SIGNAL DETECTION FOR PARETO RENEWAL PROCESSES
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1. Introduction and Summary

Several types of applications entail point processes with "dead" times after each event. One such family of stochastic processes is \( \Omega(\text{PRP}) \), the family of Pareto Renewal Processes. The i.i.d. inter-arrival times \( \{X_j\} \) satisfy, \( P(x) = P(X \leq x) = 1 - \left( \frac{A}{x} \right)^s \), \( x > A > 0 \) and \( s > 0 \).

An additional interesting property of the interarrival-time distributions is that they are all "thick-tailed" relative to the corresponding distributions for Gaussian processes and Poisson processes. Further, a variety of tail thicknesses, one for each \( s \)-value, is possible. These two properties lead to some interesting inference problems, of which one is here concerned only with signal detection.

[The Pareto distribution itself was, of course, introduced by Vilfredo Pareto (1848 - 1923). (See Reference [22]). This distribution has been used and studied by numerous other authors including Pigon (1932)]

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Both one-sample and two-sample signal detection problems with historical data will be considered here. The organization of the paper is as follows. Section 2 contains the basic properties of the inter-arrival-time distributions and the MLE's (maximum likelihood estimates) under various circumstances. Section 3 contains the distribution theory necessary for insight and inference. Most of the proofs for the results of this section are straightforward. In Section 4, one introduces the fundamental statistical concepts to be used, namely, (i) the BDT (Basic Data Transformation); (ii) the M-S-S (minimal sufficient statistic); (iii) the M-S-N (maximal statistical noise); (iv) PDF (parametric distribution-free) statistics; and (v) NPDF (nonparametric distribution-free) statistics.

Section 5 discusses the uses of the Kolmogorov-Smirnov (1933) statistic; and its extensions by Lilliefors (1967, 1969), Srinivasan (1970) and Y. Choi (1980) in signal detection. Sections 6 and 7 treat the one-sample and two-sample detection problems, respectively. Tables summarizing the results, are presented at the end of appropriate sections.

Appendix A contains proofs of some of the assertions made in the paper. Appendix B contains numerical examples illustrating each of the detection procedures derived in Sections 6 and 7.
2. Elementary Properties and Estimators of Pareto Renewal Process

Most of the results of this section are adaptations of results in Johnson and Kotz (1970). The Pareto distribution $P_a(A,s)$ is defined for each $A > 0$, $s > 0$ by

$$F(x) = \begin{cases} 1 - \left(\frac{A}{x}\right)^s & x > A \\ 0 & x \leq A \end{cases}$$

This is a special form of the Pearson type VI distribution. The Pareto density function is

$$f(x) = sA^s x^{-(s+1)} \quad \text{for } x > A > 0.$$ 

If $X \sim P_a(A,s)$, then the $r$th moment of $X$ exists if and only if $r < s$ and is given by

$$E(X^r) = \frac{sA^r}{s - r}.$$ 

The variance of $X$ is $sA^2(s - 1)^{-2}(s - 2)^{-1}$ for $s > 2$. For further details on moments, see Malik (1966).

Malik (1966, 1967) has also obtained results on the characteristic functions of the order statistics from a Pareto distribution as well as recurrence relations between the moments and covariance of the order statistics. Levy (1925) discovered a class of stable laws (Stable Pareesian) which follow the asymptotic form of the Pareto law.

If $X \sim P_a(A,s)$, $Y = \ln X$, then $Y \sim T-exp(ln A,s)$ where
T-exp(B,s) is truncated-exponential with distribution function
\[ H(x) = 1 - e^{-s(x-B)} \quad \text{for } x \geq B. \]
Also if \( Z = X^{-s} \), then \( Z \sim U(0, A^{-s}) \). Inference for the truncated-exponential has been studied by Park and U. Choi (1978), Begg (1981) and signal detection problem for the uniform process has been investigated by Y. Choi-Bell-Ahmad-Park (1982). Some of those results will be used and compared to the ones herein.

Here one is primarily interested in MLE's of the parameters \( A \) and \( s \). Other types of estimates are given in Johnson and Kotz (1970). The likelihood function for the interarrival times \( (X_1, X_2, \ldots, X_n) \) from a PRP is
\[ L = \prod_{j=1}^{n} \frac{s^A}{x_j^{s+1}}. \]
The proofs of the following theorems are straightforward.

**Theorem 2.1.** (One-parameter, \( s = s_0 \) known)

1. The MLE of \( A \) is \( \hat{A} = X(1) \), which is distributed \( \text{Pa}(A, s_0) \):
2. \( E(\hat{A}) = ns_0A(ns_0 - 1)^{-1} \) for \( ns_0 > 1 \) and \( V(\hat{A}) = ns_0A^2(ns_0 - 2)(ns_0 - 1)^{-1} \) for \( ns_0 > 2 \);
3. The MVUE (minimum variance unbiased estimator) of \( A \) is
   \[ A^* = (ns_0 - 1)(ns_0)^{-1}X(1) \quad \text{for } ns_0 > 1, \]
   \[ V(A^*) = (ns_0(ns_0 - 2))^{-1}A^2 \]
   for \( ns_0 > 2 \);
4. Both \( \hat{A} \) and \( A^* \) are consistent estimators of \( A \), i.e., \( \hat{A}/A \to 1 \), \( A^*/A \to 1 \) as \( n \to \infty \) but only \( A^* \) is unbiased.
5. The M-S-S (minimal sufficient statistic) for \( A \) is \( X(1) \) and the family \( \{\text{Pa}(A, s_0)\} \) indexed by \( A > 0 \) is complete.

**Theorem 2.2.** (One-parameter, \( A = A_0 \) known)

\[ -4 - \]
(1) The MLE of $s$ is $\hat{s} = \frac{n}{\ln \sum_{j=1}^{B} X_j - n \ln A_0}$.

(ii) $\frac{2ns}{\hat{s}} \sim \chi^2_{2n}$, $E(\frac{1}{\hat{s}}) = \frac{1}{s}$ and $V(\frac{1}{\hat{s}}) = \frac{1}{ns^2}$.

(iii) $E(\hat{s}) = \frac{n(n-1)s}{n-2}$ for $n > 1$ and $V(\hat{s}) = \frac{n^2s^2}{(n-1)^2(n-2)}$ for $n > 2$;

(iv) The MVUE of $s$ is given by $s^* = \frac{(n-1)}{n}\hat{s}$ and $V(s^*) = \frac{s^2}{n-2}$ for $n > 2$;

(v) Both $\hat{s}$ and $s^*$ are consistent estimators of $s$ but only $s^*$ is unbiased; and

(vi) the M-S-S for $s$ is $\hat{s}$ (or $s^*$).

**Theorem 2.3.** (Two parameters, $A, s$ both unknown)

(1) The MLE of $(A, s)$ is $(\hat{A}, \hat{s})$ where

$$\hat{A} = X(1) \quad \text{and} \quad \hat{s} = \frac{n}{\ln \sum_{j=1}^{B} X_j - \min X(1)}.$$

(ii) $\hat{A}$ and $\hat{s}$ are independent, $\hat{A} \sim \text{Pa}(A, ns)$, $\frac{2ns}{\hat{s}} \sim \chi^2_{2(n-1)}$.

(iii) $E(\hat{A}) = \frac{nsA}{ns - 1}$ for $ns > 1$, $V(\hat{A}) = \frac{nsA^2}{(ns-2)(ns-1)^2}$ for $ns > 2$;

(iv) $E(\hat{s}) = \frac{ns}{n - 2}$ for $n > 2$, $V(\hat{s}) = \frac{n^2s^2}{(n-2)^2(n-3)}$ for $n > 3$;

(v) The MVUE of $A$ is $A^* = \frac{(ns - 1)}{ns} X(1)$ for $ns > 1$ and $V(A^*) = \frac{A^2}{ns(ns-2)}$ for $ns > 2$. 
(vi) the MVUE of $s$ is $s^* = \frac{(n - 2)s}{n}$ and $V(s^*) = \frac{s^2}{n - 3}$;

(vii) the N-S-S for $(A, s)$ is $(\hat{A}, \hat{s})$.

The results of this section are summarized in Table 2.1 below. The last column of this table yields the N-S-N (maximal statistical noise) for various detection problems. A formal definition of the N-S-N is given in Section 4. As far as Table 2.1 is concerned, one should view the N-S-N, $N(x)$, as complementary to and statistically independent of the N-S-S, $S(x)$. Several versions of the N-S-N are given in Section 3.

3. Distribution Theory for Pareto Renewal Processes

In this section one develops some results which yield the N-S-S (to be defined in Section 4) for various one-sample and two-sample problems.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $P(a, s)$ and let $X(1) \leq X(2) \leq \ldots \leq X(n)$ denote the order statistics of the $X$'s.

**Definition 3.1.** One denotes by $G-O-S(k)$ the distribution of the order statistics induced by a random sample of size $k$ drawn from a distribution $G(\cdot)$. The following lemma is fundamental.

**Lemma 3.1.** (Rényi (1953), Pyke (1965)). Let $\xi_1, \xi_2, \ldots, \xi_k$ be
<table>
<thead>
<tr>
<th>TABLE 2.1.</th>
<th>[ \text{MLE} ]</th>
<th>[ \text{Dist of MLE} ]</th>
<th>[ \text{E}(\text{MLE}) ]</th>
<th>[ \text{Var}(\text{MLE}) ]</th>
<th>[ \text{MMLE}(\cdot) ]</th>
<th>[ \text{Var}(\text{MMLE}) ]</th>
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<th>[ \text{**}(\text{MMLE}(\cdot)) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-parameter</td>
<td>[ A \text{ is unknown} ]</td>
<td>[ \hat{A} = {1} ]</td>
<td>[ \rho(A_{\hat{A}}\hat{A}) ]</td>
<td>[ \frac{m_0A_{\hat{A}}}{m_0} ] [ \text{for } m_0 &gt; 1 ] [ \frac{m_0A_{\hat{A}}^2}{m_0^{-2}(m_0^{-1})^2} ] [ \text{for } m_0 &gt; 1 ]</td>
<td>[ A^{*} = (\frac{m_0^{-2}}{m_0}){1} ] [ \text{for } m_0 &gt; 1 ]</td>
<td>[ \text{of } \hat{A} \text{ is } m_0^{-2}(m_0^{-1})^2 ] [ \text{for } m_0 &gt; 1 ]</td>
<td>[ \text{Var}^{(*)} = \frac{m_0}{n} ] [ \text{for } (A, \hat{A}) \text{ is unknown} ]</td>
<td>[ \text{**}(\text{MMLE}(\cdot)) ]</td>
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<td>s = \sigma_k \text{ known} ]</td>
<td>[ A \text{ is known} ]</td>
<td>[ \hat{A} = {1} ]</td>
<td>[ \rho(A_{\hat{A}}\hat{A}) ]</td>
<td>[ \frac{m_0A_{\hat{A}}}{m_0} ] [ \text{for } m_0 &gt; 1 ] [ \frac{m_0A_{\hat{A}}^2}{m_0^{-2}(m_0^{-1})^2} ] [ \text{for } m_0 &gt; 1 ]</td>
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<td>[ \text{**}(\text{MMLE}(\cdot)) ]</td>
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<td>Theorem 2.1</td>
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<td>1-parameter</td>
<td>[ s \text{ is known} ]</td>
<td>[ \hat{s} = {1} ]</td>
<td>[ \rho(s_{\hat{s}}\hat{s}) ]</td>
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<td>Theorem 2.2</td>
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<td>2-parameters</td>
<td>[ (A, s) \text{ is known} ]</td>
<td>[ \hat{A} = {1} ]</td>
<td>[ \rho(A_{\hat{A}}\hat{A}) ]</td>
<td>[ \frac{m_0A_{\hat{A}}}{m_0} ] [ \text{for } m_0 &gt; 1 ] [ \frac{m_0A_{\hat{A}}^2}{m_0^{-2}(m_0^{-1})^2} ] [ \text{for } m_0 &gt; 1 ]</td>
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<tr>
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<td>[ \hat{s} = {1} ]</td>
<td>[ \rho(s_{\hat{s}}\hat{s}) ]</td>
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<td>Theorem 2.3</td>
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* \( (y_1), (y_2), (y_3), (y_4), (y_5), (y_6) \) are defined in Section 3, Definition 1.2.

** \( w_{S,N} \) is defined in Section 4.
i.i.d. $\text{Exp}(\lambda)$. Then (i) $k\xi = \sum_{j=1}^{k} \xi_j \sim \Gamma(k, \lambda)$, (ii) $\xi^* = \frac{\xi_1}{k\lambda}$, (iii) $\frac{\xi_2}{k\lambda}, ..., \frac{\xi_j}{k\lambda}$ are independent, where $U(-)$ is the $U(0,1)$ distribution. (iii) $k\xi$ and $\xi^*$ are independent.

**Remark.** Note that $\lambda$ has been eliminated in $\xi^*$.

**Lemma 3.2.** Let $\eta_1, \eta_2, ..., \eta_k$ be $\text{Exp}(\lambda) - O - S(k)$. Define $\eta_1^*, ..., \eta_k^*$ by $\eta_1^* = k\eta_1$, $\eta_2^* = \eta_1 + (k - 1)\eta_2$, ..., $\eta_j^* = \eta_1 + ... + \eta_{j-1} + (k - 1 - j)\eta_j$, ..., $\eta_k^* = k\eta_k$ and $\eta^* = \frac{\eta_1^*}{k\lambda}$, $\frac{\eta_2^*}{k\lambda}$, ..., $\frac{\eta_{k-1}^*}{k\lambda}$. Then

(i) $\eta_k^* \sim \Gamma(k, \lambda)$ is the M-S-S for $\lambda$;

(ii) $\eta^* \sim U-O-S(k - 1)$;

(iii) $\eta_k^*$ and $\eta^*$ are independent.

**Theorem 3.3.** Let $X_1, X_2, ..., X_n$ be i.i.d. $\mathcal{P}(A, s)$, then

(i) $(\ln \frac{X_1}{A}, \ln \frac{X_2}{A}, ..., \ln \frac{X_n}{A})$ i.i.d. $\text{Exp}(s)$;

(ii) $(\ln X(1), \ln X(2), ..., \ln X(n)) \sim T - \text{Exp}(\ln A, s) - O - S(n)$;

(iii) $(\ln \frac{X(1)}{A}, \ln \frac{X(2)}{A}, ..., \ln \frac{X(n)}{A}) \sim \text{Exp}(s) - O - S(n)$;
(iv) \( \{ \frac{A}{X(n)} , \frac{A}{X(n-1)} , \ldots , \frac{A}{X(1)} \} \sim P_w(s) - O-S(n) \) where \( P_w(s) \) is the power distribution defined by
\[
H(x) = \begin{cases} 
0 & x \leq 0 \\
x^s & 0 < x < 1 \\
1 & x \geq 1
\end{cases}
\]

(v) \( \{ \frac{1}{X(n)} , \frac{1}{X(n-1)} , \ldots , \frac{1}{X(1)} \} \sim U(0,A^{-S}) - O-S(n) \),

(vi) \( \{ \frac{A}{X(n)} , \frac{A}{X(n-1)} , \ldots , \frac{A}{X(1)} \} \sim U-O-S(n) \).

**Definition 3.2.** Let \( X_1, X_2, \ldots, X_n \) be i.i.d. \( P_A(A,s) \). [The following random variables \( T_r, \varphi_r, \varphi_r, 1 \leq r \leq n - 1 \) will be used frequently throughout the rest of the paper and are mentioned in Table 2.1.]

(1) \( T_r = \frac{\sum_{j=1}^{r} X_j}{r}, \quad 1 \leq r \leq n; \)

(11) \( \varphi_r = \frac{\sum_{j=1}^{r-1} \ln \left( \frac{X(j+1)}{X(1)} \right) + (n - r) \ln \frac{X(r+1)}{X(1)} }{r}, \quad 1 \leq r \leq n - 1; \)

(111) \( \varphi_r = \sum_{j=1}^{r} (n + 1 - j) \ln \left( \frac{X(j+1)}{X(j)} \right), \quad 1 \leq r \leq n - 1. \)

The following theorem is an easy consequence of Lemmas 3.1 and 3.2.

**Theorem 3.4.** (1) \( T_r = \frac{T_{1,n}}{n}, \frac{T_{2,n}}{n}, \ldots, \frac{T_{n-1,n}}{n} \sim U-O-S(n-1), \)

and is independent of \( T_n \sim \Gamma(n,s) \);
(ii) \( V = \frac{V_1}{n-1}, \frac{V_2}{n-1}, \ldots, \frac{V_{n-2}}{n-1} \) \( \sim \) U-O-S(n - 2) and is independent of \( \frac{V_{n-1}}{n-1} \sim \Gamma(n - 1, s) \);

(iii) \( E = \left( \frac{E_1}{n-1}, \frac{E_2}{n-2}, \ldots, \frac{E_{n-2}}{n-1} \right) \sim \) U-O-S(n - 2) and is independent of \( \frac{E_{n-1}}{n-1} \sim \Gamma(n - 1, s) \).

The following theorem contains various versions of the M-S-N (see Definition 4.1) when \( A \) is unknown. They will be used in the Kolmogorov-Smirnov statistics for the one-sample case in Section 6.

**Theorem 3.5.** (i) \( \left( \ln \frac{X(2)}{X(1)}, \ln \frac{X(3)}{X(1)}, \ldots, \ln \frac{X(n)}{X(1)} \right) \sim \) Exp(s)-O-S(n - 1); (ii) \( \left( \frac{X(2)}{X(1)}, \frac{X(3)}{X(1)}, \ldots, \frac{X(n)}{X(1)} \right) \sim \) Pa(1, s)-O-S(n - 1);

(iii) \( \left( \frac{X(1)}{X(n-1)}, \frac{X(1)}{X(n-2)}, \ldots, \frac{X(1)}{X(2)} \right) \sim \) U-O-S(n - 1);

(iv) \( \left( \frac{X(1)}{X(2)} \right)^{n-1}, \left( \frac{X(2)}{X(3)} \right)^{n-2}, \ldots, \left( \frac{X(n-1)}{X(n)} \right) \) i.i.d. \( U(0, 1) \);

(v) \( n \ln \frac{X(2)}{X(1)}, (n - 1) \ln \frac{X(3)}{X(2)}, \ldots, \ln \frac{X(n)}{X(n-1)} \) i.i.d. Exp(s).

**Definition 3.3.** For \( 1 \leq j \leq n - 1 \), one defines

(1) \( V_j = \ln \frac{X(j+1)}{X(1)} \), (ii) \( U_j = \frac{X(j+1)}{X(1)} \), (iii) \( W_j = \frac{X(j)}{X(n-j+1)} \);

(iv) \( Y_j = \frac{X(j)}{X(j+1)} \), (v) \( Z_j = (n - 1 - j) \ln \frac{X(j+1)}{X(j)} \).

They will be used throughout the rest of the paper.
The final lemma for this section deals with a two-sample situation.

**Lemma 3.6.** Let \( X_1, X_2, \ldots, X_n \), \( Y_1, Y_2, \ldots, Y_n \) be independent with

\[
X_j \sim \text{Pa}(A_1, s_1) \quad \text{and} \quad Y_j \sim \text{Pa}(A_2, s_2).
\]

Let \( \eta_1 = \sum_{j=1}^{n-1} \ln \left( \frac{X(j+1)}{X(1)} \right) \),

\[
\eta_2 = \sum_{j=1}^{n-1} \ln \left( \frac{Y(j+1)}{Y(1)} \right) ,
\]

then (i) \( X(1), Y(1), \eta_1, \eta_2 \) are independent;

(ii) \( X(1) \sim \text{Pa}(A_1, \frac{m}{m+1}) \), \( Y(1) \sim \text{Pa}(A_2, \frac{n}{n+1}) \); (iii) \( \eta_1 \sim \Gamma(m-1, s_1) \), \( \eta_2 \sim \Gamma(n-1, s_2) \);

(iv) 2m \ln \left( \frac{X(1)}{A_1} \right) \text{ and } 2n \ln \left( \frac{Y(1)}{A_2} \right) \text{ are i.i.d.}

\[
x_2 = \text{Exp}(1/2) ; \quad (v) 2s_1 \eta_1 \sim x_{2n-2}^2, \quad 2s_2 \eta_2 \sim x_{2n-2}^2 \text{ and } 2s_1 \eta_1 + 2s_2 \eta_2 \sim x_{2N-4}^2 \quad \text{where } N = m + n; \text{ and}
\]

\[
\frac{(N - 2) \ln \left( \frac{X(1)}{A_1} \right) - ns_2 \ln \left( \frac{Y(1)}{A_2} \right)}{2s_1 \eta_1 + 2s_2 \eta_2} \sim F(2, 2N-4).
\]

**Remark.** Lemma 3.6 (vi) is useful when \( m = n \) because if \( A_1 = A_2 \)

\( s_1 = s_2 \) then the expression on the left does not involve \( A \) nor \( s \).

[See Section 7.]

The results of this section are summarized in Table 3.1 below.

4. Basic Statistical Concepts

(A) The BDT (Basic Data Transformation) and MNS (Maximal Statistical Noise).
### TABLE 3.1. Distribution Theory for Pareto Renewal Processes

<table>
<thead>
<tr>
<th>Theorem or Lemma</th>
<th>Assumptions</th>
<th>Conclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lemma 3.1.</td>
<td>$\xi_1, \xi_2, \ldots, \xi_k$ i.i.d. Exp($\lambda$)</td>
<td>(i) $k \xi \sim \Gamma(k, \lambda)$ (ii) $\xi^* = \frac{\xi_1}{k \xi}, \frac{\xi_1 + \xi_2}{k \xi}, \ldots, \frac{\sum_{j=1}^{k-1} \xi_j}{k \xi}$ (iii) $U(0,1)-0-S(k-1)$ (iv) $\xi^*$ is independent for $\lambda$.</td>
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</table>

| Lemma 3.2.       | $\eta_1, \eta_2, \ldots, \eta_k \sim \text{Exp($\lambda$)}-0-S(k)$, and $\eta^* = kn_1 \eta_1 = \eta_1 + (k-1)\eta_2, \ldots,$ $\eta_j = \eta_1 + \eta_2 + \ldots + \eta_{j-1} + (k-j+1)\eta_j$, $\eta_k = kn$. | (i) $\eta_k^* \sim \text{M-S-S for } \lambda$, $\eta_k^* \sim \Gamma(k, \lambda)$ (ii) $\eta_k^* = \frac{\eta_k}{\eta_k^*} = \ldots = \frac{\eta_k}{\eta_k^*}$ and $\eta_k^*$ are independent. (iii) $\eta_k^* \sim U(0,1)-0-S(k-1)$. |

| Theorem 3.3.     | $X_1, X_2, \ldots, X_n$ be i.i.d. Exp($A, s$) | (i) $\ln X_1, \ln X_2, \ldots, \ln X_n$ i.i.d. Exp($s$) (ii) $\{\ln X(1), \ln X(2), \ldots, \ln X(n)\} \sim T-\text{Exp}(\ln A, s)-0-S(n)$ (iii) $\{\ln X(1), \ln X(2), \ldots, \ln X(n)\} \sim \text{Exp}(s)-0-S(n)$ (iv) $\{\frac{A}{X(n)}, \frac{A}{X(n-1)}, \ldots, \frac{A}{X(1)}\} \sim P_w(s)-0-S(n)$ (v) $\{\frac{1}{X(n)}, \frac{1}{X(n-1)}, \ldots, \frac{1}{X(1)}\} \sim U(0, A^{-s})-O-S(n)$ (vi) $\{\frac{A}{X(n)}, \frac{A}{X(n-1)}, \ldots, \frac{A}{X(1)}\} \sim U-0-S(n)$. |
### Table 3.1. (Continued)

<table>
<thead>
<tr>
<th>Theorem 3.4.</th>
<th>$X_1, X_2, \ldots, X_n$ i.i.d. $\text{Pa}(A, s)$, define and observe:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i) $T_r = \sum_{j=1}^{r-1} \ln \frac{X_j}{A}$ for $1 \leq r \leq n$</td>
</tr>
<tr>
<td></td>
<td>(ii) $D_r = \sum_{j=1}^{r-1} \ln \frac{X(j+1)}{X(1)} + (n-r) \ln \frac{X(r+1)}{X(1)}$ for $1 \leq r \leq n-1$</td>
</tr>
<tr>
<td></td>
<td>(iii) $E_r = \sum_{j=1}^{r-1} (n+1-j) \ln \frac{X(j+1)}{X(j)}$ for $1 \leq r \leq n-1$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem 3.5.</th>
<th>$X_1, X_2, \ldots, X_n$ i.i.d. $\text{Pa}(A, s)$, for $1 \leq j \leq n-1$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i) $V_j = \ln \frac{X(j+1)}{X(1)}$, (ii) $U_j = \frac{X(j+1)}{X(1)}$</td>
</tr>
<tr>
<td></td>
<td>(iii) $W_j = \left( \frac{X(1)}{X(n-j+1)} \right)^{s}$</td>
</tr>
<tr>
<td></td>
<td>(iv) $Y_j = \left( \frac{X(j)}{X(j+1)} \right)^{(n-j)s}$</td>
</tr>
<tr>
<td></td>
<td>(v) $Z_j = (n+1-j) \ln \frac{X(j+1)}{X(j)}$</td>
</tr>
</tbody>
</table>

| (i) $\mathcal{D} = \left( \frac{D_1}{n-1}, \frac{D_2}{n-1}, \ldots, \frac{D_{n-2}}{n-1} \right) \sim U-O-S(n-2)$ |
| (ii) $\mathcal{T} = \left( \frac{T_1}{n}, \frac{T_2}{n}, \ldots, \frac{T_{n-1}}{n} \right) \sim U-O-S(n-1)$ |
| (iii) $\mathcal{E} = \left( \frac{E_1}{n-1}, \frac{E_2}{n-1}, \ldots, \frac{E_{n-2}}{n-1} \right) \sim U-O-S(n-2)$ |
| (iv) $\mathcal{T}$ and $\mathcal{D}$ are independent and $\mathcal{D}$ is $\Gamma(n-1, s)$. |
| (v) $\mathcal{T}$ and $\mathcal{T}$ are independent and $\mathcal{T}$ is $\Gamma(r, s)$ for $1 \leq r \leq n$. |
| (vi) $\mathcal{E}$ and $\mathcal{E}$ are independent and $\mathcal{E}$ is $\Gamma(j, s)$ for $1 \leq j \leq n-1$. |

| (i) $V_1, V_2, \ldots, V_{n-1} \sim \text{Exp}(s)$-O-S(n-1); |
| (ii) $U_1, U_2, \ldots, U_{n-1} \sim \text{Pa}(1, s)$-O-S(n-1); |
| (iii) $W_1, W_2, \ldots, W_{n-1} \sim U-O-S(n-1)$; |
| (iv) $Y_1, Y_2, \ldots, Y_{n-1} \sim \text{i.i.d.} \ U(0, 1)$; |
| (v) $Z_1, Z_2, \ldots, Z_{n-1} \sim \text{i.i.d.} \ \text{Exp}(s)$. |
| Lemma 3.6 | \[X_1, X_2, \ldots, X_m \text{ i.i.d. } \text{Pa}(A_1, s_1)\] \[\text{independent of } Y_1, Y_2, \ldots, Y_n\] \[\text{i.i.d. } \text{Pa}(A_2, s_2)\] \[\eta_1 = \sum_{j=1}^{m-1} \ln \frac{X(j+1)}{X(1)}\] \[\eta_2 = \sum_{j=1}^{n-1} \ln \frac{Y(j+1)}{Y(1)}\] | (i) \(X(1), Y(1), \eta_1, \eta_2\) are independent; \[(ii) X(1) \sim \text{Pa}(A_1, ms_1), Y(1) \sim \text{Pa}(A_2, ns_2);\] \[(iii) \eta_1 \sim \Gamma(m-1, s_1), \eta_2 \sim \Gamma(n-1, s_2);\] \[(iv) 2ms_1 \ln \frac{X(1)}{A_1}, 2ns_2 \ln \frac{Y(1)}{A_2}\] \[\text{are i.i.d. } \chi^2_2;\] \[(v) 2s_1 \eta_1 \sim X_{2m-2}^2, 2s_2 \eta_2 \sim X_{2n-2}^2, 2s_1 \eta_1 + 2s_2 \eta_2 \sim X_{2N-4}^2;\] \[(vi) (N-2)ms_1 \ln \frac{X(1)}{A_1} - ns_2 \ln \frac{Y(1)}{A_2} \sim [2s_1 \eta_1 + 2s_2 \eta_2]^{-1}\] \[\sim F(2, 2N-4)\] |
Let the generic data point be denoted by \( Z = (X_1, \ldots, X_n) \) in the one-sample cases and by \( Z = (X_1, \ldots, X_m, Y_1, \ldots, Y_n) = (Z_1, \ldots, Z_N) \) in the two-sample cases; and let \( S(Z) \) denote the M-S-S.

**Definition 4.1.** Let \( N(Z) \) be a (vector-valued) statistic independent of \( S(Z) \) and such that \( \delta(Z) = [S(Z), N(Z)] \) is 1-1 a.e. Then, (i) \( \delta(\cdot) \) is called the BDT; and (ii) \( N(Z) \) is called the M-S-N.

It is known that \( S(Z) \) contains all relevant information about the parameter (vector); and it will be seen that \( N(Z) \) contains all relevant information about the structure of the process. From \( \delta(Z) = [S(Z), N(Z)] \), one should almost always be able to reconstruct the original data, \( Z \).

**Example 4.1.** (i) In Lemma 3.1, \( S(\xi) = k\xi \); \( N(\xi) = \xi_* \), and \( \delta(\xi) = [k\xi, \xi_*] \) is the BDT. From \( \delta(\xi) \), one can reconstruct \( \xi \).

(ii) In Lemma 3.2, \( S(\eta) = \eta_* k \) and \( N(\eta) = \eta^{**} \). Hence \( \delta(\eta) = \{\eta_* k, \eta^{**}\} \).

From the examples above, it should be clear that there are several possible versions of the M-S-S and M-S-N.

**Remark.** The importance of these concepts is that as a rule of thumb in a goodness-of-fit problem, the decision rule should involve only the M-S-S while in a class-fit problem, the decision rule should only involve M-S-N. This will be seen in the sequel.

(B) **Distribution-free-ness.**

There are two types of distribution free statistics that arise in detection procedures.
Definition 4.2. (i) A statistic \( T(Z) \) is called nonparametric distribution free (NPDF) with respect to a family \( \Omega^* \) of stochastic process laws if there exists a single distribution \( Q(\cdot) \) such that

\[
P\{T(Z) \leq t | L\} = Q(t) \quad \text{for all } L \in \Omega^*,
\]

(ii) A family of statistics, \( \{T_{1}(Z; L)\} \) indexed by the members \( L \in \Omega^* \) is called parametric distribution free (PDF) with respect to \( \Omega^* \) if there exists a distribution function \( Q_1(\cdot) \) such that,

\[
P\{T_{1}(Z; L) \leq t\} = Q_1(t) \quad \text{for all } L \in \Omega^*.
\]

It is clear that each NPDF statistic is PDF.

Example 4.2. Let \( \Omega^* = \Omega(\text{PRP}) \), and \( Z = (Z_1, \ldots, Z_N) \) be the first \( N \) interarrival times with \( Z_j = X_j \) for \( 1 \leq j \leq m \), and \( Z_m + r = Y_r \) for \( 1 \leq r \leq n \), where \( N = m + n \). Let \( \hat{s}_1 \) and \( \hat{s}_2 \) be the MLE's given by Theorem 2.2; \( T_1 = \frac{m^2}{n \hat{s}_1^2} \) and \( T_2 = \frac{2mn}{\hat{s}_2} \). Then, \( T_1 \)

is NPDF wrt \( \Omega^* \) with \( Q = F(2m, 2n) \); and \( T_2 \) is PDF wrt \( \Omega^* \) with \( Q_1 = X_{2m}^2 \). Furthermore, it can be shown that \( T_1 \) is a function of the \( Z \) only through the M-S-N, \( N(Z) \); and \( T_2 \) is a function of \( X = (X_1, \ldots, X_m) \) only through the M-S-S, \( S(X) \).

The M-S-N for the respective cases are given in Table 2.1. The relations between the M-S-S, M-S-N and DF statistics is best given by the following theorem.
Theorem 4.1. Let \( \Omega^* \) be a family of cdfs admitting a M-S-S, \( S(Z) \), for data \( Z = (Z_1, \ldots, Z_n) \). Then, (i) \( T(\psi, Z) = \psi(N(Z)) \) is NPDF wrt \( \Omega^* \) for each (measurable) function \( \psi(\cdot) \); and (ii) \( T^*(\psi^*, G, S(Z)) = \psi^*[G, S(Z)] \), when \( Z \) is governed by \( G \), is PDF wrt \( \Omega^* \) for each (measurable) function \( \psi^*(\cdot) \).

5. The K-S (Kolmogorov-Smirnov), Lilliefors and Srinivasan Statistics

(A) Kolmogorov's Original Statistic

Kolmogorov (1933) introduced the K-S statistic \( D_n(F_0) = \sup_{z} |F_n(z) - F_0(z)| \), for continuous cdfs \( F_0(\cdot) \) and empirical cdfs \( F_n(\cdot) \), where \( F_n(z) = \frac{1}{n} \sum_{j=1}^{n} \varepsilon(z - X_j) = \frac{1}{n} \sum_{j=1}^{n} \varepsilon(z - X(j)) \), and \( \varepsilon(u) = 1 \) if \( u \geq 0 \); and \( = 0 \) if \( u < 0 \).

Definition 5.1. If \( X_1, \ldots, X_n \) are i.i.d. \( F_0 \), continuous, then \( D_n(F_0) \sim K-S(n) \).

In order to apply the K-S statistic directly, one must know \( F_0(\cdot) \) completely. However, in many signal detection problems, \( F_0(\cdot) \) is known only up to a nuisance parameter, or, equivalently is known only to be a member of a specific (parametric) family. Lilliefors (1967, 1969), Srinivasan (1970), and Choi (1980) introduced modified versions of the K-S statistic for such situations.

(B) Lilliefors-type Statistics
Let \( \Omega^* = \{ F(\theta; \cdot) : \theta \in \Theta \} \) be a family of cdfs admitting a MLE, \( \hat{\theta} = \hat{\theta}(x_1, \ldots, x_n) \), for \( \theta \).

**Definition 5.2.** (1) \( \hat{F}(\cdot) \) is the cdf satisfying \( \hat{F}(z) = F(\hat{\theta}; z) \) for all \( z \); and

(ii) \( \hat{D}_n = \sup_z |F_n(z) - \hat{F}(z)| \).

Lilliefors (1967, 1969) calculated Monte Carlo tables for \( \hat{D}_n \) in the normal and exponential cases, while Y. Choi (1980) has given such a table in the uniform case.

Srinivasan (1970) replaces \( \hat{F}(\cdot) \) with the Rao-Blackwell estimate of \( F(\theta; \cdot) \) in the Lilliefors statistic.

(C) **Srinivasan-type Statistics**

Consider a family \( \Omega^* = \{ F(\theta; \cdot) : \theta \in \Theta \} \) of cdfs admitting a M-S-S, \( S(\cdot) \) for \( \theta \).

**Definition 5.3.** (1) \( \hat{\nu}(z) = F(X_1 < z | S(z)) \); and

(ii) \( \hat{D}_n = \sup_z |F_n(z) - \hat{\nu}(z)| \).

Srinivasan (1970) computed critical values of \( \hat{D}_n \) by (Monte Carlo) simulation for the exponential and normal families. Some of his numerical results however were in error, as was pointed out by Schafer, Finkelstein and Collins (1972).

**Remark.** These three statistics are in many cases, asymptotically equivalent. Calculations of \( \hat{\nu}_n(z) \) for the case when \( \lambda \) is unknown, \( s = s_0 \) known and the case when both \( \lambda \) and \( s \) are unknown are presented in Appendix A. These statistics are summarized in Table 5.1.
TABLE 5.1. Kolmogorov-Smirnov, Lilliefors, Srinivasan Statistics for Pareto Renewal Processes

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $Pa(A,s)$.

<table>
<thead>
<tr>
<th>K-S Statistic with $M$-$S$-$N$: $D_n^{(1)}$</th>
<th>Lilliefors Statistic: $\hat{D}_n$</th>
<th>Srinivasan Statistic: $\nu_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_n^{(1)} = \sup_{z} \left</td>
<td>\frac{1}{n} \sum_{j=1}^{n-1} \varepsilon (z - Q_j^{(1)}) - F_j(z) \right</td>
<td>$</td>
</tr>
<tr>
<td>$A$ unknown, $s = s_0$ known</td>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td>where $F_n(z) = 1 - \left( \frac{n-1}{n} \right) \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
</tr>
<tr>
<td>$Q_j^{(1)} = V_j$, $F_1(z) = 1 - e^{-s_0 z}$</td>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td></td>
</tr>
<tr>
<td>$Q_j^{(2)} = U_j$, $F_2(z) = 1 - z^{-s_0}$</td>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td></td>
</tr>
<tr>
<td>$Q_j^{(3)} = W_j$, $F_3(z) = z$</td>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td></td>
</tr>
<tr>
<td>$Q_j^{(4)} = Y_j$, $F_4(z) = z$</td>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td></td>
</tr>
<tr>
<td>$(V_j), (U_j), (W_j), (Y_j)$ defined in Definition 3.2.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a = A_0$ known, $s$ unknown</td>
<td>$D_n = \sup_{z} \left</td>
<td>\frac{1}{n} \sum_{j=1}^{n-1} \varepsilon (z - T_j) - z \right</td>
</tr>
<tr>
<td>$D_n = \sup_{z} \left</td>
<td>\frac{1}{n} \sum_{j=1}^{n-1} \varepsilon (z - T_j) - z \right</td>
<td>$</td>
</tr>
<tr>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td>$\hat{F}_n(z) = 1 - \left( \frac{X(1)}{z} \right)^{s_0}$ for $z &gt; X(1)$</td>
<td></td>
</tr>
<tr>
<td>$(T_j)$ is defined in Definition 3.2.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\delta = n \left( \frac{\sum_{j=1}^{n} \ln X_j - n \ln A_0}{n \ln z - n \ln A_0} \right)^{-1}$

$\delta' = \frac{n}{\sum_{i=1}^{n} \ln X_i - n \ln A_0}$

$\delta = n \left( \frac{\sum_{j=1}^{n} \ln X_j - n \ln A_0}{n \ln z - n \ln A_0} \right)^{-1}$

$\delta' = \frac{n}{\sum_{i=1}^{n} \ln X_i - n \ln A_0}$
<table>
<thead>
<tr>
<th>$A_s$ both unknown</th>
<th>$D_n^{(1)} = \sup_z \frac{1}{n^{1/2}} \sum_{j=1}^{n-2} \varepsilon (z - \frac{D_j}{n})^2 - z \varepsilon \frac{D_j}{n-1}$</th>
<th>$\hat{D}_n$ same as above with $\hat{F}<em>n(z) = 1 - \left( \frac{X(1)}{z} \right)^{\delta}$ for $z &gt; X(1)$ where $\delta = n \left[ \sum</em>{j=1}^{n} \ln X_j - n \ln X(1) \right]^{-1}$</th>
<th>$\check{D}_n$ same as above with $\check{F}<em>n(z) = \left{ \begin{array}{ll} 0 &amp; z &lt; X(1) \ 1/n &amp; z = X(1) \ 1 - \left( \frac{n-1}{n} \right) \left[ 1 - \frac{\delta}{n} \ln \frac{z}{X(1)} \right]^{n-1} &amp; X(1) &lt; z &lt; X(1)e^{\delta} \ 1 &amp; z &gt; X(1)e^{\delta} \end{array} \right.$ where $\delta = n \left[ \sum</em>{i=1}^{n} \ln X_i - n \ln X(1) \right]^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_n^{(2)} = \sup_z \frac{1}{n^{1/2}} \sum_{j=1}^{n-2} \varepsilon (z - \frac{E_j}{n})^2 - z \varepsilon \frac{E_j}{n-1}$ where ${D_j}, {E_j}$ are defined in Definition 3.2.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
One can now treat the detection problems.

6. One-Sample Detection Procedures.

In this section, one will derive detectors for deciding between PW and the alternative N + S (noise plus signal). The data set will be denoted by \( z = (x_1, x_2, ..., x_n) \) and \( \alpha \) is the PFA. There are altogether 8 problems considered. The first 4 deal with the cases when at least one of \( A \) and \( s \) is known and the last 4 concern cases in which both \( A \) and \( s \) are unknown. Numerical examples for each case are provided in the Appendix B, where they are numbered the same as the cases they illustrate. The results in this section are summarized in Table 6.1 at the end of this section.

**Case 6.1.** (A unknown, s = s\(_0\) known, \( L(A,s_0) \in \Omega(\text{PRP}) \))

\[ \text{PW: } A = A_0 \quad \text{vs.} \quad \text{N + S: } A \neq A_0 \]

The minimum PDF procedure is to decide \( N + S \) if and only if

\[ x(1) < A_0 \quad \text{or} \quad x(1) > bA_0 \]

where \( b = \alpha^{1/n_0} \). Equivalently, one can use the statistic

\[ T = 2n_0 s_0 \ln \frac{x(1)}{A_0} \]
which has a $\chi^2$ distribution under PN. Note that the procedure is based solely on the W-S-S for $A$, which is $\hat{A} = X(1)$. The one-sided detection procedure for PN: $A \leq A_0$ vs. $N+S$: $A > A_0$ may be formulated similarly.

Case 6.2. (A unknown, $s = s_0$ known; class-fit problem)

PN: $L(A, s_0) \in \Omega(\text{PRP})$ vs. $N+S$: $L(A, s_0) \notin \Omega(\text{PRP}).$

One employs the K-S statistic with size $n - 1$ (K-S(n - 1)) through anyone of the four versions of W-S-N, $\{V_j\}$, $\{U_j\}$, $\{W_j\}$, $\{Y_j\}$ given in Theorem 3.5 and Definition 3.3. Explicitly, they are

$$D_n^{(1)} = \sup_{z > 0} \frac{1}{n-1} \sum_{j=1}^{n-1} \epsilon (z - V_j) - (1 - e^{-s_0 z})|$$

$$D_n^{(2)} = \sup_{z > 1} \frac{1}{n-1} \sum_{j=1}^{n-1} \epsilon (z - U_j) - (1 - z^{-s_0})|$$

$$D_n^{(3)} = \sup_{0 < z < 1} \frac{1}{n-1} \sum_{j=1}^{n-1} \epsilon (z - W_j) - z|$$

$$D_n^{(4)} = \sup_{0 < z < 1} \frac{1}{n-1} \sum_{j=1}^{n-1} \epsilon (z - Y_j) - z|$$

where $D_n^{(1)} \sim \text{K-S(n - 1)}$ for $i = 1, 2, 3, 4$. Therefore one decides $N+S$ if and only if $D_n^{(1)} > d'$ where $d'$ is an appropriate critical value from the K-S(n - 1) distribution.

Alternatively, one may use the Srinivasan-type statistic (see Table as follows.
Case 6.3. \((A = A_0 \text{ known, } s \text{ unknown, } L(A_0, s) \in \Omega(\text{PRP}))\)

\[
\begin{align*}
D_n &= \left(\frac{n-1}{n}\right) \sup_{0 < u < 1} \left| \frac{1}{n-1} \sum_{j=1}^{n-1} (u - U_j) - (1 - u^{-s_0}) \right|.
\end{align*}
\]

Note that \(D_n \overset{d}{=} \left(\frac{n-1}{n}\right) D_{n-1}^{(2)}\).

The Lilliefors-type statistic is asymptotically equivalent to the Srinivasan-type statistic in this case.

Case 6.4. \((A = A_0 \text{ known, } s \text{ unknown; class-fit problem})\)

\[
D_n = \left(\frac{n-1}{n}\right) \sup_{0 < u < 1} \left| \frac{1}{n-1} \sum_{j=1}^{n-1} (u - U_j) - (1 - u^{-s_0}) \right|.
\]

Thus let

\[
D_n = \sup_{0 < u < 1} \left| \frac{1}{n-1} \sum_{j=1}^{n-1} (u - U_j) - (1 - u^{-s_0}) \right|.
\]

and one decides
N + S if and only if \( D_n > d' \) where \( d' \) is the appropriate critical value from the K-S(n - 1) table.

**Case 6.5.** (A, s both unknown, \( L(A, s) \in \Omega(\text{PRP}) \); Goodness-of-fit test)

\[
\text{PN: } L(A, s) = L(A_0, s_0) \quad \text{vs.} \quad N + S: \quad L(A, s) \neq L(A_0, s_0)
\]

From Theorem 3.3 (vi) and under PN situation,

\[
\{ \left( \frac{A_0}{X(n)} \right)^{s_0}, \left( \frac{A_0}{X(n-1)} \right)^{s_0}, \ldots, \left( \frac{A_0}{X(1)} \right)^{s_0} \} \sim \text{U-O-S}(n).
\]

The statistic is (as in Case 6.3).

\[
T = -2 \sum_{j=1}^{n} \ln\left( \frac{A_0}{X(j)} \right)^{s_0} \sim \chi^2_{2n} \quad \text{and the decision rule is decide } N + S \text{ if and only if } T > \chi^2_{(2n, 1 - \alpha/2)} \text{ or } T < \chi^2_{(2n, \alpha/2)}.
\]

**Case 6.6.** (A and s unknown, \( L(A, s) \in \Omega(\text{PRP}) \))

\[
\text{PN: } s < s_0 \quad \text{vs.} \quad N + S: \quad s > s_0
\]

The detection statistic here should only involve the N-S-S

\[
S'(2) = \hat{s} = \sum_{j=1}^{n} \ln\left( \frac{X(j)}{X(1)} \right)^{-1} \text{ for } s.
\]

The decision rule is: Decide N + S iff \( \frac{2s_0}{\hat{s}} < c' \), where \( c' \) is the (100\( \alpha \))th percentile of the \( \chi^2_{(2n-2)} \)-distribution.

One notes that if \( Y_j = \ln X_j \), then the Y's are i.i.d. T-exp \((\ln A_s)\). Park and U. Choi (1978) derive the minimum PFD one-sided
procedures for the shape parameter $s$. It will be shown in Appendix A.3, that the Park-Choi decision rule is equivalent to the one given above.

For the two-sided detection problem: \[ \mathbb{P}: s = s_0 \text{ vs. } \mathbb{N} + \mathbb{S}: s \neq s_0, \] the decision rule follows from that above.

Case 6.7. (A, s unknown; $L(A,s) \in \Omega(\text{PRP})$)

\[ \mathbb{P}: A < A_0 \text{ vs. } \mathbb{N} + \mathbb{S}: A > A_0 \]

One should employ the M-S-S, $(\hat{A}, \hat{s})$, and the statistic

\[
T(Z, A) = \frac{n(n - 1)(\ln X(1) - \ln A)}{\sum_{j=1}^{n} \ln X_j - n \ln X(1)} \sim F(2, 2n - 2).
\]

The decision rule is: Decide $\mathbb{N} + \mathbb{S}$ iff $X(1) > A_0$ and $T(Z, A_0) > f'$, where $f'$ is the $100(1 - \alpha)$th percentile of the $F(2, 2n - 2)$-distribution.

Again, as in Case 6.6, Park and U. Choi (1978) give a minimum PDF procedure for the truncated exponential case, which is equivalent to the decision rule above.

Case 6.8. (A, s unknown; class-fit problem)

\[ \mathbb{P}: L(A,s) \in \Omega(\text{PRP}) \text{ vs. } \mathbb{N} + \mathbb{S}: L(A,s) \notin \Omega(\text{PRP}). \]

One can use the Kolmogorov-Smirnov, Lilliefors or Srinivasan-type
statistics. From Theorem 3.4, one finds
\[
D^{(1)}_n = \sup_{0 < z < 1} \left| \frac{1}{n-2} \sum_{j=1}^{n-2} e (z - \frac{D_j}{\sqrt{n-1}}) - z \right|
\]
\[
D^{(2)}_n = \sup_{0 < z < 1} \left| \frac{1}{n-2} \sum_{j=1}^{n-2} e (z - \frac{F_j}{\sqrt{n-1}}) - z \right|
\]
then \( D^{(i)}_n \sim K-S(n-2) \) for \( i = 1, 2 \). The decision rule is decide \( N+S \) if and only if \( D^{(i)}_n > d' \) where \( d' \) is the critical value from the \( K-S(n-2) \) table.

One now considers the Srinivasan statistic. From Table 5.1, one has
\[
\hat{D}_n = \sup_z |F_n(z) - \hat{F}_n(z)|
\]
\[
= \sup_{0 < z < \hat{s} s'} |F_n(z) - \hat{F}_n(z)|
\]
where \( \hat{A} = X(1) = a, \hat{s} = s' \). Let \( V = \frac{z}{X(1)} \), \( V(j) = \frac{X(j+1)}{X(1)} \). Then
\[
\hat{D}_n = \sup_{1 < v < e^{s'/s}} \left| \frac{1}{n} \sum_{j=1}^{n-1} e \left( \frac{z}{X(1)} - \frac{X(j)}{X(1)} \right) - 1 + \left( \frac{n-1}{n} \right) \left( 1 - \frac{s'}{n} \ln v \right) n^{-1} \right|
\]
\[
= \sup_{1 < v < e^{s'/s}} \left| \frac{1}{n} \sum_{j=1}^{n-1} e \left( \ln v - \ln V(j) \right) \right| - 1 + \left( \frac{n-1}{n} \right) \left( 1 - \frac{s'}{n} \ln v \right) n^{-1} \right|
\]
Let \( u = \ln v \), \( U(j) = \ln V(j) \), then since \( s = \frac{n}{n-1} = \frac{n}{\sum_{j=1}^{n-1} \ln V_j} \),
\[
\hat{D}_n = \left( \frac{n-1}{n} \right) \sup_{0 < u < n/s'} \left| \frac{1}{n} \sum_{j=1}^{n-1} e (u - U(j)) - 1 + \left( \frac{n-1}{n} \right) \left( 1 - \frac{s'}{n} u \right) n^{-1} \right|
\]
The critical values of this statistic cannot be obtained from the known existing tables. They may be obtained by the Monte Carlo simulation method for various sample sizes and PFA. These statistics are summarized in Table 6.1.

7. Two-Samples Detection Procedures

The data set here is \( Z = (X, Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n) = (Z_1, Z_2, \ldots, Z_N) \) where \( N = m + n \). Here \( X, Y \) are two independent random samples from \( Pa(A_1, s_1) \) and \( Pa(A_2, s_2) \) respectively. The letter \( \alpha \) will denote the PFA. As in the one-sample case, numerical examples are given for each case in Appendix B with the same order and numberings as they are presented here.

**Case 7.1.** \((A_1, A_2 \text{ known}, s_1, s_2 \text{ unknown})\)

**PN:** \( s_1 = s_2 \) \text{ vs. } **N + S:** \( s_1 \neq s_2 \)

The M-S-S for \((s_1, s_2)\) is \((\hat{s}_1, \hat{s}_2) = (m\left[ \sum_{j=1}^{m} \ln \left( \frac{X_j}{A_1} \right) \right]^{-1}, n\left[ \sum_{j=1}^{n} \ln \left( \frac{Y_j}{A_2} \right) \right]^{-1})\)
<table>
<thead>
<tr>
<th>Case</th>
<th>Assumptions</th>
<th>PN</th>
<th>N + S</th>
<th>Statistics; PN-distribution</th>
<th>Decision Rule: ( \text{Decide } N + S \text{ iff} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>A unknown s (= s_0 ) known ( L(A, s_0) \in \Omega(\text{PRP}) )</td>
<td>( A = A_0 )</td>
<td>( A \neq A_0 )</td>
<td>(i) ( X(1) ) or (ii) ( T = 2ns_0 \ln \frac{X(1)}{A_0} - b )</td>
<td>( \text{(i) } X(1) &lt; A_0 \text{ or } X(1) &gt; bA_0, ) ( b = \frac{1}{ns_0} ) ( \text{(ii) } T &lt; 0 \text{ or } T &gt; \chi^2_{2n, (1-\alpha)} )</td>
</tr>
<tr>
<td>6.2</td>
<td>A unknown s (= s_0 ) known ( L(A, s_0) \in \Omega(\text{PRP}) )</td>
<td>( L(A, s_0) \notin \Omega(\text{PRP}) )</td>
<td>( L(A, s_0) \notin \Omega(\text{PRP}) )</td>
<td>The K-S statistic ( D_n^{(i)} ), i = 1, 2, 3, 4 are defined in Table 5.1</td>
<td>( D_n^{(i)} &gt; d^<em>, i = 1, 2, 3, 4, \text{d</em> appropriate critical value from K-S(n-1) table.} )</td>
</tr>
<tr>
<td>6.3</td>
<td>A = ( A_0 ) known s unknown ( L(A_0, s) \in \Omega(\text{PRP}) )</td>
<td>( s = s_0 )</td>
<td>( s \neq s_0 )</td>
<td>( T = 2s_0 \sum_{j=1}^{n} \ln \frac{X(i)}{A_0} - \chi^2_{2n} )</td>
<td>( T &gt; \chi^2_{2n, 1-\alpha/2} \text{ or } \chi^2_{2n, \alpha/2} )</td>
</tr>
<tr>
<td>6.4</td>
<td>A = ( A_0 ) known s unknown ( L(A_0, s) \in \Omega(\text{PRP}) )</td>
<td>( L(A_0, s) \notin \Omega(\text{PRP}) )</td>
<td>( L(A_0, s) \notin \Omega(\text{PRP}) )</td>
<td>( D_n = \text{sup}_{0 &lt; z &lt; 1} \left</td>
<td>1 - \frac{1}{n} \sum_{j=1}^{n-1} \epsilon(z - \frac{j}{n}) \right</td>
</tr>
<tr>
<td>6.5</td>
<td>A unknown s unknown ( L(A, s) \in \Omega(\text{PRP}) )</td>
<td>( L(A, s) = L(A_0, s_0) )</td>
<td>( L(A, s) \neq L(A_0, s_0) )</td>
<td>( T = -2 \ln \frac{A_0}{X(1)} s_0 \chi^2_{2n} )</td>
<td>( T &gt; \chi^2_{2n, 1-\alpha/2} \text{ or } \chi^2_{2n, \alpha/2} )</td>
</tr>
<tr>
<td>6.6</td>
<td>A unknown s unknown ( L(A, s) \in \Omega(\text{PRP}) )</td>
<td>( s \leq s_0 )</td>
<td>( s &gt; s_0 )</td>
<td>( T = 2s_0 \left{ \sum_{j=1}^{n} \ln X_j - n \ln X(1) \right} )</td>
<td>( T &lt; \chi^2_{2n-2, \alpha} )</td>
</tr>
</tbody>
</table>
| 6.7 | A unknown  
\( L(A, s) \in \Omega \) (PRP) | \( A \leq A_0 \) | \( A > A_0 \) | \( T = \frac{n(n-1)[\ln X(1)-\ln A_0]}{\sum_{1}^{n} \ln X_j - n \ln X(1)} \) | \( X(1) > A_0 \) and 
\( T > f_{2,2n-2;1-\alpha} \) |
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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>6.8</td>
<td>A, s unknown</td>
<td>( L(A, s) \in \Omega ) (PRP)</td>
<td>( L(A, s) \notin \Omega ) (PRP)</td>
<td>( D^{(1)} = \frac{1}{n} \sup_{0 &lt; z &lt; 1} \frac{1}{n-2} \sum_{j=1}^{n-2} \epsilon(z - \frac{D_j}{n-1}) - z )</td>
<td>( D^{(i)} &gt; d^* ) for ( i = 1 ) or ( 2 ), ( d^* ) appropriate critical value from K-S(n-2) table.</td>
</tr>
</tbody>
</table>
and under $P_N$, $T^* = \frac{\sum_{j=1}^{m} \ln \frac{X_j}{A_1} - \sum_{j=1}^{n} \ln \frac{Y_j}{A_2}}{\sum_{j=m+1}^{n} \ln \frac{A_1}{A_2}} \sim F(2m, 2n)$. Therefore, decide $N + S$ if and only if $T > f'$ or $T < f''$ where $f'$ and $f''$ are the appropriate percentiles of the $F(2m, 2n)$-distribution.

An important point arises in Case 7.1 above. The detection statistic $T^*$ involved the M-S-S ($\hat{s}_1, \hat{s}_2$) for $(s_1, s_2)$, the unknown parameter pair. However, the remark following Example 4.1, and the cases of Section 6, suggest that for Case 7.1, one should employ the M-S-N, since the particular values of $s_1$ and $s_2$ are not pertinent here. This, indeed, is the case, as will be seen from the derivation to follow.

One can directly verify (from Lemma 3.1)

**Theorem 7.1.** Let $W_j = \ln \frac{Z_j}{A_1}$ for $1 \leq j \leq m$; and $V_r = \frac{1}{\sum_{1}^{N} W_j} \left[ \sum_{1}^{N} W_j \right]^{-1}$ for $m + 1 \leq j \leq N$; and $V_r = \frac{1}{\sum_{1}^{N} W_j} \left[ \sum_{1}^{N} W_j \right]^{-1}$ for $1 \leq r \leq N - 1$. Then

(i) $(W_1, \ldots, W_N)$ are i.i.d. $\text{Exp}(s)$.

(ii) $(V_1, \ldots, V_{N-1})$ is the M-S-N; and is $\sim \text{U-O-S}(N - 1)$. Further,

(iii) $T^* = \left( \frac{N}{m} \right) \left( \frac{m}{1-V_m} \right)$.

This means that $T^*$ is a function of the data, $Z$, solely via the M-S-N of the combined sample. Hence, $T^*$ is both a function of the
M-S-S's for the individual samples and the M-S-N of the combined samples.

The one-sided detection problems may be handled analogously.

Case 7.2. \((A_1, A_2 \text{ unknown, } s_1, s_2 \text{ known})\)

PN: \(A_1 = A_2\) vs. \(N + S: A_1 \neq A_2\)

In this case, the M-S-S for \((A_1, A_2)\) is \((\hat{A}_1, \hat{A}_2) = (X(1), Y(1))\).

Since \(\ln\left(\frac{X(1)}{A_1}\right) \sim \text{Exp}(ms_1), \ln\left(\frac{Y(1)}{A_2}\right) \sim \text{Exp}(ns_2)\), one has by Lemma 3.6 the following,

Lemma 7.2. Let \(T = \ln\left(\frac{Y(1)}{X(1)}\right)\). Then under PN, the distribution of \(T\) is given by

\[
H(t) = \begin{cases} 
1 - \frac{ms_1}{(ms_1 + ns_2)} e^{-ns_2 t} & t \geq 0 \\
\frac{ns_2}{(ms_1 + ns_2)} e^{ms_1 t} & t < 0
\end{cases}
\]

The proof of this lemma is given in Appendix A. Thus the decision rule is: Decide \(N + S\) if and only if \(T > C_1\) or \(T < C_2\) where \(C_1, C_2\) are determined by \(H(C_1) = 1 - \frac{\alpha}{2}\), and \(H(C_2) = \frac{\alpha}{2}\).

In the special case where \(s_1 = s_2 = s_0\) is known, the decision rule reduces to: Decide \(N + S\) iff \(\frac{Y(1)}{X(1)} < b_1\) or \(> b_2\) where

\[
b_1 = \left[\frac{N}{n} \left(\frac{\alpha_1}{2}\right)\right]^{\frac{1}{ms_0}} \text{ and } b_2 = \left[\frac{N}{n} \left(\frac{\alpha_2}{2}\right)\right]^{\frac{1}{ns_0}}.
\]
One further notes that $X^S \sim U(0, \theta)$ with $\theta = A^{-S}$. For this uniform case, Y. Choi, Bell, Ahmad and Park (1982) present a detection procedure which coincides with the special case above.

**Case 7.3.** $(A_1 = A_2 = A, \text{ unknown}; \ s_1, \ s_2 \text{ unknown})$

$$\text{PN: } s_1 = s_2 \quad \text{vs.} \quad \text{N + S: } s_1 \neq s_2$$

One first attempts to base the decision rule on the theorem below.

**Theorem 7.3.** Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be i.i.d. $P(A,s)$. Then $(i)\ X(1), Y(1), \eta_1 = \sum_{j=1}^{n-1} \ln \frac{X(j+1)}{X(1)}$ and $\eta_2 = \sum_{j=1}^{n-1} \ln \frac{Y(j+1)}{Y(1)}$ are independent; with

(ii) $2ns|\ln X(1) - \ln Y(1)| \sim \chi^2_2$; and

(iii) $2sn_1 \overset{d}{=} 2sn_2 \sim \chi^2_{2n-2}$

The decision rule for this case, with $m = n$, would then be:

Decide N + S iff

$$Q = \frac{2ns|\eta_1 - \eta_2|}{2ns|\ln X(1) - \ln Y(1)|} = \frac{|\eta_1 - \eta_2|}{|\ln X(1) - \ln Y(1)|} > C^*$$

However, the cdf of $Q$ is not known, and, hence one seeks other approaches.

Beg (1980) derives a uniformly minimum PFD procedure for the case:

$$\text{PN: } s_1 \leq s_2 \quad \text{vs.} \quad \text{N + S: } s_1 > s_2$$

in the truncated-exponential case, which applies to the Pareto problem.
at hand even when $m \neq n$.

Let $\eta_1$ and $\eta_2$ be as above; $\eta = \sum_{i=1}^{m} \ln X_i + \sum_{j=1}^{n} \ln Y_j - NW$,

where $N = n + m$ and $W = \min\{\ln X(1), \ln Y(1)\}$. Beg proves

**Lemma 7.4.** The conditional density of $\eta_1$, given $W$ and $\eta$, is

$$h(\eta_1|w, \eta) = \frac{(m-n-2)!n_1^{m-2}(n - \eta_1)^{n-1}}{(m-2)!(n-1)! \eta^{m+n-2}}$$

for $0 < \eta_1 < \eta$.

The decision rule based on this lemma becomes: Decide $N + S$ iff

$\eta_1 < c = c(w, \eta)$ where $\int_{c}^{\eta} h(\eta_1|w, \eta) d\eta_1 = \alpha$.

If one performs the actual integration, it is easily seen that $c$ satisfies the relation $c = c'z$ where $I_c(m-1, n) = 1 - \alpha$,

$$I_c(m,n) = \frac{1}{B(m,n)} \int_{0}^{c} y^{m-1}(1 - y)^{n-1} dy$$

is the incomplete Beta-function. A table for the incomplete Beta function has been tabulated by K. Pearson (1934).

**Case 7.4.** ($A_1$, $A_2$ unknown, $s_1 = s_2 = s$ unknown)

$$PN: A_1 < A_2 \quad vs. \quad N + S: A_1 > A_2$$

Let $\eta_1$, $\eta_2$, $W$, $\eta$ be as defined in Case 7.3. Let $N = m + n.$
$$h(x^*|w, n) = \begin{cases} \frac{m}{N} & \text{if } x^* = m \\ \frac{mn(N-2)}{Nn} \left[ 1 - \frac{m(x^*-w)}{n} \right]^{N-3} & \text{if } w < x^* < w + \frac{n}{m}, \end{cases}$$

and define the number $c = c(w, n)$ by

$$\int_{M}^{W} h(x^*|w, n) dx^* = a.$$ 

From the result of Beg (1980), the decision rule is decide $N + S$ if and only if $\ln X(1) > c$. The number $c$ may be found by performing the actual integration in which case one gets

$$c = \frac{nc'}{n} + w \quad \text{where} \quad c' = 1 - \left( \frac{N}{n} \right)^{1-N}$$

**Remark:** Suppose $m = n = \frac{N}{2}$, then Lemma 3.6 (vi) may be used to test $\overline{\text{PN: } s_1 = s_2}$ vs. $\underline{\text{N + S: } s_1 \neq s_2}$ in Case 7.3 and to test $\overline{\text{PN: } A_1 = A_2}$ vs. $\underline{\text{N + S: } A_1 \neq A_2}$ in Case 7.4. In both cases, under $\overline{\text{PN,}}$ the statistic

$$T = \frac{(N-2)N|\ln X(1) - \ln Y(1)|}{4(n_1 + n_2)} \sim F(2, 2N - 4).$$

The decision rule is decide $N + S$ if and only if $T > f'$, where $f'$ is the appropriate percentile of the $F(2, 2N - 4)$-distribution.

**Case 7.5.** ($A_1, A_2, s_1, s_2$ all unknown)

$\overline{\text{PN: } s_1 = s_2}$ vs. $\underline{\text{N + S: } s_1 \neq s_2}$
Let \( n_1 = \sum_{j=1}^{m-1} \ln \frac{X(j+1)}{X(1)} \), \( n_2 = \sum_{j=1}^{n-1} \ln \frac{Y(j+1)}{Y(1)} \),

then from Lemma 3.6 (iii), under \( \text{PN} \),

\[
T = \frac{(n-1)n_1}{(m-1)n_2} \sim F(2(m-1), 2(n-1)).
\]

The decision rule is decide \( N + S \) if and only if \( T > f' \) or \( T < f'' \) where \( f' \) and \( f'' \) are appropriate percentiles of the \( F(2m-2, 2n-2) \)-distribution.

**Case 7.6.** \((A_1, A_2, s_1, s_2 \text{ all unknown})\)

\[
\begin{align*}
\text{PN:} & \quad A_1 = A_2 \quad \text{vs.} \quad N + S: \quad A_1 \neq A_2 \\
\end{align*}
\]

This detection procedure does not appear to have an elementary solution. One may try to apply the likelihood-ratio test. One has when not assuming \( A_1 = A_2 \),

\[
\hat{A}_1 = X(1), \quad \hat{s}_1 = m \left( \sum_{j=1}^{m-1} \ln \frac{X(j+1)}{X(1)} \right)^{-1}
\]

\[
\hat{A}_2 = Y(1), \quad \hat{s}_2 = n \left( \sum_{j=1}^{n-1} \ln \frac{Y(j+1)}{Y(1)} \right)^{-1}
\]

Under \( \text{PN} \), one has

\[
A^* = \min \{X(1), Y(1)\} = Z(1) \quad \text{and}
\]

\[
\hat{s}_1^* = m \left( \sum_{j=1}^{m-1} \ln \frac{X(j+1)}{Z(1)} \right)^{-1}, \quad \hat{s}_2^* = n \left( \sum_{j=1}^{n-1} \ln \frac{Y(j+1)}{Z(1)} \right)^{-1}
\]
Therefore the likelihood ratio is

\[
\frac{L_1}{L_0} = \begin{cases} 
\left(\frac{\hat{s}^2_2}{\hat{s}^2_1}\right)^{n} \left(\frac{Y(1)}{X(1)}\right)^{\frac{n}{2}} \left[ \sum_{j=1}^{n} Y_j \right]^{\frac{n}{2}} & \text{if } Z(1) = X(1) < Y(1) \\
\left(\frac{\hat{s}^2_1}{\hat{s}^2_2}\right)^{m} \left(\frac{X(1)}{Y(1)}\right)^{\frac{m}{2}} \left[ \sum_{j=1}^{m} X_j \right]^{\frac{m}{2}} & \text{if } Z(1) = Y(1) < X(1) 
\end{cases}
\]

The distribution of $\frac{L_1}{L_0}$ is complex and the above expression resembles in structure that of the Behrens-Fisher problem.

**Case 7.7.** $(A_1, A_2, s_1, s_2$ all unknown)

**PN:** $(A_1, s_1) = (A_2, s_2)$ \quad vs. \quad **N + S:** $(A_1, s_1) \neq (A_2, s_2)$

The decision rule consists of 2 steps, first deciding whether $s_1 = s_2$ and then if one decides $s_1 = s_2$, one tries to decide whether $A_1 = A_2$. The procedure is a combination of Cases 7.5 and 7.4.

Let $\eta_1 = \sum_{j=1}^{m-1} \ln \frac{X(j+1)}{X(1)}$, $\eta_2 = \sum_{j=1}^{n-1} \ln \frac{Y(j+1)}{Y(1)}$, then

under PN, \quad $T = \frac{(n-1)\eta_1}{(m-1)\eta_2}$ \quad $\sim F(2(m - 1), 2(n - 1))$.

The decision rule is:

(1) Decide **N + S** if

\[ T > f(2(m-1), 2(n-1), 1-a/4) \equiv C_1 \quad \text{or} \quad T < f(2(m-1), 2(n-1), a/4) \equiv C_2 \]
(ii) If $C_2 \leq T \leq C_1$, then decide $N \ast S$ if and only if

$$\ln X(1) > C_3 \text{ and } \ln Y(1) > C_4 \text{ where } C_3 = \frac{\eta C_1^i}{n} + w \text{ and } C_4 = \frac{\eta C_4^i}{m} + w,$$

$$w = \min \{\ln X(1), \ln Y(1)\}, \quad \eta = \sum_{j=1}^{m} \ln X_j + \sum_{j=1}^{n} \ln Y_j - Nw,$$

$$C_3' = 1 - \left(\frac{\eta \alpha}{4n}\right)^{N-2}, \quad C_4' = 1 - \left(\frac{\eta \alpha}{4m}\right)^{N-2}, \quad N = m + n.$$
TABLE 7.1. Two-Samples Signal Detection Procedures

\[
X = (X_1, \ldots, X_n) \sim i.i.d. \text{ Pa}(A_1,s_1), \ Y = (Y_1, \ldots, Y_n) \sim i.i.d. \text{ Pa}(A_2,s_2), \ X, Y \text{ independent}, \ \eta_1 = \sum_{j=1}^n \ln x_j - m \ln X(1),
\]

\[
\eta_2 = \sum_{j=1}^n \ln Y_j - n \ln Y(1), \ N = m+n, \ \eta = \sum_{j=1}^n \ln x_j + \sum_{j=1}^n \ln Y_j - NW, \ W = \min\{\ln X(1), \ln Y(1)\}
\]

<table>
<thead>
<tr>
<th>Cases</th>
<th>Assumptions</th>
<th>PN</th>
<th>N + S</th>
<th>Statistics; PN-distributions</th>
<th>Decision Rule-decide N+S iff</th>
</tr>
</thead>
</table>
| 7.1   | \((A_1,A_2) \text{ known}
(s_1,s_2) \text{ unknown} \) | \(s_1 = s_2\) | \(s_1 \neq s_2\) | \(T = \frac{\sum_{j=1}^n \ln x_j - \ln A_1}{\sqrt{2m(n-n_1)}}\) \(\sim \text{F}(2m,2n)\) \(T > f(2m,2n,1-\alpha/2)\) or \(T < f(2m,2n,\alpha/2)\) |
| 7.2   | \((A_1,A_2) \text{ unknown}
(s_1,s_2) \text{ known} \) | \(A_1 = A_2\) | \(A_1 \neq A_2\) | \(T = \ln \frac{Y(1)}{X(1)}, \ T \sim H(x) \) where \(H(x) = \begin{cases} 1 - \left(\frac{m_1}{m_1+n_2}\right) e^{-\frac{ms_1-x}{n_2}} & x > 0 \\ \frac{n_2}{m_1+n_2} e^{m_1x} & x < 0 \end{cases}\) \(T > C_1 \) or \(T < C_2 \) where \(H(C_1) = 1-\alpha/2 \), \(H(C_2) = \alpha/2\). If \(s_1 \neq s_2 \neq s_0\), then test reduces to \(Y(1)/X(1) < b_1 \) or \(> b_2\) where \(b_1 = \left[\frac{N_0/\alpha}{n_2}\right]^{1/m_0}, \ b_2 = \left[\frac{N_0\alpha}{m_2}\right]^{-1/n_0}\) |
| 7.3   | \(A_1=A_2=A \text{ unknown}
(s_1,s_2) \text{ unknown} \) | \(s_1 \leq s_2\) | \(s_1 > s_2\) | \(C' \) determined by \(I_{C'}(m-1,n) = 1 - \alpha\) \(\eta_1 > C'n\) |
| 7.4   | \((A_1,A_2) \text{ unknown}
(s_1=s_2) \neq s \text{ unknown} \) | \(A_1 < A_2\) | \(A_1 > A_2\) | \(c = \frac{nC'}{n} + \frac{n}{n}, \ c' = 1 - \left(\frac{N_0}{n}\right)^{1/(N-2)}\) \(\ln X(1) > c\) |


TABLE 7.1. (Continued)

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<th>(s_1 = s_2)</th>
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<th>(T = \frac{(n-2)\eta_1}{(m-2)\eta_2} \sim F(2(m-1), 2(n-1)))</th>
<th>(T &gt; f(2(m-1), 2(n-1), 1- \alpha/2)) or (T &lt; f(2(m-1), 2(n-1), \alpha/2))</th>
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<td>Unknown</td>
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<tr>
<td>7.7</td>
<td>((A_1, A_2, s_1, s_2)) all unknown</td>
<td>((A_1, s_1) = (A_2, s_2))</td>
<td>((A_1, s_1) \neq (A_2, s_2))</td>
<td>(c_1 = f(2(m-1), 2(n-1), 1- \alpha/4)) (c_2 = f(2(m-1), 2(n-1), \alpha/4)) (c_3 = 1 - \frac{N_0}{4n} \frac{1}{(N-2)}) (c_4 = 1 - \frac{N_0}{4m} \frac{1}{(N-2)}) (c_5 = \frac{nc_3}{n} \sim w, c_4 = \frac{nc_4}{m} \sim w)</td>
<td>decide (N + S) if (T &gt; c_1) or (T &lt; c_2). If (c_2 \leq T \leq c_1), then decide (N + S) if and only if in (X(1) &gt; c_3), and in (Y(1) &gt; c_4).</td>
</tr>
</tbody>
</table>
REFERENCES


unknown", Biometrika, 59, 222-224.


APPENDIX A: Proofs

A.1: A unknown, $s = s_0$ known, then $\hat{F}_n(z) = \begin{cases} 
1 - \left( \frac{n-1}{n} \right)^{s_0} & z > X(1) \\
0 & z \leq X(1).
\end{cases}$

Proof. $\hat{F}_n(z) = E[I_{\{X_1 \leq z\}}|S(z)] = P(X_1 \leq z|X(1))$

$$= \frac{1}{n} \sum_{j=1}^{n} P(X_j \leq z|X(1))$$

$$= \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} P(X(j+1) \leq z|X(1)) \quad z > X(1)$$

$$= \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} P\left\{ \frac{X(j+1)}{X(1)} \leq \frac{z}{X(1)} \right\}.$$

Now let $U(j) = \frac{X(j+1)}{X(1)}$, then $\{U(1), U(2), \ldots, U(n-1)\} \sim \text{Pa}(1, s_0)$

O-S(n - 1).

Therefore $\hat{F}_n(z) = \frac{1}{n} + \left( \frac{n-1}{n} \right) \left[ \frac{1}{n-1} \sum_{j=1}^{n-1} P(U_j \leq \frac{z}{X(1)}) \right]$

$$= \frac{1}{n} + \left( \frac{n-1}{n} \right) \left[ 1 - P(U_1 > \frac{z}{X(1)}) \right]$$

$$= 1 - \left( \frac{n-1}{n} \right) P(U_1 > \frac{z}{X(1)})$$

$$= 1 - \left( \frac{n-1}{n} \right) \left( \frac{X(1)}{z} \right)^{s_0}.$$
A.2. A, s both unknown, then \( \hat{F}_n(z) = \begin{cases} 
0 & z < a \\
\frac{1}{n} & z = a \\
1 - \left(\frac{n-1}{n}\right) \left[1 - \frac{s}{n} \ln \frac{z}{a}\right] & a < z < ae^{n/s} \\
1 & z = ae^{n/s} \end{cases} \)

where \( \hat{\lambda} = a, \hat{s} = s. \)

**Proof.** \( \hat{F}_n(z) = E\{I\{X_1 \leq z\}|S(Z)\} = P\{X_1 \leq z| (\hat{\lambda}, \hat{s})\} \)

\[
= \frac{1}{n} \sum_{j=1}^{n} P(X_j \leq z| (\hat{\lambda}, \hat{s})) = \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} P(X(j+1) \leq z| (\hat{\lambda}, \hat{s})). \quad z > a
\]

Let \( U(j) = \frac{X(j+1)}{X(1)}, V(j) = \ln U(j), \) then by Theorem 3.5,

\( \{U(1), U(2), \ldots, U(n-1)\} \sim \text{Pa}(1, s) - O-S(n - 1) \) and \( \{V(1), V(2), \ldots, V(n-1)\} \sim \text{Exp}(s) - O-S(n - 1). \)

Hence \( \hat{F}_n(z) = \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} P(U(j) \leq \frac{z}{a}|s) \)

\[
= \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} P(V(j) \leq \ln \frac{z}{a}|s) = \frac{1}{n} + \frac{1}{n} \sum_{j=1}^{n-1} \frac{V_j}{(n-1)\bar{V}} \leq \frac{s}{n} \frac{\ln \frac{z}{a}}{\bar{V}}
\]

since \( s = \hat{s} = \frac{n}{\sum_{j=1}^{n-1} V_j} = \frac{n}{(n-1)\bar{V}} \).
By Lemma 3.1, \( \left\{ \frac{V_1}{(n-1)V}, \frac{V_1 + V_2}{(n-1)V}, \ldots \right\} \sim U O S(n - 1) \). To continue,

\[
F_n(z) = \frac{1}{n} + \left( \frac{n-1}{n} \right) P \left\{ \frac{V_1}{(n-1)V} \leq \frac{s}{n} \ln \frac{z}{a} \right\}
\]

\[
= 1 - \left( \frac{n-1}{n} \right) P \left\{ \frac{V_1}{(n-1)V} \geq \frac{s}{n} \ln \frac{z}{a} \right\}
\]

\[
= 1 - \left( \frac{n-1}{n} \right) [1 - \frac{s}{n} \ln \frac{z}{a}]^{n-1}
\]

In the above derivation, since \( V_j \leq (n - 1)V = \frac{n}{s} \) for every \( j \), one may restrict \( z \) to \( \ln \frac{z}{a} < \frac{n}{s} \) or \( z \leq ae^{n/s} \).

A.3. One verifies here that the u.m.p. procedure found by Park and U. Choi (1978) is equivalent to that in Cases 6.6 and 6.7. Park and Choi considered the p.d.f. of the truncated exponential,

\[
f(x, \lambda, \nu) = \lambda e^{-\lambda(x-\nu)} I_{(\nu, \infty)}(x) \quad 0 < \nu, \quad \lambda < \infty
\]

where \( \lambda, \nu \) are unknown. Let \( Y_1, Y_2, \ldots, Y_n \) be random sample from a truncated exponential distribution. Let \( S = \sum_{i=1}^{n} Y_i \), then given \( Y(1) = y \),

\[
f(s|y) = \frac{\lambda^{n-1}(-s+ny)^{n-2} e^{-\lambda(-s+ny)}}{\Gamma(n-1)} I_{(ny, \infty)}(s)
\]

while given \( S = s \),
\[ f(y|s) = \frac{n(n-1)(s-ny)^{n-2}}{(s-ny)^{n-1}} I_{(\nu,s/n)}(y). \tag{2} \]

**Theorem A.** (Park and U. Choi). For testing hypothesis \( H_\nu: \nu \leq \nu \) against \( K_\nu: \nu > \nu_0 \), the u.m.p. unbiased test is given by

\[
\phi(Y(1)) = \begin{cases} 
1 & \text{if } Y(1) \geq C(s) \\
0 & \text{if } Y(1) < C(s)
\end{cases}
\]

where \( C(s) \) is uniquely determined by \( P\{Y(1) \geq C(s)|S = s, \nu = \nu_0\} = \alpha \).

**Theorem B.** (Park and U. Choi). For testing hypothesis \( H_\lambda: \lambda \leq \lambda_0 \) against \( K_\lambda: \lambda > \lambda_0 \), the u.m.p. unbiased test is given by

\[
\phi(S) = \begin{cases} 
1 & \text{if } S \leq C(Y(1)) \\
0 & \text{if } S > C(Y(1))
\end{cases}
\]

where \( C(Y(1)) \) is uniquely determined by \( P\{S \leq C(Y(1))|Y(1) = y, \lambda = \lambda_0\} = \alpha \).

From Theorem A and (2), one concludes \( C = C(S) \) is determined by

\[
\frac{n(n-1)}{(s-n\nu_0)^{n-1}} \int_0^s \frac{(s-ny)^{n-2}}{C} \, dy = (n-1) \int_0^1 (1-u)^{n-2} \, du
\]

where \( U = \frac{n(Y(1) - \nu_0)}{S-n\nu_0} \sim B(1, n-1) \). But

\[
U = \frac{n(Y(1) - \nu_0)}{S-nY(1)+nY(1)-n\nu_0} = \frac{T'}{1 + T'} \quad \text{where } T' = (n-1)T,
\]
T is the statistic given in Section 6, Case 6.7. Since the function \( f(t') = \frac{t'}{1 + t'} \) is strictly increasing for \( t' > 0 \), the two tests are equivalent. It is also easily verified that if \( U \sim B(1, n - 1) \), then \( \frac{T'}{n - 1} \sim F(2, 2(n - 1)) \). The proof of the equivalence between Theorem B and the test procedure in Case 6.6 follows from (1) in the same manner and will be omitted.

A.4. (Proof of Lemma 7.1). From the hypothesis, let \( U = \ln Y(1) \), \( V = \ln X(1) \) and \( T = U - V \), then

\[
P(T \leq t) = P(U - V \leq t) = \int_{0}^{\infty} P(U \leq v + t) (ms_1)e^{-ms_1v} dv
\]

\[
= ms_1 \int_{\max(-t,0)}^{\infty} [1 - e^{-ns_2(v+t)}] e^{-ms_1v} dv.
\]

Considering separately the case \( t > 0 \) and \( t < 0 \), the above integral can be evaluated and equals to \( H(t) \) given in Lemma 7.2.
APPENDIX B: Numerical Examples

The numerical examples correspond to the various techniques developed in the main body of the report, and have corresponding numbers. The data is divided into three sets.

Table 1. This consists of simulated interarrival times from Pa(1;1); Pa(2;1); Pa(3;1); and Pa(5;5) distributions.

Table 2. This consists of simulated interarrival times from Pa(1;2); Pa(1;5); Pa(2;2); and Pa(5;2) distributions.

Table 3. This consists of real data related to 24 complete heart-beat cycles. [Special attention will be paid to the waiting times for the "R-peaks". See sketch below the table.]
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TABLE 2  Pareto Data (Interarrivals)

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Example 6.1: A unknown, \( s = s_0 \) known, \( L(A,s_0) \in \Omega(\text{PRP}) \)

PN: \( A = A_0 \) vs. \( N + S: A \neq A_0 \).

Detector Statistics: (i) \( X(1) \) or (ii) \( T = 2n s_0 \ln \frac{X(1)}{A_0} \sim X^2_2 \).

Decision Rule: Decide \( N + S \) iff

(i) \( X(1) < A_0 \) or \( X(1) > bA_0 \), \( b = \alpha^{-1/n s_0} \)

(ii) \( T < 0 \) or \( T > X^2_2, (1-\alpha) \).

Decision rule (i), the MP procedure, will be illustrated using the following PN situation:

PN: \( A = 2 \) vs. \( N + S: A \neq 2 \) (s known, \( \alpha = .01 \))

Data Sets: (See Tables 1 and 2)

1. \( V_1, \ldots, V_{50} \) i.i.d. \( \text{Pa}(1;1) \)
2. \( W_1, \ldots, W_{50} \) i.i.d. \( \text{Pa}(2;1) \)
3. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(3;1) \)
4. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(2;2) \)
5. \( Z_1, \ldots, Z_{50} \) i.i.d. \( \text{Pa}(1;5) \)

The Decision Rules are:

1. Decide \( N + S \) iff \( V(1) < 2 \) or \( V(1) > 2.193 \). Since \( V(1) = 1.010 \) one decides \( N + S \).
2. Decide \( N + S \) iff \( W(1) < 2 \) or \( W(1) > 2.193 \). Since \( W(1) = 2.033 \) one decides \( PN \).
3. Decide \( N + S \) iff \( X(1) < 2 \) or \( X(1) > 2.193 \). Since \( X(1) = 3.001 \) one decides \( N + S \).

4. Decide \( N + S \) iff \( Y(1) < 2 \) or \( Y(1) > 2.094 \). Since \( Y(1) = 2.044 \) one decides \( P N \).

5. Decide \( N + S \) iff \( Z(1) < 2 \) or \( Z(1) > 2.037 \). Since \( Z(1) = 1.008 \) one decides \( N + S \).

Example 6.2: A unknown, \( s = s_0 \) known

\[
\text{PN: } L(A, s_0) \in \Omega(\text{PRP}) \quad \text{vs. } \quad N + S: \quad L(A, s_0) \notin \Omega(\text{PRP})
\]

Detector Statistics: Four Kolmogorov-Smirnov statistics

\( (D_n^{(1)}, D_n^{(2)}, D_n^{(3)}, D_n^{(4)}) \) and a Srinivasan-type statistic \( D_n \) are available.

Decision Rules: (1-4) Decide \( N + S \) iff \( D_n > d_{n-1,\alpha} \) where \( d_{n-1,\alpha} \) is value from Kolmogorov-Smirnov table and \( D_n = D_n^{(1)}, D_n^{(2)}, D_n^{(3)}, D_n^{(4)} \). (5) decide \( N + S \) iff \( D_n > \left(\frac{n-1}{n}\right)d_{n-1,\alpha} \).

Data Sets: (see Tables 1 and 2)

1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(5;5) \)
<table>
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<th>statistic value (X)₀</th>
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<td>PN</td>
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<td>PN</td>
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Example 6.3: A = A₀ known, s unknown, \( L(A₀, s) \in \Omega(\text{PRP}) \)

PN: \( s = s₀ \) vs. N + S: \( s \neq s₀ \)

Detector Statistic: \( T = 2s₀ \sum_{j=1}^{n} \ln \frac{X(j)}{A₀} \sim X^2_{2n} \)

Decision Rule: Decide N + S iff

\[ T > X^2_{2n,1-\alpha/2} \quad \text{or} \quad T < X^2_{2n,\alpha/2} \]

Test the following PN situation using generated Pareto data:

PN: \( s = 2 \) vs. N + S: \( s \neq 2 \) (A known, \( \alpha = 0.01 \))

Data Sets: (see Tables 1 and 2)
1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(5;5) \)

**Decision Rule:** Decide \( N + S \) iff \( T > 140.2 \) or \( T < 82.4 \)

From data set 1: \( T = 91.017 \), so one decides \( PN \).

From data set 2: \( T = 48.737 \), so one decides \( N + S \).

**Example 6.4:** \( A = A_0 \) known, \( s \) unknown

**PN:** \( L(A_0, s) \in \Omega(\text{PRP}) \) vs. \( N + S: \ L(A_0, s) \notin \Omega(\text{PRP}) \)

**Detector Statistic:**
\[
D^* = \sup_{0 < u < 1} \left| \frac{1}{n-1} \sum_{i=1}^{n-1} \epsilon \left( u - \frac{T_i}{n} \right) - u \right|
\]

where \( T_r = \sum_{j=1}^{r} \ln \left( \frac{X_j}{A_0} \right) \)

**Decision Rule:** Decide \( N + S \) iff \( D^*_n > d_{n-1, \alpha} \) where \( d_{n-1, \alpha} \) is appropriate value from Kolmogorov-Smirnov table.

**Data Sets:** (see Tables 1 and 2)
1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(5;5) \)

Decide \( N + S \) iff \( D^*_{50} > .233 \)

Since \( D^*_{50}(X) = 0.101 \), one decides \( PN \).

Since \( D^*_{50}(Y) = 0.053 \), one decides \( PN \).
Example 6.5. A unknown, s unknown, \( L(A,s) \in \Omega(\text{PRP}) \).

\[
\text{PN: } L(A,s) = L(A_0,s_0) \quad \text{vs.} \quad \text{N+} \text{S: } L(A,s) \neq L(A_0,s_0)
\]

Detector Statistic: 
\[
T = -2 \sum_{j=1}^{n} \ln \left[ \frac{A_0}{X(j)} \right] \sim \chi^2_{2n}
\]

Decision Rule: Decide N + S iff 
\[
T > \chi^2_{2n,1-\alpha/2} \quad \text{or} \quad T < \chi^2_{2n,\alpha/2}
\]

Test the following PN situation using generated Pareto data:

\[
\text{PN: } (A,s) = (1,2) \quad \text{vs.} \quad \text{N+S: } (A,s) \neq (1,2)
\]

Data Set: (see Table 2)

1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(1;2) \)

Decide N + S iff \( T > 140.2 \) or \( T < 82.4 \)

Since the calculated value of the test statistic is \( T = 91.02 \), one decides PN.

Example 6.6. A unknown, s unknown, \( L(A,s) \in \Omega(\text{PRP}) \).

\[
\text{PN: } s < s_0 \quad \text{vs.} \quad \text{N+S: } s > s_0
\]

Detector Statistic: 
\[
T = 2s_0 \left[ \sum_{j=1}^{n} \ln X_j - n \ln X(1) \right] \sim \chi^2_{2n-2,\alpha}
\]

Decision Rule: Decide N + S iff \( T < \chi^2_{2n-2,\alpha} \)
Test the following PN situation using generated Pareto data:

\[ \text{PN: } s \leq 3 \quad \text{vs. } N + S: \quad s > 3 \quad (\alpha = 0.05) \]

**Data Sets:** (see Tables 1 and 2)

1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(5;5) \)

The critical value is \( X^2_{(98, 0.05)} = 76.5 \)

The calculated statistic values are

From data set 1: \( T = 129.32 \), so one decides PN.
From data set 2: \( T = 72.64 \), so one decides N + S.

**Example 6.7.** \( A \) unknown, \( s \) unknown \( L(A, s) \in \Omega(\text{PRP}) \)

\[ \text{PN: } A < A_0 \quad \text{vs. } N + S: \quad A > A_0 \]

Detector Statistic: \( T = \frac{n(n-1)[\ln X(1) - \ln A_0]}{\sum_{j=1}^{n} \ln X_j - n \ln X(1)} \sim F_{(2, 2n-2)} \)

**Decision Rule:** Decide \( N + S \) iff

i. \( X(1) > A_0 \) and

ii. \( T > f_{(2, 2n-2, 1-\alpha)} \)

Test the following PN situation using generated Pareto data:

\[ \text{PN: } A \leq 3 \quad \text{vs. } N + S: \quad A > 3 \]
Data Sets: (see Tables 1 and 2)

1. \(X_1, \ldots, X_{50}\) i.i.d. \(\text{Pa}(1;2)\)
2. \(Y_1, \ldots, Y_{50}\) i.i.d. \(\text{Pa}(5;5)\)

The critical value for \(T\) \((\alpha = .01)\) is

\[
f_{(2,98,.99)} = 4.87
\]

Since \(X(1) = 1.024\), one decides \(PN\) for data set 1.

Since \(Y(1) = 5.006\), and \(T_Y = 103.7\), one decides \(N + S\) for Data Set 2.

**Example 6.8.** \(A, s\) unknown

\(PN: L(A,s) \in \Omega(\text{PRP})\) vs. \(N + S: L(A,s) \notin \Omega(\text{PRP})\)

Detector Statistics:

i. \(D_n^1 = \sup_z \frac{1}{n-2} \sum_{j=1}^{n-2} \epsilon (z - \frac{V_j}{n-1}) - z \sim \text{K-S}(n-2)\)

ii. \(D_n^2 = \sup_z \frac{1}{n-2} \sum_{j=1}^{n-2} \epsilon (z - \frac{E_j}{n-1}) - z \sim \text{K-S}(n - 2)\)

where

\[
P_r = \sum_{j=1}^{r-1} \ln \left[ \frac{X(j+1)}{X(1)} \right] + (n - r) \ln \left[ \frac{X(r+1)}{X(1)} \right] \quad 1 \leq r \leq n - 1
\]

and

\[
E_r = \sum_{j=1}^{r} (n + 1 - j) \ln \left[ \frac{X(j + 1)}{X(j)} \right] \quad 1 \leq r \leq n - 1
\]
Decision Rules: Decide N + S iff

i. \( D_n^1 > d_{n-2, \alpha} \)

ii. \( D_n^2 > d_{n-2, \alpha} \)

where \( d(n - 2, \alpha) \) is appropriate value from Kolmogorov-Smirnov table.

Data Sets: (see Tables 1, 2 and 3)

1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{P}(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{P}(5;5) \)
3. \( W_1, \ldots, W_{24} \) waiting times for "peaks" (i.e. R's) of heart-beat cycles.

The critical value for data sets 1 and 2 is:

\[ d(48,.01) = 0.22 \]

The critical value for data set 3 is:

\[ d(22,.01) = 0.314 \]

The calculated statistic values are:

\[ D_{50}^1(X) = 0.058 \quad \text{Decide PN.} \]
\[ D_{50}^2(Y) = 0.042 \quad \text{Decide PN.} \]
\[ D_{50}^1(Y) = 0.043 \quad \text{Decide PN.} \]
\[ D_{24}^1(W) = 0.614 \quad \text{Decide N+S.} \]
\[ D_{24}^2(W) = 0.605 \quad \text{Decide N+S.} \]
Example 7.1. $A_1, A_2$ known; $s_1, s_2$ unknown

PN: $s_1 = s_2$ vs. $N + S$: $s_1 \neq s_2$

Detector Statistic: 
\[
T = \frac{n \sum_{j=1}^{m} \ln(X_j / A_1)}{m \sum_{j=1}^{n} \ln(Y_j / A_2)} \sim F(2m, 2n)
\]

Decision Rule: Decide $N + S$ iff

\[T > F_{(2m, 2n, 1-\alpha/2)} \quad \text{or} \quad T < F_{(2m, 2n, \alpha/2)}\]

Data Sets: (See Tables 1 and 2)
1. $X_1, \ldots, X_{50}$ i.i.d. Pa(1;2)
2. $Y_1, \ldots, Y_{50}$ i.i.d. Pa(5;2)
3. $Z_1, \ldots, Z_{50}$ i.i.d. Pa(5;5)

The critical values are:

\[f_{(100, 100, .995)} = 1.68\]
\[f_{(100, 100, .005)} = 0.595\]

The calculated statistic values for each pair of data sets are given below:

From data sets 1 and 2: $T = 0.788$, so one decides PN.

From data sets 1 and 3: $T = 1.868$, so one decides $N + S$.

From data sets 2 and 3: $T = 0.422$, so one decides $N + S$. 
Example 7.2. \( A_1, A_2 \) unknown; \( s_1, s_2 \) known

PN: \( A_1 = A_2 \) vs. \( N + S: A_1 \neq A_2 \)

Detector Statistic: \( T = \ln \left( \frac{Y(1)}{X(1)} \right) \sim F(x) \)

where

\[
F(x) = \begin{cases} 
1 - \frac{ms_1}{(ms_1 + ns_2)} e^{-ns_2 x} & x \geq 0 \\
\frac{ns_2}{(ms_1 + ns_2)} e^{ms_1 x} & x < 0 
\end{cases}
\]

Decision Rule: Decide \( N + S \) iff

\( T > c_1 \) or \( T < c_2 \) where \( F(c_1) = 1 - \alpha/2 \)

\( F(c_2) = \alpha/2 \)

Data Sets: (see Tables 1 and 2)

1. \( X_1, \ldots, X_{50} \) i.i.d. \( P(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( P(5;2) \)
3. \( Z_1, \ldots, Z_{50} \) i.i.d. \( P(5;5) \)

The first two data sets constitute a special case \((s_1 = s_2 = s_0 \text{ known})\) and will be treated in Example 7.2a. For the remaining two pairs of data sets, one has the following critical values: \((m = n = 50, s_1 = 2, s_2 = 5, \alpha = .01)\)

\[
c_1 = 0.016 \quad \text{and} \quad c_2 = -0.050
\]
The appropriate decision rule is:

Decide $N + S$ iff $T > 0.016$ or $T < -0.050$.

The calculated statistic values are given below:

From data sets 1 and 3: $T = \ln\left(\frac{Z(1)}{X(1)}\right) = 1.587$, so one decides $N + S$.

From data sets 2 and 3: $T = \ln\left(\frac{Z(1)}{Y(1)}\right) = -0.008$, so one decides $PN$.

Example 7.2a. $A_1, A_2$ unknown; $s_1 = s_2 = s_0$ known

$PN: A_1 = A_2$ vs. $N + S: A_1 \neq A_2$

Detector Statistic: $T = \frac{Y(1)}{X(1)}$

Decision Rule: Decide $N + S$ iff $T < b_1$ or $T > b_2$

where $b_1 = \left[\frac{N}{n} \frac{\sigma_2^2}{2}\right]^{1/2s_0}$, $b_2 = \left[\frac{N}{m} \frac{\sigma_2^2}{2}\right]^{-1/2s_0}$

Data Sets: (See Table 1)

1. $X_1, \ldots, X_{50}$ i.i.d. $\text{P}(1;2)$
2. $Y_1, \ldots, Y_{50}$ i.i.d. $\text{P}(5;2)$

The critical values, for $\alpha = 0.01$ are:

$b_1 = 0.95$ and $b_2 = 1.05$

The calculated statistic value is

$T = \frac{Y(1)}{X(1)} = 4.929$ Decide $N + S$. 
Example 7.3. \( A_1 = A_2 = A \) unknown, \( s_1, s_2 \) unknown

\[ \text{PN: } s_1 < s_2 \quad \text{vs.} \quad \text{N + S: } s_1 > s_2 \]

Detector Statistic:
\[ T = \frac{n_1}{n} \]

where
\[ n_1 = \sum_{j=1}^{m} \ln X_j - m \ln X(1) \]

and
\[ \eta = \sum_{j=1}^{m} \ln X_j + \sum_{j=1}^{n} \ln Y_j - \eta W \]

where \( W = \min \{ \ln X(1), \ln Y(1) \} \)

Decision Rule: Decide \( N + S \) iff \( T < c \) where \( c \) is determined by
\[ I_c(m - 1, n) = 1 - \alpha \]

Data Sets: (See Tables 1 and 2)
1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(5;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(5;5) \)

Computation of critical value:
\[ I_c(49,50) = 0.99 \quad \text{(take} \quad \alpha = .01) \]
\[ c = 0.61 \quad \text{(from tables of Incomplete Beta function)} \]

The decision rule is:
Decide \( N + S \) iff \( T < 0.61 \)

Since \( T = \frac{n_1}{n} = \frac{28.396}{40.91} = .694 \)
one decides PN.

Example 7.4. \( A_1, A_2 \) unknown; \( s_1 = s_2 = s \) unknown

PN: \( A_1 \leq A_2 \)  vs.  \( N + S: A_1 > A_2 \)

Detector Statistic: \( T = \ln X(1) \)

Decision Rule: Decide \( N + S \) iff \( T > c \) where \( c = \frac{nc'}{n} + W, \)

\[ W = \min \{ \ln X(1), \ln Y(1) \} \]

\[ \eta = \sum_{j=1}^{m} \ln X_j + \sum_{j=1}^{n} \ln Y_j - NW \]

and \( c' \) is determined by

\[ \left( \frac{n}{N} \right) (1 - c')^{N-2} = \alpha \]

Data Sets: (See Tables 1 and 2)
1. \( X_1, \ldots, X_{50} \) i.i.d. \( \text{Pa}(1;2) \)
2. \( Y_1, \ldots, Y_{50} \) i.i.d. \( \text{Pa}(5;5) \)

Computation of critical value (\( \alpha = .01 \)):

\[ \left( \frac{50}{100} \right) (1 - c')^{98} = .01 \]

\[ c' = .04 \]

since \( \eta = 129.73 \) and \( W = \ln X(1) = .024 \), one has

\[ c = \frac{(129.73)(.04)}{50} + .024 = .128 \]
since $T = .024$, one decides PN.

Example 7.5. $A_1, A_2$ unknown; $s_1, s_2$ unknown

$PN: s_1 = s_2$ \text{ vs. } $N + S: s_1 \neq s_2$

Detector Statistic: \[ T = \frac{(n - 2)\eta_1}{(m - 2)\eta_2} \sim F(2(m-1), 2(n-1)) \]

where \[ \eta_1 = \sum_{j=1}^{m-1} \ln \frac{X(j+1)}{X(1)} \]

and \[ \eta_2 = \sum_{j=1}^{n-1} \ln \frac{X(j+1)}{X(1)} \]

Decision Rule: Decide $N + S$ iff

\[ T > f(2(m-1), 2(n-1), (1-\alpha/2)) \]

or

\[ T < f(2(m-1), 2(n-1), \alpha/2) \]

Data Sets: (see Tables 1 and 2)

1. $X_1, \ldots, X_{50}$ i.i.d. $Pa(1;2)$
2. $Y_1, \ldots, Y_{50}$ i.i.d. $Pa(5;2)$
3. $Z_1, \ldots, Z_{50}$ i.i.d. $Pa(5;5)$

Critical values:

\[ f(98,98,0.99) = 1.69 \]

\[ f(98,98,0.01) = 0.592 \]
The decision rule is:
Decide \( N + S \) iff \( T > 1.69 \) or \( T < 0.592 \)

For data sets 1 and 2: \( T = 0.760 \), so one decides \( PN \).
For data sets 1 and 3: \( T = 1.779 \), so one decides \( N + S \).
For data sets 2 and 3: \( T = 0.427 \), so one decides \( N + S \).

**Example 7.6.** No techniques for Case 7.6 have been developed by the authors.

**Example 7.7.** \( A_1, A_2 \) unknown; \( s_1, s_2 \) unknown

\[ PN: \quad (A_1, s_1) = (A_2, s_2) \quad \text{vs.} \quad N + S: \quad (A_1, s_1) \neq (A_2, s_2) \]

**Detector Statistics**

(1) \( T_1 = \frac{(n - 2) \eta_1}{(m - 2) \eta_2} \sim F(2(m-1), 2(n-1)) \)

\((m = n)\)

(ii) \( T_2 = \frac{N(N-2) \ln X(1) - \ln Y(1)}{4(\eta_1 + \eta_2)} \sim F(2, 2N-4) \)

**Decision Rule:** Decide \( N + S \) iff

(1) \( T_1 > f_{(2(m-1), 2(n-1), 1-a/4)} \) or
\[ T_1 < f_{(2(m-1), 2(n-1), a/4)} \]

(ii) If \( f_{a/4} \leq T_1 \leq f_{1-a/4} \)

decide \( N + S \) iff
\[ T_2 > f_{(2, 2N-4, 1-a/2)} \]
Data Sets: (see Tables 1 and 2)

1. $X_1, \ldots, X_{50}$ i.i.d. $\text{Pa}(1;2)$
2. $Y_1, \ldots, Y_{50}$ i.i.d. $\text{Pa}(5;2)$
3. $Z_1, \ldots, Z_{50}$ i.i.d. $\text{Pa}(5;5)$

Critical values ($\alpha = .01$):

(i). $f(98,98,.9975) = 1.77$

(ii) $f(2,196,.995) = 5.35$

Decide $N + S$ iff

(i) $T_1 > 1.77$ or $T_1 < .565$

(ii) If $.565 < T_1 < 1.77$, decide $N + S$ iff $T_2 > 5.35$

1. For $(X, Y)$, $T_1 = 0.760$, $T_2 = 39.102$ so one decides $N + S$.
2. For $(X, Z)$, $T_1 = 1.779$, $T_2 = 57.682$ so one decides $N + S$.
3. For $(Y, Z)$, $T_1 = 0.427$, $T_2 = 0.251$ so one decides $N + S$. 
SIGNAL DETECTION FOR PARETO RENEWAL PROCESSES

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14. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Pareto distribution, renewal process, distribution-free; sufficient statistic; statistical noise; order statistics; Kolmogorov-Smirnov statistic; conditional distribution; unbiased, Lilliefors's statistic, Srinivasan statistic.

15. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Employing minimal sufficient statistics maximal statistical noise, several Kolmogorov-type statistics; and conditional distributions, optimal detection procedures are constructed for various one-and-two-sample problems involving Pareto Renewal processes. Cases with and without nuisance parameters are treated. Optimal parametric and distribution-free procedures are developed.