A LOWER BOUND FOR THE BAYES RISK FOR TESTING SEQUENTIALLY
THE SIGN OF THE DRIFT PARAMETER OF A WIENER PROCESS

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ABSTRACT

Let $X(t)$ be a Wiener process with drift $u$ and variance 1 per unit of time. For testing $H: u \leq 0$ vs $H: u > 0$ with the loss function $|u|$ if the wrong decision is made and 0 otherwise, $c$ cost of observation per unit time and $u$ has a prior distribution which is normal with mean 0 and variance $\sigma^2_u$, we followed an idea of Bickel and Palu to obtain a lower bound for the Bayes risk and showed that this lower bound is strict as $\sigma^2_u \to \infty$ for all $c$.

Key Words: Sequential tests, S.P.R.T. Bayes, stopping times, lower bound, asymptotic expansion.

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By considering the above testing problem with the additional information of the magnitude of \( u \), Bickel and Yahav [1] obtained a lower bound for the Bayes risk for the case of \( u \) having the improper prior distribution and conjectured that the lower bound can be attained as \( c \to 0 \). In this note we assume that \( u \) has a normal prior distribution with mean 0 and variance \( \sigma^2 \). By using similar techniques as in Bickel and Yahav [1], we obtained a lower bound for the Bayes risk, then showed that this lower bound is not asymptotically achievable as \( \sigma^2 \to 0 \) for all \( c > 0 \).

2. Lower Bound For Bayes Risk: From Chernoff [3], the posterior cost of wrong decision is given by

\[
T_c = \left( c + \frac{1}{2} \right) \left[ \frac{1}{2} \log \left( \frac{1}{2} \right) + \frac{1}{2} \log \left( \frac{1}{2} \right) \right] + \frac{1}{2} \log \left( \frac{1}{2} \right)
\]

where \( \alpha = \left( c + \frac{1}{2} \right) \delta(t) \). Let the posterior risk at time \( t \) be,

\[
R(c, t) = T_c + \beta t
\]

We are interested in a stopping rule \( T_0 \) for which

\[
E[R(c, T_0)] = \inf_{T \in Y} E[R(c, T)]
\]

where \( Y \) is the class of all stopping times.

Using the idea of Bickel and Yahav [1], let us consider the following problem of testing,

\[
H_0: u = \nu_0 \quad \text{vs} \quad H_1: u = \nu_0
\]

with \( |u_0| \) for cost of wrong decision and prior distribution \( P(u = \nu_0) = P(u = -\nu_0) = \frac{1}{2} \). Then the posterior cost of wrong decision is

\[
\tilde{T}_c = |u_0| \cdot P(X(t) = \nu_0 | X(t))
\]

Let

\[
R(c, t) = \tilde{T}_c + \beta t
\]

To solve the above Bayes problem, we have to find a stopping rule \( \tilde{T}^* \) such that

\[
E[R(c, \tilde{T}^*)] = \inf_{T \in Y} E[R(c, T)]
\]

From the property of S.P.R.T we have the following lemma.
Lemma 2.1: The stopping rule \( I \): Stop the first \( R(t) = a \) where \( a \) is determined by the minimization of

\[
E_0 \left( (1 + \exp(2a_0))^{-1} + c_0 a_0^{-1} (1 - (1 + \exp(2a_0)))^{-1} \right)
\]

is the optimal stopping rule for the above problem.

Proof: Let

\[
x' = \left( 2v \right)^{1/2} \int_0^1 E_0 \left( 2v \right)^{-1/2} \exp(-v^2/2v_0^2) dv \leq E_0 (c, c_0^2)
\]

where \( a' \) is the solution of the minimization problem in Lemma 2.1. Then \( a' \) should satisfy the relation

\[
2a' = c(2a'^{-1} + 2 \ln a')
\]

We have by using (2.3), (2.4) and Lemma 2.1,

\[
\int_0^1 E_0 \left( 2v \right)^{-1/2} \exp(-v^2/2v_0^2) dv
\]

\[
= \left( 2^{1/2} \pi^{1/2} \right) \int_0^1 (2a'^{-1} + 2 \ln a') \frac{1}{1 + 2a'^{-1} + a'^2} \exp(-a'^2/2a_0^2) a'^{-1/2} \, da'
\]

Theorem: Let

\[
E_0 \left( (1 + \exp(2a_0))^{-1} + c_0 a_0^{-1} (1 - (1 + \exp(2a_0)))^{-1} \right)
\]

From it the lemma follows.
Let
\[ \gamma = 2^{-5/3} e^{2/3} \rho^{-2} \]
\[ f(s) = (s-x^{-1} + 2 \ln s)^{-2/3}(1 + \ln s-x^{-1} + 2 \ln s^{-1} + x^{-2}) \]

We have

(2.5) \[ \int_0^1 \frac{h(s,x^2 s^2)}{x^2 \rho} \exp(-\frac{s^{-2}}{2 \rho^2 \rho^2}) ds = 2^{1/3} \gamma^{-1} e^{2/3} \]

Lemma 2.2:
\[ \int_1^{1/\gamma} f(s) \exp(-\gamma(s-x^{-1} + 2 \ln s)^{-2/3}) ds \]
\[ = \int_1^{1/\gamma} f(s) ds + \gamma^{-1} \frac{1}{2} \left( -\frac{1}{3} \ln \gamma - \frac{1}{2} + \frac{1}{3} + \rho^{-2/3}(1 + \rho^{-1/3}(\gamma - 1)) \right) \]

Proof:
\[ \int_1^{1/\gamma} f(s) \exp(-\gamma(s-x^{-1} + 2 \ln s)^{-2/3}) ds \]
\[ = 2^{-5/3} e^{2/3} \rho^{-2} \gamma^{-1} \frac{1}{2} \left( -\frac{1}{3} \ln \gamma - \frac{1}{2} + \frac{1}{3} + \rho^{-2/3}(1 + \rho^{-1/3}(\gamma - 1)) \right) \]

\[ = \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \left( \frac{1}{3} \ln \gamma - \frac{1}{2} + \frac{1}{3} + \rho^{-2/3}(1 + \rho^{-1/3}(\gamma - 1)) \right) \]

Proof: Let \( w = \gamma(s-x^{-1} + 2 \ln s)^{-2/3} \)
\[ = \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \left( \frac{1}{3} \ln \gamma - \frac{1}{2} + \frac{1}{3} + \rho^{-2/3}(1 + \rho^{-1/3}(\gamma - 1)) \right) \]

\[ = \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \left( \frac{1}{3} \ln \gamma - \frac{1}{2} + \frac{1}{3} + \rho^{-2/3}(1 + \rho^{-1/3}(\gamma - 1)) \right) \]

Proof: Let \( w = \gamma(s-x^{-1} + 2 \ln s)^{-2/3} \)
\[ = \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \int_1^{1/\gamma} f(s) ds - \gamma^{-1} \left( \frac{1}{3} \ln \gamma - \frac{1}{2} + \frac{1}{3} + \rho^{-2/3}(1 + \rho^{-1/3}(\gamma - 1)) \right) \]
From (1.1), Lemma 2.2 and the Theorem, we have the following corollary to the Theorem.

**Corollary:** \( \text{for } \lambda > 0 \)

**Remark:** Consider the case of \( u \) having a prior distribution of Lebesgue measure. For any stopping rule \( T \),

\[
\int_{\Omega} R(u, v) \, du = \lim_{\omega \to \infty} \left( \frac{1}{2} - \int_{0}^{\infty} R(u, v) \, du \right)
\]

So the Bayes risk with respect to Lebesgue measure

\[
\inf_{\omega} \int R(u, v) \, du = \omega \cdot \frac{1}{2} \cdot \pi^{3/2}
\]

for all \( \omega > 0 \).

Here, \( \omega \cdot \pi^{3/2} \) is the lower bound derived in [11].

Therefore, we have shown that Bickel and Yahav's lower bound cannot be attained.
References


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*Key Words:*
Sequential tests, S.P.R.T, Bayes, stopping times, lower bound, asymptotic expansion

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See reverse side.
Let $x(t)$ be a Wiener process with drift $\mu$ and variance $\sigma^2$ per unit of time. For testing $H_0: \mu < 0$ vs $H_1: \mu > 0$ with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, $c$ cost of observation per unit time and $\mu$ has a prior distribution which is normal with mean 0 and variance $\sigma^2$. We followed an idea of Bickel and Yahav to obtain a lower bound for the Bayes risk and showed that this lower bound is strict as $\sigma_0 \to \infty$ for all $c$.
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