How to Divide a Polygon Fairly

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Abstract

Let $P$ be a simple polygon with $N$ vertices, each assigned a weight $c_i$ ($0 \leq i \leq N$) with $c_i \in \{0, 1\}$. We define the weight $C$ of $P$ as the added weight of its vertices, i.e., $C = c_1 + \ldots + c_N$. Making the assumption that the vertices of $P$ have been sorted along some axis—which can be done in $O(N \log N)$ time—, we prove that it is possible, in $O(N)$ time, to find two vertices $a, b$ in $P$, such that the segment $ab$ lies entirely inside the polygon $P$ and partitions it into two polygons, each with a weight not exceeding $2C/3$. We also give a list of problems which can be solved efficiently with a divide-and-conquer strategy based on that result.
1. Introduction

It is a fact that fast algorithms often owe their efficiency to a handful of mathematical truths that bring to light unsuspected specificity — or singularity — about the problem under consideration. But besides recognizing their mere aptitude at serving computational purposes, one may wish to add a bit of classification into the nature of those subservient truths.

To ease the matter, let us take two examples: proving that a minimum convex decomposition of a polygon could always be found free of interior parts was a major step in deriving a polynomial algorithm for that problem [CH80]. Similarly, establishing the unimodality of the vertex-coordinate function of a convex polygon was instrumental in the setting of logarithmic intersection-algorithms for convex objects [CD80]. There is, however, a notable difference in methodology between these two cases. Whereas the first example presents us with a mathematical fact which seems inherently helpful for deriving any algorithm, the latter brings out a compelling flavor of binary search-like technique which strongly suggests the proper algorithmic treatment.

Another case which witnesses both phenomena is the well-known near-neighbor problem, in which all nearest neighbor pairs in a given set of points are to be computed. One of the most efficient algorithms for this problem relies on an elegant geometric construction known as the Voronoi diagram [SH77]. Although the existence of this diagram was undoubtedly the keystone in the elaboration of the algorithm, it fell short of even suggesting an effective method, since there was no immediate evidence that constructing the Voronoi diagram was to be any easier than computing the neighbor pairs directly. Another mathematical fact, i.e., the decomposability of the Voronoi construction, was indeed needed to fire the final blow and crack the problem. There we should observe that although both facts are geometric in nature, the latter is combinatorial in spirit, as it draws its motivation from the algorithmic, all-purpose, divide-and-conquer technique.

Lipton and Tarjan's planar separator theorem [LT77,LI77] is a notable example of a systematic technique for introducing a computational tool, i.e., divide-and-conquer, into a whole class of related problems, i.e., planar graph problems. Drawing its inspiration from this philosophy, this paper presents a theoretical result on polygon decomposition which can be applied to derive a number of efficient algorithms for geometric problems, in particular, problems of convex decompositions, triangulation, visibility, and internal distance.

2. The Polygon-cutting Theorem

Let \( P \) be a simple\(^1\) polygon with vertices \( v_1, \ldots, v_N \) in clockwise order. Let \( OXY \) be an orthogonal system of

\(^1\)A polygon is said to be simple if only adjacent edges intersect.
reference. Wlog, we can always assume that no two vertices have the same X-coordinate. We define a partial, so-called vertical, order as follows:

We say that "edge $e > edge f$" iff their projections onto the X-axis overlap and, restricting ourselves to the overlapping area, $e$ lies entirely above $f$ (Fig. 1).

[FIGURE 1]

Figure 1: The partial order among the edges of $P$.

Throughout this paper, we will assume that along with a description of the boundary of $P$, provided by a doubly-linked list $L_P$, we have available both a doubly-linked list $L_V$ of all the edges in topological (vertical) order, and a doubly-linked list $L_H$ of all the vertices sorted by X-coordinates [KN73]. The preprocessing involved in setting up these lists requires $O(N \log N)$ time and $O(N)$ space. The decomposition algorithm which we will describe later on runs in linear time, with this preprocessing in hand. Note that it is legitimate to separate both tasks, since in the applications which we will mention, the decomposition algorithm will be called recursively several times, while the preprocessing will be needed initially, once and for all.

The goal of this paper is to prove the following theorem:

**Theorem 1:** The Polygon-cutting Theorem. Let $P$ be a simple polygon with $N$ vertices $v_1, \ldots, v_N$, each assigned a weight $c_i$ ($c_i = 0, 1$). Let $C(P)$ denote the total weight of $P$, defined as the sum $c_1 + \ldots + c_N$, and assume that $C(P) > 2$. With the lists $L_P, L_H, L_V$ in hand, it is possible to find, in $O(N)$ time, a pair of vertices $v_i, v_j$ such that the segment $v_i v_j$ lies entirely inside the polygon $P$ and partitions it into two simple polygons $P_1, P_2$ satisfying:

$$C(P_1) \leq C(P_2) \leq 2C(P)/3$$

The weights of the vertices in $P_1$ and $P_2$ are the same as in $P$, except for $v_i$ and $v_j$, for which we will assume that in both $P_1$ and $P_2$, these weights become 0. This assumption is made only for the sake of simplicity, and other conventions (e.g., keeping the same weights $c_i, c_j$ in both $P_1$ and $P_2$) are indeed acceptable, if we are ready to add a term $+2$ to $2C(P)/3$ in the inequality of Theorem 1. To facilitate our task, we will first prove the theorem with slightly relaxed requirements.

**2.1. An existence theorem**

To begin with, we will prove the existence, not of two vertices, but of two points on the boundary of $P$, satisfying the inequalities of Theorem 1.

**Theorem 2:** Same assumptions as Theorem 1. There exists a pair of points $A, B$ on the boundary of $P$, such that the segment $AB$ is parallel to the Y-axis, lies entirely in the polygon $P$, and partitions it into two simple polygons $P_1, P_2$ satisfying:

$$C(P_1) \leq C(P_2) \leq 2C(P)/3$$
If $A$ and $B$ are not vertices of $P$, for consistency, we assign them a 0-weight. The underlying notion of "equal size" decomposition expressed in the polygon-cutting theorem motivates the introduction of a distance function $d(A,B)$, defined between two points $A,B$ on the boundary of $P$ as the minimum path weight between $A$ and $B$. More precisely, let $v_{j+1}v_j$ (resp. $v_{j+1}$) be the edge of $P$ containing $A$ (resp. $B$). If $A$ (resp. $B$) is a vertex of $P$, it is assumed to be $v_j$ (resp. $v_j$). We introduce the function $h$, defined as follows:

$$h(A,B) = c_{j+1} + c_{j+2} + ... + c_i$$

from which we can define $d(A,B)$:

$$d(A,B) = \min [ h(A,B), C(P) - h(A,B) ]$$

Starting at the edge $v_1v_2$, we label each edge of $P$ recursively, as follows:

$$\lambda(v_1v_2) = c_1$$
$$\lambda(v_{j+1}) = \lambda(v_{j+1}) + c_{j+1}$$

Note that this labeling gives us an alternate way of defining the distance between two boundary points $A,B$:

$$d(A,B) = \min [ |\lambda(v_{j+1}) - \lambda(v_{j+1})|, C(P) - |\lambda(v_{j+1}) - \lambda(v_{j+1})| ]$$

We are now in a position first to prove the existence of the segment $AB$, as defined in Theorem 2, then to describe an efficient method for finding it. As we will see, the first step is not superfluous; it is an essential ingredient in ensuring the correctness of the algorithm.

Choose the leftmost point of $P$ as the starting point of the left-to-right sweep of a vertical segment $S = AB$ ($A$ below $B$). $S$ will always stretch vertically so as to keep its endpoints $A,B$ in permanent contact with the boundary of $P$. It will thus be able to move continuously to the right, until it must either expand (fig.2.1) or split (fig.2.2). At any time during the course of the motion, $S$ will be assigned a value $\Delta = h(A,B)$ to indicate how close it is to being the desired segment. We observe that initially, $\Delta = C(P)$, and that as long as $S$ moves continuously, $\Delta$ decreases monotonously by unit steps. When either situation depicted in Figure 2 arises, we can always write

$$\Delta = h(A,B) = h(A,C) + h(C,B)$$

from which we can derive a decision procedure for redefining $S$.

Starting from the leftmost vertex of $P$, move $S$ from left to right, stretching or shrinking this segment so that it entirely lies in $P$, and its endpoints always lie on the boundary of $P$. As long as the motion of $S$ is continuous, check whether $\Delta > 2C(P)/3$, in which case continue, else stop. When falling in either case of fig.2, reset $S$ to $AC$ if $h(A,C) > h(C,B)$, or to $CB$ otherwise. Note that if $S$ is reset to $CB$ in the case of fig.2.1, the motion must reverse its direction.

\[2\text{All arithmetic on indices is done [mod } N].\]
Relation (1) shows that every discontinuity causes $\Delta$ to decrease by at most half, while otherwise $\Delta$ decreases at most by unit steps, since we have assumed that no two vertices may lie on the same vertical line. Since $\Delta$ eventually vanishes and $C(P)>2$, it must take on some value in the interval $[C(P)/3, 2C(P)/3]$, at which point the procedure will stop and return the desired segment $AB$. This completes the proof of Theorem 2. □

![Figure 2](image)

**Figure 2:** Proving the existence of a separator $S$.

### 2.2. A relaxed version of the polygon-cutting theorem

Unfortunately, Theorem 2 falls short of providing an efficient algorithm for computing $AB$. We can, however, graft to it a binary search-like structure to improve the performance of a naive implementation. The purpose of this section is thus to prove the following result:

**Theorem 3:** Same assumptions as Theorem 1. It is possible, in $O(N)$ time, to find a pair of points $A, B$ on the boundary of $P$, such that the segment $AB$ lies entirely in the polygon $P$, and partitions it into two simple polygons $P_1, P_2$ satisfying:

$$C(P_1) \leq C(P_2) \leq 2C(P)/3$$

The algorithm which we will describe in order to prove Theorem 3 is recursive; it requires $O(N)$ time to cut down the size of the problem by half, therefore its overall performance is linear.

1. Recall that initially, we have available both the vertical topological order ($LV$) of the edges of $P$ and the horizontal order ($LH$) of its vertices. The latter list permits us to determine a median vertical line $L$ in $O(N)$ time, i.e., a line parallel to the $Y$-axis that separates the vertices of $P$ into two sets of size $[C(P)/2]$ and $[C(P)/2]$ respectively. Once again, this is always possible since no two vertices may have the same $X$-coordinate (fig. 3).

2. Compute all the intersection points of $L$ with the boundary of $P$, and sort them by $Y$-coordinates, doing all of this in $O(N)$ time with the list $LV$. Next, form the intersection segments $A_1B_1,...,A_kB_k$ in ascending order (fig. 3).

3. If $M$ is a point lying on an edge $\gamma_i\gamma_{i+1}$ of $P$, distinct from its endpoints, by extension, we define $\lambda(M)$ as $\lambda(\gamma_i\gamma_{i+1})$. Check whether any of the segments $A_iB_i$ satisfies the relation

$$C(P)/3 \leq |\lambda(A_i) - \lambda(B_i)| \leq 2C(P)/3$$

for $i=1,...,k$.

If yes, the corresponding segment $A_iB_i$ can be chosen as $AB$, and the procedure can stop. Otherwise, go to 4.

4. Transform the list $LP$ as follows. For every $A_iB_i$ ($i=1,...,k$) in turn, if $|\lambda(A_i) - \lambda(B_i)| < C(P)/3$, start at $A_i$ or $B_i$ (say, $A_i$) whichever has the smaller label $\lambda(A_i)$ or $\lambda(B_i)$, and proceed to traverse $LP$ in clockwise order, until $B_i$ is reached, deleting all the vertices and links visited. Finally, close the chain by inserting $A_i$ and $B_i$ into $LP$, setting a double link between the two points. If $|\lambda(A_i) - \lambda(B_i)| > 2C(P)/3$, perform the same sequence of operations, now starting at the point $A_i$. 


or \( B_i \) with the larger label \( \lambda(A_i) \) or \( \lambda(B_i) \).

5. Using the new description \( LP \) of the pruned polygon, update the lists \( LV \) and \( LH \) by deleting the edges and vertices no longer existing in \( LP \).

6. Apply the algorithm recursively until termination.

[FIGURE 3]

Figure 3: The recursive algorithm for computing the separator \( S \).

A few words of explanation may be needed to justify the algorithm. To begin with, observe that once the intersection points are available in vertical order, setting up the list of segments \( A_iB_i \) is straightforward, since the list of consecutive segments formed by the intersection points alternates inside and outside \( P \). Also, it is easy to verify that the deletion of edges is accompanied by the setting of links which, in particular, ensures that no edge is visited more than twice. It may be the case that handling a segment \( A_iB_i \) may also handle several others at the same time. For example, see the effect in fig.4 of handling \( A_3B_3 \) before \( A_4B_4 \), or vice-versa. In this figure, the dashed area is to be removed from \( P \). Finally, we note that deletions are indeed permissible since any part of \( P \) cut off by \( A_iB_i \) weighs at most \( d(A_iB_i) < C(P)/3 \). Thus any vertical segment \( AB \) in that part has a distance \( d(AB) < C(P)/3 \), and may therefore be dismissed. Since, when iterating down the recursion level, all the segments \( A_iB_i \) are being cut off and their attached part removed, there remains only one connected part which lies entirely on one side of the median line, and to which, therefore, \( P \) can be reset before calling the algorithm recursively. Of course, the labeling \( \lambda \) remains unchanged. The correctness of the algorithm follows from Lemma 2 and from the fact that we remove only non-candidates. The reason for choosing a median line to operate the cut-offs is now apparent. Since all the vertices on one side of the line are bound to disappear, the recursion will be invoked on a problem of size reduced by at least half. This completes the proof of Theorem 3.

[FIGURE 4]

Figure 4: Pruning the polygon \( P \).

2.3. Completing the proof of the polygon-cutting theorem

We may now turn our attention back to Theorem 1. Let \( v_i v_{i+1} \) and \( v_j v_{j+1} \) be the edges of \( P \) that contain the points \( A \) and \( B \) of Theorem 3, respectively. To prove the desired result, one may be tempted to slide \( A \) and \( B \) towards the endpoints of \( v_i v_{i+1} \) and \( v_j v_{j+1} \) respectively, until one of the configurations \( v_i v_j v_{i+1} v_{j+1} \), or \( v_{i+1} v_{j+1} \) has been reached. Unfortunately, obstacles may prevent this from ever happening (fig.5), so our next step will be to take a closer look at these possible obstacles.

Since the quadrilateral \( v_i v_{i+1} v_j v_{j+1} \) contains the segment \( AB \), it is a simple polygon, and \( AB \) partitions it
into two polygons $Q_1 = Av_{i+1} v_j BA$ and $Q_2 = AB v_{j+1} v_i A$. As a corollary of the Jordan Curve Theorem [HS55], it is easy to show that the only obstacles encountered in $Q_1$ (resp. $Q_2$) are vertices in the set \{$v_{i+1}, v_{i+2}, \ldots, v_j$\} (resp. \{$v_{j+1}, v_{j+2}, \ldots, v_i$\}). Moreover, the segment $S = AB$ can encounter only vertices on the convex hull $H_1$ (resp. $H_2$) of the vertices of $P$ lying inside $Q_1$ (resp. $Q_2$) (fig.5), as is shown in following result.

Lemma 4: The segment $AB$ intersects any edge of $P$ (outside of $A$ or $B$) if and only if it intersects the boundary of either $H_1$ or $H_2$.

Proof: Since $H_1$ (resp. $H_2$) lies entirely inside $Q_1$ (resp. $Q_2$), $AB$ intersects any edge of $P$ lying in $Q$ iff the infinite line passing through $AB$ does, hence iff $AB$ lies outside of $H_1$ and $H_2$. □

The next task is to compute the convex hulls $H_1$ and $H_2$. We only give the details of the algorithm for $H_1$, the other case being strictly similar. Since we cannot afford to use a standard $O(M \log N)$ algorithm to simply compute the convex hull of the vertices of $P$ in $Q_0$, we must exploit the fact that these vertices lie on a polygonal line in order to achieve linear time.

[FIGURE 5]

Figure 5: Defining the domain of safety for $AB$.

To begin with, let us give an informal description of the algorithm. The goal is, in a first stage, to produce a polygon $K_1$ which lies entirely in $Q_1$, and whose convex hull is exactly $H_1$. $K_1$ consists essentially of polygonal chains $^4$ $L_i$ made of consecutive edges from the set:

$$ L = \{ v_{i+1} v_{i+2}, v_{i+2} v_{i+3}, \ldots, v_{j-1} v_j \} $$

Each chain has the property that it lies in $Q_1$ and intersects $v_{i+1} v_j$ in two points, $U_k$ and $V_k$. Moreover, no two segments $U_k V_k$ overlap (fig.6.1). To compute these chains, we must distinguish between two types of edges in $L$. An edge $v_k v_{k+1}$ is said to be entering (resp. exiting) if it intersects $v_{i+1} v_j$ and $v_{k+1}$ lies inside (resp. outside) $Q_1$. The algorithm proceeds as follows:

Wlog, assume that there is at least one vertex $v_{i+2}$ from $v_{i+1}$ to $v_j$ in clockwise order. Traverse $L$ from $v_{i+1} v_{i+2}$ to $v_{j-1} v_j$ stopping at entering and exiting edges and taking the following actions. If the current edge is entering, it may be the endpoint $U_k$ of a new chain $L_k$. To decide of this, look at the next exiting edge in $L$; if it intersects $v_{i+1} v_j$ in a point $V_k$ on the segment $U_k v_j$, we have indeed a new chain $L_k$ from $U_k$ to $V_k$. Otherwise, not only don’t we have a new chain $L_k$, but the chain just visited may enclose previously computed chains, which must then be deleted. For that purpose, we use a stack to hold the pairs $(U_k V_k, L_k)$, so that

$^3$The Jordan Curve Theorem states that a closed curve in the plane partitions the plane into two connected regions: the inside and the outside.

$^4$We define a polygonal chain as any connected sequence of segments such that only consecutive segments may intersect.
deletions may be done efficiently. The algorithm is straightforward, so we may omit the details.

Initially, the stack is empty, and the current edge $e$ is the first entering edge of $L$.

\begin{verbatim}
begin
Let $f$ be the next exiting edge in $L$ following $e$, and let $U, V$ be respectively the intersections of $e$ and $f$ with $v_{i+1}v_j$.

if $V$ lies on $Uv_j$
then
  Let $L_u$ be the chain between $e$ and $f$ in $L$.
  Push $\langle UV, L_u \rangle$ onto stack.
  Go to next entering edge $e'$ in $L$, whose intersection with $v_{i+1}v_j$ lies on $Vv_{j+1}$, then iterate.
else
  Go to next entering edge $e'$ whose intersection $U'$ with $v_{i+1}v_j$ lies on $v_{i+1}V$.
  Pop all pairs $\langle U_k, V_k \rangle$ off the stack as long as $V_k$ lies on $Uv_j$.
  Iterate.
end
\end{verbatim}

[ FIGURE 6 ]
Figure 6: Computing the polygon $K_j$

To prove the correctness of the algorithm, we begin by observing that the intersection of $L$ with $Q_j$ consists of chains whose endpoints lie on $v_{i+1}v_j$ and that $K_1$ consists of exactly all the maximal chains. A chain is said to be maximal if it does not lie in the enclosure of any other chain with $v_{i+1}v_j$. A maximal chain is also characterized by the fact that the segment formed by its endpoints does not lie inside any other such segments.

From the Jordan Curve Theorem [HS55], we derive the fact that a maximal chain from $U$ to $V$, in clockwise order, has its endpoint $V$ lying above $U$ (i.e., on the segment $Uv_j$). In consequence, only the chains which move towards $v_j$ are candidates for being part of $K_1$, which justifies the selection criterion of the algorithm. On the other hand, we can also show that a non-maximal chain which moves towards $v_j$ is necessarily enclosed by another chain moving away from $v_j$. This explains the deletion rule. Finally, the last observation to make is that a chain from $U$ to $V$ which moves away from $v_j$ (i.e., $U$ lies on $Vv_j$) must be enclosed by a subsequent maximal chain, therefore since maximal chains are computed "towards" $v_j$, we may skip directly to the next entering edge in $L$ that intersects $v_{i+1}v_j$ at a point $U'$ below $V$ (i.e., $U'$ lies on $v_{i+1}V$) - see illustration of the various cases in fig.6.1. This completes the proof of correctness, and shows that all the chains $L_k$ may be
computed in sorted order along the segment \( v_{i+1} v_j \), all these computations requiring \( O(N) \) time.

The final step in computing \( K_1 \) is to connect all the chains \( L_k \) together in the order in which they appear in the stack. To do so, we borrow segments from \( v_{i+1} v_j \), as shown in fig. 6.2. We may now apply any standard linear convex hull algorithm for simple polygons [LE80, SH77] in order to obtain \( H_1 \) in \( O(N) \) time.

Assuming that both \( H_1 \) and \( H_2 \) are available, we are now in a position to give an algorithm for finding the two vertices of Theorem 1. The idea is to connect the vertices of \( H_1 \) with those of \( H_2 \), so as to triangulate the polygon \( H^* \) defined as the area between \( H_1 \) and \( H_2 \) containing \( AB \). We claim that at least one of the edges of the triangulation will provide the desired pair of vertices, with the property of Theorem 1. The algorithm proceeds as follows:

Let \( h_1, \ldots, h_p \) and \( k_1, \ldots, k_q \) be the vertices of \( H_1 \) and \( H_2 \), respectively, as we traverse them from \( v_{i+1} v_1 \) to \( v_j v_{j+1} \), i.e., \( h_1 = v_{i+1}, h_p = v_j, k_1 = v_1, k_q = v_{j+1} \). We maintain two pointers, \( h \) on \( H_1 \) and \( k \) on \( H_2 \), moving them from \( h_1 \) to \( h_p \) and \( k_1 \) to \( k_q \), respectively, and computing the triangulation on the fly. Note that, at all times, the segment \( hk \) intersects \( H_1 \) and \( H_2 \) only at its endpoints (fig. 7.1).

![FIGURE 7]

Figure 7: Triangulating the domain of safety.

The simplest way of describing the algorithm is recursively. Initially, \( hk = v_{i+1} v_1 \); the algorithm terminates with \( hk = v_j v_{j+1} \).

Let \( hk = h k \), and consider the quadrilateral \( hh_1 k u_{u+1} k h \). We will show in Lemma 5 that at least one of its diagonals, \( h k_{u+1} \) or \( k h_{1+1} \), connects \( H_1 \) and \( H_2 \) without intersecting these polygons outside of its endpoints, i.e., lies entirely in \( H^* \). Moreover, this diagonal can be found in constant time. We may then determine that diagonal, add it to the triangulation, set \( h k \) to it, and iterate.

The algorithm clearly runs in linear time. Also, the assurance that it effectively produces a triangulation of \( H^* \) comes from the fact that it keeps only edges which lie entirely in \( H^* \), and that the pointers \( h \) and \( k \) pass a vertex only after a diagonal has been assigned to it. Thus there only remains to prove the following lemma:

**Lemma 5:** If the segment \( h k \) connects \( H_1 \) and \( H_2 \) and lies entirely inside \( H^* \), so does one of the diagonals \( h u_{u+1} \) or \( k h_{1+1} \). Moreover, this diagonal can be found in constant time.

**Proof:** Consider the line passing through \( hk \), oriented from \( h \) to \( k \). If a point lies to the right (resp. left) of this line, we will say that it lies below (resp. above) \( hk \). Since \( H_1 \) and \( H_2 \) are convex, at

\[5\text{For consistency, we define } h_{p+1} \text{ and } k_{q+1} \text{ as } v_{j+1} \text{ and } v_j \text{ respectively.}\]
least one of the vertices $v_j$ or $v_{j+1}$ lies above $hk$, therefore it is impossible that both $h_{t+1}$ and $k_{u+1}$ lie below $hk$. Indeed this would involve the existence of at least three intersection points between a line and a convex boundary, leading to a contradiction. If only one segment, say $h_{t+1}$ lies above $hk$, it can be easily determined in constant time, and since in that case, all of $H_1$ lies below $hk$, the diagonal $h_{t+1}k$ does not intersect $H_2$ (nor $H_1$ either) outside of its endpoints, and may thus be chosen as the next segment of the triangulation (fig.7.2). If, on the other hand, both $h_{t+1}$ and $k_{u+1}$ lie above $hk$, the quadrilateral $h_{t+1}k$ is a simple polygon, therefore it contains at least one of its diagonals entirely (fig.7.3), and this diagonal can be found in constant time. Note that, because of its convexity, $H_1$ (resp. $H_2$) lies totally on one side of the line passing through $hh_{t+1}$ (resp. $kk_{u+1}$), therefore the whole quadrilateral, hence the chosen diagonal, lies inside the polygon $H^*$, which completes the proof.

The purpose of triangulating the polygon $H^*$ will become apparent with the following result.

**Lemma 6:** There exists an edge $uv$ in the triangulation of $H^*$ which satisfies the relation: $C(P)/3 < h(u,v) < 2C(P)/3$.

**Proof:** From Theorem 3, we know that $AB$ partitions $P$ into two polygons with weights between $C(P)/3$ and $2C(P)/3$, which gives the relations

$$C(P)/3 < \min \{ h(A,B), h(B,A) \} \leq \max \{ h(A,B), h(B,A) \} < 2C(P)/3$$

As a result, any pair of vertices $a,b$ on $H_1$ (resp. $H_2$), with $b$ following (resp. preceding) $a$ in the list $\{h_1, \ldots, h_p\}$ (resp. $\{k_1, \ldots, k_q\}$) satisfies the relation:

$$h(b,a) < 2C(P)/3$$

(1)

On the other hand, each triangle $abc$ of the triangulation has one side $ab$ on the boundary of either $H_1$ or $H_2$, with the two others $ac$, $bc$ constructed by the triangulation algorithm. Wlog, let $ac$ be the segment of the triangle constructed first (i.e., $ac$ lies below $bc$). We always have

$$h(c,a) = h(b,a) + h(c,b)$$

(2)

Now we can show that a simple upward scan through the faces of the triangulation, i.e., starting at the triangle adjacent to $v_j$ and ending at the triangle adjacent to $v_{j+1}$, will inevitably lead to the desired segment of Theorem 1. To see that, we may obviously assume that none of the edges $ab$ of $H_1$ or $H_2$ satisfies the relations:

$$C(P)/3 < h(b,a) < 2C(P)/3,$$

otherwise, we have achieved our goal.

In that case, Relation (1) shows that for any triangle abc visited, the edge on the boundary of $H^*$, say $ab$, satisfies the stronger inequality

$$h(b,a) < C(P)/3,$$

which, combined with Relation (2), leads to

$$h(c,b) > h(c,a) - C(P)/3.$$

Since $h(v_j,v_{j+1}) = C(P) - c_{j+1}$ and $h(v_{j+1},v_j) = c_{j+1}$, it follows that if $a_1b_1a_2b_2 \ldots$ is the sequence of interior edges visited in the traversal of the triangulation, with the points $a_m$ (resp. $b_m$) on $H_1$ (resp. $H_2$), the sequence $h(a_1b_1), h(a_2b_2), \ldots$ is monotonously decreasing from $C(P) - c_{j+1}$ to $c_{j+1}$ by jumps of at most $C(P)/3$. In consequence, it must take on at least one value in the interval...
\[ C(P)/3, 2C(P)/3, \] which can be chosen as the pair \( u, v. \)

The proof of Theorem 1 is now complete. Computing \( H_1 \) and \( H_2 \) definitely constitutes the most difficult part of the algorithm to implement. We may observe, however, that this overhead will often be unnecessary since, in practice, it may be seldom the case that the segment \( AB \) of Theorem 3 is prevented from sliding towards the endpoints of its supporting edges.

3. Applications to polygon decomposition problems

It is intuitive that the polygon-cutting theorem should lead to efficient methods for partitioning a polygon into convex pieces. We will examine two instances of this problem: in one, what is desired is a partition of the polygon into a small number of convex pieces, while in the other, only a triangulation of the polygon is sought, without consideration of optimality.

3.1. Convex decompositions

Given a simple, non-convex polygon \( P \), find a minimum number of convex, pairwise disjoint polygons, whose union is \( P \).

This problem has been well-studied [CH80,CD79,FS81,GJ78,SC78,SV80,TO80], and several algorithms have been discovered for producing minimal or near-minimal decompositions. Here we consider only decompositions which do not introduce new points, i.e., all the vertices of the polygons are vertices of \( P \).

In connection with the previous section, we will assign to each vertex \( v_i \) of \( P \) a weight \( c_i = 1 \) if its adjacent edges form a reflex angle (in which case, \( v_i \) is called a notch), and a weight \( c_i = 0 \) otherwise. Thus we can apply the polygon-cutting theorem (Theorem 1) iteratively to decompose \( P \) into smaller polygons. Note that since, with our convention, the endpoints of the splitting segment lose their weights, the number of notches \( C_1, C_2 \) of the two parts is each bounded by \( 2C/3 + 2 \), where \( C \) is the number of notches in the original polygon. As a result, we must stop the iteration when the algorithm ceases to reduce the number of notches, i.e., when all the parts have a number of notches satisfying: \( C \leq 2C/3 + 2 \), i.e., \( C \leq 6 \). Finally, to resolve the remaining reflex angles, we consider each of them in turn, proceeding as follows:

Let \( Q \) be the polygon (with at most 6 reflex angles), and \( v \) be the notch exhibiting the reflex angle to be resolved. Let \( L \) (resp. \( R \)) denote the ray (i.e., semi-infinite line) starting at \( v \) in the direction of the edge ending (resp. starting) at \( v \) (fig.8). Compute the intersection(s) of \( L \) (resp. \( R \))

\[ \text{Applying a quality criterion to the triangulation of a polygon is common practice in numerical analysis, where an area distribution or a shape function is often to be optimized. There are many good reasons, however, for making the availability of any triangulation desirable. See [Cl182], for example.} \]
with the boundary of \( Q \), and keep only the intersection point \( A \) (resp. \( B \)) closest to \( v \). If \( A \) and \( B \) lie on different edges, there exists at least one vertex on the part of the boundary of \( P \) between \( A \) and \( B \) which can be joined to \( v \), so as to resolve the reflex angle at \( v \) (fig.8.1). For example, we can choose the vertex \( w \) between \( A \) and \( B \) that minimizes the angle \( \angle(vBvw) \), while keeping it positive. If, on the other hand, \( A \) and \( B \) lie on the same edge \( v_Av_B \) (fig.8.2), we compute the vertex \( a \) of the list \( (v, ... , v_A) \), given in clockwise order, which lies in the triangle \( v_Av_Bv \) and minimizes the angle \( \angle(v_Av_B) \). Similarly, we compute the vertex \( b \) which lies in \( v_Bv_B \) and minimizes the angle \( \angle(v_Bv_B) \). Both of these operations can be executed in linear time. Note that minimizing the angles ensures that both \( w \) and \( vb \) lie entirely in \( Q \). It is also easy to show that the combination of these two segments resolves the reflex angle at \( v \) by splitting \( Q \) into 3 polygons (note that in most cases, \( a \) and \( b \) will be \( v_A \) and \( v_B \), respectively).

![Figure 8](image)

**Figure 8:** Completing the convex decomposition of \( P \).

The decomposition algorithm thus consists of a recursive “cutting” phase which relies on the algorithm given for the polygon-cutting theorem. The recursion stops when the polygon currently examined has fewer than 7 notches, at which point the procedure just described is called upon to finish off the decomposition. We observe that if either the vertex \( v_1 \) or \( v_2 \), say \( v_i \), in Theorem 1 is a notch, i.e., \( c_i = 1 \), \( v_i \) appears in both of the resulting polygons \( P_1 \) and \( P_2 \), but is a notch for at most one of them. Therefore if, by extension, we let \( C(N) \) denote the weight of \( P \) and \( C(N_1) \) (resp. \( C(N_2) \)) be the weight of \( P_1 \) (resp. \( P_2 \)), we can write:

\[
C(N) \leq C(N_1) + C(N_2)
\]

(1)

\[
C(N_1) \leq C(N_2) \leq 2C(N)/3 + 2
\]

(2)

Note that \( C(N_1) \) and \( C(N_2) \) are actual weights, i.e., they count exactly the number of notches in \( P_1 \) and \( P_2 \), as opposed to the weights of \( P_1 \) and \( P_2 \) as defined in Theorem 1, which did not account for the endpoints of the splitting segment. It is easy to see that, in the worst case, we will end up with \( C(N)/6 \) polygons with, each, 6 notches, and the final phase will use 2 cuts for the resolution of each reflex angle. This will result in \( 13C(N)/6 \) convex pieces, which is to be compared with the minimum number of convex pieces, shown to be always greater than or equal to \( C(N)/2 + 1 \) in \([CH80]\).

Next we turn to the complexity of our decomposition algorithm. While it clearly needs \( O(N) \) space, evaluating its run-time \( T(N) \) calls for further investigation. If we neglect the preprocessing phase for the time being, we have the relations

\[
T(N) = T(N_1) + T(N_2) + O(N), \text{ if } C(N) > 6
\]

(3)

\[
N_1 + N_2 = N + 2
\]

(4)

\[
T(N) = O(N), \text{ if } C(N) \leq 6
\]

(5)
Consider the recursion tree, and label each node with the number \( p \) of vertices in the corresponding polygon. At the leaves, we have \( C(P) \leq 6 \), while for their ancestors, \( C(P) > 6 \). Relations (3) and (5) show that up to within a constant factor, \( T(N) \) is equal to the sum of the labels in the tree, while from Relation (4), it follows that if \( L(i) \) counts the sum of all labels at level \( i \), we have \( L(0) = N \) and

\[
L(i) \leq L(i-1) + 2^i
\]

which gives \( L(i) \leq N + 2^{i+1} \). If \( k \) is the height of the tree, we easily find that

\[
L(0) + \ldots + L(k-1) \leq Nk + 2^{k+1},
\]

and since, from (2), we have \( k = O(\log C(N)) = O(\log N) \), including the \( O(M \log N) \) preprocessing in the running time, we can conclude:

Theorem 7: In \( O(M \log N) \) time and with \( O(N) \) space, it is possible to decompose a simple \( N \)-gon \( P \) into fewer than \( 4.333\ldots \times \text{OPT} \) convex pieces, without introducing new vertices, where \( \text{OPT} \) is the minimum number of convex pieces necessary to partition \( P \).

3.2. Triangulation

When all the pieces of a convex decomposition are triangles and no new vertices are introduced, the decomposition is called a triangulation of the polygon. An \( O(M \log N) \) algorithm for computing a triangulation of a simple polygon has been given in [GJ78]. The method requires the somewhat cumbersome use of AVL-trees or similar logarithmic search trees. This can be avoided by using a strategy based on the polygon-cutting theorem.

We may choose to assign a weight \( =1 \) to each vertex of \( P \) and apply the polygon-cutting theorem recursively, until the polygon under consideration has fewer than 7 vertices, at which point it is straightforward to complete the triangulation. We omit the details. An alternative consists of computing a convex decomposition of \( P \) as described in the previous section, then triangulate each convex polygon. To do so, pick any vertex of the polygon and join it to every other.

In both cases, a triangulation of \( P \) can be explicitly computed in \( O(M \log N) \) time, which matches the performance of [GJ78]. We recall that it is yet unknown whether \( k \log N \) is optimal for this problem.

Theorem 8: Using the polygon-cutting theorem, it is possible to triangulate a simple \( N \)-gon in \( O(M \log N) \) time and \( O(N) \) space.

It is shown in [CH82] how the additional information provided by an arbitrary triangulation of a simple polygon is sufficient to derive optimal algorithms for a number of geometric problems. Using the previous result to compute a triangulation of \( P \), it is then possible to determine the area visible from any point inside \( P \) in linear time. Also, internal path problems, i.e., problems involving the computation of the shortest path
between two points inside $P$ can then be solved optimally [CH82].

4. Conclusions and future research

The decomposition principle in geometry expresses the feasibility of local treatments for the solution of general problems on arbitrary figures. The polygon-cutting theorem presented in this paper asserts the applicability of this principle in the case of simple polygons, and by doing so, leads to efficient, simple divide-and-conquer methods for solving a variety of geometric problems.

The merit of this approach lies primarily in the versatility of its applications as well as in the increased efficiency which it affords. The most immediate open question is whether sorting the vertices in preprocessing is indeed required. If not, the algorithm would automatically become linear. In this paper, we have deliberately chosen simplicity and practicality over generality by restricting the weights attached to the vertices to take on the values 0, 1. This was motivated by the fact that this restriction still allowed us to apply divide-and-conquer to an arbitrarily chosen subset of vertices, while adding simplicity to the exposition. The reader will observe, however, that it is straightforward to extend the theorem to a more general weight function.

The applications mentioned in this work are only a few examples among a number of other problems which can benefit from the polygon-cutting theorem. Enlarging the list of applications given here is certainly a worthwhile endeavor.
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Figure 1
Figure 2
Figure 3

\[ \left\lfloor \frac{L}{x} \right\rfloor = 8 \]

\[ \left\lfloor \frac{L}{x} \right\rfloor = 9 \]
Figure 8