n \)th simply supported beam mode given in Eq. 63}

\( \delta \) \text{ variation}

\( \delta_{jk} \) \text{ delta function}

\( \nabla \) \text{ operator, see Eq. 55}

\( \zeta_i \) \text{ critical damping ratio of \( i \)th mode without fluid}

\( \lambda_k \) \text{ mode shape constant of Eq. 35}

\( \mu \) \text{ mass ratio } = \frac{m_f}{m_f + m}

\( \nu \) \text{ viscous damping parameter } = 7.04 \zeta_1 \sqrt{1-\mu}

\( \xi \) \text{ nondimensional coordinate } = \frac{x}{\lambda}

\( \phi \) \text{ nondimensional displacement } = \frac{y}{r}

\( \sigma_k \) \text{ mode shape constant of Eq. 35}

\( \sigma_x \) \text{ stress (x direction) in psi}

\( \tau \) \text{ nondimensional time } = \sqrt{\frac{EI}{(m_f+m)^4}} t

\( \theta_1, \theta_2 \) \text{ angles of deflection}

\( \omega \) \text{ frequency}

\( \dot{\omega}_1 \) \text{ natural frequency of \( i \)th mode without fluid}

\( \Omega \) \text{ nondimensional frequency } = \sqrt{\frac{(m_f+m)^4}{EI}} \omega

= \frac{3.52}{\sqrt{1-\mu}} \frac{\omega}{\dot{\omega}_1}
Subscripts and Superscripts

\( c \) cantilevered

\( f \) fluid

\( ss \) simply supported at both ends

\( T \) true

\( ^* \) \( \frac{d}{d \tau} ( \ ) \)

\( ( \ )' \) \( \frac{d}{d \xi} ( \ ) \)
PART I: INTRODUCTION
As early as 1950, the observation of bending vibration of a simply supported petroleum pipeline prompted scientific investigations (2). More recently, it was observed that a short pipeline vibrates in its second circumferential mode at fairly high frequency (300-800 Hz) above a certain critical flow velocity (see Figure 1). The vibrations may sometimes generate a shrill sound (3). If the pipeline is sufficiently long, this instability is then superposed on the flexural oscillatory instabilities (Figure 2). High performance liquid-propellant launch vehicles may require rapid transfer of large quantities of fluid from tanks to pumps through pipes with relatively thin walls. Thus, a study of the influence of fluid velocity on the static and dynamic stabilities of propellant lines (tubes) is necessary (4). Other applications occur in hydraulic lines and human lung airways (5).

The purpose of this paper is to analyze the observed motion of an elastic tube conveying fluid using a mathematical model based upon the following assumptions:

(a) The three basic classes of operative forces are inertial, elastic and aerodynamic forces coupled by the elastic deformations of the tube.

(b) The fluid being conveyed in the tube is inviscid, incompressible and non-heat conducting.

(c) The tube may be approximated by a uniform beam.

Earlier investigations have developed linearized mathematical models capable of describing the behavior of the tube up to and including the threshold of instability. Section 1 of Part II briefly discusses the approach taken by Alan S. Greenwald and John Dugundji (1). To predict the large
amplitude periodic motion after the threshold of instability is exceeded, nonlinearities have to be taken into consideration. One such nonlinearity is due to tension induced by bending and consequent stretching of the line. Its existence is closely examined in Part II, Section 1.2. So far, this study is believed to be the only one conducted on the nonlinear tension effect.

Computer simulation is used to solve the set of ordinary differential equations obtained by applying Galerkin's method to the partial differential equations of motion. Data have been analyzed and the results are presented graphically for two sets of tube boundary conditions, cantilevered (clamped-free) and simply supported-simply supported. The clamped-free tube undergoes limit-cycle oscillations of constant peak amplitude and frequency at a fixed fluid velocity. Unlike what was expected, the maximum stress does not always develop at the root of the cantilevered tube (the clamped end). In fact, the point of maximum stress travels along the length of the tube during one period of the limit cycle oscillation. The stress and deflection of the tube increases as fluid velocity increases. For a simply supported-simply supported tube, a static deflection occurs above the threshold of instability whose amplitude increases as the flow velocity increases. At still higher velocity, the tube enters into a dynamically unstable region.
PART II: CANTILEVERED TUBE
Section 1
Derivation of Equations of Motion

1.1 General Approach*

To obtain a differential equation of motion describing the tube, an energy approach has been utilized to account for all the major forces acting on the nonconservative system. Hamilton's principle states

\[
\delta \int_{t_1}^{t_2} (T-U) \, dt + \int_{t_1}^{t_2} \delta W \, dt = 0
\]

The total kinetic energy of the system is the sum of the kinetic energy due to the motion of the line, and the kinetic energy due to the flowing fluid. In other words,

\[
T = T_{\text{line}} + T_{\text{fluid}}
\]

where

\[
T_{\text{line}} = \frac{1}{2} \int_0^L \rho \langle \frac{\partial \mathbf{u}}{\partial t} \rangle^2 \, dx,
\]

\[
T_{\text{fluid}} = \frac{1}{2} \int_0^L \rho_f |\mathbf{V}|^2 \, dx
\]

In this paper, the x and y components of the fluid velocity are approximated as**

\[
\begin{align*}
V_x & \sim V \\
V_y & \sim \frac{\partial y}{\partial t} + V \frac{\partial y}{\partial x}
\end{align*}
\]

* A similar discussion was previously reported in reference 1.
** See figure 3.
Then
\[ T_{\text{fluid}} = \frac{1}{2} \int_0^\lambda m_f [V^2 + (\frac{\partial y}{\partial t})^2] \, dx \] (6)

and
\[ T = \frac{1}{2} \int_0^\lambda \{m(\frac{\partial y}{\partial t})^2 + m_f[V^2 + (\frac{\partial y}{\partial t})^2 + 2V \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} + V^2(\frac{\partial y}{\partial x})^2]\} \, dx \] (7)

The total potential energy of the system consists of elastic strain energy of bending and the elastic energy stored in tension due to gravity. By assumption (c) of Part I, the concept of simple beam theory may be applied. In mathematical notation the total potential energy may be written as

\[ U = \frac{1}{2} \int_0^\lambda E I (\frac{\partial^2 y}{\partial x^2})^2 \, dx + \frac{1}{2} \int_0^\lambda mg(x) (\frac{\partial y}{\partial x})^2 \, dx \] (8)

A structural damping force and a shear force are assumed to be the two major nonconservative forces. Furthermore it is assumed that this structural damping is a viscous damping force of the form

\[ f_{\text{damp}} = D \frac{\partial y}{\partial t} \] (9)

\[ = 2\zeta_1 \bar{w}_1 m \frac{\partial y}{\partial t} \]

From momentum considerations, the shear force at the end of the line may be given as *

\[ R = m_f V [\frac{\partial y}{\partial t} (\lambda) + V \frac{\partial y}{\partial x} (\lambda)] \] (10)

It then follows that

\[ \delta W = - \int_0^\lambda 2\zeta_1 \bar{w}_1 m \frac{\partial y}{\partial t} \delta y \, dx - m_f [\frac{\partial y}{\partial t} (\lambda) + V \frac{\partial y}{\partial x} (\lambda)] \delta y(\lambda) \] (11)

* See Figure 3.
To derive the general equations of motion from Hamilton's Principle, equations (7), (8) and (1) are substituted into equation (1) first. The procedures of the calculus of variation are then performed. Appendix I details the mathematics involved. After considerable algebra, the partial differential equation of motion is found to be

\[
\begin{align*}
\frac{EI}{4} \frac{\partial^4 y}{\partial x^4} + (m + m_f) \frac{\partial^2 y}{\partial t^2} + 2m_f V \frac{\partial^2 y}{\partial t \partial x} + m_f V^2 \frac{\partial^2 y}{\partial x^2} \\
+ 2\alpha_1 \omega_1 m \frac{\partial y}{\partial t} - mg \frac{\partial}{\partial x} \left[(\ell-x) \frac{\partial y}{\partial x}\right] = 0
\end{align*}
\]  

(12)

and the boundary conditions are*

\[
\begin{align*}
\frac{EI}{4} \frac{\partial^2 y}{\partial x^2} (\ell) &= 0 \\
\frac{\partial y}{\partial x} (0) &= 0 \\
\frac{3y}{\partial x} (\ell) &= 0 \\
\frac{3y}{\partial x} (0) &= 0
\end{align*}
\]  

(13) (14) (15) (16)

1.2 The Nonlinear Tension Effect**

A very important physical characteristic of the tube undergoing flutter is its large amplitude periodic motion. This can only be explained by taking nonlinear effects into account because no linearized model can predict such a periodic solution.

* In Part III these are changed to those appropriate for simply supported edges.

** Up until now, it has been believed that this is the only study on the nonlinear tension effect on the dynamics of a fluttering propellant line (tube).
Imagine that the tube is deflected as shown in the free body diagram for a differential length, \( ds \) (Fig. 4). For small displacements, the approximations

\[
\theta_1 \approx \sin \theta_1 \approx \frac{\partial y}{\partial x} \quad (17)
\]
\[
\theta_2 \approx \sin \theta_2 \approx \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \right) \, dx \quad (18)
\]

may be used. Summing forces in the y-direction due to tension yields a net force of

\[
T_0 \frac{\partial y}{\partial x} - T_0 \left( \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} \right) \, dx = -T_0 \frac{\partial^2 y}{\partial x^2} \, dx \quad (19)
\]

for a tube element of differential length, \( dx \). Assuming that the propellant line is Hookean ideal elastic, then

\[
T_0 = \sigma_x a \quad (20)
\]
\[
E = \frac{\sigma_x}{\varepsilon_x} \quad (21)
\]

where

\[
\varepsilon_x = \frac{\Delta l}{l} \quad (22)
\]

The elongated length can be written as

\[
ds = \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial^2 y}{\partial x^2}\right)^2} \quad (23)
\]

and

\[
\int_0^l ds = \int_0^l \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \, dx
\]

\[
= \int_0^l \left[ 1 + \frac{1}{2} \left( \frac{\partial^2 y}{\partial x^2} \right)^2 + \ldots \right] \, dx \quad (24)
\]

Neglecting higher order terms, equation (22) may be written as

\[
\varepsilon_x = \int_0^l \left[ 1 + \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2 \right] \, dx - \frac{\varepsilon_x}{l} = \frac{1}{2l} \int_0^l \left( \frac{\partial y}{\partial x} \right)^2 \, dx \quad (25)
\]

Combining equations (25), (21) and (20) give
With this tension term and using (19) and (26), the equation of motion becomes

\[
EI \frac{d^4 y}{dx^4} + (m + m_f) \frac{d^2 y}{dt^2} + 2m_f V \frac{d^2 y}{dt dx} + m_f V^2 \frac{d^2 y}{dx^2} + \\
2 \zeta \omega_n m \frac{dy}{dt} - mg \frac{dy}{dx} \left[(\lambda - x) \frac{dy}{dx}\right] \\
- \frac{Ea}{2} \int_0^L \left[(\frac{dy}{dx})^2 \right] dx \cdot \left[\frac{d^2 y}{dx^2}\right] = 0
\]

In most cases, the gravity force is small compared to the other forces and may be neglected. Thus,

\[
EI \frac{d^4 y}{dx^4} + (m + m_f) \frac{d^2 y}{dt^2} + 2m_f V \frac{d^2 y}{dt dx} + m_f V^2 \frac{d^2 y}{dx^2} + \\
2 \zeta \omega_n m \frac{dy}{dt} - \frac{Ea}{2} \int_0^L \left[(\frac{dy}{dx})^2 \right] dx \frac{d^2 y}{dx^2} = 0
\]
Section 2

Methods of Solution

2.1 Non-dimensional Equations of Motion

The coordinate system $(\xi, \phi)$ and time $\tau$ chosen to nondimensionalize the equations of motion are defined as

\begin{align*}
\xi &= x/l \\
\phi &= y/r, \quad (r^2 = R^2/2) \quad \text{(*)}
\end{align*}

(29)

\[ \tau = \sqrt{\frac{EI}{(m+m_f)k^4}} t \]  \quad (30)

With the above dimensionless quantities substituted into equation (28), the equation of motion is found to be

\[ \frac{\partial^4 \phi}{\partial \xi^4} + \frac{\partial^2 \phi}{\partial \tau^2} + \frac{2Vm_f\phi}{\sqrt{EI(m+m_f)}} \frac{\partial^2 \phi}{\partial \xi^2 \partial \tau} + \frac{m_fV^2\phi^2}{EI} \frac{\partial^2 \phi}{\partial \xi^2} \]

\[ + \frac{2\xi \omega_m k \phi}{\sqrt{EI(m+m_f)}} \frac{\partial \phi}{\partial \tau} - \frac{\alpha^2}{2I} \int_0^1 (\frac{\partial \phi}{\partial \xi})^2 d\xi \frac{\partial^2 \phi}{\partial \xi^2} = 0 \]

(30)

With

\[ \beta \equiv \sqrt{\frac{m_fk^2}{EI}} V \] (velocity parameter)

\[ \mu \equiv \frac{m_f}{m + m_f} \] (mass ratio)

* See Appendix III for the derivation.
\[ \nu \equiv \frac{2\tilde{\omega}_1 \omega \eta l^2}{\sqrt{EI(m+m_f)}} \] (damping parameter)

and the fact that for a thin cylindrical tube, \( I = a r^2 \), equation (30) can be expressed as

\[ \frac{\partial^4 \phi}{\partial \xi^4} + \frac{\partial^2 \phi}{\partial t^2} + 2\beta \mu \frac{\partial^2 \phi}{\partial \xi \partial \tau} + \beta^2 \frac{\partial \phi}{\partial \tau} - \frac{1}{2} \int_0^1 \left( \frac{\partial \phi}{\partial \xi} \right)^2 d\xi + \frac{\partial^2 \phi}{\partial \xi^2} = 0 \] (31)

Equation (31) has associated boundary conditions

at \( \xi = 0 \) \hspace{1cm} \phi = 0, \hspace{1cm} \frac{\partial \phi}{\partial \xi} = 0 \] (32)

at \( \xi = 1 \) \hspace{1cm} \frac{\partial^2 \phi}{\partial \xi^2} = 0, \hspace{1cm} \frac{\partial^3 \phi}{\partial \xi^3} = 0 \] (33)

2.2 Galerkin's Method

The partial differential equation discussed in the previous section may be reduced to a set of ordinary differential equations by using Galerkin's method. It is assumed that the solution is of the form

\[ \phi(\xi, \tau) = \sum_{k=1}^{m} \gamma_k(\xi) q_k(\tau) \] (34)

where \( q_k(\tau) \)'s are unknown functions of time, and \( \gamma_k(\xi) \)'s are assumed functions satisfying all boundary conditions of the problem. For the uniform cantilever beam considered in this study, \( \gamma_k(\xi) \)'s are taken to be

\[ \gamma_k(\xi) = \cosh \lambda_k \xi - \cos \lambda_k \xi - \sigma_k[\sinh \lambda_k \xi - \sin \lambda_k \xi] \] (35)

Values of \( \lambda_k \) and \( \sigma_k \) are given in Table 1. Substituting equation (34) into equation (31), and using the notations

\[ \tilde{\omega}_1 = 3.52 \sqrt{EI/ml^4} \]
\[
\frac{\partial q_k(\tau)}{\partial \tau} = \ddot{q}_k(\tau) \quad \frac{\partial \gamma_k(\xi)}{\partial \xi} = \gamma_k''(\xi) \tag{36}
\]

The following equation is obtained

\[
\sum_{k=1}^{m} q_k(\tau) \gamma_k'(\xi) + \sum_{k=1}^{m} \ddot{q}_k(\tau) \gamma_k(\xi) + 2 \beta \sqrt{\mu} \sum_{k=1}^{m} \gamma_k'(\xi) \ddot{q}_k(\tau) + \beta^2 \sum_{k=1}^{m} q_k(\tau) \gamma_k''(\xi) 
\]

\[+ \nu \sum_{k=1}^{m} \gamma_k(\xi) \ddot{q}_k(\tau) - \frac{1}{2} \left[ \int_{0}^{1} \left( \sum_{k=1}^{m} \gamma_k'(\xi)\right)^2 d\xi \right] \cdot \left[ \sum_{k=1}^{m} q_k(\tau) \gamma_k''(\xi) \right] = 0 \tag{37}\]

The integral in the last term of equation (37) may be re-written as

\[
\int_{0}^{1} \left( \sum_{k=1}^{m} q_k(\tau) \gamma_k'(\xi) \right)^2 d\xi = \sum_{k=1}^{m} \sum_{h=1}^{m} \left[ q_h(\tau) q_i(\tau) \cdot \int_{0}^{1} \gamma_h(\xi) \gamma_i'(\xi) d\xi \right] \tag{38}\]

The mode shape integral is evaluated in Appendix II with its values tabulated as \( c_{hi} \) in Table 2. Equation (38) then becomes

\[
\int_{0}^{1} \left( \sum_{k=1}^{m} q_k(\tau) \gamma_k'(\xi) \right)^2 d\xi = \sum_{k=1}^{m} \sum_{h=1}^{m} c_{hi} q_h(\tau) q_i(\tau) \tag{39}\]

Applying Galerkin's method to equation (37) yields

\[
\sum_{k=1}^{m} \left[ \int_{0}^{1} \gamma_j \gamma_k d\xi \right] \ddot{q}_k + \int_{0}^{1} \gamma_j \gamma_k d\xi \ddot{q}_k 
\]

\[+ 2 \beta \sqrt{\mu} \left[ \int_{0}^{1} \gamma_j \gamma_k d\xi \right] \dddot{q}_k + \beta^2 \left[ \int_{0}^{1} \gamma_j \gamma_k'' d\xi \right] q_k 
\]

\[+ \nu \left[ \int_{0}^{1} \gamma_j \gamma_k d\xi \right] \ddot{q}_k - \frac{1}{2} \sum_{h=1}^{m} \sum_{i=1}^{m} q_h(\tau) q_i(\tau) c_{hi} \left[ \int_{0}^{1} \gamma_j \gamma_k'' d\xi \right] q_k = 0 \]

\[ (j = 1, 2, \ldots, m) \tag{40}\]
To simplify equation (40), define

\[ a_{jk} = \int_0^1 \gamma_j \gamma_k \, d\xi \]  

\[ (41) \]

\[ b_{jk} = \int_0^1 \gamma_j \gamma_k \, d\xi \]  

\[ (42) \]

\[ a_{jk} \text{ and } b_{jk} \text{ have been evaluated in the same manner as } c_{hi}. \text{ Their values are also tabulated in Table 2. Furthermore, } \]

\[ \int_0^1 \gamma_j \gamma_k \, d\xi = \delta_{jk} \]  

\[ (43) \]

\[ \int_0^1 \gamma_j \gamma_k^{iv} \, d\xi = \delta_{jk} \lambda_k^4 \]  

\[ (44) \]

where \( \delta_{jk} \) is the delta function. Equation (40) may be further reduced to

\[ \sum_{k=1}^{m} \{ \delta_{jk} q_k + [2\beta \sqrt{\mu} a_{jk} + \nu \delta_{jk}] \dot{q}_k \}

+ \{ \delta_{jk} \lambda_k^4 + \beta^2 b_{jk} - \frac{1}{2} \sum_{h=1}^{m} \sum_{i=1}^{m} q_h q_i c_{hi} b_{jk} \} q_k \} = 0 \]

\[ (j = 1, 2, \ldots, m) \]  

\[ (45) \]

2.3 Numerical Methods

The time history of the tube motion may be determined by solving equation (45) numerically. For simplicity, first consider a two mode analysis for equation (45) without the nonlinear term, that is
\[ \sum_{k=1}^{2} \left\{ \delta_{jk} \dot{q}_k + [2 \beta \sqrt{\bar{\mu}} a_{jk} + \delta_{jk} \nu] \dot{q}_k + [\delta_{jk} \lambda_k^4 + \beta^2 b_{jk}] q_k \right\} = 0 \quad (j = 1, 2) \] (46)

In another form,

\[ \begin{align*}
\dot{q}_1 &= [2 \beta \sqrt{\bar{\mu}} a_{11} + \nu] \dot{q}_1 + [\lambda_1^4 + \beta^2 b_{11}] q_1 + 2 \beta \sqrt{\bar{\mu}} a_{12} \dot{q}_2 + \beta^2 b_{12} q_2 = 0 \\
2 \beta \sqrt{\bar{\mu}} a_{21} \dot{q}_1 + \beta^2 b_{21} q_1 + \dot{q}_2 + [2 \beta \sqrt{\bar{\mu}} a_{22} + \nu] \dot{q}_2 + [\lambda_2^4 + \beta^2 b_{22}] q_2 &= 0
\end{align*} \] (47) (48)

If one defines

\[ q_3 = \dot{q}_4 \Rightarrow \dot{q}_1 = \dot{q}_3 \] (49)
\[ q_4 = \dot{q}_2 \Rightarrow \dot{q}_2 = \dot{q}_4 \] (50)

equations (47) and (48) may be written in matrix form as

\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-(\lambda_1^4 + \beta^2 b_{11}) & -\beta^2 b_{12} & -(2 \beta \sqrt{\bar{\mu}} a_{11} + \nu) & -2 \beta \sqrt{\bar{\mu}} a_{12} \\
-\beta^2 b_{21} & -(\lambda_2^4 + \beta^2 b_{22}) & -2 \beta \sqrt{\bar{\mu}} a_{21} & -(2 \beta \sqrt{\bar{\mu}} a_{22} + \nu)
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
\]

(51)

In short,

\[ \dot{\mathbf{q}} = A \dot{\mathbf{q}} \] (52)

But

\[ \dot{\mathbf{q}}(\tau) \sim \frac{\dot{\mathbf{q}}(\tau + \Delta \tau) - \dot{\mathbf{q}}(\tau)}{\Delta \tau} \] (53)

then

\[ \dot{\mathbf{q}}(\tau + \Delta \tau) = A \mathbf{q}(\tau) \Delta \tau + \ddot{\mathbf{q}}(\tau) \] (54)

for \( \Delta \tau \) sufficiently small.
The fact that equation (54) is easily expandable for an m-mode analysis makes it the preferred form for writing the equations of motion.

In the case of four mode nonlinear analysis, the nonlinear term of equation (45) alone generates 256 terms. However, only 20 of them are unique. The size of the matrix can thus be reduced to (8,28).*

The error of the approximated solution can be reduced by using the more accurate finite difference relation.**

\[ q(T+\Delta T) \approx q(T) + \Delta T(1 + \frac{1}{2} V + \frac{5}{12} V^2 + \ldots) \ddot{q}(T) \]  \hspace{1cm} (55)

where

\[ Vq(T) = \dot{q}(T) - \dot{q}(T-\Delta T) \]  \hspace{1cm} (56)

and

\[ V^{r+1} \ddot{q}(T) = V^r \ddot{q}(T) - V^r \ddot{q}(T-\Delta T) \]  \hspace{1cm} (57)

A listing of WATFIV computer programs for four mode linear and nonlinear analysis is given in Appendix IV. They are based on the algorithms discussed in this sections and improved by equation (55). Initial conditions and data used in the program are given in Table 3.

The above approach to determination of nonlinear limit cycle oscillations is similar to that employed by Dowell for plates and shells (7).

* Note that the vector and the matrix have to be conformal.

** The formulas given can be found in Reference 6.
3.1 Comparison of Numerical and Previous Solutions

The three most interesting questions posed are, for a given mass ratio and damping ratio,

i) What is the fluid velocity at which the tube begins to flutter?

ii) What is the flutter frequency? and,

iii) What is the limit cycle amplitude of vibration?

Answers to all the questions above may be obtained from the raw computer data. The most primitive method is the "binary search" method.* The computer program is run at several values of the fluid velocities until the flutter behavior is observed. Binary search is then applied to determine the fluid velocity at which the system is neutrally stable.

Another method is to process the raw data. Frequencies and damping ratios of the different vibration modes are plotted against the nondimensional fluid velocity on the same graph. Flutter velocity and frequency may be readily detected from the graph. Examples of two and four-mode linear analysis are shown in Figures 5 and 6.

To make sure that the numerical results obtained reflect the actual dynamic instabilities of the tube, rather than numerical instabilities, the results from the four mode linear analysis are double checked against the previous solutions derived by Alan S. Greenwald and John Dugundji (1). Although the approaches used are significantly different, the two results are found to be in very close agreement. They are shown in Figures 7a and 7b.

* Concept of Binary Search is given in detail in Reference 8.
3.2 Theoretical Limit Cycle Predictions

From the nonlinear model, one can predict the behavior of a cantilevered tube conveying fluid at low fluid velocity. For a given mass ratio, the amplitude of the vibrations is expected to decay to zero with time. However, above a certain critical velocity, the amplitude would increase first as if there is no upper bound. After a few cycles, the amplitude of the vibrations drops back by a fraction and stays at a constant magnitude indefinitely. This is called a limit cycle. The amplitude of the limit cycle increases with the fluid velocity. Graphical presentations of the relationship between the amplitude and the fluid velocity are given in Figure 8 and 9 for mass ratios of 0.1 and 0.5 respectively.

First and second mode frequencies are identical at the threshold of instability. Beyond that, both frequencies increase with the fluid velocity. This is shown in figures 10 and 11 for mass ratios of 0.1 and 0.5 respectively.

3.3 Mode Shapes and Stress Distributions

Mode shapes of the fluttering tube can be obtained by holding $\tau$ in equation (34) constant. Altogether five values of $\tau$ are judiciously chosen to represent equal time increments over one-half of the period of the limit cycle oscillations. The mode shapes of the tube at these instances are plotted on the same scale for ease of comparison. Figures 12, 13, 14 show mode shapes of tube of mass ratio 0.1 for three different fluid velocity, $\beta$. Figures 15, 16, 17 show the cases where mass ratio is 0.5, and the mode shapes are at $1/2, 5/8, 3/4, 7/8$ and the end of the period.

Stress distribution may be determined by using the following equation
\[ \sigma_x = Er^2 \frac{\partial^2 Y}{\partial x^2} \]  

or \[ \sigma_x = \frac{E r^2}{\lambda^2} \frac{\partial^2 \phi}{\partial \xi^2} \]

\[ = \frac{E r^2}{\lambda^2} \sum_{k=1}^{m} q_k(\tau) \gamma_k''(\xi) \]

\[ = \frac{E r^2}{\lambda^2} \sum_{k=1}^{m} q_k(\tau) \gamma_k'' \left[ \cosh \lambda_k \xi + \cos \lambda_k \xi - \sigma_k (\sinh \lambda_k \xi + \sin \lambda_k \xi) \right] \]  

Figure 18 shows the stress distribution in the tube for \( \mu = 0.5 \) and \( \beta = 10.4 \) at five different time steps. Unlike what was expected, the maximum stress does not occur at the root of the tube (the clamped end). It appears to be shifting along the length of the tube during the entire cycle of oscillation. This is because the maximum curvature of the tube does not necessarily occur at the root of the tube.

* \( \sigma_x \) is in psi in Fig. 18.
PART III: SIMPLY SUPPORTED TUBE
Section 1

Equations of Motion

The partial differential equation of motion of a simply supported tube is the same as the one for a cantilever tube, i.e., equation 37. The only difference is the set of boundary conditions

\[ \begin{align*}
\text{at } \xi = 0, & \quad \phi = 0 \\
& \quad \frac{\partial^2 \phi}{\partial \xi^2} = 0 \\
\text{at } \xi = 1, & \quad \phi = 0 \\
& \quad \frac{\partial^2 \phi}{\partial \xi^2} = 0
\end{align*} \tag{60} \]

It is again assumed that the solution is of the form

\[ \phi(\xi, \tau) = \sum_{n=1}^{m} \gamma_n(\xi) q_n(\tau) \tag{62} \]

To satisfy the boundary conditions, the \( \gamma_n(\xi)'s \) are taken to be

\[ \gamma_n(\xi) = \sin n\pi \xi \tag{63} \]

\[ n = 1, \ldots, m \]

Since everything else is the same as for the cantilevered tube except for \( \gamma_n(\xi) \), equation (40) can be reused. The only things that have to be rederived are the mode shape integrals. Mode shape integrals for a tube simply-supported at both ends are evaluated in Appendix V.

As before, define

\[ a_{jk} = \frac{1}{\gamma_j} \gamma_k \tag{64} \]
and the equations of motion become

\[ \sum_{k=1}^{m} \left\{ \frac{1}{2} \delta_{jk} q_k + \left[ 2 \beta \sqrt{\mu} a_{jk} + \frac{1}{2} \nu \delta_{jk} \right] q_k \right\} + \left[ \frac{k^4 \pi^4}{2} \delta_{jk} + \beta^2 b_{jk} - \frac{1}{2} \sum_{h=1}^{m} \sum_{i=1}^{m} q_h q_i c_{hi} b_{jk} \right] q_k \right\} = 0 \]  

\( j = 1, 2, \ldots, m \)  

The computer program written for evaluating the time histories of a cantilevered tube may be readily modified to solve equation (67) by the following changes

\[ \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases} \] 

\[ \lambda_k = k \pi \] 

A listing of the modified program in FORTRAN is given in Appendix VI. Other alternations were made to conserve core storage space and computer time.
Section 2

Predicted Characteristics

For $0 < \beta < \pi$, the tube undergoes damped stable vibrations. Examples of these damped vibrations are shown in Figures 19 and 20 for the two limiting cases. The magnitude of the true damping ratio, $\zeta_1^T$, increases with flow velocity. This relation is shown in Figure 21a. First mode frequency, $\omega_1$, as a function of $\beta$ is shown in Figure 21b. The frequency of the first mode vibration decreases as $\beta$ is increased $\pi$. In the neighborhood of $\beta = \pi$, the first mode damping ratio becomes very large and the frequency approaches zero. In Figure 22, the product $\zeta_1 \omega_1$ is shown. Note that it varies smoothly with $\beta$ as $\beta$ passes through $\pi$.

Beyond $\beta = \pi$, the tube was found to undergo divergence by previous investigators, Ref. 5. This is only true for $\beta$ less than $2\pi$. For $\pi < \beta < 2\pi$, the magnitude of $q_1(\tau)$ grows exponentially with time initially. $q_1(\tau)$ then oscillates near some constant value, say $q^*$. These transient oscillations decay with time and eventually die out, and $q_1(\tau)$ converges to the steady state amplitude, i.e., $q^*$. Higher mode oscillations are not very significant and the amplitudes decay to zero. Examples of such behavior for four different values of $\beta$ are given in Figures 23, 24, 25 and 26. To show the relationship between $\beta$ and the amplitude of $q_1(\tau)$ more clearly, the non-oscillatory average of $q_1(\tau)$ is plotted against time for the four $\beta$'s in Figure 27.

Near $\beta = 2\pi$, all four modes become dynamically unstable. However, the first and second modes are dominant. Four of the relevant time histories are illustrated in Figures 28 to 31.
It was speculated that if at $\beta = \pi$, the first mode becomes statically unstable, and at $\beta = 2\pi$, the second mode becomes important and changes the static instability to a dynamic instability, then at $\beta = 3\pi$, the third mode may become very significant and drastically alter the time history of motion again. However, this is not true. The third and fourth modes are dynamically unstable, but they do not dominate the total motion. For $\beta > 3\pi$, no drastic change in time history is found. It appears to have the same "pulse" or "beat" characteristics as for $\beta < 3\pi$. These are shown in Figures 32 and 33.

A stress analysis was done for $\beta = 5.92$. As was expected, the maximum stress is in the middle between the two ends. Figure 34 shows the stress distribution in the tube.

Although there is no drastic change in the time history for $\beta > 3\pi$, the third mode does appear in the deformation shape during the cycle. At $\beta = 10.0$, the deformation shape of the tube changes from a two mode sine wave to a three mode sine wave and back to a two mode sine wave over a complete cycle of motion. Deformation shapes at 20 equal time increments over the period of the modulated sine wave are shown in Figures 35a-35e. Graphs showing stress distribution in the tube for the 20 corresponding deformation shapes are given in Figures 36a-36e.

The effects of fluid velocity $\beta$ on the first and second mode peak limit cycle amplitudes are shown in Figure 37a and 37b respectively. These further illustrate the dramatic changes in modal content of the motion which occur near $\beta = 2\pi$.

The first correct linear stability analysis of a simply supported-simply supported pipe was given by Housner (9). The present discussion extends his results to the nonlinear regime.
PART IV: CONCLUDING REMARKS
This study has developed a mathematical model capable of describing the large amplitude periodic behavior of a cantilever or simply-supported propellant line (tube) conveying high velocity liquid. The study has also illustrated the importance of the nonlinear tension effect induced by bending of the tube. Thresholds of instability predicted by the theory are in good agreement with those previously obtained by other investigators. Plans have been made for setting up experiments to confirm the predicted magnitudes of the limit cycles.

The theoretical model might be improved by considering other nonlinearities, e.g., nonlinear modulus of elasticity or nonlinear curvature. Instead of viscous damping, one might consider viscoelastic damping, or better still, a combination of viscous and viscous-elastic damping. The uniform beam assumption could be replaced by the thin cylindrical shell approximation. For cases in which the hanging tubes are very flexible, the gravity term should not be neglected.
References


### Table 1

**MODE SHAPE PARAMETERS**

(Cantilever Beam)

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\sigma_k$</th>
<th>$\lambda_k$</th>
<th>$\lambda_k^4$</th>
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<tr>
<td>1</td>
<td>0.73410</td>
<td>1.8751</td>
<td>12.362</td>
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<td>2</td>
<td>1.01847</td>
<td>4.6941</td>
<td>485.52</td>
</tr>
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<td>3</td>
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<td>3806.5</td>
</tr>
<tr>
<td>4</td>
<td>1.000034</td>
<td>10.9955</td>
<td>14617</td>
</tr>
</tbody>
</table>
Table 2

**MODE SHAPE INTEGRALS**

(Cantilever Beam)

\[
a_{jk} = \int_0^1 \gamma_j(\xi) \gamma_k(\xi) \, d\xi
\]

<table>
<thead>
<tr>
<th>j/k</th>
<th>1</th>
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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>3.7840</td>
<td>-4.1200</td>
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<td>2.0000</td>
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<td>2.0000</td>
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<tr>
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</table>

\[
b_{jk} = \int_0^1 \gamma_j(\xi) \gamma_k''(\xi) \, d\xi
\]

<table>
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</tr>
<tr>
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</tr>
<tr>
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<td>1.0870</td>
<td>5.5410</td>
<td>4.2540</td>
<td>-98.9200</td>
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</tbody>
</table>

\[
c_{hi} = \int_0^1 \gamma_i(\xi) \gamma_i(\xi) \, d\xi
\]

<table>
<thead>
<tr>
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<td>-35.6482</td>
<td>142.4605</td>
</tr>
</tbody>
</table>
Table 3
INITIAL CONDITIONS AND DATA USED

\[ \dot{q} = 0.1 \]
\[ \ddot{q} = 0 \]
\[ \ell = 9.5 \text{ ins.} \]
\[ m = 1.21 \times 10^{-6} \text{ lb-sec}^2/\text{in}^2 \]
\[ I = 46.6 \times 10^{-6} \text{ in}^4 \]
\[ \bar{\omega}_1 = 25.1 \text{ rad/sec} \]
\[ E = 1.07 \times 10^4 \text{ lbs/in}^2 \]
\[ \xi_1 = 0.080 \]
\[ \nu = 0.400 \]
\[ \mu = 0.1, 0.5 \]
Table 4

**MODE SHAPE INTEGRALS**

(Simply Supported Beam)

\[ a_{jk} = \int_0^1 \gamma_j(\xi) \gamma_k'(\xi) \, d\xi \]

<table>
<thead>
<tr>
<th>j/k</th>
<th>1</th>
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</tr>
</thead>
<tbody>
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<td>0.0000</td>
<td>-0.5333</td>
</tr>
<tr>
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<td>1.3333</td>
<td>0.0000</td>
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<td>0.0000</td>
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<td>0.0000</td>
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Figure 1. Observed Flutter of Short Tube  
(From Reference 3)

Figure 2. Flexural Oscillatory Instabilities  
(From Reference 1)
Absolute velocity of fluid,

\[ V_x \approx V \]
\[ V_y \approx \frac{\partial y}{\partial t} + V \frac{\partial y}{\partial x} \]

Figure 3. Tube with Flowing Fluid
(From Reference 1)
Figure 4. Deflected Beam
Figure 5. The Effects of Velocity on Damping ($\mu = 0.5$)
(Two Modes)
Figure 6a. The Effects of Velocity on Frequency ($\mu = 0.1$) (Two Modes)
Figure 6b. The Effects of Velocity on Damping
($\mu = 0.1$, Four Modes)
Previous Solution
(From Ref.1)

Four Mode Analysis

Figure 7a. Comparison of Previous and Numerical Solutions
(Frequency vs. Mass Ratio)
Figure 7b. Comparison of Previous Solution and Numerical Solutions (Velocity vs. Mass Ratio)
Figure 8. Effects of Velocity on Amplitude of Limit Cycle
($\mu = 0.10$, Four Modes)
Figure 9. The Effects of Velocity on the Amplitude of Limit Cycle \( (\mu = 0.50, \text{ Four Modes}) \)
Figure 10. Variation of Frequency Due to Velocity

$\mu = 0.1$, Four Modes
Figure 11. Variation of Frequency Due to Velocity
($\mu = 0.5$, Four Modes)
Figure 12. Tube Deformation Shapes at Various Times During Limit Cycle
Figure 13. Tube Deformation Shapes at Various Times During Limit Cycle
Figure 14. Tube Deformation Shapes at Various Times During Limit Cycle
Figure 15. Tube Deformation Shapes at Various Times During Limit Cycle
Figure 16. Tube Deformation Shapes at Various Times During Limit Cycle
Figure 17. Tube Deformation Shapes at Various Times During Limit Cycle
Figure 18. Tube Stress Distribution at Various Times During Limit Cycle.
Figure 19a. First Mode Time History of Tube Motion (Damped Vibration)
Figure 19b. Second Mode Time History of Tube Motion (Damped Vibrations)
Figure 20a. First Mode Time History of Tube Motion (Damped Vibrations)
Figure 20b. Second Mode Time History of Tube Motion (Damped Vibrations)

\[ \beta = 2.14 \]
\[ \mu = 0.5 \]
Figure 21a. Effects of Beta on First Mode Damping Ratios ($\mu = 0.1$)
Figure 21b. Effects of Beta on First Mode Frequencies ($\mu = 0.1$)
Figure 22. Effects of $\beta$ on $\zeta_1^T \omega_1$
Note different scales for positive and negative $\zeta_1^T \omega_1$
Figure 23a. First Mode Time History of Tube Motion (Divergence)
Figure 24a. First Mode Time History of Tube Motion (Divergence)

$p = 4.14$

$\mu = 0.5$
Figure 25b. Second Mode Time History of Tube Motion (Divergence)
Figure 26a. First Mode Time History of Tube Motion (Divergence)
Figure 26b. Second Mode Time History of Tube Motion (Divergence)
Figure 27. Influence of Velocity, $\beta$, on Average $q_1(\tau)$
Figure 28b. Dynamic Instability of a Simply Supported Tube (Second Mode)
Figure 29a. First Mode Instability
\( (\mu = 0.5, \quad \beta = 7.4) \)
Figure 29b. Second Mode Instability
($\mu = 0.5$, $\beta = 7.4$)
Figure 31c. Third Mode Instability
\( \mu = 0.5, \beta = 8.88 \)
Figure 32a. First Mode Instability

$\left( \mu = 0.5, \beta = 9.42 \right)$
Figure 32b. Second Mode Instability
(μ = 0.5, β = 9.42)
Figure 32c. Third Mode Instability
($\mu = 0.5, \beta = 9.42$)
Figure 33a. First Mode Instability
\( (\mu = 0.5, \ \beta = 10.0) \)
Figure 33b. Second Mode Instability
(μ = 0.5, β = 10.0)
Figure 34. Tube Stress Distribution at Divergence ($\beta = 5.92$)
Figure 35a. Tube Deformation Shapes at Various Time Intervals ($\beta = 10.0$)
Figure 35b. Tube Deformation Shapes at Various Time Intervals ($\beta = 10.0$)
Figure 35c. Tube Deformation Shapes at Various Time Intervals ($\beta = 10.0$)
Figure 35d. Tube Deformation Shapes at Various Time Intervals ($\beta = 10.0$)
Figure 35e. Tube Deformation Shapes at Various Time Intervals (β = 10.0)
A THEORETICAL ANALYSIS OF NONLINEAR EFFECTS ON THE Flutter AND DIVERGENCE. (U) PRINCETON UNIV N J DEPT OF AEROSPACE AND MECHANICAL SCIENCES. E CH'NG 02 AUG 77
UNCLASSIFIED PUAMS-1343
Figure 36a. Tube Stress Distribution at Various Times ($\beta = 10.0$)
Figure 36b. Tube Stress Distribution at Various Times ($\beta = 10.0$)
Figure 36c. Tube Stress Distribution at Various Times \( (\beta = 10.0) \)
Figure 36d. Tube Stress Distribution at Various Times ($\beta = 10.0$)
Figure 36e. Tube Stress Distribution at Various Times ($\beta = 10.0$)
Figure 37a. Effects of $\beta$ on Peak Amplitude ($\mu = 0.5$)
Figure 37b. Effects of $\beta$ on Peak Amplitude
($\mu = 0.5$)
APPENDIX I

Solving the Energy Equation

Substituting equations (7), (8), and (11) into equation (1) gives

\[
\frac{1}{2} \oint_{t_1}^{t_2} \left( m(\frac{\partial y}{\partial t})^2 + m_f [(\frac{\partial y}{\partial t})^2 + 2V \frac{\partial y}{\partial x} \frac{\partial y}{\partial x} + y^2 (\frac{\partial y}{\partial x})^2 + v^2] \right) dt - EI \left( \frac{\partial^2 y}{\partial x^2} \right)^2 + mg(l-x) (\frac{\partial y}{\partial x})^2 \right) \right) dx \right] dt
- \oint_{t_1}^{t_2} 2 \int_{t_1}^{t_2} m \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \delta y \right) dx \right] dt

\[= \oint_{t_1}^{t_2} \frac{\partial}{\partial t} \left( m_f \left[ \frac{\partial y}{\partial t} (x) + V \frac{\partial y}{\partial x} (x) \right] \right) dx \right) dt \delta y (x) = 0 \quad (A-1)\]

In the calculus of variations, \( \delta \) may be considered as a linear differential operator. Examples:

\[
\oint_{t_1}^{t_2} \frac{\partial}{\partial x} m(\frac{\partial y}{\partial t})^2 \right) dx \right] dt = \oint_{t_1}^{t_2} \frac{\partial}{\partial t} \left( 2m \frac{\partial y}{\partial t} \delta \frac{\partial y}{\partial t} \right) dx \right] dt \quad (A-2)
\]

\[
\oint_{t_1}^{t_2} \frac{\partial}{\partial t} m_f V^2 \right) dx \right] dt = \oint_{t_1}^{t_2} \delta (m_f V^2) \right) dx \right] dt = 0 \quad (A-3)
\]

\[
\oint_{t_1}^{t_2} \frac{\partial}{\partial x} \left( 2m_f V \frac{\partial y}{\partial x} \frac{\partial y}{\partial x} \right) dx \right] dt = \oint_{t_1}^{t_2} \frac{\partial}{\partial t} \left( 2m_f V \left( \frac{\partial y}{\partial t} \delta \frac{\partial y}{\partial x} + \frac{\partial y}{\partial y} \delta \frac{\partial y}{\partial t} \right) \right) dx \right] dt \quad (A-4)
\]

Similarly, performing variation on equation (A-1) and then rearranging the terms yields

95
\[ t_2 \int \frac{1}{t_1} \left[ (m+m_f) \frac{\partial y}{\partial t} \delta \frac{\partial y}{\partial x} \right] dx dt + m_f V \frac{\partial y}{\partial t} \delta \frac{\partial y}{\partial x} + m_f V \frac{\partial y}{\partial x} \delta \frac{\partial y}{\partial t} \]

\[ + m_f V^2 \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial^2 y}{\partial x^2} - EI \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial^2 y}{\partial x^2} - mg (l-x) \frac{\partial y}{\partial x} \delta \frac{\partial y}{\partial x} \]

\[ - 2t_1 \bar{\omega}_m \frac{\partial y}{\partial t} \delta y \right) dx dt - \int_{t_1}^{t_2} m_f V \left[ \frac{\partial y}{\partial t} (l) + V \frac{\partial y}{\partial x} (l) \right] \delta y(l) \] \[ \int_{t_1}^{t_2} dt = 0 \] (A-5)

Due to the tedious mathematics involved, only representative terms of equation (A-5) will be worked out completely and shown below.

Integration by parts:

\[ \int_{t_1}^{t_2} \frac{\partial y}{\partial t} \delta y \right) dx dt = \int_{t_1}^{t_2} \left( \frac{\partial y}{\partial t} \right) \delta y \right) dx dt \]

\[ \left[ \frac{\partial y}{\partial t} \right]_{t_1}^{t_2} \int_{t_1}^{t_2} \right] \delta y \right) dt \]

\[ \int_{t_1}^{t_2} \frac{\partial^2 y}{\partial x^2} \delta y \right) dx dt \]

\[ \int_{t_1}^{t_2} \frac{\partial^2 y}{\partial x^2} \delta y \right) dx dt \]

\[ \int_{t_1}^{t_2} \frac{1}{t_1} \left[ (m+m_f) \frac{\partial y}{\partial t} \delta \frac{\partial y}{\partial x} \right] dx dt + m_f V \frac{\partial y}{\partial t} \delta \frac{\partial y}{\partial x} + m_f V \frac{\partial y}{\partial x} \delta \frac{\partial y}{\partial t} \]

\[ + m_f V^2 \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial^2 y}{\partial x^2} - EI \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial^2 y}{\partial x^2} - mg (l-x) \frac{\partial y}{\partial x} \delta \frac{\partial y}{\partial x} \]

\[ - 2t_1 \bar{\omega}_m \frac{\partial y}{\partial t} \delta y \right) dx dt - \int_{t_1}^{t_2} m_f V \left[ \frac{\partial y}{\partial t} (l) + V \frac{\partial y}{\partial x} (l) \right] \delta y(l) \] \[ \int_{t_1}^{t_2} dt = 0 \] (A-5)

Due to the tedious mathematics involved, only representative terms of equation (A-5) will be worked out completely and shown below.

Integration by parts:
\[ EI \left\{ \int_{t_1}^{t_2} \frac{\partial^2 y}{\partial x^2} \delta \left( \frac{\partial y}{\partial x} \right)^2 \, dt - \int_{t_1}^{t_2} \frac{\partial^3 y}{\partial x^3} \delta y \, dt + \int_{t_1}^{t_2} \frac{\partial^4 y}{\partial x^4} \delta y \, dx \, dt \right\} \quad (A-8) \]

\[ \int_{t_1}^{t_2} \int_{x_1}^{x_2} \delta \frac{\partial v}{\partial x} \delta \frac{\partial y}{\partial x} \, dx \, dt = \int_{t_1}^{t_2} \left\{-m_fV \frac{\partial y}{\partial x} \delta y \right\} - \int_{t_1}^{t_2} m_fV \frac{\partial^2 y}{\partial x^2} \delta y \, dt \quad dx \quad (A-9) \]

\[ \int_{0}^{L} \left. m_fV^2 \frac{\partial y}{\partial x} \delta \frac{\partial y}{\partial x} \right|_{x=0}^{L} + \int_{0}^{L} m_fV^2 \frac{\partial^2 y}{\partial x^2} \delta y \, dx \quad (A-10) \]

\[ \int_{0}^{L} \left. \left( -m_g(l-x) \frac{\partial y}{\partial x} \delta \frac{\partial y}{\partial x} \right) \right|_{x=0}^{L} + \int_{0}^{L} m_g \frac{\partial}{\partial x} [(l-x) \frac{\partial y}{\partial x}] \delta y \, dx \quad (A-11) \]

and

\[ [\delta y] = 0 \quad (\text{See Ref. 6}) \]

To satisfy Eq. (A-5), the integrands have to be zero. The sum of all the integrands of the double integrals give the partial differential equation for the problem,

\[ EI \frac{\partial^4 y}{\partial x^4} + (m+m_f) \frac{\partial^2 y}{\partial t^2} + 2m_fV \frac{\partial^2 y}{\partial t \partial x} + m_fV^2 \frac{\partial^2 y}{\partial x^2} + 2m \frac{1}{\omega_1} \frac{\partial y}{\partial t} - mg \frac{\partial}{\partial x} [(l-x) \frac{\partial y}{\partial x}] = 0 \quad (A-12) \]

Also the several end point terms must be zero. Thus

\[ EI \left. \frac{\partial^2 y}{\partial x^2} \delta \left( \frac{\partial y}{\partial x} \right) \right|_{x=0, L} = 0 \quad (A-13) \]

For a cantilevered tube (A-13) is satisfied by
\[ \frac{\partial y}{\partial x} \bigg|_{x=0} = 0 \]  
\[ (A-13c) \]

and

\[ EI \frac{\partial^2 y}{\partial x^2} \bigg|_{x=L} = 0 \]

For a simply-supported tube,

\[ EI \frac{\partial^2 y}{\partial x^2} \bigg|_{x=0, L} = 0 \]  
\[ (A-13s) \]

The other end point conditions are

\[ [EI \frac{\partial^3 y}{\partial x^3} - mg(x-x) \frac{\partial^2 y}{\partial x^2} ] \bigg|_{x=0, L} = 0 \]  
\[ (A-14) \]

For a cantilevered tube, \((A-14)\) is satisfied by

\[ y\big|_{x=0} = 0 \]  
\[ (A-14c) \]

and

\[ [EI \frac{\partial^3 y}{\partial x^3} - mg(x-x) \frac{\partial^2 y}{\partial x^2} ]_{x=0} = 0 \]

For a simply-supported tube,

\[ y\big|_{x=0, L} = 0 \]  
\[ (A-14ss) \]

It should be noted that in obtaining \((A-14)\) from \((A-5)\) to \((A-11)\), there has been a cancellation of terms involving the last term in \((A-5)\) and the first terms on the right hand side of \((A-7)\) and \((A-10)\) for \(x=L\) to give the same form in \((A-14)\) for the boundary conditions at both \(x=0\) and \(x=L\).
APPENDIX II

Mode Shape Integrals of a Cantilevered Beam

Integrals containing the characteristic functions and their derivatives are referred to as mode shape integrals in this paper. They can be evaluated by the method of partial integration along with their orthogonality properties. Special formulas for those mode shape integrals which occur frequently in engineering application may be found in references 10 and 11. They are given in terms of \( \alpha_n \) and \( \beta_n \) for five common boundary conditions: clamped-clamped, clamped-free, clamped-supported, free-free, and free-supported.

As an illustration, consider

\[
\chi_{hi} = \int_0^1 \gamma_h(\xi) \gamma_i(\xi) \, d\xi
\]

(A-15)

for a cantilevered uniform beam. Combining formulas (6) and (16) of reference (7) yields.

\[
\chi_{hi} = \begin{cases} 
\alpha_h \beta_h (2 + \alpha_h \beta_h), & h = i \\
\frac{4 \beta_h \beta_i}{\beta_h^4 - \beta_i^4} \left[ (-1)^{h+i} (\alpha_i \beta_h^3 - \alpha_h \beta_i^3) - \beta_h \beta_i (\alpha_i \beta_h - \alpha_h \beta_i) \right], & h \neq i
\end{cases}
\]

The corresponding \( \alpha 's \) and \( \beta 's \) can be found in reference 11. They are

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha_n )</th>
<th>( \beta_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7341</td>
<td>1.8751</td>
</tr>
<tr>
<td>2</td>
<td>1.0185</td>
<td>4.6940</td>
</tr>
<tr>
<td>3</td>
<td>0.9992</td>
<td>7.8546</td>
</tr>
<tr>
<td>4</td>
<td>1.0000</td>
<td>10.9955</td>
</tr>
</tbody>
</table>
Numerical values of $c_{h_1}$ are given in Table 2. The computations were done on the IBM 360/91.
APPENDIX III

Radius of gyration, $r$

(i) For a thin cylindrical tube, the area moment is

$$I = \int z^2 \, da \equiv r^2 a$$

Now

$$a = 2\pi R t$$

and

$$da = R \cdot t \cdot d\theta$$

Also

$$z = R \cos \theta$$

Thus

$$I = \int_{0}^{\pi} \int_{R_1}^{R_2} R^2 \cos^2 \theta \ R \, dR \, d\theta \ t$$

$$= R^3 \ t \int_{0}^{\pi} \cos^2 \theta \ d\theta$$

$$= R^3 \ t \pi$$

Hence

$$r^2 \ a = R^3 \ t \pi$$

Solving,

$$r^2 = \frac{R^3 t \pi}{2\pi R t}$$

or

$$r^2 = \frac{R^2}{2}$$

$$r = \sqrt{\frac{R^2}{2}}$$

(II) For a tube with a thick wall,

$$I = \int_{R_1}^{R_2} \int_{0}^{\pi} \frac{R^2}{R_2} \cos^2 \theta \ R \, dR \, d\theta$$
\[ = \int_0^{2\pi} \left[ (R_2^4 - R_1^4)/4 \right] \cos^2 \theta \, d\theta \]

\[ = \frac{\pi}{4} [R_2^4 - R_1^4] \]

\[ a = \int_0^{2\pi} \int_{R_1}^{R_2} \rho \, d\rho \, d\theta \]

\[ = \int_0^{2\pi} \frac{1}{2} (R_2^2 - R_1^2) \, d\theta \]

\[ = \pi [R_2^2 - R_1^2] \]

Hence

\[ r^2 = \frac{1}{a} \]

\[ = \frac{\pi [R_2^4 - R_1^4]}{4\pi [R_2^2 - R_1^2]} \]

\[ = \frac{1}{4} [R_2^4 + R_1^4] \]
APPENDIX IV

A LISTING OF WATFIV PROGRAMS FOR CANTILEVERED TUBE

(Linear and Nonlinear)
**NUMERICAL SOLUTION FOR STATIC AND DYNAMIC INSTABILITIES OF A PROPELLANT LINE. (FOUR MODES ANALYSIS)**

INTEGER I, J, M1, M2, N1, N2, I1, I2, J1, J2
REAL DC(8, 1506)
REAL DEL, MAT
REAL RMU
REAL ETA, MU, NU, M, MP, E
REAL C1, Q2, Q3, Q4, L1, L2, VF
REAL Q0, Q5, Q6, Q7, Q8
REAL L3, LN
REAL L, Z1, I, TMAX, DELT, A11, A12
REAL A11, A12, B12, B22, B21, E11
REAL A13, A14, A23, A24
REAL A31, A32, A33, A34
REAL D1, D2, D3, A44
REAL E13, E14, E23, E24
REAL B31, B32, B33, B34
REAL E41, E42, B43, B44
REAL K(8, 8), L(3, 1506), AT(1506), V(8)
REAL TO1(500), TO2(500), TO(500)
REAL IC3(500), IC4(500)

C C1, C2, Q3, ... ARE INITIAL CONDITIONS
C DC(n)'s ARE PSEUDO INITIAL CONDITIONS
C I1, L2 ARE YOUR SHAFT PARAMETERS
C L = LENGTH OF PIPE
C Z1 = CRITICAL DAMPING RATIO
C E = MASS OF PIPE
C MF = MASS OF FLUID
C VP = FLUID VELOCITY
C I = INERTIA
C E = MODULUS OF PLASTICITY
C A11, A12, ... ARE MODE SHAPE INTEGRALS

REAL (5, 1006) Q1, Q2, Q3, Q4, Q5, Q6
REAL (5, 1006) Q7, Q8, L1, L2, L3, L4

1000 FORMAT (6F10.4)
1001 FORMAT (MF10.4)
998 FORMAT (5, 1111, 'cI=9(f4.0)' V F, L, T1, T, DELT, TMAX
1111 FORMAT (F12.4, 5F10.4)
1002 FORMAT (41, 10.3)

**DELT = DIMENSIONLESS FLUID VELOCITY**

**MF = MASS RATIO**
C

MU = VISCOUS DAMPING COEFFICIENT

32

BETA = VF*SQRT((MP*L**2)/(I*Z))

33

MU = W/E*(MF*VF)

34

NU = 7.04*Z1*SQRT(1.0-MU)

C

35

WRITE (6,1012)

36

WRITE (6,1063) A11,A12,A13,A14,A21,A22,A23,A24

37

1003 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

38

WRITE (6,1011) A31,A32,A33,A34,A41,A42,A43,A44

39

3011 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

40

WRITE (6,3004) E11,B12,B13,B14,B21,B22,B23,B24

41

3001 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

42

WRITE (6,1011) E31,B32,B33,B34,B41,B42,B43,B44

43

3111 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

44

WRITE (6,1004) C1,Q2,Q3,Q4,Q5,Q6,Q7,Q8

45

1004 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

46

WRITE (6,1005) BETA,MU,NU

47

1005 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

48

WRITE (6,1006) VF,L1,L2,L3,L4,L5

49

1006 FORMAT (' ', ' ', ' ', (1,J)=',4F20.4,/', ' ', 'A (J)=',4F20.4)

C

50

RMU = SQRT(MU)

C

51

+ + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + +

52

 + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + +

53

 + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + + +

C

54

C

55

DC (1,1) = 8.0

56

DC (1,2) = 0.0

57

DC (2,1) = 0.0

58

DC (2,2) = 0.0

59

DC (3,1) = 0.0

60

DC (3,2) = 0.0

61

DC (4,1) = 0.0

62

DC (4,2) = 0.0

63

DC (5,1) = 0.0

64

DC (5,2) = 0.0

65

DC (6,1) = 0.0

66

DC (6,2) = 0.0

67

DC (7,1) = 0.0

68

DC (7,2) = 0.0

69

DC (8,1) = 0.0

70

DC (8,2) = 0.0

71

K (1,1) = 0.0

72

K (1,2) = 0.0

73

K (1,3) = 0.0

74

K (1,4) = 0.0

75

K (1,5) = 0.0

76

K (1,6) = 0.0

77

K (1,7) = 0.0

78

K (1,8) = 0.0

79

K (2,1) = 0.0
K(2,2) = 0.0
K(2,3) = 0.0
K(2,4) = 0.0
K(2,5) = 0.0
K(2,6) = 1.0
K(2,7) = 0.0
K(2,8) = 0.0
K(3,1) = 0.0
K(3,2) = 0.0
K(3,3) = 0.0
K(3,4) = 0.0
K(3,5) = 0.0
K(3,6) = 0.0
K(3,7) = 1.0
K(3,8) = 0.0
K(4,1) = 0.0
K(4,2) = 0.0
K(4,3) = 0.0
K(4,4) = 0.0
K(4,5) = 0.0
K(4,6) = 0.0
K(4,7) = 0.0
K(4,8) = 0.0
K(5,1) = (L1**4 + BETA**2 * B11)
K(5,2) = -BETA**2 * B12
K(5,3) = -BETA**2 * B13
K(5,4) = -BETA**2 * B14
K(5,5) = -(2 * BETA**2 * M11 + NU)
K(5,6) = -2 * BETA**2 * M12
K(5,7) = -2 * BETA**2 * M13
K(5,8) = -2 * BETA**2 * M14
K(6,1) = -BETA**2 * B21
K(6,2) = -(L2**4 + BETA**2 * B22)
K(6,3) = -BETA**2 * B23
K(6,4) = -BETA**2 * B24
K(6,5) = -2 * BETA**2 * M21
K(6,6) = -(2 * BETA**2 * M22 + NU)
K(6,7) = -2 * BETA**2 * M23
K(6,8) = -2 * BETA**2 * M24
K(7,1) = -BETA**2 * A11
K(7,2) = -BETA**2 * A12
K(7,3) = -(L3**4 + BETA**2 * B33)
K(7,4) = -BETA**2 * B34
K(7,5) = -2 * BETA**2 * M31
K(7,6) = -2 * BETA**2 * M32
K(7,7) = -(2 * BETA**2 * M33 + NU)
K(7,8) = -2 * BETA**2 * M34
K(8,1) = -BETA**2 * B41
K(8,2) = -BETA**2 * B42
K(8,3) = -BETA**2 * B43
K(8,4) = -(L4**4 + BETA**2 * B44)
K(8,5) = -2 * BETA**2 * M41
K(8,6) = -2 * BETA**2 * M42
K(8,7) = -2 * BETA**2 * M43
K(8,8) = -(2 * BETA**2 * M44 + NU)

C = WRITE (5,1009) ((K(I,J), JJ=1,8), II=1,8)
C = 1009 FC5M44 ("",&F12.4)
C = 137 K1 = 1
AC(1,N1) = Q1
AC(2,N1) = Q2
AC(3,N1) = Q3
AC(4,N1) = Q4
AC(5,N1) = Q5
AC(6,N1) = Q6
AC(7,N1) = Q7
AC(8,N1) = Q8

CONTINUE

DC 1C1 I1 = 1,8
V(I1) = AC(I1,N1)
CONTINUE

NI(N1) = T
NNN = N1 + 2
N1 = N1 + 1

*****************************************************************************
* MATRIX MULTIPLICATION *
*****************************************************************************

DC 1C2 I3 = 1,8
DUMMY = 0.
DC 1C2 I2 = 1,9
MATM = K(I3,I2)*V(I2)
DUMMY = DUMMY + MATM
CONTINUE

DC(I3,NNN) = DUMMY
AC(I3,N1) = DELT*(1.9167*DQ(I3,NNN) - 1.3333*DQ(I3,NNN-1)
A + 0.4167*DQ(I3,NNN-2) + V(I3)
CONTINUE

I = T + DELT
IF (ITMAX.NE.T) GO TO 100
WRITE (6,1C10)
1010 FCEMAI (/1,5X,T,Q1,Q2,Q3,Q4 ARE AS FOLLOWED:/)
NOTE: ONLY EVERY FIFTH VALUE IS PAINTED
WRITE (5,1C11) (AC(J), (AO(IZ,J), IZ = 1,4), J = 1, N1, 5)
1011 FCEMAI ('1',5EZ15.4)

*****************************************************************************
* FLOATING FUNCTION *
*****************************************************************************

NOTE: ONLY EVERY FIFTH VALUE IS PLOTTED
NZ = 0
DC 1C4 NZ = 1,1,5
N3 = NZ + 1
IT(N3) = NI(N2)
TC1(N3) = AO(1,N2)
TC2(N3) = AO(2,N2)
TC3(N3) = AQ(1,N2)
108

175 IC4(K3) = LQ(4,N2)

176 104 CONTINUE

C

177 1012 FORMAT (i11)

178 CALL WEL07 (TT,TQ1,N3,12,'DISPLACEMENT')

180 CALL WEL07 (TT(1),TMAX,DELT)

181 1013 FORMAT (40X,'DISPLACEMENT (1) AS A FUNCTION OF TAU',//,20X,
11 'T=', '1E7.4,10X,'TMAX=', '1E7.4,10X,'DELT=', '1E7.4)

182 CALL WEL07 (TT(2),TMAX,DELT)

183 CALL WEL07 (TT(3),TMAX,DELT)

184 CALL WEL07 (TT(4),TMAX,DELT)

185 1014 FORMAT (40X,'DISPLACEMENT (2) AS A FUNCTION OF TAU',//,20X,
11 'T=', '1E7.4,10X,'TMAX=', '1E7.4,10X,'DELT=', '1E7.4)

186 CALL WEL07 (TT(5),TMAX,DELT)

187 CALL WEL07 (TT(6),TMAX,DELT)

188 CALL WEL07 (TT(7),TMAX,DELT)

189 1015 FORMAT (40X,'DISPLACEMENT (3) AS A FUNCTION OF TAU',//,20X,
11 'T=', '1E7.4,10X,'TMAX=', '1E7.4,10X,'DELT=', '1E7.4)

190 CALL WEL07 (TT(8),TMAX,DELT)

191 CALL WEL07 (TT(9),TMAX,DELT)

192 CALL WEL07 (TT(10),TMAX,DELT)

193 1016 FORMAT (40X,'DISPLACEMENT (4) AS A FUNCTION OF TAU',//,20X,
11 'T=', '1E7.4,10X,'TMAX=', '1E7.4,10X,'DELT=', '1E7.4)

194 CALL WEL07 (TT(11),TMAX,DELT)

195 GC TC 998

196 999 CONTINUE

197 SICF

198 END

SICETR
**NUMERICAL SOLUTION PCF STATIC AND DYNAMIC INSTABILITIES OF A PROPELLANT LINE. (FOUR MODES ANALYSIS)**

**NB: WITH FORMULA OF OPEN TYPES NONLINEAR AND IMPROVED**

```plaintext
INTEGER II
INTEGER IA, IB, IC, ID1, ID2
INTEGER IT, I1, I2, I3, I4, I5, I6
INTEGER JT
INTEGER J1, J2, J3, J4, J5, J6
REAL DUMMY, MAT
REAL EMU
REAL ETA, MU, NU, M, MF, T, E
REAL C1, Q2, Q3, Q4, Q5, Q6, Q7, Q8
REAL LAM(4), VF
REAL L, Z1, T, TMAX, DELT
REAL DEL(4,4)
REAL CO(8,3200)
REAL K(4,4), B(4,4), C(4,4)
REAL K(8,23), AQ(8,3200), AP(3200), V(28)
REAL TC1(500), TC2(500), T03(500), T04(500)
REAL TT(500)

G1, Q2, Q3, ... ARE INITIAL CONDITIONS
L0(N) S ARE PSEUDO INITIAL CONDITIONS
LAM(N) ARE MODE SHAPE PARAMETERS
VF = FLUID VELOCITY
L = LENGTH OF PIPE
Z1 = CRITICAL DAMPING RATIO
K = MASS OF PIPE
MF = MASS OF FLUID
I = INERTIA
E = MODULUS OF ELASTICITY
A(K), B(N), C(Y) ARE MODE SHAPE INTEGRALS

REAL (5,1000) G1, Q2, Q3, Q4, Q5, Q6, Q7, Q8
1000 FORMAT (6F10.4)
005 REAL (5,1005) (LAM(I1), I1=1,4)
1005 FORMAT (4F10.4)
006 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
007 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
008 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
009 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
000 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
001 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
002 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
003 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
004 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
005 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
006 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
007 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
008 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
009 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
010 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
011 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
012 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
013 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
014 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
015 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
016 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
017 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
018 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
019 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
020 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
021 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
022 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
023 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
024 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
025 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
026 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
027 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)
028 REAL (5,1005) (C(I1,J1), J1=1,4, I1=1,4)

DATA = DIMENSIONLESS FLUID VELOCITY
MU = MASS RATIO
NU = VISCOUS DAMPING COEFFICIENT
```
BEIA = VP*SQRT(MP*L*2./((I*I))
MU = MF/(MF*+)
MU = 7.0*Z1*SQRT(1.-MU)

WRITE (6,1015)
1015 FORMAT ('**')
WRITE (6,1020) ((L(I2,J2), J2=1,4), I2=1,4)
102C FORMAT (',',A(I,J)=',4F20.4,(/6X,4F20.4,/')
WRITE (6,1025) ((L(I2,J2), J2=1,4), I2=1,4)
1025 FORMAT ('**',B(I,J)=',4F20.4,(/6X,4F20.4,/')
WRITE (6,1030) ((C(I2,J2), J2=1,4), I2=1,4)
1030 FORMAT ('**',C(I,J)=',4F20.4,(/6X,4F20.4,/')
WRITE (6,1035) Q1,02,03,C4,05,06,C7,09
1035 FORMAT ('**',Q(I,J)=',4F20.4,(/6X,4F20.4,/')
WRITE (6,1040) BETA,MU,MU
1040 FORMAT ('**')
WRITE (6,1045) VP,L,Z1,((I=1,J), I=1,4)
1045 FORMAT ('**',VP=','1F15.4,5X,'**',I=1,J=1,4)
WRITE (6,1050) M,MP,IE
1050 FORMAT ('**')
WRITE (6,1055) VP,L,Z1,((I=1,J), I=1,4)
1055 FORMAT ('**',VP=','1F15.4,5X,'**',I=1,J=1,4)
WRITE (6,1060) M,MP,IE
1060 FORMAT ('**')
WRITE (6,1065) MP,MP,IE
1065 FORMAT ('**')

C

AKU = SCI(MU)

***************+*******************************
+ ENTRIES OF MATRIX +
+ ***************+*******************************

AKUL: ISLTA FUNCTION

EC 110 ID1=1,4
DC 110 ID2=1,4
DEI (ID1,ID2) = 0.0
100 CONTINUE
DEI (ID1,ID1) = 1.0
110 CONTINUE

C

EC 120 I4=1,8
DC (14,1) = 0.0
61 DC (14,2) = 0.0
62 D2O CONTINUE
C

NOTE: DO-LOCUS TO FILL IN 0'S & 1'S

IC = 4
64 DC 140 IA=1,4
65 DC 130 IB=1,23
66 K (IA,IB) = 0.0
67 13C CONTINUE
68 IC = IC + 1
69 K (IA,IC) = 1.0
70 14C CONTINUE
C
DC 160  IS=5,6
I= J5-4
DC 150 J5=1,4
K(I5,J5) = -(DL(IT,J5)*LAM(IT)**4.0*BETA**2.0*B(IT,J5))
CONTINUE
DC 155 J3=5,8
JT= J3-4
K(I5,J3) = -(2.0*BETA*RMU*K(IT,JT)*DL(IT,JT)*NU)
CONTINUE
K(I5,9) = B(IT,1)*C(1,1)
K(I5,10) = B(IT,1)*C(2,2)+B(IT,2)*(C(1,2)+C(2,2))
K(I5,11) = B(IT,1)*C(3,3)+B(IT,2)*(C(1,3)+C(3,1))
K(I5,12) = B(IT,1)*C(4,4)+B(IT,2)*(C(4,1)+C(1,4))
K(I5,13) = B(IT,1)*(C(1,2)+C(2,1))+B(IT,2)*C(1,1)
K(I5,14) = B(IT,1)*(C(1,3)+C(3,1))+B(IT,2)*C(1,1)
K(I5,15) = B(IT,1)*(C(2,3)+C(3,2))+B(IT,2)*C(1,1)+C(3,1)
K(T5,3) = C(1,2)+C(2,1)
K(5,16) = B(IT,1)*(C(1,4)+C(4,1))+B(IT,2)*C(1,1)
K(I5,17) = B(IT,1)*(C(2,4)+C(4,2))+B(IT,2)*C(1,1)+C(4,1)
K(I5,18) = B(IT,1)*(C(3,4)+C(4,3))+B(IT,2)*C(1,1)+C(4,1)
K(I5,19) = B(IT,1)*(C(2,2)+C(2,2))
K(I5,20) = B(IT,2)*(C(3,3)+B(IT,3)*(C(3,2)+C(2,3))
K(I5,21) = B(IT,2)*(C(2,3)+C(3,2))+B(IT,3)*C(2,2)
K(I5,22) = B(IT,2)*(C(4,4)+B(IT,4)*(C(4,2)+C(2,4))
K(I5,23) = B(IT,4)*C(2,2)+B(IT,2)*(C(2,4)+C(4,2))
K(I5,24) = B(IT,2)*(C(3,4)+C(4,3))+B(IT,3)*(C(2,4)+C(4,2))
K(I5,25) = B(IT,3)*(C(3,3))
K(I5,26) = B(IT,4)*(C(4,2)+B(IT,4)*(C(4,3))
K(I5,27) = B(IT,4)*(C(3,3)+B(IT,3)*(C(3,4)+C(4,3))
K(I5,28) = B(IT,4)*(C(4,4)
CONTINUE
WRITE (6,1050) ((K(IA,JK), IA=1,8), JK=1,28)
EC3MAT (!,8ELL4.4)
CONTINUE
103 N1 = 1
AC(1,N1) = C1
AC(2,N1) = C2
AC(3,N1) = C3
AC(4,N1) = C4
AC(5,N1) = C5
AC(6,N1) = C6
AC(7,N1) = C7
AC(8,N1) = C8
CONTINUE
DC 160 121=1,9
V(I21) = AC(I21,1)
CONTINUE
NOTE: HAVE TO ADD NON LINEAR EQUATIONS
\begin{verbatim}
C
V(9) = AC(1,N1)**3
117 V(10) = AC(1,N1)**2*Q(2,N1)**2
118 V(11) = AC(1,N1)**2*Q(3,N1)**2
119 V(12) = AC(1,N1)**2*Q(4,N1)**2
120 V(13) = AC(1,N1)**2*Q(5,N1)**2
121 V(14) = AC(1,N1)**2*Q(6,N1)**2
122 V(15) = AC(1,N1)**2*Q(7,N1)**2
123 V(16) = AC(1,N1)**2*Q(8,N1)**2
124 V(17) = AC(1,N1)**2*Q(9,N1)**2
125 V(18) = AC(1,N1)**2*Q(10,N1)**2
126 V(19) = AC(1,N1)**2*Q(11,N1)**2
127 V(20) = AC(1,N1)**2*Q(12,N1)**2
128 V(21) = AC(1,N1)**2*Q(13,N1)**2
129 V(22) = AC(1,N1)**2*Q(14,N1)**2
130 V(23) = AC(1,N1)**2*Q(15,N1)**2
131 V(24) = AC(1,N1)**2*Q(16,N1)**2
132 V(25) = AC(1,N1)**2*Q(17,N1)**2
133 V(26) = AC(1,N1)**2*Q(18,N1)**2
134 V(27) = AC(1,N1)**2*Q(19,N1)**2
135 V(28) = AC(1,N1)**2*Q(20,N1)**2

C
AI(K1) = T
137 WNN = "1 + 2
138 N1 = N1+1

+++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ M A T R I X   M U L T I P L I C A T I O N +
+++++++++++++++++++++++++++++++++++++++++++++++++++++++

DC 2CG  IT2=1,8
140 DUMMY = 0
141 DC 1CG  IT3=1,28
142 MATK = K(IT2,IT3)*V(IT3)
143 DUMMY = DUMMY + MATK
144 190 CONTINUE
145 DC (IT2,WNN) = DUMMY
146 AC(IT2,N1) = DELT*(1.0167*DQ(IT2,1NN)-1.3333*DQ(IT2,WNN-1)+1.4167*DQ(IT2,WNN-2)+AQ(IT2,N1-1)
147 200 CONTINUE

C
148 I = I+1
149 IF (I MAX, G1, T) GO TO 170

C
WRITE (6,1065)
151 1065 FORMAT (/5X,'T,Q1,Q2,Q3,Q4 ARE AS FOLLOWED:',/)

C
NOTE: ONLY THE FIRST 2CG VALUES AFTER 1.3 SEC ARE PRINTED

C
WRITE (6,1070) (AT(JP), (AQ(IP,JP), IP=1,4), JP=1000,1200)
153 1070 FORMAT ("1,5E15.4")

+++++++++++++++++++++++++++++++++++++++++++++++++++++++
+ F L O A T I N G   F U N C T I O N +
+++++++++++++++++++++++++++++++++++++++++++++++++++++++}
\end{verbatim}
NOTE; ONLY EVERY TENTH VALUE IS PLOTTED

154 N3 = 0
155 DC 210 N2 = 1, N1, 10
156 N3 = N3 + 1
157 TI(N3) = AT(N2)
158 TC1(N3) = AC(1, N2)
159 TC2(N3) = AC(2, N2)
160 TC3(N3) = AO(3, N2)
161 TC4(N3) = AO(4, N2)
162 210 CONTINUE
163 WRITE (6, 1015)
164 CALL WPLCT1 (T1, TQ1, N3, 12, 'DISPLACEMENT')
165 WRITE (6, 1071) T1(1), TMAX, DELT
166 1071 FORMAT (4X, 'DISPLACEMENT (1) AS A FUNCTION OF TAU', //, 20X,
171 T1 = ', 1F7.4, 10X, 'TMAX = ', 1F7.4, 10X, 'DELT = ', 1F7.4)
167 WRITE (5, 1015)
168 CALL WPLCT1 (T2, TQ2, N3, 12, 'DISPLACEMENT')
169 WRITE (6, 1075) T2(1), TMAX, DELT
170 1075 FORMAT (4X, 'DISPLACEMENT (2) AS A FUNCTION OF TAU', //, 20X,
175 T2 = ', 1F7.4, 10X, 'TMAX = ', 1F7.4, 10X, 'DELT = ', 1F7.4)
171 WRITE (5, 1015)
172 CALL WPLCT1 (T3, TQ3, N3, 12, 'DISPLACEMENT')
173 WRITE (6, 1060) T3(1), TMAX, DELT
174 1060 FORMAT (4X, 'DISPLACEMENT (3) AS A FUNCTION OF TAU', //, 20X,
179 T3 = ', 1F7.4, 10X, 'TMAX = ', 1F7.4, 10X, 'DELT = ', 1F7.4)
175 WRITE (5, 1015)
176 CALL WPLCT1 (T4, TQ4, N3, 12, 'DISPLACEMENT')
177 WRITE (6, 1085) T4(1), TMAX, DELT
178 1085 FORMAT (4X, 'DISPLACEMENT (4) AS A FUNCTION OF TAU', //, 20X,
183 T4 = ', 1F7.4, 10X, 'TMAX = ', 1F7.4, 10X, 'DELT = ', 1F7.4)
179 WRITE (5, 1015)
180 GC TC 598
181 999 CONTINUE
182 STCF
183 END

SENTRY

COPY AVAILABLE TO DTIC DOES NOT PERMIT FULLY LEGIBLE REPRODUCTION
APPENDIX V

Mode Shape Integrals of a Simply Supported Beam

\[ \int_0^1 \gamma_j \gamma_k \, d\xi = k^4 \pi^4 \int_0^1 \sin j\pi \xi \sin k\pi \xi \, d\xi \]  
\[ = \begin{cases} \frac{k^4 \pi^4}{2} & j=k \\ 0 & j \neq k \end{cases} \]

\[ = \frac{k^4 \pi^4}{2} \delta_{jk} \]

\[ \int_0^1 \gamma_j \gamma_k \, d\xi = \int_0^1 \sin j\pi \xi \sin k\pi \xi \, d\xi \]  
\[ = \begin{cases} \frac{1}{2} & j=k \\ 0 & j \neq k \end{cases} \]

\[ = \frac{1}{2} \delta_{jk} \]

\[ \int_0^1 \gamma_j' \gamma_k \, d\xi = k\pi \int_0^1 \sin j\pi \xi \cos k\pi \xi \, d\xi \]  
\[ = a_{jk} \quad \text{(values of } a_{jk} \text{ are tabulated in Table 4)} \]
\[ \int_0^1 \gamma_j \gamma_k \, d\xi = -k^2 \pi^2 \int_0^1 \sin j\pi \xi \sin k\pi \xi \, d\xi \quad (A-24) \]

\[
\begin{cases}
-k^2 \pi^2 \\
0
\end{cases}
\]

\[
\begin{cases}
\frac{1}{2} \quad j=k \\
0 \quad j \neq k
\end{cases}
\]

\[= \frac{-k^2 \pi^2}{2} \delta_{jk} \]

\[= b_{jk} \]

\[\int_0^1 \gamma_h \gamma_i \, d\xi = h^2 \pi^2 \int_0^1 \cos h\pi \xi \cos i\pi \xi \, d\xi \quad (A-25) \]

\[
\begin{cases}
h^2 \pi^2 \\
0
\end{cases}
\]

\[
\begin{cases}
\frac{1}{2} \quad h=i \\
0 \quad h \neq i
\end{cases}
\]

\[= \frac{-k^2 \pi^2}{2} \delta_{hi} \]

\[= c_{hi} \]
APPENDIX VI

A LISTING OF FORTRAN IV PROGRAM FOR SIMPLY SUPPORTED TUBE
REAL A(4,4), D(4,4), C(4,4)

**SIMPLY-SUPPORTED**

**SPECIAL EDITION**

0001 REAL A(4,4), D(4,4), C(4,4)
0002 REAL DEL(4,4)
0003 REAL MATM, NU, NU, MU, NF, T
0004 REAL LAM(4), L
0005 REAL DO(8,256)
0006 REAL K(8,28), AO(8, 2020), AT(2020), V(28)

0008 READ (5,1000) (AQ(IU,1), IU=1,8)
0009 1000 FORMAT (9F10.4)
0010 READ (5,1005) ((A(I1,I1), J1=1,4), I1=1,4)
0011 1005 FORMAT (4F10.4)
0012 READ (5,1010) NF, MF, T, E
0013 1010 FORMAT (4F10.3)
0014 READ (5,1011) VF, L, Z1
0015 1011 FORMAT (1F12.4, 2F10.4)
0016 N3 = 0

0017 BETA = VF*SQR(T(MF*L**2./(E*1))
0018 MU = MF/(MF*T)
0019 NU = 7.04*21*SQR(1-MU)

0020 WRITE (6,1015)
0021 1015 FORMAT ('1')

**NOTE: DELTA FUNCTION**

S(J,K), C(E,I) ... ARE DIFFERENT

0022 DO 110 IE=1,4
0023 DO 100 ID=1,4
0024 DEL(ID1, ID2) = 0.0
0025 B(ID1, ID2) = 0.0
0026 C(ID1, ID2) = 0.0
0027 100 CONTINUE
0028 DEL(ID1, ID1) = 0.5
0029 LAM(ID1) = 3.14159*ID1
0030 B(ID1, ID1) = -0.5*LAM(ID1)**2
0031 C(ID1, ID1) = 0.5*LAM(ID1)**2
0032 110 CONTINUE

0033 FMU = SQRT(MU)
**C**

***************

**C**

ENTRIES OF MATRIX

**C**

***************

**C**

DO 120 IC=1,8
DO (I4,1) = 0.0
DO (I4,2) = 0.0
CONTINUE

**C**

NOTE: DO-LOOP TO FILL IN 0's & 1's

**C**

IC = 4
DO 140 IA=1,4
DO 130 IB=1,2
K(IA,IB) = 0.0
CONTINUE

**C**

IC = IC + 1
K(IA,IC) = 1.0
CONTINUE

**C**

NOTE: ENTRIES WITH RECOGNIZED PATTERN

**C**

DO 160 IS=5,8
IT = IS-4
DO 150 JS=1,4
K(IS,JS) = -(DEL(IT,J5) * LAM(IS)*4.0 + BETA**2.0) * 3(IS,J5))
CONTINUE

**C**

J5 = JS-4
K(IS,J3) = -(2.0 * BETA * P**4) * (IT,JT) + DEL(IT,JT) = IN)
CONTINUE

**C**

K(IS,9) = B(IT,1) * C(1,1)
K(IS,10) = B(IT,1) * C(2,2) + B(IT,2) * (C(1,2) + C(2,2))
K(IS,11) = B(IT,1) * C(3,3) + B(IT,3) * (C(1,3) + C(3,1))
K(IS,12) = B(IT,1) * C(4,4) + B(IT,4) * (C(1,4) + C(4,1))
K(IS,13) = B(IT,1) * (C(1,2) + C(2,1)) + B(IT,2) * C(1,1)
K(IS,14) = B(IT,1) * (C(1,3) + C(3,1)) + B(IT,3) * C(1,1)
K(IS,15) = B(IT,1) * C(2,3) + C(3,2) + B(IT,2) * (C(1,3) + C(3,1))
K(IS,16) = B(IT,1) * (C(1,4) + C(4,1)) + B(IT,4) * C(1,1)
K(IS,17) = B(IT,1) * (C(2,4) + C(4,2)) + B(IT,2) * (C(1,4) + C(4,1))
K(IS,18) = B(IT,1) * (C(3,4) + C(4,3)) + B(IT,3) * (C(1,4) + C(4,1))
K(IS,19) = B(IT,1) * (C(1,3) + C(3,1))
K(IS,20) = B(IT,2) * C(2,2)
K(IS,21) = B(IT,2) * (C(2,3) + C(3,2)) + B(IT,3) * (C(2,3) + C(3,2))
K(IS,22) = B(IT,2) * (C(2,4) + C(4,2)) + B(IT,4) * (C(2,4) + C(4,2))
K(IS,23) = B(IT,2) * (C(3,4) + C(4,3)) + B(IT,3) * (C(3,4) + C(4,3))
K(IS,24) = B(IT,2) * (C(4,4) + C(4,4)) + B(IT,4) * (C(4,4) + C(4,4))
K(IS,25) = B(IT,3) * C(3,3)
I
c
2^1 = 2
K(15,26) = B(I5,3)*C(4,4) + F(I5,4)* (C(3,4) + C(4,3))
K(15,27) = B(I5,4)*C(3,3) + F(I5,3)* (C(3,4) + C(4,3))
K(15,28) = F(I5,4)*C(4,4)

C
16C CONTINUE
C
2 CONTINUATION CARDS JUST IN CASE
C
CONTINUE
C
READ (5,1012,END=979) T,DELT,TMAX
1012 FORMAT (3F10.4)
WRITE (6,5554) T,DELT,TMAX
5554 FORMAT ( ' ',3F20.4)
N1 = 1
C
CONTINUE
C
DO 180 IZ1=1,9
V(IZ1) = A0(IZ1,N1)
180 CONTINUE
C
NOTE: HAVE TO ADD NON LINEAR EQUATIONS
C
V(9) = A0(1,N1)**3
V(10) = A0(1,N1)*A0(2,N1)**2
V(11) = A0(1,N1)*A0(3,N1)**2
V(12) = A0(1,N1)*A0(4,N1)**2
V(13) = A0(1,N1)**2*A0(2,N1)
V(14) = A0(1,N1)**2*A0(3,N1)
V(15) = A0(1,N1)**2*A0(4,N1)
V(16) = A0(1,N1)**2*A0(4,N1)
V(17) = A0(1,N1)**2*A0(4,N1)
V(18) = A0(1,N1)**2*A0(3,N1)
V(19) = A0(1,N1)**2*A0(2,N1)
V(20) = A0(2,N1)**3
V(21) = A0(2,N1)**2*A0(3,N1)
V(22) = A0(2,N1)**2*A0(4,N1)
V(23) = A0(2,N1)**2*A0(4,N1)
V(24) = A0(2,N1)**2*A0(3,N1)
V(25) = A0(2,N1)**2*A0(4,N1)
V(26) = A0(2,N1)**2*A0(3,N1)
V(27) = A0(2,N1)**2*A0(4,N1)
V(28) = A0(2,N1)**3

C
AT(N1) = T
NNN = N1 + 2
N1 = N1 + 1
C
+-------------------------------------------------------------+
+ MATRIX MULTIPLICATION +
+-------------------------------------------------------------+
DO 200 IZ2=1,9
DUMMY = 0.
DO 190 IZ3=1,2
MATM = K(IZ2,IZ3)*V(IZ3)
DUMMY = DUMMY + MATM
19C CONTINUE
IO(IZ2,NNN) = DUMMY
A0(IZ2,N1) = DELT*(1.9167*EC(IZ2,NNN)-1.3333*DO(IZ2,NNN-1) -0.4167*DO(IZ2,NNN-2)) + A0(IZ2,N1-1)
20C CONTINUE
T = T+DELT
IF (IMAX.GT.T) GO TO 170
C
DO 210 N2=1,N1,10
N3 = N3+1
T(N3) = AT(N2)
TO1(N3) = A0(1,N2)
TO2(N3) = A0(2,N2)
21C CONTINUE
DO 220 IS=1,3
DO(IS,2) = DO(IS,NNN-1)
DO(IS,1) = DO(IS,NNN-2)
22C CONTINUE
N3 = N3-1
GO TO 165
999 CONTINUE
*
PLOTTING FUNCTION*
*
**********
NOTE: ONLY EVERY TENTH VALUE IS PLOTTED
*
**********
C
WRITE (6,5555) N3,T(T1),T(T3)
5555 FORMAT (' ',I6,2F10.4)
CALL DFIIPS1(T,T01,N3,1,10.0)
CALL DFIIPS1(T,T02,N3,1,10.0)
WRITE (6,1015)
STCP
END