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ON COLLOCATION SCHEMES FOR QUASILINEAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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ABSTRACT

A numerical method for boundary value problems for quasilinear systems of
singularly perturbed ordinary differential equations is presented. The method
is based on collocation with polynomial splines. The stability properties of
the associated difference operator are examined and a stepsize algorithm to
achieve a certain over-all accuracy is developed. The number of gridpoints
required by the algorithm is estimated.

AMS (MOS) Subject Classifications: 34E99, 65L10

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Many high-order discretization methods for the solution of two-point boundary value problems for systems of ordinary differential equations are already circulated. However, these methods can behave quite poorly in case the solution has large derivatives, unless severe restrictions on the mesh are imposed. The reason for these restrictions is mainly a stability problem.

In this paper a strongly A-stable difference method based on polynomial collocation is developed for a class of quasilinear, singularly perturbed, two-point boundary value problems. Many problems of practical interest are included in this class, for instance the nonlinear deformation of thin beams or one-dimensional models of carrier transport in semiconductor devices. The method combines the advantages of having the same stability properties as the lower order methods which are used already, with the high order of convergence of collocation methods. It is shown that the number of gridpoints (and therefore the amount of computing time and required storage) is of the same order of magnitude as the one required for solving unperturbed problems.
ON COLLOCATION SCHEMES FOR QUASILINEAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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1. Introduction.

We consider quasilinear systems of singularly perturbed O.D.E.'s on the interval [0,1] of the form

\[ \begin{align*}
F_0(y,z) &= 0, \\
F_1(y,z) &= \begin{pmatrix}
y'(t,\varepsilon) - f(t,y,z,\varepsilon) \\
z'(t,\varepsilon) - g(t,y,z,\varepsilon)
\end{pmatrix} \\
b_0(y,z) &= b(y(0,\varepsilon),y(1,\varepsilon),z(0,\varepsilon),z(1,\varepsilon),\varepsilon)
\end{align*} \tag{1.1} \]

where \( y \) and \( z \) are vectors of dimension \( n \) and \( m \). The prime denotes the derivative with respect to \( t \). (1.1) is quasilinear in the following sense:

\[ f(t,y,z,\varepsilon) = f_1(t,z)y + f_2(t,y,z,\varepsilon) \tag{1.2} \]

\[ b(y_0,y_1,z_0,z_1,\varepsilon) = b_1(z_0,z_1)y_0 + b_2(y_0,y_1,z_0,z_1,\varepsilon) \]

\( f_1 \) and \( b_1 \) are matrices of appropriate dimensions. We assume that the derivatives \( \frac{\partial f_2}{\partial y} \) and \( \frac{\partial b_2}{\partial y_0,y_1} \) are uniformly of order \( \varepsilon \) for \( y,z,y_0,y_1,z_0,z_1 \) in any bounded domain of the appropriate real spaces. For \( f_2,b_2 \) and \( g \) there exist asymptotic expansions in powers of \( \varepsilon \). The \( n \times n \)-matrix \( f_1 \) is a block-diagonal-matrix of the form

\[ f_1(t,z) = \begin{pmatrix}
f_{1}^{-}(t,z) & 0 \\
0 & f_{1}^{+}(t,z)
\end{pmatrix} \tag{1.3} \]

The square-matrices \( f_{1}^{-} \) and \( f_{1}^{+} \) are of dimension \( n_- \) and \( n_+ \) (where \( n_- + n_+ = n \) holds). \( f_{1}^{-} \) has only strictly stable and \( f_{1}^{+} \) has only strictly unstable eigenvalues for

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t ∈ [0,1] and z in a certain domain D of ℝ (defined more precisely in chapter 3). From the analysis of singular perturbations (see c.f. Hoppenstaedt (1971), O'Malley (1975), Eckhaus (1979), Flaherty and O'Malley (1980) and Ringhofer (1981) we know that under these assumptions we can expect a smooth solution in the interior of [0,1] while at the endpoints boundary layers will occur. Therefore the usual discretization methods are not applicable unless we impose conditions on the mesh. (See c.f. Ascher and Weiss (1980) for the behaviour of collocation methods of Gauss, Radau and Lobatto-type.) The standard theory for discretization methods for general grids is not applicable unless the maximal stepsize is smaller than ε. Flaherty and O'Malley (1980) avoid this difficulty by solving only the reduced problem (which is independent of ε) instead of solving the full problem (1.1): As ε tends to zero the solution (y,z) of (1.1) will converge to a function (y(z),z(t)) uniformly in compact subsets of (0,1) with a convergence rate O(ε). Obviously (y, z) satisfies

\[ 0 = f(t, y, z, 0), \quad z' = g(t, y, z, 0) \]  

To represent (y, z) as the solution of a boundary-value problem (the reduced problem) we need m boundary conditions for the equation (1.4). In the quasilinear case this can be done by a "cancellation law" (see Flaherty and O'Malley (1980)). In compact subsets of (0,1) this gives the solution (y, z) up to the global discretization error we make by approximating the reduced solution and a term of order O(ε). Inside the layer this gives us no approximation. Using some information about the analytical structure of the solution we construct a method which is designed especially for problems of the type (1.1): It shall solve (1.1) directly. So we need not compute the boundary conditions for the reduced problem numerically. The amount of labour (i.e. of gridpoints) which is required to obtain a uniform approximation of the solution on [0,1] shall be "reasonable" (that means not proportional to ε⁻¹). If we use a general (so for instance a uniform) mesh with maximal stepsize N > > ε the approximation shall be as good as the one obtained by solving the reduced problem only. The solution will consist of a smooth part and two "layer-parts" exponentially decaying from the endpoints. More precisely we have

-2-
\[ y^- (t, \varepsilon) = y^-(t) + \mu(\varepsilon) + o(\varepsilon) \]
\[ y^+ (t, \varepsilon) = y^+(t) + \nu(\varepsilon) + o(\varepsilon) \]
\[ x(t, \varepsilon) = x(t) + o(\varepsilon) \]

\( y^- \) and \( y^+ \) are the first \( n \) and last \( n+ \) components of \( y \). \( \mu \) and \( \nu \) decay exponentially as their arguments tend to infinity. \( \mu(\varepsilon) \) behaves roughly like the solution \( \zeta \) of

\[ \varepsilon \zeta' (t, \varepsilon) = \lambda \zeta (t, \varepsilon), \zeta (0, \varepsilon) = 1, \text{ Re } \lambda < 0 \]

A discretisation method which gives us a good approximation of \( y^- \) for an arbitrary mesh should also be able to deal with (1.6) on a uniform grid. We now compare the behaviour of the Box-scheme and the implicit Euler scheme at the problem (1.6). We have

\[ \frac{\varepsilon}{h} \zeta_{1}^{(1)} - \zeta_{0}^{(1)} = \frac{1}{2} \left( \zeta_{1}^{(1)} + \zeta_{0}^{(1)} \right), \quad \varepsilon \frac{\zeta_{1}^{(2)} - \zeta_{0}^{(2)}}{h} = \lambda \zeta_{1}^{(2)} \]

(1.7) shows that only the implicit Euler scheme gives us a good approximation of the solution. The reason for that is, that the growth function of the Box-scheme

\((1+z)(1-z)^{-1}\)

tends to \(-1\) as \( z \) tends to infinity whereas the growth function of the implicit Euler scheme \((1-z)^{-1}\)
tends to zero. This is the basic idea of a method developed by Kreiss and Nichols (1975): They use the implicit Euler scheme for \( y^- \). Since the eigenvalues of \( f^x_1 \) are strictly unstable, they use the explicit Euler scheme for \( y^+ \). For \( z \) they use the Box-scheme. We extend this idea to higher order methods and our approach for this extension is polynomial collocation. For a positive integer \( q \) we choose \( q+1 \) reference points \( 0 = u_0 < ... < u_q = 1 \) on the interval \([0,1]\) according to

\[ u_j = \frac{1}{q}, \quad j = 0(1)q \]

For a given mesh \( T_N := \{ 0 = t_0 < ... < t_N = 1, \ N = \text{max}(t_{i+1} - t_i) \} \) we choose the collocation points \( t_{ij} \) according to
Note that \( t_{iq} = t_{i+1,0} \) holds. On the subinterval \( (t_i, t_{i+1}) \) we now use for \( y^- \) the collocation points \( t_{i1}, \ldots, t_{iq} \). For \( y^+ \) we use \( t_{i0}, \ldots, t_{i,q-1} \). For \( z \) we use \( t_{i0}, \ldots, t_{i,q} \). For \( q = 1 \) this gives the method of Kreiss and Nichols except that we use the trapezoidal rule for \( x \).

We show that if the reduced problem has an isolated solution, the operator built by \( F_{ch} \) and the boundary conditions is stable in the sense of Keller (1975) and that the stability constant is independent of \( \varepsilon \) and the grid \( \mathcal{T}_N \). So if we use a mesh selecting strategy to achieve an over all accuracy \( \delta \) we only have to control the local discretization error. We develop such a stepsize algorithm and show that the amount of gridpoints required to achieve an over-all-accuracy \( \delta \) is essentially independent of \( \varepsilon \) and \( 1 \) proportional to \( \delta^{-q} \). This is comparable with the amount of gridpoints our method would need for an unperturbed problem. If we do not use a stepsize algorithm we can show the following result: For an arbitrary mesh we define the global error \( e_h \) by

\[
e_h = (y_h - y(t_h, \varepsilon), z_h - z(t_h, \varepsilon)) \quad \text{where} \quad (y(t_h, \varepsilon), z(t_h, \varepsilon)) \quad \text{is the solution of the continuous problem restricted to the grid} \quad \mathcal{T}_N.
\]

Then we have

\[
\|e_h\| \leq c_0 h + h^{q+1} + \delta + \exp\left(-\frac{\lambda}{\varepsilon}\right) + \delta \exp\left(\frac{t_{i+1,0} - 1}{\varepsilon}\right)
\]

for some positive constants \( c, \delta, \lambda \) (here \( H \) denotes the maximal stepsize). So away from the boundary we have the normal order of convergence \( O(H^{q+1}) \) of our method plus a term of order \( O(\varepsilon) \). Thus the approximation is as good as if we had solved the reduced problem only.

This paper is organized as it follows. In chapter 2 we define the collocation scheme and introduce some basic notations. In chapter 3 we present some analytical results of singular perturbation theory. For the proofs we only refer to the literature. Furthermore
we impose the main assumptions to be satisfied by the continuous problem. In chapter 4 we state our main results and explain them in a more detailed manner. In chapter 6 we prove them. In chapter 5 we demonstrate our method on a model problem where the continuous solution is known.

2. The method.

In this chapter we define the collocation method and introduce some basic notations which we will use throughout the rest of the paper.

(2.1) Definition: We call a set of $q+1$ points $0 < u_0 < u_1 < \ldots < u_{q-1} < u_q$ a set of reference points in the unit interval.

(2.2) Definition: Given $N+1$ gridpoints $0 = t_0 < \ldots < t_N = 1$ and a set of $q+1$ reference points we define for each subinterval $[t_i, t_{i+1}]$ $q+1$ collocation points by

$$ t_{ij} = t_i + h_i u_j, \quad h_i = t_{i+1} - t_i, \quad j = 0(1)q, \quad i = 0(1)N-1. \quad (2.3) $$

(Note that $t_{10} = t_{1-1,q} = t_1$ holds.)

(2.4) Notation: We denote the mesh consisting of the gridpoints and the collocation points and its maximal stepsize by

$$ T_h = \{ t_{ij}, j = 0(1)q, \quad i = 0(1)N-1 \}, \quad h = \max h_i. \quad (2.5) $$

(2.5) Definition: For a given mesh $T_h$ we define the scalar (vector-, matrix-) grid function $x_h$ as a sequence of numbers (vectors, matrices) by

$$ x_h = \{ x_{ij}, j = 0(1)q, \quad i = 0(1)N-1, \quad x_{iq} = x_{i+1,0} \}. \quad (2.6) $$

(2.6) Definition: In the space of grid functions we introduce a norm by

$$ \| x \|_h = \max\{ \|x_{ij}\|, j = 0(1)q, \quad i = 0(1)N-1 \}, \quad (2.7) $$

where $\| \cdot \|$ denotes the modulus or the max-norm in $\mathbb{R}^2$ or the matrixnorm introduced by the max norm.

(2.7) Notation: We denote the first $n_-$ components (rows) and the last $n_+$ components (rows) of the vector (matrix) $x$ with $x^-$ and $x^+$ where $n_-$ and $n_+$ are the dimensions of $f_1^-$ and $f_1^+$ in (1.3).
(2.8) Definition: To apply a (polynomial) collocation method one generally seeks a polynomial spline function, which satisfies the given differential equation at certain points (the collocation points) as well as the boundary conditions. In order to approximate the solution \( y = (y^-, y^+, z) \) of (1.1) we construct a vector-spline function \( (p_y, p_z) \), \( p_y = (p_y^-, p_y^+)\) which satisfies:

a) \( p_y, p_z \in C([0,1]) \)

b) \( p_y^- \) and \( p_y^+ \) are polynomials of degree \( q \); \( p_z \) is a polynomial of degree \( q+1 \) in each subinterval \([t_i, t_{i+1}]\).

c) To generalize the implicit and explicit Euler method we request that \( p_y^- \) satisfies the differential equation at all collocation points \( t_{ij} \) except the first one in each subinterval and \( p_y^+ \) satisfies the differential equations at all collocation points except the last one in each subinterval.

\[
\begin{align*}
\varepsilon (p_y^-)' & = f(t, p_y^-, p_z, \varepsilon) \quad \text{for} \quad t = t_{ij}, \quad j = 1(1)q, \quad i = 0(1)N-1 \\
(2.9) \quad \varepsilon (p_y^+)' & = f^+(t, p_y^+, p_z, \varepsilon) \quad \text{for} \quad t = t_{ij}, \quad j = 0(1)q-1, \quad i = 0(1)N-1
\end{align*}
\]

For \( z \) we take all \( q+1 \) collocation points except for 2 subintervals. For reasons which will be explained in chapter 6 it is necessary to modify the method used for \( z \) in order to obtain a higher order method. Given a mesh \( T_h \) we assume, that \( I_1 \) steps \( h_1 \) on the left hand side and \( I_2 \) steps on the right hand side are of order \( O(\varepsilon) \) (where we also allow \( I_1 \) and \( I_2 \) to be zero):

\[
\begin{align*}
h_1 & = O(\varepsilon) \quad i = 0(1)I_1 - 1, \quad i = I_2(1)N-1 \\
h_{I_1} & > \varepsilon \quad h_{N-I_1-1} > \varepsilon
\end{align*}
\]

For the subinterval \([t_{I_1}, t_{I_1+1}]\) we take the collocation points \( t_{I_1, I_1+1, \cdots, I_1+q} \) (as for \( p_y^- \)). For \([t_{N-I_2-1}, t_{N-I_2}]\) we take \( t_{N-I_2-1}, t_{N-I_2-2}, \cdots, t_{N-I_2}, q - 1 \) so we have:
If \( t = t_{ij} \),

\[ y_j = q(t, p_y, p_\xi, \xi) \]

for \( t = t_{ij} \),

\[ j = 0(1)q \text{ if } i = \frac{1}{2} - 1 \]

\[ j = 0(1)q - 1 \text{ if } i = \frac{1}{2} - 1 \]

\[ j = 0(1)q \text{ else } . \]

For our analysis it will be convenient to rewrite (2.4) into a difference scheme. If we denote the values of \( p_y \) and \( p_\xi \) at \( t = t_{ij} \) by \( y_{ij} \) and \( z_{ij} \) (2.4) can be written (see Weiss (1974)) as

\[
\begin{align*}
\frac{y_{ij} - y_{i0}}{h_1} &= \sum_{k=1}^{q} w_{jk} f(t, y_{i+1}, z_{i+1}, \xi) \\
\frac{y_{ij} - y_{i+1}}{h_1} &= \sum_{k=0}^{q-1} w_{jk} f(t, y_{i+1}, z_{i+1}, \xi) \\
\frac{z_{ij} - z_{i0}}{h_1} &= \sum_{k=0}^{q} w_{jk} g(t, y_{i+1}, z_{i+1}, \xi) \\
\end{align*}
\]

\( j = 1(1)q, \ i = 1(1)N-1 \)

\[
\begin{align*}
w_{jk} &= \int_{0}^{u_j} w(s) \frac{(a_k - a_0)}{(s-a_k)(s-a_0)} ds, \quad \nu_{jk} = \int_{0}^{u_j} w(s) \frac{(a_k-a_0)}{(s-a_k)(s-a_0)} \\
\nu_{0j} &= \int_{0}^{u_j} w'(s-a_k) ds \\
w(s) &= (s-a_0) \cdots (s-a_q) \]

(2.6) Notation: We define the discrete operators \( D_n, S_n, S_n^+, S_n^- \) by
\[
(D_h x_h)_{ij} = \frac{x_{ij} - x_{i0}}{h_1}
\]
\[
(s_h^+ x_h)_{ij} = \sum_{j=1}^q w_{jk} x_{ik} (s_h^- x_h)_{ij} = \sum_{j=0}^{q-1} w_{jk} x_{ik}
\]
\[
(s_{h I_1}^- x_h)_{ij} = \begin{cases} (s_h^- x_h)_{ij} & \text{if } i = I_1 \\ (s_h^+ x_h)_{ij} & \text{if } i = N-I_2^{-1} \end{cases}
\]
\[
\sum_{k=0}^q w_{jk} x_{ik} \quad \text{otherwise .}
\]

With this notation the discrete problem can be written as

(2.7) \[ F_{ch}(y_h, x_h) = 0 \]
\[
\epsilon D_h y_h^+ - S_h^- f^-(t_h, y_h, x_h, \epsilon)
\]
\[
F_{ch}(y_h, x_h) = \epsilon D_h y_h^+ - S_h^- f^+(t_h, y_h, x_h, \epsilon)
\]
\[
D_h x_h^0 - S_h g(t_h, y_h, x_h, \epsilon) \ .
\]

(2.8) Remark: To this point we did not restrict ourselves to a certain set of reference points. A natural choice would be Lobato-points in order to achieve the highest possible order of accuracy for \( z \). However, for the rest of this paper we restrict ourselves to equidistant reference points.
3. Analytical results and main assumptions

Quasilinear singularly perturbed b.v.p.'s of the form (1.1) have been studied by several authors (see c.f. O'Malley (1974), Flaherty and O'Malley (1980), Howes (1980), Ringhofer (1981)). In this section we present their results in a manner which is appropriate for our purposes. One can find the proofs either directly in the above mentioned papers or can prove the results of this section analogously using the special structure of problem (1.1).

The solution \((y,z)\) of (1.1) has a uniformly valid asymptotic expansion of the form

\[
y(t;\epsilon) \sim \sum_{j=0}^{\infty} \left( \tilde{y}^*_j(t) + \frac{\tilde{y}^*_j(t)}{\epsilon} \right) + \sum_{j=1}^{\infty} \left( \tilde{z}^*_j(t) + \frac{\tilde{z}^*_j(t)}{\epsilon^2} \right)
\]

where \(\tilde{y}^*_j, \tilde{z}^*_j, \alpha_j, \beta_j\) are exponentially decaying as their arguments tend to infinity. (We call a function \(\Psi(t)\) exponentially decaying if it satisfies \(|\Psi(t)| \lesssim c_1 \exp(-c_2 t)\) for positive constants \(c_1\) and \(c_2\).) To derive this expansion it is necessary to define the reduced problem. This is done in Flaherty and O'Malley (1980) for the case of separated boundary conditions. In our case we proceed analogously.

\((\tilde{y}_0^*, \tilde{z}_0^*), \tilde{y}_0^*, \tilde{z}_0^*\) satisfy

\[
0 = f(t, \tilde{y}_0^*, \tilde{z}_0^*, \tilde{v}_0^*), \quad \tilde{y}_0^* = g(t, \tilde{y}_0^*, \tilde{z}_0^*)
\]

\[
\frac{d}{dt} \tilde{v}_0^*(t) = f_1(0, \tilde{y}_0^*(0)), \quad \frac{d}{dt} \tilde{v}_0^*(t) = -f_1(1, \tilde{y}_0^*(1)) \tilde{v}_0^*(t)
\]

\[
b_1(\tilde{y}_0^*(0), \tilde{z}_0^*(1))^T \left( \tilde{y}_0^*(0) + \tilde{v}_0^*(0) \right) + b_2(0, \tilde{y}_0^*(0), \tilde{z}_0^*(1), 0) = 0.
\]

Since \(v_0^*\) and \(u_0\) must decay exponentially \(v_0^*(0) = 0, \tilde{v}_0^*(0) = 0\) must hold. We split up \(b_1\), into

\[
b_1(x_0^*, x_1) \begin{pmatrix} y_0^* \\ y_1^* \end{pmatrix} = b_{11}(x_0^*, x_1) \begin{pmatrix} y_0^* \\ y_1^* \end{pmatrix} + b_{12}(x_0^*, x_1) \begin{pmatrix} y_0^- \\ y_1^- \end{pmatrix}.
\]
(Here \( y^- \) and \( y^+ \) are defined as in (1.5).) So, \( b_{11} \) and \( b_{12} \) are \((n+m) \times n\) matrices. We assume that \( b_{11} \) has maximal rank in a domain of \( \mathbb{R}^{2m} \) containing \((\tilde{z}_0(0),\tilde{z}_0(1))\) and that there exists a regular \((n+m)\times(n+m)\) matrix \( W(z_0,z_1) \) such that
\[
W(z_0,z_1)b_{11}(z_0,z_1) = \begin{pmatrix} V(z_0,z_1) & 0 \\ 0 & 0 \end{pmatrix}
\]
holds where \( V \) is a regular \( n \times n \) matrix. In that case (3.5) is equivalent to
\[
W^+(z_0(0),z_0(1))(b_{12}(z_0(0),z_0(1)) \begin{pmatrix} y_0(0) \\ -y_1(-1) \end{pmatrix} + b_2(0,0,z_0(0),z_0(1),0)) = 0
\]
(3.8)
\[
\begin{pmatrix} \mu_0^-(-1) \\ \nu_0^+(-1) \end{pmatrix} = -V^{-1}(z_0(0),z_0(1))W^-(z_0(0),z_0(1))b(y(0),y(1),z_0(0),z_1(0),z_1(0)) .
\]
(3.9)
(Here \( W^- \) and \( W^+ \) denote the first \( n_- \) and the last \( n_+ \) rows of \( W \).) (3.8) together with (3.3) gives the reduced problem. If the reduced problem has an isolated solution we can show the uniform validity of the asymptotic expansion (3.1), (3.2). This leads to the following assumptions: There exists a domain \( D \) of \( \mathbb{R}^m \) so that
\[
(3.10) \ H_1: \ f_1(t,z) \ is \ a \ block \ diagonal \ matrix. \ So
\]
\[
f_1(t,z) = \begin{pmatrix} f_1^-(t,z) & 0 \\ 0 & f_1^+(t,z) \end{pmatrix}
\]
holds. There exist positive constants \( \bar{\lambda}_- \) and \( \bar{\lambda}_+ \) so that all eigenvalues \( \lambda \) of the \( n_- \) dimensional block \( f_1^-(t,z) \) satisfy \( \text{Re} \lambda < -\bar{\lambda}_- \) and all eigenvalues of the \( n_+ \) dimensional block \( f_1^+(t,z) \) satisfy \( \text{Re} \lambda > \bar{\lambda}_+ \) for all \( t \in [0,1] \) and \( z \in D \).
\[
(3.11) \ H_2: \ There \ exist \ matrices \ W(z_0,z_1) \ and \ V(z_0,z_1) \ as \ defined \ in \ (3.7) \ for \ all \ z_0,z_1 \in D.
\]
\[
(3.12) \ H_3: \ The \ reduced \ problem \ (3.3), \ (3.8) \ has \ an \ isolated \ solution \ (\tilde{y}_0(t),\tilde{z}_0(t)). \ \tilde{z}_0(t) \ lies \ within \ a \ compact \ subset \ of \ D \ for \ all \ t \in [0,1].
\]
\[
(3.13) \ H_4: \ f,g, \ and \ b \ are \ as \ often \ differentiable \ as \ necessary \ with \ respect \ to \ all \ of \ their \ arguments.
Under these assumptions the asymptotic expansion (3.1), (3.2) is uniformly valid.

(3.14) Notation: For the further we denote with \((Wb)(y, z) = 0\) the boundary condition.
\[ W(z(0, \varepsilon), z(1, \varepsilon))b(y(0, \varepsilon), y(1, \varepsilon), z(0, \varepsilon), z(1, \varepsilon)) = 0 \]

4. Main results

We first show, that under the assumptions (3.10)-(3.13) there exists also a solution of the discrete problem

\[(4.1) \quad F_{ch}(y_h, z_h) = 0, \quad b_{ch}(y_h, z_h) = 0\]

and this solution is stable in the sense of Keller (1975). We start with constructing a uniform \(O(\varepsilon + H)\) approximation of \((y_h, z_h)\): We define the discrete reduced problem by

\[(4.2) \quad F^R_{ch}(y_{h}^R, z_{h}^R) = 0, \quad b^R_{ch}(y_{h}^R, z_{h}^R) = 0\]

Because of hypothesis (3.12), (4.2) has an isolated solution \((\tilde{y}_h^R, \tilde{z}_h^R)\) (see Keller (1975)). We now define \(y_h^R, z_h^R\) by

\[
\begin{align*}
\frac{\partial}{\partial h} \tilde{y}_h^R &= S_h \tilde{y}_h^R, \quad \frac{\partial}{\partial h} \tilde{z}_h^R = S_h \tilde{z}_h^R, \\
(\tilde{y}_h^R, \tilde{z}_h^R) &= \text{reg} (\tilde{y}_h^R, \tilde{z}_h^R, y_{h0}, z_{h0}), \quad W_1(\tilde{y}_h^R, \tilde{z}_h^R) b(\tilde{y}_h^R, \tilde{z}_h^R, y_{h0}, z_{h0}, 0) \\
\end{align*}
\]

\[y_h^0 = \tilde{y}_h^R, \quad z_h^0 = \tilde{z}_h^R.\]

\((y_h^0, z_h^0)\) is a uniform \(O(\varepsilon)\) approximation of the solution \((y_h, z_h)\) of (1.18). To show the existence and the stability of \((y_h, z_h)\) we need that the linearization of the operator built by \(F_{ch}\) together with the boundary conditions (3.14) at \((y_h^0, z_h^0)\) has an inverse bounded uniformly in \(\varepsilon\) and \(H\). This gives a linear difference operator \((L_{ch}, B)\) of the
Since \( y_h^0 \) will exhibit a rapid transition at the endpoints the matrices \( A_h^{11} \) will consist of a smooth part and a part exponentially decaying away from the boundary i.e. they will satisfy

\[
A_h^{11} = A_h^{11} + A_h^{12}, \quad k = k + 1, k \leq c, k, i = 1, 2, 3, 4, 5 \Rightarrow 0
\]

\[
(A_h^{12} - 1) \leq c_2 (1 + \frac{\gamma}{\epsilon})^{-1}, \quad i(A_h^{12} - 1) \leq c_2 (1 + \frac{\gamma}{\epsilon})^{-1}, \quad j = 0(1)N, \quad i = 0(1)N-1
\]

Because \( B \) is the linearization of the boundary conditions (3.14) \( B \) will satisfy

\[
\det(B^{11}) \neq 0.
\]

For the problem

\[
L_h(y_h, z_h) = A_h, \quad B(y_h, z_h) = B
\]

(4.6)

(4.7)

(4.8)

For the problem

\[
L_h(y_h, z_h) = A_h, \quad B(y_h, z_h) = B
\]

(4.6)

(4.7)
Theorem: Let $L_{h}$, $\mathcal{A}_{h}$, $\mathcal{B}_{h}$ be defined as in (4.3), (4.4), (4.8), (4.9). If there exists a constant $c_{1}$, so that $L_{h}$ satisfies

$$I((y_{h,0},z_{h})_{h}) \leq c_{1}(L_{h}(y_{h,0},z_{h})_{h}) + ||B(y_{h,0},z_{h})||_{h}$$

then the problem

$$L_{h}(y_{h,0},z_{h}) = \mathcal{A}_{h}, \mathcal{B}(y_{h,0},z_{h}) = \mathcal{B}$$

has a solution $(y_{h,0},z_{h})$ for all vector-grid functions $\phi_{h}$ and all vectors $\mathcal{B}$. There exists a constant $c_{2}$ independent of $\epsilon$ so that $(y_{h,0},z_{h})$ satisfies:

$$I((y_{h,0},z_{h})_{h}) \leq c_{2}(\|\mathcal{A}_{h}\|_{h} + ||\mathcal{B}||)$$

Since the continuous reduced problem has an isolated solution the linearization of

$$F_{h}(y_{h,0},z_{h}), (\mathcal{B})_{h}(y_{h,0},z_{h})$$

satisfies the hypotheses of theorem (4.10). (Here $(\mathcal{B})_{h}(y_{h,0},z_{h})$ is defined analogously to (3.14).) Using theorem (4.10) we can show the existence of a solution of (4.1). Moreover the operator defined by $F_{h}$ and the boundary conditions $\mathcal{B}$ is stable in the sense of Keller (1975) in a neighborhood of $(y_{h,0},z_{h})$ of (4.1). The neighborhood and the stability constant are independent of $\epsilon$. So we have

(4.15) Theorem: If the continuous problem (1.1) satisfies the hypotheses in chapter 2 then there exist some positive constants $c_{0}, N_{0}$ so that (4.1) is solvable for all $\epsilon < c_{0}$ and $N < N_{0}$. The solution $(y_{h,0},z_{h})$ of (4.1) satisfies

$$I((y_{h,0},z_{h})_{h}) - (y_{h,0},z_{h})_{h} \leq c_{3}\epsilon$$

for some positive constant $c_{3}$ independent of $\epsilon$. Here $(y_{h,0},z_{h})$ are defined as in
The stability-inequality

\[ \| (y_{h}^{1}, x_{h}^{1}) - (y_{h}^{2}, x_{h}^{2}) \|_{h} \leq c_{4} \| \delta_{h} \|_{h} + \| \delta_{h}^{2} \|_{h} + \| \delta_{h}^{2} \|_{h} \]

is satisfied for all \((y_{h}^{1}, x_{h}^{1}), i = 1, 2\) satisfying \(\| (y_{h}^{0}, x_{h}^{0}) \|_{h} \leq \varepsilon\) and all \(N < N_{0}, \varepsilon < \varepsilon_{0}\).

Since \((y_{h}^{0}, x_{h}^{0})\) in (4.3) is an \(O(\varepsilon)\) approximation of \((y_{h}^{0}, x_{h}^{0})\) the global error \((y_{h}^{0}, x_{h}^{0}) - (y_{h}^{0}, x_{h}^{0})\) is up to terms of order \(O(\varepsilon)\) given by the difference between \((y_{h}^{0}, x_{h}^{0})\) and the \(O(1)\) terms in the asymptotic expansion (2.1), (2.2) of \((y(t, \varepsilon), z(t, \varepsilon))\):

\[ (y(t, \varepsilon), z(t, \varepsilon)) \]

Then there exist positive constants \(c\) and \(\kappa\) so that

\[ \| y(t_{ij}, \varepsilon), z(t_{ij}, \varepsilon) - (y_{h}^{0}, x_{h}^{0}) \|_{h} \leq c_{6}(N^{i+1} + \varepsilon) \]

holds for all \(N < N_{0}\) and \(\varepsilon < \varepsilon_{0}\) where \(\gamma_{-}\) and \(\gamma_{+}\) are the (in modulus) smallest real parts of \(f_{-}(0,2(0))\) and \(f_{+}(1,2(1))\).

From Theorem (4.18) follows immediately:

(4.20) Theorem: Let \((y(t, \varepsilon), z(t, \varepsilon))\) and \((y_{h}^{0}, x_{h}^{0})\) be the solutions of (1.1) and (3.1). Then there exist positive constants \(c_{6}, \gamma\) and \(\kappa\) so that

\[ \| y(t_{ij}, \varepsilon), z(t_{ij}, \varepsilon) - (y_{h}^{0}, x_{h}^{0}) \|_{h} \leq c_{6}(N^{i+1} + \varepsilon) \]

holds for all \(\varepsilon < \varepsilon_{0}\) and all grids \(T_{N}\) with \(N < N_{0}\).
Away from the boundary we will have an approximation of order $O(h^{q+1})$ for an arbitrary (so for instance also uniform) mesh. So in this case we have the order of convergence which we would obtain by solving the reduced problem plus a term of order $O(\varepsilon)$. In that sense our method is equivalent to solving only the reduced problem except that we need not compute the boundary conditions of the reduced problem. We now consider the case where we want a better approximation than the one we obtain by using an arbitrary mesh. This is of some interest if either $\varepsilon$ is rather big (let's say $\varepsilon = 10^{-2}$) or if we are interested in the solution inside the layers.

If we can show now that $(y(t_h, \varepsilon), z(t_h, \varepsilon))$ (the restriction of the exact solution $(y(t, \varepsilon), z(t, \varepsilon))$ on the mesh $T_H$) lies within the ball $K_{y_h, z_h}$, theorem (4.15) tells us that we only have to control the local discretization error to obtain a uniform approximation of the solution. In other words if we choose the mesh $T_H$ so that

$$h(y(t_h, \varepsilon), z(t_h, \varepsilon))_h < \delta$$

holds we have $h(y, z_h) = (y(t_h, \varepsilon), z(t_h, \varepsilon)) = O(\delta)$.

(4.22) Lemma: Let the local discretization error $(\hat{y}_h^j, \hat{z}_h^j)$ be defined by

$$(4.22) \quad (\hat{y}_h^j, \hat{z}_h^j) = P_{t_h}(y(t_h, \varepsilon), z(t_h, \varepsilon))$$

where $(y(t_h, \varepsilon), z(t_h, \varepsilon))$ denotes the restriction of the solution of (1.1) to the grid $T_H$. Then there exists a positive constant $c_7$ such that

$$h(y(t_h, \varepsilon), z(t_h, \varepsilon))_h < c_7(h_{t_h}^q + h_{t_h}^{q+1} + \frac{h_{t_h}^q}{\varepsilon} \exp(-\frac{\lambda \varepsilon}{q \varepsilon}))$$

$$+ \frac{h_{t_h}^{q+1}}{\varepsilon} \exp(-\frac{\lambda \varepsilon}{(q+1) \varepsilon}) + \frac{h_{t_h}^q}{\varepsilon} \exp(\frac{\lambda \varepsilon}{q \varepsilon}) + \frac{h_{t_h}^{q+1}}{\varepsilon} \exp(\frac{\lambda \varepsilon}{(q+1) \varepsilon})$$

$$+ \exp(\lambda \frac{\varepsilon}{(q+1) \varepsilon})$$

holds for $j = O(1)q, i = O(1)q+1$ and some $t_{i+1} - t_i = t_{i+1} - t_i$.

So if we want to guarantee that $h(y(t_h, \varepsilon), z(t_h, \varepsilon))_h = O(\delta)$ holds for a certain desired accuracy $\delta$ we have to choose the mesh $T_H$ according to
If we choose \( T_H \) according to (3.24) we can show that
\[ (y_h, z_h) - (y(t_h, \varepsilon), z(t_h, \varepsilon)) = O(\varepsilon^6) \]
holds. Therefore \((y, z)\) lies within the ball with radius \( \rho \) and center \((y_0, z_0)\)
where \( F \) is stable (see Theorem (4.15)) if only \( \varepsilon \) and \( \delta \) are sufficiently small.

Applying Theorem (4.15) gives

(4.25) Theorem: If \( T_H \) is chosen according to (4.24) then there exist positive constants
\( c_q, c_0 \) and \( \delta_0 \) so that
\[ (y_h, z_h) - (y(t_h, \varepsilon), z(t_h, \varepsilon)) \leq c_0 \delta \]
holds for any \( \delta < \delta_0 \) and \( \varepsilon < c_0 \).

To construct the mesh \( T_H \) we proceed as it follows:

Step 1: Starting with \( h_0 \) we choose \( h_1 \) according to
\[ h_1 \leq \min \left( c_0^q \exp \left( \frac{\lambda_1 t_1}{q \varepsilon} \right) \right) \]
and
\[ \frac{1}{c_0^q} \exp \left( \frac{\lambda_1 t_1}{q \varepsilon} \right) \] as long as \( h_1 < \delta^{q+1} \) holds. (For a reasonable \( \delta > \varepsilon^{q+1} \)
\[ \frac{1}{\delta^{q+1}} < \frac{1}{c_0^q} \] \( \varepsilon^q \) holds.)

Step 2: Starting with \( h_{n-1} \) we choose \( h_n \) according
\[ h_n \leq \min \left( c_0^q \exp \left( \frac{-\lambda_n (t_{n+1}-1)}{q \varepsilon} \right), c_0^{q+1} \exp \left( \frac{-\lambda_n (t_{n+1}-1)}{(q+1) \varepsilon} \right) \right) \]
as long as \( h_n < \delta^{q+1} \) holds.

Step 3: For the rest of the interval choose \( h_i < \delta^{q+1} \).

Therefore it is necessary to know the constant \( \lambda_0 \) and \( \lambda_q \) which are the smallest
moduli of the real parts of \( f_i^-(0, z(0)) \) and \( f_i^+(1, z(1)) \). These can either be computed by
solving the problem on a uniform mesh first or estimates for them can be obtained, if one uses a continuation method.

An immediately arising question is: How big is the additional amount of labour inside the layers? That means: How many grid-points do we need in steps 1 and 2? What one would like to have is an estimate for the number of grid-points, which is independent of \( \varepsilon \). Unfortunately the number of grid-points required to achieve a certain accuracy \( \delta \) tends to infinity as \( \varepsilon \) goes to zero, but the growth is so slow that for reasonable ranges of \( \varepsilon \) and \( \delta \) it appears to be independent of \( \varepsilon \). More precisely we have

\[ (4.27) \text{Lemma:} \text{ Let } N(\delta, \lambda, q, \varepsilon) \text{ denote the number of grid-points produced by step 1 (or 2) (so } \lambda \text{ either equals } \lambda_- \text{ or } \lambda_+ \text{). Then we have} \]

\[ (4.28) \quad |N(\delta, \lambda, q, \varepsilon_1) - N(\delta, \lambda, q, \varepsilon_2)| < 3 \]

for \( \delta, \lambda, q \) and \( \varepsilon_1 \) and \( \varepsilon_2 \) satisfying

\[
\frac{1}{1q^2} > 10^{-3}, \quad 10^{-10} \leq \varepsilon_i \leq 10^{-2}, \quad i = 1, 2 .
\]

Furthermore

\[ (4.29) \quad N(\delta, \lambda, q, \varepsilon) \leq \text{const} \frac{1}{\delta \lambda} q^2 \ln(\ln(\frac{1}{\varepsilon})) \]

holds.

\[ (4.30) \text{Remark: The estimate } (4.29) \text{ is not sharp, but together with } (4.28) \text{ it shows, that for reasonable choices of } \varepsilon \text{ and } \delta \text{ the amount of additionally needed grid-points is of the same order or magnitude as a } q \text{-th order collocation method would need for an unperturbed problem.} \]

5. Numerical Example

We now illustrate our results with a third order quasilinear model problem where we know the exact solution. For the computations a modified version of the packages LOBATO and SOLVEBLOC (see de Boor-Weiss (1976)) has been used. We consider the system
$$ey_1' = -(x-ey_1+e^t+5e^t)(y_1-ey_2-e^t+5e^t)$$

$$ey_2' = (x-ey_1+e^t+5e^t)(y_2-5e^t)+5e^t$$

$$y' = -e^t(y_1-ey_2-e^t+5e^t)$$

(5.1)

$$y_1(0,ε) + y_2(0,ε) + z(0,ε) = 0$$

(5.2)

$$y_1(1,ε) + y_2(1,ε) + z(1,ε) - 7ε - 1 - ε = 0$$

$$y_1(1,ε) + z(1,ε) = 0$$

Here the matrix \( f_1(t,s) \) is of the form

(5.3)

$$f_1(t,s) = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$$

and we have the reduced problem

(5.4)

$$0 = \ddot{y}_1 + z \dot{y}_1, \quad 0 = \ddot{y}_2 + 5z \dot{y}_2, \quad \dot{z} = z - e^t(y_1 - e^t)$$

(5.5)

$$\ddot{z}(1) + \dot{y}_2(1) = 2e$$

which has the solution \( y_1 = e^t, y_2 = 5e^t, z = e^t \).

(5.1), (5.2) has the solution

$$y_1 = e^t + \exp\left(\frac{1-t}{ε}\right) + ε \exp\left(\frac{t-1}{ε}\right)$$

(5.6)

$$y_2 = 5e^t + \exp\left(\frac{1-t}{ε}\right)$$

$$z = e^t + ε \exp\left(\frac{1-t}{ε}\right)$$

We solved (5.1), (5.2) using the mesh selecting strategy (4.8). We varied the desired accuracy \( δ \) by halving \( δ^{q+1} \) starting with \( δ^1 = \frac{1}{2} \). Thus the maximal error should decrease like \( \frac{1}{2^q} \). We varied \( ε \) from \( 10^{-2} \) to \( 10^{-10} \) in steps of \( 10^{-2} \). By varying \( ε \) for fixed \( δ \) and \( q \), the norm of the errors differed by less than 1%. 

-18-
Maximal errors of:

<table>
<thead>
<tr>
<th>1/q^2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>.25</td>
<td>.11</td>
<td>.81E-2</td>
<td>.64E-3</td>
<td>.32E-4</td>
</tr>
<tr>
<td>.13</td>
<td>.63E-1</td>
<td>.21E-2</td>
<td>.81E-4</td>
<td>.29E-5</td>
</tr>
<tr>
<td>.63E-1</td>
<td>.35E-1</td>
<td>.54E-3</td>
<td>.1E-4</td>
<td>.12E-6</td>
</tr>
<tr>
<td>.18E-1</td>
<td>.14E-3</td>
<td>.13E-5</td>
<td>.13E-5</td>
<td>.73E-8</td>
</tr>
</tbody>
</table>

Note that the error-constants are decreasing for increasing q. (So for instance for q = 4 the error constant is $0(10^{-2})$). To achieve the accuracy the following amounts of meshpoints have been required:

Required meshpoints:

<table>
<thead>
<tr>
<th>1/q^2</th>
<th>q</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>.25</td>
<td></td>
<td>17</td>
<td>25</td>
<td>33</td>
<td>41</td>
</tr>
<tr>
<td>.13</td>
<td></td>
<td>26</td>
<td>43</td>
<td>50</td>
<td>74</td>
</tr>
<tr>
<td>.63E-1</td>
<td></td>
<td>44</td>
<td>77</td>
<td>109</td>
<td>140</td>
</tr>
<tr>
<td>.31E-1</td>
<td></td>
<td>78</td>
<td>144</td>
<td>208</td>
<td>272</td>
</tr>
</tbody>
</table>

So for a fixed q the number of required meshpoints grows like $\delta^4$. Varying $\epsilon$ from $10^{-2}$ to $10^{-6}$ the amount of grid-points for fixed $q$ and $\delta$ varied only by $\pm 1$.

Now we solve (4.1), (4.2) using a uniform mesh ($h_i = H, i = 0(1)N-1$). Because of the theory in the preceding chapter we expect an error of order $(N^{-1}+\epsilon)$ away from the boundary layers. In the next table the norm of the error at $t = 0.5$ is listed for $q = 3$. 

-19-


<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>.1</th>
<th>.5\varepsilon - 1</th>
<th>.25\varepsilon - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1\varepsilon - 2</td>
<td>.5\varepsilon - 5</td>
<td>.55\varepsilon - 3</td>
<td>.51\varepsilon - 3</td>
</tr>
<tr>
<td>.1\varepsilon - 4</td>
<td>6\varepsilon - 5</td>
<td>6\varepsilon - 5</td>
<td>6\varepsilon - 5</td>
</tr>
<tr>
<td>.1\varepsilon - 6</td>
<td>4\varepsilon - 7</td>
<td>5\varepsilon - 7</td>
<td>57\varepsilon - 7</td>
</tr>
</tbody>
</table>

So we see that the error is of order \( \varepsilon \) independent of \( H \) except for \( \varepsilon = .1\varepsilon - 6 \) and \( H = .1 \) where \( c\varepsilon q^{q + 1} > \varepsilon \) holds. (c is the error constant of our method for \( q = 3 \).)

6. Proofs:

To prove Theorem (4.10) we first need some auxiliary results. We start with scalar constant coefficient problems and estimate the growth function of the method we use for the component \( y^* \) in (1.18):

6.1 Lemma: Given the complex numbers \( x_0 \) and \( z \) and the vector \( a = (a_1, \ldots, a_q) \) the solution \( x = (x_1, \ldots, x_q) \) of the system

\[
x_j = x_0 + z \sum_{k=1}^q a_j w_j x_k + a_j , \quad j = 1(1)q
\]

(6.2)

satisfies

\[
x_j = y_j(z)x_0 + \beta_j(z)a .
\]

(6.3)

\( \beta_j \) are vectors of dimension \( q \). \( y_j \) and the components of \( \beta_j \) are rational functions of \( z \). For \( q = 1(1)10 \) there exist angles \( \Theta_q \) satisfying \( 0 < \Theta_q < \frac{\pi}{2} \) and positive constants \( c_q \) such that

\[
|y_j(z)|, |1s_j(z)|, |s\gamma_j(z)|, |s\beta_j(z)| < c_q
\]

holds for \( z \) in \( S_q = \{z|\text{Re}z < 0, |\text{Im}(\text{Re}z)^{-1}| < \tan \Theta_q \} \) and \( j = 1(1)q, y_q \) satisfies \( |y_q| < 1 \) in \( S_q \) and \( y_q(0) = 1 \).

Proof: (5.2) can be written in the form

\[
x = -[1]^* x_0 + z\hat{\Omega}^* x + a
\]

where \( [1]^* \) denotes the vector \( (1, \ldots, 1)^T \) and \( \hat{\Omega}^* \) the \( q \times q \) matrix \( (w_{1j,j}^*, j=1,q) \). If
we solve this system by Cramer's rule and look at the involved determinants as functions of \( z \) we can write the solution \( (x_1, \ldots, x_n)^T \) in the form (6.2). The \( \gamma_j \) and \( \delta_j \) turn out to be rational functions where the degree of the denumerator-polynomial is \( q - 1 \) and the degree of the denominator polynomial is \( q \). Therefore they behave like \( z^{-1} \) at \( z = \infty \).

The growth functions \( \gamma_q \) have been analyzed numerically by Uberhuber (1979) and are less then 1 in modulus within the sector \( S_q \).

Lemma: Let all eigenvalues \( \lambda \) of the \( n \times n \) matrices \( A_h \) lie within a compact subset of the interior of \( S_q \) and satisfy \( \Re \lambda < -\bar{\lambda} < 0 \) for a positive constant \( \bar{\lambda} \). Let

\[
\text{Id} A_h \text{Id} < c_{11} \text{ hold for a positive constant } c_{11}. \]

Then the solution \( y_h \) of

\[
\begin{align*}
D_h y_h &= \gamma_h (A_h y_h)
\end{align*}
\]

(6.16)

\[
y_{h, 0} \leq c_{12} (1 + |f_h|)
\]

for some positive constant \( c_{12} \) independent of \( h \). (Here \( y_h \) and \( f_h \) are vector grid functions of dimension \( n \).)

Proof: We rewrite (6.16) in

\[
\varepsilon (D_h y_h)_{ij} = A_{ij} (x_h y_h)_{ij} + \gamma_h (D_h (A_h y_h))_{ij} + f_{ij} - (1)q, \ i = 0(1)n-1 .
\]

Since \( D_h A_h \) is bounded we get, by using Lemma (6.1)

\[
y_{1+1, 0} = \varepsilon \left( A_{10} y_h \right) y_{1+1, 0} + \frac{1}{\varepsilon} \left( \gamma_h (A_{10} y_h) + f_{10} \right) - 1(1)q, \ i = 0(1)n-1 .
\]

Here the rational functions \( \gamma_q \) and \( \delta_q \) are applied to the nonsingular matrix \( A_{10} \) in the usual manner i.e.

\[
\gamma_q (A_{10}) = \int \gamma_q (A_{10}) dA
\]

where \( \Gamma \) is a closed curve in the complex domain and all eigenvalues of \( A_{10} \) lie within \( \Gamma \). (See c.f. Dunford Schwartz (1957).)
\[ l y_{ij} l < l o l + \text{const}(\sum_{j=0}^{N} \sum_{k=-j}^{j} \left(1 - \frac{\xi}{c}\right)^{-1} h_i \frac{1}{\varepsilon}) \]

(6.17)

by estimating \( \left(1 - \frac{\xi}{c}\right)^{-1} \frac{1}{\varepsilon} \) with some suitable constant \( \varepsilon \) we get

(6.18)

Since the integral is bounded and the other sum in (6.17) is geometric we get the desired estimate.

(6.19) Lemma: Let \( A_h \) be a \( n \times n \) matrix and \( \hat{f}_h \) and \( \hat{f}_h \) be \( n \)-dimensional vector grid functions. Let \( A_h \) satisfy the conditions for the matrices \( A_h \) in Lemma (6.15). We assume that

\[ d_A A_h h < c_{13}, \quad d_{\hat{f}_h h} \hat{f}_h < c_{13} \]

(6.20)

holds for some positive constants \( c_{13} \), \( c_{14} \) and \( \varepsilon \). Then the solution \( y_h \) of

(6.21)

\[ d_{\hat{f}_h h} \hat{f}_h h \hat{f}_h + E = 0, \quad y_0 = \alpha \]

satisfies

(6.22)

for some positive constants \( c_{15} \) and \( \sigma \) where \( \sigma < \min(\kappa, c_{12}^{-1}) \) holds. (Here \( c_{12} \) is defined as in Lemma (6.15).)

Proof: We define \( y_h \) and \( u_h \) by

(6.23)

\[ d_{\hat{f}_h h} \hat{f}_h h \hat{f}_h, \quad j = 1(1)q, \quad i = 0, 1, \ldots, y_0 = \alpha + \lambda_0^{-1} c_0 \]

(6.24)

\[ u_h = y_h + \hat{f}_h h \hat{f}_h, \quad \text{then } u_h \text{ satisfies} \]

(6.25)

\[ d_{\hat{f}_h h} u_h = \hat{f}_h h \hat{f}_h + d_{\hat{f}_h h} \hat{f}_h \hat{f}_h \]
Because of Lemma (6.15) we obtain that $|v_h|_h = 0(\varepsilon)$ holds. We define $\varphi_h$ by

\begin{equation}
\varphi_{i,j}^h = Y_{i,j}^h \frac{\Omega}{1 + \frac{\epsilon}{r_0}}, \quad j = (1)(q), i = 0, 1, \ldots, v_{00} = \varphi_0^h,
\end{equation}

for some positive $\sigma$ less than $\min(c, \epsilon^{-1})$. $v_h$ satisfies

\begin{equation}
\varphi_{i,j}^h = \varepsilon v_{00} + \Phi_{i,j}^h(IH + \rho_{i,j}^h), \quad \rho_{i,j}^h = \frac{\Omega}{1 + \frac{\epsilon}{r_0}} j = (1)(q), i = 0, 1, \ldots.
\end{equation}

Because of (6.20) $|v_h|_h < c_{14}$ holds. So $v_h$ is bounded and therefore (6.22) holds.

(6.28) Lemma: Let the operator associated with the problem

\begin{equation}
D_h s_h = X_h(\varepsilon C_h + \phi_h), \quad H(s_{00}, \varepsilon)^\alpha = \eta
\end{equation}

have a uniformly bounded inverse i.e. let $|s_h|_h < c_{16}(\alpha H + \|s_{10}\|)$ hold for some constant $c_{16}$. (Here $\varepsilon_h$ and $\Phi_h$ denote vector grid functions of dimension $m$, $\eta$, $\gamma$, and $C_h$ are of appropriate dimensions.) Let the $m$-dimensional vectors $f_h$ satisfy

\begin{equation}
\epsilon_{i,j}^h = \frac{1}{\eta} (1 + \frac{c_{18}^h \varepsilon^{-1}}{\epsilon_0}), \quad j = (1)(q), i = 0(1)N
\end{equation}

for some positive constants $c_{17}$ and $c_{18}$. Then the solution $\varphi_h$ of

\begin{equation}
D_h s_h - \phi_h^0 (\varepsilon C_h + \phi_h) = 0, \quad H(s_{00}, \varepsilon)^\alpha = \Phi_h satisfies
\end{equation}

\begin{equation}
|s_h|_h < c_{19}(\alpha H + \eta)
\end{equation}

for some positive $c_{19}$ independent of $\epsilon$.

(6.36) Lemma: Let $L_{\Phi_h}$ be defined as in (3.3). Let the coefficient matrices $\Phi_{i,j}^h$ satisfy (3.5). Assume that the operator associated with the reduced problem (3.7) has a uniformly bounded inverse, i.e. that there exists a constant $c_{22}$ so that

\begin{equation}
I(y_h, \varepsilon y_h)_{\Phi_h} < c_{22}(H^R(y_h, \varepsilon y_h)_{\Phi_h} + \|s_{10}\|)
\end{equation}

holds for all grids $T_h$. Then there exists a solution $(y_h, s_h)$ of the problem

\begin{equation}
L_{\Phi_h}(y_h, \varepsilon y_h) = f_h, \quad y_0 = \beta_0, \quad \beta_0, \quad \beta_0, \quad s_{10} = \beta_1, \quad s_{00} = \beta_2
\end{equation}

for all $(m+n)$-dimensional vectors $\Phi = (\beta_0, \beta_1, \beta_2)$ and all grid functions $\Phi_h$. $(y_h, s_h)$ satisfies

-23-
for a positive constant $c_{23}$ independent of $\varepsilon$.

Proof: Let the $n \times n$ matrices $T_h$ be defined by

$$

\begin{align*}
\mathbf{c}D_h(T_h)^{\pm} - \mathbf{c}h^{-1}((A_{11})^{\pm}T_h + (A_{12})^{\pm}) = 0, & \quad (T_0)^{\pm} = 0 \\
\mathbf{c}D(T_h)^{\pm} - \mathbf{c}h^{-1}((A_{11})^{\pm}T_h + (A_{12})^{\pm}) = 0, & \quad (T_{N0})^{\pm} = 0.
\end{align*}

(Here $T_h(A_{11}^{\pm}, (A_{12}^{\pm})$ and $T_h(A_{11}^{\pm}, (A_{12}^{\pm})$ denote the first $n_+$ and last $n_-$ rows of $T_h(A_{11}, A_{12})$). Because of Lemma (6.19) $T_h$ satisfies $T_h = -(A_{11})^{-1} 2(A_{12}) + T_h^{\pm} 0(\varepsilon)$ and

$$

\begin{align*}
\mathbf{T}_{1j}^{\pm} & \ll \mathbf{I} (1 + \frac{\varepsilon}{\xi}), \quad j = 1(1)q, \quad \mathbf{T}_{1j}^{\pm} \ll \mathbf{I} (1 + \frac{\varepsilon}{\xi}), \\
& \ll \mathbf{I}(1)q-1, \quad i = 0(1)N-1.
\end{align*}

We define $u_h = y_h - T_h z_h$ and obtain

$$

\begin{align*}
\mathbf{c}D u_h & = \mathbf{c}h^{\pm}((A_{11})^{\pm}u_h) + \mathbf{c}h^{\pm}((A_{22})^{\pm}x_h) + \mathbf{c}h^{-1}(A_{11})^{\pm}T_h + (A_{12})^{\pm}) - \mathbf{c}h^{-1}D x_h
\end{align*}

\begin{align*}
\mathbf{D} z_h & = \mathbf{D}((A_{11})h u_h) + \mathbf{D}((A_{12})h z_h) + \mathbf{D}((A_{11})h T_h + (A_{12})h x_h) - 0
\end{align*}

\begin{align*}
\mathbf{b}_0 & = \mathbf{b}_0, \quad \mathbf{u}_0 = \mathbf{b}_1, \quad \mathbf{b} = \mathbf{b}^{\mp}((A_{11})h z_h, z_h) = \mathbf{b}_2 - \mathbf{b}^{\mp}(T_h, u_h, x_h).
\end{align*}

So for $\varepsilon$ and $N$ sufficiently small (6.42) can be solved by using contraction and (6.39) holds.

With the aid of Lemma (6.36) we now construct the general solution of

$$

L = y_h, z_h = \Phi_h \ i.e. \ a \ solution \ which \ depends \ on \ (n+m) \ free \ parameters. \ We \ start \ with \ the \ homogenous \ problem:

(6.43) Lemma: Let the assumptions of Lemma (6.36) hold. Then there exist solutions $(y_h^{\pm}, z_h^{\pm})$ of the problems

$$

\begin{align*}
L(y_h^{\pm}, z_h^{\pm}) = 0, (y_{00}^{\pm}, z_{00}^{\pm}) = 0, & \quad (y_h^{\pm}, z_h^{\pm}) = 0
\end{align*}

$$

for any $n$ dimensional parameter vector $\eta = (\eta, \tilde{\eta})^\top$. For any constant $\kappa < c_{23}$
(where \( c_{23} \) is defined as in Lemma (6.36) there exists a \( K(\varepsilon) \) so that)

\[
I(y_{i_j}^{q})^{-1} \leq K(\varepsilon) \ln^{-1} \text{IP}(0,1,\varepsilon), \quad j = 1(1)q
\]

(6.45)

\[
I((y_{i_j}^{q})^{+}, x_{i_j}^{q})^{-1} \leq K(\varepsilon) \ln^{-1} \text{Imin}(\varepsilon, \text{IP}(0,1-1,\varepsilon), \quad j = 0(1)q-1
\]

(6.46)

\[
I(y_{i_j}^{q})^{-1} \leq K(\varepsilon) \ln^{-1} \text{IP}(i, N-1, \varepsilon), \quad j = 0, 1, q-1
\]

\[
P(k, \varepsilon) = \prod_{i=k}^{1} \left( 1 + \frac{c_{i_j}}{\varepsilon} \right)
\]

holds independent of \( \varepsilon \).

**Proof:** We define \( u_{i_j}^{q} = (y_{i_j}^{q})^{-1} \text{IP}(0,1,\varepsilon)^{-1}, \quad v_{i_j}^{q} = x_{i_j}^{q} \text{IP}(0,1,\varepsilon)^{-1}, \quad j = 1(1)q, \)

(6.47)

\[
\xi v_{i_j}^{q} = S_{i_j}^{q}((h_{i_j}^{q})^{-1} u_{i_j}^{q} + (h_{i_j}^{q})^{-1} v_{i_j}^{q}) + aw_{10}^{q}
\]

(6.48)

Then \( u_{h}^{q} \) and \( v_{h}^{q} \) satisfy

\[
\begin{align*}
\xi v_{i_j}^{q} & = S_{i_j}^{q}((h_{i_j}^{q})^{-1} u_{h}^{q} + (h_{i_j}^{q})^{-1} v_{h}^{q}) + \varepsilon(1 + \frac{c_{i_j}}{\varepsilon})^{-1} u_{i_j}^{q} + \\
\xi v_{i_j}^{q} & = \varepsilon v_{i_j}^{q} + \xi S_{i_j}^{q}((h_{i_j}^{q})^{-1} u_{h}^{q} + (h_{i_j}^{q})^{-1} v_{h}^{q}) + \varepsilon(1 + \frac{c_{i_j}}{\varepsilon})^{-1} u_{i_j}^{q}.
\end{align*}
\]

(6.49)

\[
\begin{align*}
\xi u_{i_j}^{q} & = S_{i_j}^{q}((h_{i_j}^{q})^{-1} u_{h}^{q} + (h_{i_j}^{q})^{-1} v_{h}^{q}) + \varepsilon(1 + \frac{c_{i_j}}{\varepsilon})^{-1} u_{i_j}^{q} + \\
\xi u_{i_j}^{q} & = \varepsilon u_{i_j}^{q} + \xi S_{i_j}^{q}((h_{i_j}^{q})^{-1} u_{h}^{q} + (h_{i_j}^{q})^{-1} v_{h}^{q}) + \varepsilon(1 + \frac{c_{i_j}}{\varepsilon})^{-1} u_{i_j}^{q}.
\end{align*}
\]

(6.49)

\[
\begin{align*}
u_{h}^{q} = h \ , \quad v_{0}^{q} = 0, \quad v_{h}^{q} = 0
\end{align*}
\]

\[
\begin{align*}
0 & = 1 \ , \quad v_{0}^{q} = 0, \quad v_{h}^{q} = 0
\end{align*}
\]

\[
\begin{align*}
\xi^{1} = (1 + \frac{c_{i_j}}{\varepsilon})^{-1}, \quad j = 1(1)q-1, \quad \xi^{1} = 1, \quad \xi^{1} = 1, \quad j = 1(1)q
\end{align*}
\]

(6.50)

\[
\begin{align*}
\xi^{1} & = 1 + \frac{c_{i_j}}{\varepsilon}, \quad D_{i_j}^{1} = \text{diag}(d_{i_j}^{1}, \ldots, d_{i_j}^{1}^{-1}, (d_{i_j}^{1})^{-1}, \ldots, (d_{i_j}^{1})^{-1})
\end{align*}
\]

(To derive (6.48) the equations \( \xi_{h}^{q} = \xi_{h}^{q} - \xi_{h}^{q}, \xi_{h}^{q} = \xi_{h}^{q} - \xi_{h}^{q}, \quad \xi_{h}^{q} = \xi_{h}^{q} - \xi_{h}^{q}, \quad \xi_{h}^{q} = \xi_{h}^{q} - \xi_{h}^{q}, \) have to be used.) Since we can look at (6.49) as a perturbation of the explicit Euler-scheme the terminal value problem for (6.49) is stable. Thus we obtain (using a perturbation argument for (6.47) and (6.48)) that \( I(u_{h}^{q}, v_{h}^{q}) \leq \text{const} \) holds if \( \varepsilon \) is less \( c_{23}^{-1} \) (where \( c_{23} \) is defined as in Lemma (6.36)). Because of Lemma (6.28)
Ish  holds and therefore \((y^h_1)^+\) is also of order 0(c). Therefore (6.45) holds. (6.46) is proved analogously.

Proof of Theorem (3.10): To complete the construction of the general solution of

\[
L_{ch}(y_h^0,\tau_h^0) = \delta_h,
\]
we first define a solution \((\tilde{y}_h^0,\tilde{\tau}_h^0)\) of the inhomogeneous problem which depends on the \(m\)-dimensional free parameter vector \(\tilde{\eta}\):

\[
(6.52)
L_{ch}(\tilde{y}_h^0,\tilde{\tau}_h^0) = \delta_h,
\]

\(\tilde{y}_0^0 = 0\), \(\tilde{\tau}_0^0 = 0\), \(\delta^\tau(\tilde{y}_h^0,\tilde{\tau}_h^0) = \tilde{\eta}\).

Let \((y_h^0,\tau_h^0), (y_h^p,\tau_h^0), \eta^-, \eta^+\) be defined as in Lemma (5.45). Then \((\tilde{y}_h^0,\tilde{\tau}_h^0)\) is the general solution of \(L_{ch}(y_h^0,\tau_h^0) = \delta_h\). To prove the existence of a solution of (3.6) and the stability of the associated operator we show that the equation, which arises when we insert the general solution in the boundary conditions (3.4) is soluble. We obtain

\[
\begin{align*}
(6.53) \\
\tilde{\eta} + 0(\varepsilon(\eta^-,\eta^+)) = \beta_1
\end{align*}
\]

This system is invertible and the norm of the inverse is bounded uniformly in \(\varepsilon\). So there exists a constant \(c_2\) such that \(1(\eta^-,\eta^+,\eta)\leq c_2\). Therefore \((y_h^0,\tau_h^0) \leq c_2\). \(1(\eta^-,\eta^+)\) holds independent of \(\varepsilon\) and the grid \(T_h^0\).

(6.54) Proof of Theorem (4.15): We start with constructing an \(0(\varepsilon)\) approximation of the solution of (3.1): Let \((\tilde{y}_h^0,\tilde{\tau}_h^0), (y_h^0,\tau_h^0), (y_h^0,\tau_h^0)\) be defined by

\[
(6.55) \\
(\tilde{y}_h^0,\tilde{\tau}_h^0) = 0, \quad L(\tilde{y}_h^0,\tilde{\tau}_h^0) = 0, \quad \delta^\tau(\tilde{y}_h^0,\tilde{\tau}_h^0) = \tilde{y}_h^0 = 0,
\]

\[
\tilde{y}_h^0 = y_h^0 + \tilde{y}_h^0, \quad \tilde{\tau}_h^0 = \tilde{\tau}_h^0,
\]

\[
\tilde{y}_h^0 = -L(\tilde{y}_h^0,\tilde{\tau}_h^0), \quad (y_h^0 = ((y_h^0,\tau_h^0) = 0), \quad \tilde{y}_h^0 = 0),
\]

\[
\tilde{y}_h^0 = -L(\tilde{y}_h^0,\tilde{\tau}_h^0) = 0,
\]

Here \(V\) and \(W\) are defined as in (2.7). Because of (2.4) there exists a solution of

(5.55). Let \(u_h^0\) and \(v_h^0\) be defined by

\[
u_h^0 = y_h^0 - v_h^0, \quad v_h^0 = x_h^0 - x_h^0.
\]

Then \(u_h^0\) and \(v_h^0\)
satisfy
\[ CD_{n}u_{h} = s_{h}^{2}u_{h} + \frac{2}{\delta_{s}} (u_{h})v_{h} + H_{1}(u_{h},v_{h}) + O(\varepsilon^{2}) \]
\[ CD_{n}v_{h} = s_{h}^{2}v_{h} + \frac{2}{\delta_{s}} (u_{h})v_{h} + H_{2}(u_{h},v_{h}) + O(\varepsilon^{2}) \]
\[ CD_{n}w_{h} = s_{h}^{2}w_{h} + \frac{2}{\delta_{s}} (u_{h})v_{h} + H_{3}(u_{h},v_{h}) + O(\varepsilon^{2}) \]
\[ \| (u_{h})^{(n)}(v_{h})^{(n)}(w_{h})^{(n)} + H_{1}(u_{h},v_{h}) + H_{2}(u_{h},v_{h}) + H_{3}(u_{h},v_{h}) \| = 0. \]

Here \( u_{h} \) denotes \((u_{h},v_{h},w_{h})\). \( H_{1}(u_{h},v_{h}) \) satisfies \( H_{1}(u_{h}^{1},v_{h}) = H_{1}(u_{h}^{2},v_{h}) \)
\( c \leq \max (u_{h}^{1}v_{h}^{1})^{1} + (u_{h}^{1}v_{h}^{1})^{1} + H_{1}(u_{h},v_{h})^{1} \), for some constant \( C_{25} \)
independent of \( \varepsilon \). Because of Theorem (3.10) we can now use nonlinear Picard-iteration starting with \((u_{h},v_{h}) = (0,0)\) to show the existence of a solution of (3.1). We obtain
\( l(u_{h},v_{h})^{1} = 0(\varepsilon) \) since the first step of the iteration is of order \( 0(\varepsilon) \). Keller (1975) shows, that if a nonlinear operator has a Lipschitz continuous linearization within a sphere \( K_{0}(y_{0},x_{0}) \), then the operator is stable in a sphere \( K_{0}(y_{0},x_{0}) \) and \( \rho_{1} \) and the stability constant depend only on the Lipschitz constant and the norm of the inverse of the linearization of the operator at \((y_{0},x_{0})\).

(6.57) Proof of Theorem (6.18): Let \( y_{0}(t), z_{0}(t), y_{0}^{(q)}(t), v_{0}^{(q)}(t) \) be defined as in (2.1), (2.2) and let \((y_{h},z_{h})\), \( y_{h} \) be defined as above. Since (5.55) is the discrete counterpart of the reduced problem and we use a method of order \( q+1 \) we obtain
\( l(y_{h}^{0},z_{h}^{0}) - (y_{0}^{(q)}(t), z_{0}^{(q)}(t)) = 0(\varepsilon^{q+1}) \)
by using (2.7) and standard arguments. Because of Lemma (5.19) we obtain for \( H \)
sufficiently small
\( 1 \leq j \leq \text{const} \leq 1 + \frac{\varepsilon}{\varepsilon}, j = 1(1)q, k_{j} \leq \text{const} \times (1 + \frac{\varepsilon}{\varepsilon}), j = 0, q+1. \)

Estimating \( l(y_{h}^{0},z_{h}^{0}) - (y_{0}^{(q)}(t), z_{0}^{(q)}(t)) = 0(\varepsilon^{q+1}) \) gives the result.
(6.60) Proof of Lemma (3.22): Let \((y(t_h,s), z(t_h,s))\) denote the solution of (1.1) restricted on the grid \(T_h\). Then we have

\[ e_{ij} y(t_h,s) = u_i^0 \int_0^1 e y'(t_h^0 + h h') ds. \]

If we interpolate \(y'(t_i^0 + h h')\) at \(u_1, \ldots, u_q\) by a polynomial of degree \(q-1\) (which is integrated exactly by \(S_{ij}^-\)) and use the formula for the remainder term of polynomial interpolation we get

\[ ||y_{ij}^-|| < \text{const} \, h_i \| y^{(q+1)}\|_{(t_i, t_{i+1})}. \]

Similarly we get
\[
||z_{ij}^-|| = 0(h_i^{q+1}) \, ||y^{(q+1)}||_{(t_i, t_{i+1})},
\]

follows from the asymptotic expansion (2.1), (2.2) that \(y^{(q+1)}\), \(z^{(q+2)}\) satisfy

\[ ||y_{ij}^{(q+1)}(t,s), z^{(q+2)}(t,s)|| \leq \text{const} \left(1 + \frac{\exp(-\frac{t}{\varepsilon}) + \exp(+\frac{t}{\varepsilon})}{\varepsilon^{q+1}}\right). \]

(6.64) Proof of Theorem (4.25): Let \(y(t,s)\) be defined by

\[ \hat{e} y = A(t, z_0(t,s)), \hat{y} = \hat{y}(0, s) = y_0^e(0), \hat{y}^{(1)}(t, s) = y_0^e(0) \]

where \(z_0\), \(y_0\), \(v_0\) are defined as in (2.1). As it is easy to show \(\hat{y} \), \(\hat{v} \) and \(\hat{v} \) satisfy

\[ \int_0^t \hat{y} \int_0^1 \hat{v} = 0(c), \int_0^t \hat{v} \int_0^1 \hat{v} = 0(c). \]

Let now \(y_h\) and \(z_h\) be defined as in the proof of Theorem (4.15). Since \(I_h = z_0(t_h, s_h)\) holds and the initial and terminal value problems in (5.55) for \(y_h\) are stable we obtain

\[ \hat{y}_h - \hat{y}(t_h, s_h) \leq \text{const} \left(\int_0^t \hat{y}_h \right)_h + \max(h_i^{q+1}) \| y^{(q+1)}\|_{(t_i, t_{i+1})}, \]

if we choose the mesh according to (3.24). Summarising (6.65) and (6.66) we obtain that \((y_h, \hat{y}_h) = (y(0, s_0), z(0, s_0))\) holds, where \((y_0, s_0)\) is defined as in Theorem (3.18). Therefore \(I(\hat{y}_h, s_h) = 0(\varepsilon + 6)\) holds. So for \(\varepsilon \) and \(\delta\) sufficiently small \((y(t_h, s), z(t_h, s))\) lies within the ball \(R_p(y_h, z_h)\). So we can apply Theorem (3.15) and obtain that \((y_h, \hat{y}_h) = (y(t_h, s), z(t_h, s))\) \(\text{const} \delta \) holds independent of \(\varepsilon\).
Proof of Theorem (4.27): We choose \( h_1 = \frac{1}{c^2} \exp\left(\frac{-\lambda}{q\varepsilon}\right) \) as long as \( h_1 < \frac{1}{q^3} \) holds i.e. as long as \( t_1 < \frac{\ln(1/\varepsilon)}{\ln(1/\varepsilon)} \) holds. By setting \( \tilde{h}_1 = \frac{h_1}{q\varepsilon}, \tilde{t}_1 = \frac{t_1}{q\varepsilon} \) Step 1 in (4.25) becomes

Choose \( \tilde{h}_1 = \frac{1}{c^2} \) as long as \( \tilde{t}_1 < \ln(1/\varepsilon) \) holds. (With \( \alpha = \frac{1}{c^2} \))

Given \( \varepsilon_0 \) and \( \alpha \) assume that we would need \( N \) points, i.e. \( t_N > \frac{1}{\varepsilon_0} \) holds. If we add 1 point \( t_{N+1} > \frac{1}{\varepsilon_0} \exp\left(\frac{-a}{\varepsilon_0}\right) \). Therefore \( N+1 \) is the number of points we would need to solve the same problem with \( \varepsilon_1 = \alpha \varepsilon_0 \). Computing the sequence \( \varepsilon_{n+1} = \varepsilon_n \exp\left(-\frac{a}{\varepsilon_n}\right) \) shows that for \( a > 10^{-3} \) it decreases from \( \varepsilon = 10^{-2} \) to \( \varepsilon = 10^{-18} \) in 3 steps.

To obtain estimate (4.29) we state

\[
h_1 > c^2 \exp(ia)\]

\[
t_1 > c^2 \sum_{j=0}^{i-1} \exp(ja) = c^2 \frac{\exp(ia) - 1}{\exp(a) - 1}.
\]

Therefore

\[
N < \frac{1}{a} \ln\left(\ln\left(\frac{1}{\varepsilon}\right) \frac{\exp(a) - 1}{\exp(a) - 1} + 1\right)
\]

holds.

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References


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**Abstract**
A numerical method for boundary value problems for quasilinear systems of singularly perturbed ordinary differential equations is presented. The method is based on collocation with polynomial splines. The stability properties of the associated difference operator are examined and a stepsize algorithm to achieve a certain over-all accuracy is developed. The number of gridpoints required by the algorithm is estimated.

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