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TECHNICAL REPORT

DETERMINING POINTS OF A CIRCULAR REGION
REACHABLE BY JOINTS OF A ROBOT ARM

John Hopcroft
Deborah Joseph
Sue Whitesides*

TR 82-516
October 1982

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On leave from Mathematics Department, Dartmouth College, Hanover NH

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Abstract

An "arm" is a sequence of links whose endpoints are connected consec-
tively by movable joints. The location of the first endpoint is fixed. This
report gives a polynomial time algorithm for determining the regions that each
joint can reach when the arm is restricted to a circular region of the plane.

1. Introduction

In an earlier report [1], we gave a polynomial time algorithm for deter-
mining whether the end of an arm can reach a given point from a given initial
configuration when the arm is restricted to a circular region of the plane.
In this report, we give a polynomial time algorithm for computing the bound-
daries of the regions each joint can reach. The presentation assumes some

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familiarity with the earlier report.

An arm consists of a sequence of $n$ links $L_1, \ldots, L_n$ that are hinged together consecutively at their endpoints. The links may rotate freely about their joints and are allowed to cross over one another. The endpoints are consecutively labeled $A_0, \ldots, A_n$, and the length of $L_i$ is denoted by $l_i$. The location of $A_0$ remains fixed in the plane.

Consider an embedding of the arm inside a circle $C$ with center $O$, radius $r$ and diameter $d$. Let $S_j$ be the set of points to which $A_j$ can be moved, and let $E_j = C \cap S_j$ be the points that $A_j$ can reach on the circle $C$. In the earlier report [1], we showed that $R_j$ consists of at most two arcs of $C$. A point $p$ is on the boundary of $S_j$ if and only if its neighborhood of $p$ contains a point that belongs to $S_j$ and a point inside or on $C$ that does not belong to $S_j$. Normally the only points of $E_j$ that are boundary points are the endpoints of the arcs.

We will show that for each $j$, $E_j \cap S_j$, the boundary of $S_j$ can be covered by a finite number of circles. Furthermore, the number of circles needed for a given joint is bounded above by a constant that does not depend on the arm in any way. The entire process of computing the centers and radii of the circles needed for each joint in turn is $p(n)$, where $p(n)$ is a polynomial in the number of links. It is then straightforward to piece together the actual boundary of each region $S_j$ once a set of covering circles has been found. One begins with the circle $C$ and selects one of the possible boundary circles, tests whether the circle is in fact a boundary (using the algorithm described in [1]), and if so intersects it with the circle $C$ and then proceeds with the next possible boundary circle.
The proof that the boundary of any $S_j$ can be covered by a constant number of circles is quite technical and involves handling many special cases. We have organized the details into several sections whose contents we now briefly outline.

Section 2 considers the possibility that $A_0$ is not the only "fixed" joint. By our definition of "arm", $A_0$ is the only joint that is fastened to the plane. However, it may be that other joints are effectively fixed for geometric reasons. For example, $A_0$ may be located on $C$, and the first link $L_1$ may have length equal to the diameter $d$ of $C$ so that the location of joint $A_1$ cannot change. Theorem 1 shows that there is a joint index $j$ such that the location of $A_j$ can change if, and only if, $j < i/n$. Furthermore, this index can be found quickly. This result takes care of regions consisting of single points and allows us to assume without loss of generality that $A_0$ is the only fixed joint.

Section 3 introduces "basic" circles and "elbows". Basic circles are natural candidates for inclusion in a cover of the boundary of a region $S_j$ by circles. For example, $C$ itself will be called a basic circle. Joints are called "elbows" if they lie off $C$ but are neither straight nor folded. If there is an elbow between $A_i$ and $A_k$, then the location of $A_i$ can often be held fixed while $A_k$ is moved to points in an open set containing the original location of $A_k$. Hence elbows are important because they can tell us that a joint is not on the boundary of its region. The lemmas given in Section 3 are concerned with the occurrence of elbows.

In Section 4, it is shown in Theorem 2 that as long as a joint $A_m$ is not on a "basic" circle, then some joint $A_j$, $0 < j < m$, must be on $C$ if $A_m$ is on the boundary of its region. These joints on $C$ can be thought of as dividing
the part of the arm between $A_0$ and $A_n$ into "segments". Theorem 3 shows that the intermediate segments consist of straight lines of links while the initial and final segments may each have one joint that is folded.

In Section 5, the possibility that the final segment lies on a diagonal of $C$ is handled as a special case. This special case motivates the definition of "supplementary" circles, which are then assumed to belong to the covering sets for the boundaries of the regions. The main result of this section is Theorem 5, which lists the possible configurations for the part of the arm between $A_0$ and $A_n$ when $A_n$ is on the boundary of its region but is located at a point not covered by a basic or supplementary circle.

The main result of the entire paper, the fact that the number of covering circles needed is bounded by a constant independent of the arm, is given in Theorem 6 of Section 6. The proof consists of handling each of the configurations enumerated in Theorem 5. The machinery needed to do this is given in Lemmas 6-8. These Lemmas state that either certain inequalities in the lengths of links hold, or certain configurations cannot place $A_n$ on the boundary of $S_n$. These inequalities restrict the possibilities for the last joint $A_j$ on $C$ before $A_n$. The possible locations for $A_j$ then become centers for covering circles of radii determined by the final segment before $A_n$.

Before we begin the technical sections, we mention two notational matters. First, an expression such as "Joint $A_j$ is between $A_i$ and $A_k$" usually means that $i < j < k$, not that $A_j$ lies between $A_i$ and $A_k$ on the line they determine in the plane. Another example of this usage is "Joint $x$ is beyond or past joint $y." meaning $x$ has the higher index. It should be clear from the context when the words "between" and "beyond" are used in a geometric way. Second, if joints $A_i$ and $A_j$ are connected by a straight line made up of links
Given an arm with $A_0$ fixed, it may be the case that certain other $A_i$ are immovable. For example, if $A_0$ is fixed on $C$ and $l_1 = d$, then certainly the location of $A_1$ cannot change. Another example of an immovable joint beyond $A_0$ is shown in Figure 1.

In general, if $A_0$ lies at a point distance $d_0$ from $C$, $d_0 = 0$ or $r$, and if $l_i$ satisfies $l_i = d - d_0 + l_1 + \ldots + l_{i-1}$, then joints $A_0, \ldots, A_i$ are effectively fixed. In Theorem 1, we prove that the immovable joints form a consecutive sequence from $A_0$ up to some $A^*$, where $k$ can be computed in time on the order of a polynomial in $n$. After proving this we will assume without loss of generality that $k=0$ so that $A_1$ and its successors are free to move.

**Theorem 1:** Suppose an arm is constrained to move inside a circle $C$ and that $z$ is the last fixed joint of the arm, i.e., the last joint whose set of reachable points contains only one element. Then the joints (if any) between $A_0$
and \( x \) must be fixed. Furthermore, \( x \) can be found in \( p(n) \) time, where \( p \) is a polynomial in the number \( n \) of links.

**Proof:** In [1,1], it was shown that an arm can always be moved to a certain "normal form" in which a straight line of links \([A_0, A_1]\) stretches from \( A_0 \) along a radius toward \( C \), where \( l_1 + \ldots + l_i \) is equal to or less than the distance \( d_0 \) from \( A_0 \) to \( C \) but \( l_1 + \ldots + l_{i+1} \) is greater than \( d_0 \). (See Figure 2.)

![Diagram](image)

**Figure 2.** Arms in normal form \( A_{i+1} \) is the first joint for which \( l_1 + \ldots + l_{i+1} \) exceeds \( d_0 \), the distance between \( A_0 \) and \( C \).

The joint \( A_1 \) may or may not lie on \( C \), depending on whether \( l_1 + \ldots + l_i \leq d_0 \) holds with equality. However, in this normal form all joints beyond \( A_1 \) lie on \( C \).
Note that if $l_1 > d_o$, perhaps because $A_o$ is on $C$, then $A_1 = A_o$ and the line of links $[A_o, A_1]$ is empty. Also note that $L_{i+1}$ may lie on the diagonal through $A_o$ even though it is too long to fit on the radius. (See Figure 1.) Obviously, the location of any immovable joint must be consistent with this normal form. This fact is useful in determining which joints are immovable.

If $A_o$ is positioned at 0, the center of $C$, or if $l_1 + \ldots + l_n > d_o$ so that the arm at best can just reach $C$, it is obvious that the last immovable joint $A_k$ is $A_0$ itself. This is because the entire arm can be rotated about $A_0$. We assume from now on that $A_o$ is not at 0 and that $l_1 + \ldots + l_n > d_o$. We also assume that the arm has been moved to normal form.

We begin by showing that if $A_o$ is not the only immovable joint, that is, if $A_k \neq A_o$, then $A_{i+1}$ and all its predecessors must be fixed, where again, $A_i$ is the last joint such that $L_i$ through $L_k$ form a straight segment that lies on the radius through $A_o$ when the arm is in normal form. The cases $A_i = A_o$ and $A_i \neq A_o$ will be treated separately.

If $A_k \neq A_o$ but $A_i = A_o$ so that $A_o$ itself is the last joint on the radius, then either $L_1$ must cross over 0 in reaching from $A_o$ to $A_1$ on $C$ or $L_2$ must be as long as the diameter $d$ of $C$. Otherwise, all joints beyond $A_o$ would be able to move, contradicting the assumption that $A_k \neq A_o$. If $L_1$ crosses over 0 or if $l_2 = d$, then $A_1$ is fixed.

If $A_k \neq A_o$ and $A_i \neq A_o$ so that there is at least one link on the radius, $L_{i+1}$ must cross 0 in reaching from $A_i$ to $A_{i+1}$ on $C$. Otherwise, all joints beyond $A_o$ would be able to move. The fact that $L_{i+1}$ crosses 0 implies that $A_{i+1}$ is fixed on $C$, as it cannot move closer to $A_o$. Also, all joints between $A_o$ and $A_{i+1}$ are fixed because they can move neither farther from $A_o$ nor closer to
This completes the proof that if $A_0 \neq A$, $A_{i+1}$ and all its predecessors are fixed.

Now note that if $A_j$ is any immovable joint lying on $C$, all joints beyond $A_j$ can move unless $l_{j+1}=d$ or $l_{j+2}=d$. Also it either of these conditions is satisfied, then $A_{j+1}$ is immovable. This implies that each successor of $A_{i+1}$ up to $A_k$ must be immovable.

It is easy to see that $A_k$ can be found in polynomial time. \]

3. Some Preliminary Lemmas

We now begin the determination of the boundaries of $S_j$. In light of Theorem 1, let us adopt the notation that $A_0$ is the last fixed joint of the arm.

Given a joint $A_1$ beyond $A_0$, we want to build up a small collection of circles whose union covers the boundary of the set $S_1$ of points $A_1$ can reach. In order to find such a collection quickly and easily, we do not insist that each circle in the collection contain a boundary point.

There are four circles, not necessarily distinct, that it is natural to place in the collection immediately -- namely, the two circles centered at 0 whose radii correspond to the minimum and maximum distances that $A_1$ can move off $C$, and two circles centered at $A_0$ whose radii are obvious bounds for the minimum and maximum distances that $A_1$ can move from $A_0$. These circles, which we call basic circles, are discussed in more detail below.

In [1] it was shown that the minimum and maximum distances that a joint can move from $A_0$ can be computed in $p(n)$ steps, where $p$ is a polynomial in the number of links. Hence the first two basic circles can be found quickly.
(Note that one of these is often \( C \) itself and that the other may consist of the point \( 0 \) considered as a circle of zero radius.) Summing the lengths of the links preceding \( A_m \) gives an upper bound for the maximum distance \( A_m \) can move from \( A_0 \). If \( A_m \) is preceded by a link \( L_j \) that is so long that

\[
\hat{l}_j - \sum_{i=1, i \neq j}^{m} l_i > 0,
\]

then this difference gives a positive lower bound for the minimum distance between \( A_0 \) and \( A_m \); otherwise, 0 is a bound. The two remaining basic circles are defined to be the circles centered at \( A_0 \) with these radii, which are easy to compute.

Before continuing to build up a collection of circles covering the boundary points of \( S_m \), we first need to observe some facts about joints. These are developed in Lemmas 1 through 3 below.

Consider a joint \( A_j \) that does not lie on the circle \( C \). If \( L_j \) and \( L_{j+1} \) form a \( 0^\circ(=360^\circ) \) \( \times \) \( 180^\circ \) angle, \( A_j \) is said to be a fold or a straight joint, respectively. If \( A_j \) does not lie on the circle and is open to any other angle, it is called an elbow. (It is important to note that the definition of an elbow requires that the joint not be on \( C \).) The next lemma gives a simple but fundamental observation about elbows.

**Lemma 1:** Suppose that no joint strictly between \( A_i \) and \( A_k \) lies on circle \( C \) but that some joint \( A_j \) between them is an elbow (\( A_i \) and \( A_k \) may or may not lie on \( C \)). Then the location of \( A_i \) can be held fixed while \( A_k \) is moved to all those points in some open ball centered at \( A_k \) that do not violate the minimum and maximum distances that \( A_k \) can be located from the circle. (See Figures 3a and b.)
Figure 3a. The elbow at $A_i$ enables $A_k$ to reach the points in the shaded area while the location of $A_i$ remains fixed.

Figure 3b. Link $L$ is so long that $A_k$ cannot reach any points inside the dashed circle.

Proof: Note that the distance between $A_i$ and $A_k$ can be both increased and decreased by adjusting the angle at $A_j$. Simultaneously, the entire configuration of links between $A_i$ and $A_k$ can be rotated about $A_i$, provided that the links beyond $A_k$ do not prevent this motion. But [1] showed that the links beyond a given joint never constrain its motion along any path that stays within the minimum and maximum distances that the joint can be located off $C$. $\blacksquare$
Another basic observation is that a fold can sometimes be turned into an elbow.

**Lemma 2:** Suppose that \( u \) and \( v \) are two joints of an arm enclosed in circle \( C \) and that all joints between \( u \) and \( v \) are straight with one exception, \( x \), which is a folded joint not lying on \( C \). If the lines of links \( L_{xu} \) and \( L_{xv} \) from \( x \) to \( u \) and \( x \) to \( v \) are not equal in length and if the longer contains at least two links, then an elbow can be created at \( x \) without changing the locations of \( u \) and \( v \). If the lines \( L_{xu} \) and \( L_{xv} \) (possibly of equal length) each contain at least two links, then again, \( x \) can be turned into an elbow without moving \( u \) and \( v \). (See Figure 4.)

**Proof:** If \(|L_{xu}| = |L_{xv}|\), the second statement is obvious. If \(|L_{xu}| \neq |L_{xv}|\), let the line \( L_{xu} \) be the longer one, and let \( y \) be a joint between \( x \) and \( u \). With the locations of \( u \) and \( v \) fixed, the line \( L_{yu} \) can be rotated about \( u \), forcing \( y \) to move away from \( v \). This can be done because \( y \) can be moved away from \( v \) by opening \( x \), and at the same time, the configuration between \( v \) and \( y \) can be rotated about \( v \) to keep \( y \) on a circle of radius \(|L_{yu}|\) centered at \( u \).

Figure 4. Creating an elbow at \( x \).

A final basic observation is that an elbow can be created from two folds that are joined by a straight line of links unless the line consists of a single "long" link.
Lemma 3: Let u and v be joints of an arm embedded in a circle C. Suppose all joints strictly between u and v are straight with two exceptions, which are folds. Then the locations of u and v can be held fixed while the arm is moved to create an elbow between u and v unless the folds are joined by a single link that is at least as long as the sum of the lengths of all the other links between u and v.

Proof: Let x and y be the folds, and let u, x, y, v be the order of the joints in the arm. First, suppose that \(|ux| \geq |xy|\). (See Figure 5a.)

\[
\begin{align*}
\text{a)} \quad & |ux| \geq |xy| \\
\text{b)} \quad & |ux| < |xy| \quad \text{and} \quad |xy| < |ux| + |yv| \\
\text{c)} \quad & |xy| \geq |ux| + |yv|
\end{align*}
\]

Figure 5. A Pair of folds. An elbow can be created between u and v except in c) when \(|xy|\) is a single link.

Then the locations of u and v can be kept fixed, and also all angles except those at u, x, y, and v can be kept fixed, while \(|yv|\) is rotated about v. This is because y will be moving away from u, which can be accomplished by opening the folded joint x into an elbow. The configuration between u and y can be rotated about u to give y the proper angular displacement with respect to u and v.

Now suppose that \(|ux| < |xy|\) and that \(|xy| < |ux| + |yv|\). (See Figure 5b.) Then while the locations of u and v are held fixed, \(|ux|\) can be
rotated about \( u \), which moves \( x \) away from \( v \) by unfolding \( y \) into an elbow and rotating the configuration between \( v \) and \( x \) about \( v \).

The only remaining possibility is that \( ||xy|| \geq ||ux|| + ||yv|| \), as shown in Figure 5c. Then an elbow can be created unless \( x \) and \( y \) are joined by a single link. \( \Box \)

Corollary: Let \( u \) and \( v \) be joints of an arm embedded in a circle. Suppose all joints strictly between \( u \) and \( v \) are straight joints or folds. If there are three or more folds between \( u \) and \( v \), then the locations of \( u \) and \( v \) can be held fixed while the arm is moved to create an elbow between \( u \) and \( v \).

Proof: Let \( e, f, \) and \( g \) be the first three folds past \( u \) between \( u \) and \( v \). Let \( h \) be the next fold past \( g \) if one exists; otherwise, let \( h \) be \( v \). If an elbow cannot be created between \( u \) and \( v \), then Lemma 3 applied to \( u, e, f, \) and \( g \) shows that \( ||ef|| > ||fg|| \). But Lemma 3 applied to \( e, f, g, \) and \( h \) shows that \( ||fg|| > ||ef|| \), a contradiction. \( \Box \)

A. Segments

Now we are ready to continue studying the boundary of \( S_m \), \( m > 0 \). In this section, we show that when a configuration of the arm places \( A_m \) on the boundary of \( S_m \) but not on a basic circle, the part of the arm between \( A_0 \) and \( A_m \) divides into an initial segment reaching from \( A_0 \) to \( C \), possibly some intermediate segments between joints on \( C \), and a final segment reaching from a joint on \( C \) to \( A_m \). The intermediate segments consist of straight lines of links, but the initial and final segments may each contain a joint that is not straight. The next theorem shows that there must be a joint between \( A_0 \) and \( A_m \) that lies on \( C \).
Theorem 2: If the arm has been moved so that $A_a$ lies on the boundary of $S_a$ but does not lie on a basic circle, then some joint strictly between $A_0$ and $A_a$ must lie on circle $C$.

Proof: Suppose that there is no joint $A_i$, $0<i<m$, that lies on $C$. Since $A_a$ is on the boundary of $S_a$ but not at its minimum or maximum distance from $C$, Lemma 1 implies that there can be no elbows between $A_0$ and $A_a$, and the corollary to Lemma 3 implies that there can be no more than two folds. There cannot be exactly two folds between $A_0$ and $A_a$, since it follows from Lemma 3 that $A_a$ would be as close as possible to $A_0$, contradicting the assumption that $A_a$ is not on a basic circle. There cannot be exactly one fold between $A_0$ and $A_a$ as the longer straight line of links would be either a single link, putting $A_a$ on a basic circle, or a multiple-link line, which according to Lemma 2 allows the formation of an elbow. On the other hand, if all the joints between $A_0$ and $A_a$ were straight, $A_a$ would be on a basic circle. \[ \]

Suppose that the arm has been moved so that $A_a$ is on the boundary of $S_a$ but not on a basic circle. Then $A_a$ is not on $C$ but, according to Theorem 2, we can find some last joint between $A_0$ and $A_a$ that is on $C$. We will say that the links between this last joint and $A_a$ form the final segment of the configuration before $A_a$, or simply, the final segment. Similarly, we will say that the links between $A_0$ and the first joint beyond $A_0$ on $C$ form the initial segment of the configuration. (Here, $A_0$ may or may not be on $C$.)

It is clear from Lemmas 1, 2, and 3 that the final segment is made up of either a straight line of links from a joint on $C$ to $A_a$ or a single link from a joint on $C$ to a fold that is followed by a straight line of links to $A_a$. In either case, the final segment lies along a line. The next lemma will help us to show that if $A_a$ is on the boundary of $S_a$ but not on a basic circle, then
the configuration of the arm has no elbows anywhere before $A_i$ and only two joints that might be folds. Also, it will enable us to treat the possibility that the final segment lies on a diagonal of $C$ as a special case.

**Lemma A:** Suppose that an arm has been moved to a position that places $A_i$ on the boundary of $S_i$, but not on a basic circle. If the final segment before $A_i$ begins at $A_i$ on $C$ and lies on a diagonal of $C$, then $R'$, the arc of $S$ to which $A_i$ belongs, consists of a single point.

**Proof:** Since $A_i$ does not lie on a basic circle, $A_i$ cannot be located at 0. For the same reason, if $A_i$ lies on the opposite side of $O$ from $A_i$, the final segment must be a multiple-link straight line. But then $A_i$ cannot be on the boundary of $S_i$ unless $R'$ is a single point. To see this, note that the distance between $A_i$ and $C$ can be decreased by rotating the final segment about $A_i$ and that it can be increased by bending a joint in the final segment. (See Figure 6a.) Also, $[A_iA_j]$ can be rotated about $A_i$, and $A_i$ can be repositioned along $R'$ to make $A_i$ sweep out arcs of circles. (See Figure 6b.) Taken together, these circular arcs cover an open ball centered at the original position of $A_i$ unless $R'$ is a single point. (It may not be possible to slide $A_i$ along $R'$ without removing $A_i$ from $C$. Nevertheless, $A_i$ can be repositioned anywhere along $R'$, and the orientation of $[A_iA_j]$ can be restored to achieve the same effect. See [1].)

Suppose $A_i$ lies between $A_i$ and 0 and that the final segment is a single link. If $R'$ is not a single point, then $A_i$ can be moved along arcs of circles centered at 0 and having radii at least $r - |A_iA_j|$. (See Figure 7a.) This is because $[A_iA_j]$ can be rotated about $A_i$ to move $A_i$ farther from 0, and then $A_i$ can be repositioned along $R'$ with $[A_iA_j]$ in its new angular position. To contradict the assumption that $A_i$ is on the boundary of $S$, it remains only to
a) \( A_1 \) can be moved to points on the dashed arcs by bending \( A_{i+1} \) and then rotating the configuration between \( A_i \) and \( A_m \) back and forth about \( A_i \) while keeping the location of \( A_i \) fixed.

b) \( \{A_i A_{i+1} \} \) can be rotated about \( A_i \) while the location of \( A_i \) remains the same. Then if \( A_i \) is not at the ccw endpoint of \( R_i' \), \( A_i \) can be repositioned ccw of its initial position while \( \{A_i A_{m} \} \) is held at some fixed angle. In this way, \( A_m \) can reach the points on the dashed arcs.

Figure 6. A final segment that is a multiple-link straight line on the diagonal with \( O \) between \( A_i \) and \( A_m \), where \( R_i' \) extends counterclockwise (and possibly clockwise) from the initial location of \( A_i \).

show that \( A_m \) can reach the neighbors of its initial location that lie outside the circle of radius \( ||A_i A_m|| \) centered at the initial location \( p_i \) of \( A_i \).

To prove that \( A_m \) can reach its neighbors outside the circle centered at
Figure 7. \( A_n \) lies between \( A_i \) and \( 0 \), and the final segment is a single link.

If \( L_1 \) does not lie on the diagonal, or if \( L_1 \) does lie on the diagonal but has length no greater than \( ||A_i A_n|| \), then \( [A_i A_n] \) can be rotated about \( A_i \) while the configuration between \( A_{i-1} \) and \( A_n \) is rotated about \( A_{i-1} \). In this way, \( A_n \) can move along arcs of circles centered at \( A_{i-1} \), as shown in Figures 7b and 7c.

Now we can assume that \( L_1 \) lies on the diagonal and that \( L_1 > ||A_i A_n|| \). Since we are assuming that \( A_n \) is not on a basic circle, \( A_{i-1} \neq A_0 \) and \( L_1 \neq d \).
otherwise, $A_n$ would be as close as possible to $A_0$ or to 0, respectively. If $A_{i-1}$ is an elbow, then the configuration between $A_{i-2}$ and $A_n$ can be rotated about $A_{i-2}$ while $[A_1A_n]$ is rotated about $A_1$. This situation is similar to the one shown in Figure 7b. If $A_{i-1}$ is a straight joint, this joint can be bent to move $A_i$ closer to 0. If $[A_1A_n]$ is simultaneously rotated about $A_1$, $A_n$ reaches its neighbors outside the circle centered at $p_i$, as shown in Figure 7d.

Now we can assume that $A_{i-1}$ is a fold and lies at the end of a straight line of links $[A_xA_{i-1}]$ that begins at some predecessor $A_x$ of $A_{i-1}$. By Lemmas 1 and 3, $A_n$ is not on the boundary of $S_n$ if $A_x$ is a fold. If $A_x$ is an elbow, we are again in a situation similar to the one shown in Figure 7b. We can assume that $A_x$ does not lie on C (on top of $A_i$) because in that situation, we could keep the location of $A_i$ and $A_x$ fixed and rotate $L_i$ and $[A_xA_{i-1}]$ off the diagonal, a situation that has already been discussed. This completes the proof of the theorem for the situation in which $A_n$ lies between $A_i$ and 0 and the final segment is a single link.

![Figure 8](image-url)

**Figure 8.** The final segment has a fold at $A_{i+1}$ and places $A_n$ between $p_i$ and 0.
If $A_m$ lies between $A_i$ and 0 and the final segment is a straight line of links containing more than one link, then $A_m$ cannot be on the boundary of $S_m$. This is because the argument that we just presented for a single link shows how to reach points in a neighborhood outside the circle centered at $p_i$ of radius $|[A_{i-1} A_m]|$ and the argument given earlier when $A_m$ reached beyond 0 shows how to reach points closer to $p_i$.

Finally, suppose that $A_m$ lies between $A_i$ and 0 and that the final segment has a fold at $A_i$. If $R_i'$ is not a single point, then $A_m$ can be moved to reach the neighbors of its initial location. To see this, rotate the final segment about $p_i$ to an off-diagonal position. Now rotate $[A_{i+1} A_m]$ about $A_{i+1}$ to increase and decrease the distance between $A_m$ and 0 while simultaneously repositioning $A_i$ along $p_i'$. (See Figure 3.)

In every case, we have found that $R_i'$ must be a single point. \|

Now we can describe the general form of a configuration that places $A_m$ on the boundary of $S_m$ but not on a basic circle.

**Theorem 3:** Suppose that an arm has been moved to a configuration that places $A_m$ on the boundary of $S_m$ but not on a basic circle. Let $A_i$ be the first joint beyond $A_0$ on C, and let $A_j$ be the last joint before $A_m$ on C. Then all joints between $A_0$ and $A_m$ that do not lie on C are straight, with the possible exceptions of $A_{i-1}$ and $A_{j+1}$. If $A_{i-1}$ and $A_{j+1}$ are not straight, then they must be folds.

**Proof:** As noted previously, Lemmas 1, 2 and 3 imply that the final segment is as described in the statement of the theorem. If an elbow appears before $A_j$, then it leads to a joint on C that by Lemma 1 is not at an endpoint of the arc(s) it can reach on C. This means that $A_j$ is also at an interior
point of its arc. By Lemma 4, the final segment cannot lie on a diagonal of 
C. But the final segment can be rotated about $A_j$ to both increase and 
decrease the distance between $A_m$ and C. Then since $A_j$ is at an interior point 
of its arc, $A_j$ can be repositioned along C so that $A_m$ sweeps out arcs of cir-
cles that cover a neighborhood of its original position. (See Figure 9.) This 
contradicts the assumption that $A_m$ is on the boundary of $S_m$. Hence there are 
no elbows before $A_m$.

Figure 9. $A_j$ is at an interior point of $R_i$, and the final segment is off 
the diagonal, so $A_m$ does not lie on the boundary of $S_m$.

For the same reason that there are no elbows before $A_m$, there can be no 
folds connected by straight lines of equal length to superimposed joints on C. 
Lemma 3 and its corollary show that there cannot be two or more folds between 
joints on the circle or between $A_0$ and $A_i$. Finally, Lemma 2 shows that if 
there is a fold between $A_0$ and $A_i$, it must be $A_{i-1}$. 

1. Configurations with $A_m$ on the Boundary of $S_m$

The next theorem will help us to treat the situation in which the final 
segment lies on a diagonal of C as a special case. After that case has been
handled, we will enumerate the configurations having \( A_n \) on the boundary of \( S_n \).

**Theorem 4:** Suppose that an arm has been moved to a position that places \( A_n \) on the boundary of \( S_n \) but not on a basic circle and that the final segment before \( A_n \) lies on a diagonal of \( C \). Then the initial segment consists of a single link, \( L_1 \), which is connected directly to the final segment.

**Proof:** By Theorem 2, there is at least one joint between \( A_0 \) and \( A_n \) that lies on \( C \). Let \( A_i \) and \( A_j \) be the first and last such joints. According to Theorem 3, \( A_{i-1} \) and \( A_{j+1} \) are the only joints of \( C \) that may not be straight, in which case they must be folds. Also, the arc of \( R_i \) containing \( A_i \) must be a single point because by Lemma 4, the arc of \( R_j \) containing \( A_j \) is a single point. This has several consequences. First, if the initial segment does not lie on a diagonal, it must consist of a single link. Second, \( A_0 \) cannot be at \( 0 \). Third, if the initial segment lies on a diagonal of \( C \), it cannot be a multiple-link straight line longer than \( r \), the radius of \( C \), nor can it have \( A_0 \) positioned between \( A_i \) and 0 and a fold at \( A_{i-1} \).

Assume that the initial segment lies on the diagonal. If 0 lies between \( A_i \) and \( A_0 \), then the initial segment cannot have a fold at \( A_{i-1} \) or be a single link because \( A_i \) would be fixed, contradicting the assumption that \( A_0 \) is the last fixed joint. Since we have already ruled out a multiple-link straight line longer than \( r \), we conclude that 0 cannot lie between \( A_i \) and \( A_0 \). Thus if the initial segment lies on the diagonal, \( A_0 \) must lie between \( A_i \) and 0. In this situation we have already ruled out the possibility that \( A_0 \) is at 0 and the possibility that the initial segment has a fold at \( A_{i-1} \). Hence if the initial segment lies on the diagonal, it must be a straight line of length less than \( r \).
Whether the initial segment is a line of links lying on a radius of \( C \) or a single off-diagonal link, \( A_j \) (the last joint before \( A_m \) on \( C \)) must be \( A_1 \). Otherwise, the first intermediate segment would have to be a diagonal chord in order to prevent its second endpoint from being able to move on \( C \). In fact, this diagonal chord would have to be the single link \( L_{i+1} \), but then \( A_{i+1} \) would be a fixed joint.

To prove the theorem, we now need only to rule out the possibility that the initial segment is a multiple-link straight line lying on a radius. Suppose, for the purpose of contradiction, that this is the case. If the final segment had a fold at \( A_{j+1} = A_{i+1} \), then Lemma 3 could be applied even though \( A_j \) lies on \( C \), because \( A_j \) cannot move beyond \( C \) even when \( C \) is removed. But Lemma 3 would imply that \( A_m \) lies on a basic circle. It is obvious that the final segment could not be a multiple-link straight line. If the final segment consisted of a single link \( L_{j+1} = L_m \), \( A_m \) would lie on a basic circle if \( L_{j+1} \) reached as far as \( A_0 \), and on the other hand, \( L_{j+1} \) could be rotated about \( A_m \) to form an elbow if \( L_{j+1} \) did not reach to \( A_0 \). This rules out all possibilities for the final segment. Hence the initial segment consists of one link if it lies on a diagonal. \( \square \)

It follows from Theorem 4 that if the final segment lies on a diagonal of \( C \), then \( A_m \) lies on one of at most four "supplementary" circles that we are about to describe. Note that in this situation, there are at most two possible locations for \( A_1 \), corresponding to the two possible orientations for \( L_1 \) (see [11] for the definition of orientation). Then, for a fixed position of \( A_1 \), \( A_m \) lies on a circle centered at \( A_1 \) of radius either \( \frac{2}{2} L_j \) or, when positive, \( L_{j+1} \). This defines at most four circles, which we call supplementary.
From now on, we assume that \( A_m \) is neither on a basic circle nor on a supplementary circle so that we need only concern ourselves with situations in which the final segment before \( A_m \) does not lie on a diagonal of \( G \).

In the next lemma, we list several configurations that can be moved to form elbows without ever changing the location of their endpoints. Thus by Theorem 3, these configurations cannot occur between \( A_o \) and \( A_m \) if \( A_m \) is on the boundary of \( S_m \) but not on a basic circle. Then we will use these forbidden configurations in enumerating the possibilities for the whole configuration of the arm from \( A_o \) to \( A_m \).

In what follows, we will use the expression \( xy \) to denote the infinite line determined by points or joints \( x \) and \( y \). As before, \( [xy] \) will denote a straight line of links connecting joints \( x \) and \( y \).

**Lemma 5**: In each of the configurations shown in Figure 10, the locations of the two endpoints can be kept fixed while the configuration is moved to form an elbow.

![Diagram](image)

**La.** \([uv]\) lies off the diagonal.

**Ib.** \([uv]\) lies off the diagonal.

\( v \) is folded, and \(|[xy]| > |[uv]|\) it \([uv]\) is a single link.
IIa. \( [vx] \) lies off the diagonal.

IIb. \( [vx] \) lies off the diagonal, \( v \) and \( x \) are folded, and \( y \) lies beyond \( u \).

IIc. \( [vx] \) lies off the diagonal. III. \( [xy] \) lies off the diagonal, and \( v \) is folded.

Figure 10. Configurations that give rise to elbows. Arrows indicate the angular range for a line of links. A sharp tip indicates that the endpoint of the arc belongs to the range, and a round tip indicates that it does not. No order is implied by the letters at joints; \( u \) could come before or after \( v \). There may be additional joints between the ones that appear in the figure. A dashed extension of a link indicates that its endpoint may lie on \( C \).

Proof: In Ia and Ib, the locations of \( u \) and \( x \) can be held fixed while \( [uv] \) is rotated about \( u \). This requires that \( v \) move closer to \( x \), which can be accomplished by creating an elbow between \( v \) and \( x \). In Ib, if \( [uv] \) contains more than one link, \( v \) can be moved straight toward \( u \) and \( x \) by creating an elbow between \( x \) and \( v \) and between \( u \) and \( v \). Hence \( [xy] \) need not be longer than
In IIIa, b, and c, the locations of $u$ and $y$ can be held fixed while $[xy]$ is rotated about $y$. This requires that $x$ move away from $u$, which can be accomplished by opening the joint at $v$ while simultaneously rotating $[uv]$ about $u$.

In III, the locations of $u$ and $x$ can be held fixed while $[xy]$ is rotated about $x$. If $u$ and $0$ lie on opposite sides of $xy$ (the infinite line determined by $x$ and $y$), then $y$ must move away from $u$. This can be accomplished by opening the joint at $v$ while simultaneously rotating $[uv]$ about $u$. If $u$ and $0$ lie on the same side of $xy$, then $y$ must move toward $u$. This can be accomplished by closing the joint at $v$ while simultaneously rotating $[uv]$ about $u$. If $u$ lies on $xy$ between $x$ and $y$, at $x$, or on the opposite side of $x$ from $y$, then $y$ must either move away from $u$, remain at the same distance from $u$, or move closer to $u$, respectively. The important point is that the required rotation of $[uv]$ about $u$ moves $v$ closer to $0$.]

Lemma 5 gives configurations that cannot appear between $A_0$ and $A_m$ in any configuration of the arm that places $A_m$ on the boundary of $S_m$ but not on a basic circle. We now use this information to enumerate the possibilities for the configuration of the arm from $A_0$ to $A_m$. Then from this list we will be able to determine the additional circles that are needed to cover the boundary of $S_m$. It is important to note that in Figure 10, $u$ can come before or after $x$ and $y$. For example, $u$ can correspond to $A_m$ as well as to $A_0$.

According to the notation we have been using, the expression $[xy]$ denotes a straight line of links between $x$ and $y$ whereas the expression $wx$ denotes an infinite line (not necessarily containing any links) through $w$ and $x$. Suppose
the line wx passes through C. We will say that [xy] lies under wx if [xy] and 0 are on opposite sides of wx and that [xy] lies above wx if 0 and [xy] lie on the same side of wx. (See Figure 11.)

Figure 11. [xy] lies "under" line wx in a) and "above" wx in b).

Theorem 1: Suppose an arm has been moved to a configuration that places A_m on the boundary of S_m but not on a basic or supplementary circle. Then if the final segment of the arm before A_m contains more than one link, it contains one of the configurations shown in Figure 12.

Configuration 1: L_{j+1} lies off the diagonal, and A_{j+1} is folded. If [A_{j+1} A_j] lies on L_{j+1}, then A_o does not reach A_m.

Configuration 2: L_{j+1} lies off the diagonal.
Configuration 3

Configuration 4

Configuration 5: Either $A_{i-1}$ is $A_0$ or $A_{i-1}$ folds back to $A_0$.

Configuration 6

Configuration 7: Either $A_j$ or $L_{j-1}$ is the last link before $A_m$ with length $= \max(l_1, \ldots, l_m)$.

Configuration 8
Figure 12. The final segment contains more than one link. Notation is the same as for Figure 10.

Proof: Since \( A_{m} \) does not lie on a basic circle, \( A_{m} \) does not lie on \( C \). Also, Theorem 2 implies that there is some joint strictly between \( A_{0} \) and \( A_{m} \) that lies on \( C \). Let \( A_{j} \) be the last such joint. Since \( A_{m} \) does not lie on a supplementary circle, \( A_{j} \neq A_{1} \), so \( 1 < j < m \). This also means, according to Theorem 4, that the final segment before \( A_{m} \) cannot lie on a diagonal of \( C \).

We will break the proof into two parts. First we will assume that the final segment before \( A_{m} \) has a fold at \( A_{j+1} \), and second we will assume that the final segment is a straight line \( [A_{j}A_{m}] \) consisting of more than one link.

(Note that the hypothesis of the theorem rules out the possibility of the final segment consisting of a single link.)

Proof for the case of a fold at \( A_{j+1} \) in the final segment:

First suppose that \( A_{j} \) is the only joint strictly between \( A_{0} \) and \( A_{m} \) that lies on \( C \). Configuration 1 of Figure 12 covers the situation in which \( A_{0} \) is connected to \( A_{j} \) by a straight line \( [A_{0}A_{j}] \), where by assumption \( j > 1 \). This is because configuration 1a of Figure 10 cannot be a configuration in the arm, so
\([A_0A_j]\) must lie either on or beneath the infinite line \(A_jA_{j+1}\). If \([A_0A_j]\) lies on \(A_jA_{j+1}\), then by IIb, \(A_\alpha\) does not lie between \(A_0\) and \(A_j\). Also, \(A_0\) does not coincide with \(A_\alpha\) since \(A_\alpha\) does not lie on a basic circle. Configuration 2 of Figure 12 covers the situation in which \(A_{j-1}\) is a fold. This is because configurations IIb and IIc are forbidden, so \(L_j\) must lie on the diagonal or on the opposite side of the diagonal from \(L_{j+1}\). (Note that either \(A_0\) or \(A_\alpha\) could correspond to \(u\) in configurations IIb and IIc.)

Next, suppose that \(A_j\) is not the only joint between \(A_0\) and \(A_\alpha\) on \(C\), so that for some \(i\), \(0<\alpha<i\), \([A_\alpha A_j]\) is a chord of \(C\). Since IIb is a forbidden configuration, \([A_\alpha A_j]\) cannot lie on \(A_jA_{j+1}\). Configuration 3 covers the situation in which \([A_\alpha A_j]\) lies "above" \(A_jA_{j+1}\). This is because IIc is forbidden, so \([A_\alpha A_j]\) cannot lie in the open wedge bounded by \(A_jA_{j+1}\) and the infinite line \(OA_j\) and \(L_\alpha\) is forbidden, so \([A_\alpha A_j]\) must be a single link. Configurations 4 and 5 cover the situation in which \([A_\alpha A_j]\) lies "under" line \(A_jA_{j+1}\). To see this, note that \(A_{i-1}\) and \(A_j\) cannot lie on opposite sides of \(OA_i\) because III is forbidden, and \(L_i\) cannot lie above \(A_\alpha A_j\) because IIa is forbidden. Then note that \(A_{i-1}\) can't be a straight joint because IIa is forbidden. Therefore, either \(A_{i-1} = A_0 \lor A_{i-1}\) is a fold, as in configuration 5, or \(A_{i-1}\) lies on \(C\), as in configuration 4.

Proof for the case of a multiple-link straight line final segment:

Because IIa and Ib are forbidden, \(A_j\) cannot be immediately preceded by a multiple-link straight line. Hence either \(A_{j-1}\) is a fold or \(A_{j-1}\) lies on \(C\). Let us consider these possibilities separately.

Configuration 6 covers the situation in which \(A_{j-1}\) is a fold. This is because \(L_j\) must lie in the closed wedge bounded by \(A_jA_\alpha\) and \(OA_j\) as \(IIa\) is for-
bidden, and furthermore, as we are about to see, \( L_j \) cannot lie on \( A_j A_m \) or on \( OA_j \). If \( L_j \) lay on \( A_j A_m \), then \( A_m \) would have to lie on \( A_o \) or between \( A_o \) and \( A_j \) because \( \text{IIb} \) is forbidden. But this would place \( A_m \) on a basic circle. If \( L_j \) lay on \( OA_j \) with \( 0 \) between \( A_{j-1} \) and \( A_j \), \( A_j \) would be fixed, and if \( A_{j-1} \) lay at \( 0 \) or between \( A_j \) and \( 0 \), \( A_j \) would not lie at the endpoint of an arc of \( R_j \), implying that \( A_m \) did not lie on the boundary of \( S_m \).

Several configurations are needed to cover the situation in which \( A_{j-1} \) lies on \( C \). Certainly \( L_j \) lies in the closed wedge bounded by \( A_j A_m \) and \( OA_j \) in that situation, as \( \text{Ia} \) is forbidden. There are two cases to consider, namely whether \( A_{j-1} \) is, or is not, the first joint past \( A_o \) on \( C \).

Configuration 7 covers the situation in which \( A_{j-1} \) is the first joint past \( A_o \) on \( C \). The possibility that the initial segment is a single link or has a fold at \( A_{j-2} \) is clearly covered. The only remaining possibility is that the initial segment is a multiple-link straight line \( [A_o A_{j-1}] \). In that case \( [A_o A_{j-1}] \) must lie on or below \( L_j \), because \( \text{Ia} \) is forbidden. Again, configuration 7 applies.

Now suppose that \( A_{j-1} \) is not the first joint past \( A_o \) on \( C \) so that for some \( A_i \) where \( j-1 > i > 0 \), \( [A_i A_{j-1}] \) is a chord. Configuration 7 covers the possibility that \( L_j \) is a diagonal chord, so we may assume that \( L_j \) is off the diagonal from now on.

If chord \( [A_i A_{j-1}] \) lies above \( L_j \), then it must be a single link since \( \text{Ia} \) is forbidden, and furthermore, this link cannot lie in the open wedge bounded by \( L_j \) and \( OA_{j-1} \) because \( \text{IIa} \) is forbidden. Hence, configuration 8 covers the situation in which \( [A_i A_{j-1}] \) lies above \( L_j \).

Chord \( [A_i A_{j-1}] \) cannot lie on top of \( L_j \) because \( L_j \) and \( [A_i A_{j-1}] \) could be
rotated about the coincident joints $A_j$ and $A_i$ to form a fold at $A_{j-1}$, violating Theorem 3.

The last possibility is that $[A_i A_{j-1}]$ lies beneath $L_j$. Since $i > 0$, there is a link $L_i$ to consider. Since IIa is forbidden, $L_i$ cannot lie on or beneath chord $[A_i A_{j-1}]$. Therefore, since III is forbidden, $L_i$ must lie either on $OA_i$ or in the open wedge bounded by $OA_i$ and $[A_i A_{j-1}]$. Since Ia is forbidden, $A_{i-1}$ cannot be straight, so either $A_{i-1}$ is a fold and configuration 9 applies, or $A_{i-1}$ lies on $C$ and configuration 10 applies. \[ \]

The basic idea for handling configurations 1-10 of Theorem 5 is this: In each case, we will show that there are only a constant number (independent of the arm) of possibilities for $A_j$, the last joint before $A_m$ on $C$. Then a constant number of circles, at most 8 for each choice of $A_j$, can be added to the basic and supplementary circles to form a collection that covers the boundary of $B_m$. This is because $A_m$ must lie on a circle of radius either $\frac{m}{2}$, $l_k$ or $k = j+1$

$\sum_{k=j+1}^{m} l_j$ about $A_j$, and $A_j$ must lie at one of the endpoints of $R_j$, of which there are at most four. The possibilities for $A_j$ will be determined by inequalities involving the link lengths that can only be satisfied in a few ways. We have already seen a simple example of this in configuration 7. There, either $A_j$ is the higher numbered endpoint of the last link of longest length before $A_m$, or $A_j$ is the next joint after that endpoint. For several of the configurations, somewhat more complicated length inequalities will generate the possibilities for $A_j$. The next three lemmas will be used to find the additional inequalities that are needed.

Lemma 4: Consider a configuration of straight lines of links $[vu]$ and $[vw]$ joined at $v$ and constrained to move inside circle $C$, where $v$ is on $C$. $vu$ does
not lie on the diagonal through v, and |_vw| lies beneath v. Joints u and w may or may not lie on C. (Figure 13 shows that the mirror image of _vu_ and |_vw| with respect to the line through u and v need not lie inside C.) If ||vw|| > 2||vu||, then the location of either endpoint of the configuration can be kept fixed while _vu_ and |_vw| are moved to their mirror images with respect to the line determined by the initial locations of u and v. During this motion, the distance from the moving endpoint to C can be kept within its initial value.

Figure 13. |_uv| lies off the diagonal. In a), the mirror image falls outside C whereas in b), the image lies inside C.

**Proof:** It is easy to see that the mirror image of the configuration with respect to the line uw lies within the circle with v off C if, and only if, v and 0 lie on opposite sides of uw, so that uw intersects 0v at a point strictly between v and 0. We will first show that ||vw|| > 2||vu|| forces this situation to occur, and then we will show that the configuration can be moved to its mirror image while either endpoint is held fixed and the other is kept within the proper distance from C.
Assume now that $|vw| \geq 2|vu|$. We want to show that $uw$ intersects $0v$ at a point strictly between $v$ and $0$. For notational convenience, we will think of $0$ as lying at the origin of a Cartesian coordinate system with $v$ on the negative horizontal axis at $(-r,0)$. $P$ will denote the point $(r,0)$. Note that $0v$ is the horizontal axis.

Suppose that $uw$ has non-negative slope. If it intersects the horizontal axis $0v$, it does so to the right of $v$. Then if $uw$ is parallel to $0v$ or if it intersects $0v$ at $P$ or to the right of $P$, the perpendicular from $v$ to $uw$ meets $uw$ on or outside $C$. Consequently $|vw| < |vu|$, a contradiction.

Consider a line $M$ with positive slope that intersects $0v$ at $0$ or between $0$ and $P$. If $u$ and $w$ are placed on this line to minimize the ratio of $|vu|$ to $|vw|$, then $u$ lies at the intersection of $M$ with the perpendicular from $v$ and, since $vw$ lies under $vu$, $v$ lies at the point $Q$ where $M$ meets $C$. (See Figure 13.) Hence we want to minimize the sine of the angle between $M$ and $vQ$, where $Q$ is on $C$. For any given $Q$, the optimum choice for $M$ is the vertical line through $Q$ if $Q$ is in the fourth quadrant, and is the line $0Q$ if $Q$ is in the third quadrant. Among the vertical lines and the lines through $0$, the minimum ratio is achieved by the vertical line through $0$. Hence, the ratio of $|vu|$ to $|vw|$ is at least $1/\sqrt{2}$, so $|vw| \leq \sqrt{2}|vu|$, a contradiction.

Next, suppose that $uw$ has negative slope, so it intersects $0v$ to the left of $P$. If the intersection lies outside $C$, then $|vu| > |vw|$, a contradiction. The intersection cannot occur at $v$ since $vu$ and $vw$ do not lie on top of one another. If the point of intersection lies at $0$ or to the right of $0$, then $|vu| \geq r$ and $|vw| < 2r$, so $|vw| < 2|vu|$, a contradiction. This completes the proof that $|vw| \geq 2|vu|$ implies that $uw$ intersects $0v$ at a point strictly between $v$ and $0$ and hence, that the mirror image of the
configuration lies inside \( C \) with \( v \) not on \( C \).

Now we need to show that when \( ||vw|| \geq 2||vu|| \), the configuration can reach its mirror image inside \( C \) by motions that keep the moving endpoint within its initial distance from \( C \).

First, suppose that \( v \) is to be held fixed during the motions. Rotate the line of links \([vu]\) about \( v \) down to \([vw]\), moving \( u \) closer to \( C \). Then force \( u \) to retrace its path back to its original location but move \([vw]\) to the other side of \([vu]\), so that \([vw]\) rotates to its mirror image position. Note that the distance between \( u \) and \( C \) never exceeds its initial value during these motions.

Finally, suppose that \( u \) is to be held fixed during the move to the mirror image. Note that \( ||vu|| < r \) since \( ||vw|| \geq 2||vu|| \). Rotate \([vu]\) about \( v \) so that \( v \) moves away from \( 0 \) down to \( C \). Then fold joint \( v \) while simultaneously rotating the configuration about \( u \) to keep \( v \) on \( C \). Note that \( v \) is constrained by \([vu]\) to remain within distance \( 2||vu|| \) of \( C \) so that \( v \) can stay on \( C \) until \( v \) folds. Then force \( v \) to retrace its path back to its original location but move \([vw]\) to the other side of \([vu]\). \( \square \)

**Lemma 2:** Suppose that \( u, v, \) and \( w \) are joints lying on circle \( C \) that are connected by straight lines of links \([uv]\) and \([vw]\), and that either \([vw]\) lies under \([uv]\) or \( v = w \) so that \([vw]\) contains no links. Also suppose that \([wx]\) is a straight line of links connecting \( w \) to a joint \( x \) that lies on \( C \) above \([vw]\) and on the opposite side of \( 0w \) from \( u \). Then if \( ||uv|| + ||wx|| \leq d \), the configuration between \( u \) and \( x \) can be moved to its mirror image with respect to the line determined by the initial locations of \( u \) and \( x \) without ever moving \( u \) or removing \( x \) from \( C \). (See Figure 14.)
Suppose that $|uv| + |wx| \leq d$. Then $ux$ lies between $uv$ and $Ou$, so the mirror image of the configuration lies inside $C$ with the images of $w$ and $v$ off $C$.

The original configuration can be moved to a position in which $u$, $w$, $v$, and $x$ are collinear and $v$ and $w$, if distinct, are folds. This can be done in the following way. If $w=v$, simply straighten the joint $v$ while keeping $u$ fixed and moving $x$ around $C$. If $w \neq v$, fold $w$ while keeping $u$ fixed and moving $v$ and $x$ around $C$. (This causes $[uv]$ to rotate about $u$.) Next, keeping $v$ folded, fold $v$ while moving $x$ around $C$ and rotating $[uv]$ about $u$. This brings the joints to the desired collinear configuration. The fact that $|uv| + |wx| \leq d$ guarantees that these motions can be made and that $x$ does not cross $Ou$.

Note that at each moment during the motions just described, the mirror image of the configuration with respect to the moving line $ux$ lies inside $C$. Hence the motions induce legal motions of the mirror image of the initial configuration that carry the mirror image to the configuration in which $u$, $v$, $w$,
and x are collinear. Hence the mirror image of the original configuration can be reached by moving to the collinear configuration, where the joints coincide with their images, and then reversing the induced motion of the mirror image.

Lemma 8: Suppose that u, v, w and x satisfy the hypotheses of Lemma 7, except that x need not lie on C. Then at least one of the following conditions holds:

i) \(||uv|| > r;\)

ii) \(||wz|| > r;\)

iii) The configuration can be moved to its mirror image with respect to the line through the initial positions of u and x while u is held fixed and x is kept within its initial distance from C.

iv) The configuration between v and x can be moved to its mirror image with respect to the line through the initial positions of v and x while u and v are held fixed and x is kept within its initial distance from C.

Proof: We will show that iii) or iv) holds when i) and ii) do not. Note that if \(||uv|| \leq r \) and \(||wz|| \leq r\), then the mirror images of w and v with respect to the line through the initial locations of u and x lie strictly inside C.

First, suppose that \(||wz|| > ||uv||\). Then \(||wz||\) can be rotated about w so that x moves away from 0 until x reaches C. Note that this induces a legal motion of the mirror image of the configuration. By Lemma 7, the mirror image of the configuration between u and x with x on C can be reached. Finally, the
reverse of the induced motions can be applied to carry the configuration the rest of the way to its initial mirror image.

Next, suppose that \(|\|wx\| \leq |lwv|\) and that \([wx]\) crosses \([uv]\). Rotate \([wx]\) about \(w\) until \(x\) reaches \([uv]\), and note that the mirror image with respect to \(ux\) stays inside \(C\). Hence it suffices to show that the present configuration with \(x\) located on \([uv]\) can be moved to its mirror image with respect to \([uv]\). But \(u, v,\) and \(x\) already coincide with their mirror images, so we only need to move \(w\) to its image. To do this, fold \(w,\) moving \(x\) to \([vw]\). Then make \(x\) retrace its path, but with \([wx]\) crossed over to the other side of \([vw]\) so that \(w\) will reach the image of its initial position.

Finally, suppose that \(|\|wx\| \leq |lwv|\) and that \([wx]\) does not cross \([uv]\). Rotate \([wx]\) about \(w\) so that \(x\) moves closer to \(C\) until \(x\) reaches \([vw]\). Then force \(x\) to retrace its path, but with \([lwv]\) crossed over to the other side of \([wx]\).

The heart of the proof of the next theorem, which is the main result of this section, uses Lemmas 6-8 to show that configurations 1-10 of Theorem 5 generate only a constant number of circles to be added to the collection covering the boundary of \(S_m\).

**Theorem 6:** If \(A_m\) is a non-tied joint of an arm confined to move inside a circle \(C\), then the boundary of \(S_m\) can be covered by a finite collection of circles. This collection contains at most 148 circles and can be determined in \(p(m)\) time, where \(p\) is a polynomial in \(m\).

**Proof:** It follows from Theorem 5 that the boundary of \(S_m\) can be covered by a collection of circles consisting of the basic circles (at most 4), the supplementary circles (at most 4), and circles of radius \(1_m\) centered at the
endpoints of $R_{m-1}$ (at most 4) together with circles covering configurations 1-10. In configurations 1-5, $A_m$ lies on a circle of radius $l_{j+1} - ||A_{j+1}A_m||$ centered at the endpoint of an arc of $R_j$, where $A_j$ is the last joint on $C$ between $A_o$ and $A_m$. In configurations 6-10, $A_m$ lies on a circle of radius $||A_{j+1}A_m||$, where $A_j$ has the same definition. Thus each possibility for $j$ in each of the configurations gives rise to at most four new circles to add to the collection because $R_j$ has at most four endpoints. Therefore, it suffices to show that the total number of possibilities for $A_j$ is small, and that these possibilities can be determined in polynomial time. We will do this one configuration at a time.

In what follows, we will say that a link index is feasible with respect to a set of inequalities if the corresponding link provides a solution.

Configuration 1: If $[A_oA_j]$ lies on $L_{j+1}$ then $A_m$ must lie on a basic circle.
If $[A_oA_j]$ does not lie on $L_{j+1}$, then Lemma 6 implies that $||A_oA_j|| < 2l_{j+1}$.
Of course $l_{j+1} > ||A_{j+1}A_m||$. If there are solutions to these inequalities, let $s$ be the smallest feasible choice for index $j$. It is easy to check that there can be at most one feasible choice for $j$ that is larger than $s$. Thus, there are all together three possibilities for $A_j$ in configuration 1.

Configuration 2: The last link of longest length between $A_o$ and $A_m$ must be either $L_j$ or $L_{j+1}$, so there are at most two choices for $A_j$.

Configuration 3: Lemma 8 implies that $l_j + l_{j+1} > r$, and of course $d > l_{j+1} > ||A_{j+1}A_m||$. If there are solutions to these inequalities, let $z$ be the largest feasible choice for index $j+1$. Then note that there can be at most three feasible choices for $j+1$ that are smaller than $z$, giving a total of four choices for $A_j$. 


Configuration 4: The number of choices for the pair of indices $i, j+1$ is at most $m \choose 2$, and of course $l_{i,j+1} > ||A_{j+1}A_m||$ and $l_i > ||A_iA_j||$.

First, suppose that $l_i > r$, and if there are solutions to the inequalities including this new one, let $s$ be the smallest index such that for some $t$, the pair $s,t$ is a feasible choice for $i,j+1$. Note that there can be at most three links beyond $L_s$ that are longer than $r$. Hence there are at most four feasible choices for $i$. But choosing a value for $i$ leads us back to configuration 1, so there are at most twelve choices for $j$ if $l_i > r$.

Next, suppose that $l_i \leq r$ and that $l_{j+1} > r$. Since $r \geq l_i > ||A_iA_j||$ and $l_{j+1} > ||A_{j+1}A_m||$, $L_{j+1}$ must be the unique link of longest length between $A_{i-1}$ and $A_m$. If there are solutions to the inequalities, including $l_i \leq r$ and $l_{j+1} > r$, let $s$ be the smallest index such that for some $t$, the pair $s,t$ is a feasible choice for $i,j+1$. We have just noted that choosing $i = s$ forces $j+1 = t$. Now note that any other feasible choice for $i$ between $s$ and $t$ also forces $j+1 = t$. Since there can be at most one link in $[A_{j+1}A_m]$ that is longer than $r$, there can be at most one more choice for $j+1$. Hence there are at most two choices for $j$ if $l_i \leq r$ and $l_{j+1} > r$.

Lemma 7 rules out the last possibility, that $l_i \leq r$ and $l_{j+1} \leq r$, so all together there are at most $12 + 2 = 14$ choices for $A_j$.

Configuration 5: Let $L$ denote the link of maximum length between $A_o$ and $A_m$ that has highest index. Then either $L = L_i$ or $L$ belongs to $[A_iA_j]$ or $L = L_{j+1}$. The possibility that $A_j$ is the lower numbered joint of the last link of longest length was already taken into account when configuration 1 was considered. The possibility that $L = L_i$ or that $L$ belongs to $[A_iA_j]$ can be handled by returning to configuration 1 and letting the endpoint of $L$ with the
higher index play the role of $A_o$. This generates at most two possibilities for $A_j$ because $[A_iA_j]$ cannot lie on $L_{j+1}$ in configuration 5.

Configuration 6: According to Lemma 6, $|[A_jA_m]| < 2l_i$, and of course, $l_j > |[A_oA_{j-1}]|$. If there are solutions to these inequalities, let $s$ be the smallest feasible choice for index $j$. Then it is easy to see that there can be at most one feasible choice for $j$ that is greater than $s$. Hence there are at most two choices for $A_j$.

Configuration 7: There are two choices for $A_j$. $A_j$ could be the higher indexed endpoint of the highest indexed link of longest length, or $A_j$ could be the next joint after that.

Configuration 8: According to Lemma 7, $l_{j-1} + l_j > d$, and of course $l_j > |[A_jA_m]|$. If these inequalities can be satisfied, then consider the largest feasible choice for $j$, and note that there can be at most one smaller feasible choice.

Configuration 9: The longest link that has the highest index is either $L_j \lor L_i$. The first possibility was handled in configuration 7. In the second possibility, $A_j$ is uniquely determined by the fact that $L_j$ is the longest link after $A_i$.

Configuration 10: According to Lemma 7, $l_i + l_j > d$. Of course, $l_j > |[A_jA_m]|$, $l_j > |[A_iA_{j-1}]|$, and $l_i > |[A_iA_{j-1}]|$. If these inequalities can be satisfied, let $s$ be the smallest feasible choice for $i$, and let $t > s$ be a feasible choice for $j$ when $i = s$. Note that $t$ is determined by the fact that $L_j$ must be the longest link between $A_i$ and $A_m$. All feasible choices for $i$ between $s$ and $t-1$ force $j = t$. Since $l_i + l_j > d$, there can be at most one more feasible choice for $i$, namely $i = t$, and then the $j$ is again uniquely
determined. In all, there are two possibilities for \( j \).

The total number of choices for \( A_j \) is at most 34, and since \( R_j \) may have as many as four endpoints, this generates at most 136 circles. There were at most 12 circles initially, so the total number of circles needed is at most 148.

The bound of 148 is very generous. The important point, though, is that the bound does not depend on the arm.

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An "arm" is a sequence of links whose endpoints are connected consecutively by movable joints. The location of the first endpoint is fixed. This report gives a polynomial time algorithm for determining the regions that each joint can reach when the arm is restricted to a circular region of the plane.