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TIME DELAY ESTIMATION: FOR
KNOWN & UNKNOWN SIGNALS

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I. Introduction

Various optimum and sub-optimum filters have been added to the basic generalized cross-correlator in order to enhance its detection and estimation capabilities in the presence of noise. An experimental comparison of the optimum and sub-optimum filters was discussed in Hassab and Boucher[1]. For the random signals, the optimum window $W_{opt}(w)$ which they derived has the results of time-delay estimation considerably.

Basically, the solution consists of cross-correlation of the sensor outputs. The time argument that corresponds to the maximum peak in the output is the time delay (Fig. 1). In this report, time delay estimation under several assumptions about signals is obtained by averaging the periodograms of the data segments[1]. When the signal is unknown, the maximum entropy power spectrum estimation of Burg's algorithm is used to estimate the signal power spectrum from the measurement of separated sensors. The Akaike FPE (Final power error) criterion is also concerned in deciding the order of AR model and signal & noise power spectrum estimation of Burg's algorithm.

II. Theory

Let the sensor output be the form

\[
\begin{align*}
Z_1(t) &= y(t) + n_1(t) \\
Z_2(t) &= ay(t-T) + n_2(t)
\end{align*}
\]

where the signal $y(t)$ and noises $n_1(t), n_2(t)$ are uncorrelated and jointly stationary zero mean Gaussian random processes over an observation interval $T >> \tau$. Here $\tau$ is the difference in arrival time or the time delay between the two sensor elements.

In the correlator, the multiplier output is given by[2]

\[
Z_1^*(w)Z_2(w) = a \phi_y(w) e^{-jwT} + \phi_{n1n2}(w) + \phi_2(w)
\]

where
where

\[ \phi_y(w) = |Y(w)|^2, \quad \phi_{n1n2}(w) = N_1^*(w) N_2(w) \]

\[ \phi_c(w) = Y^*(w)N_2(w) + aY(w)e^{-j\omega t}N_1^*(w) \]

The lower case \( \phi \) refers to a single realization of the function and the upper case indicates its expected value over the ensemble, i.e. \( <\phi_y(w)> = |Y(w)|^2 = \Phi_y(w) \).

The window \( W_{II}(w) \) derived by Hassab and Boucher [?)] is based on the optimal detection criterion of maximizing the ratio of mean correlator output due to signal and noise present. The window function can be expressed in the simple form as

\[ W_{II}(w) = \frac{\Phi_y(w)}{\Phi_{n1}(w)\Phi_{n2}(w)} \]  \hspace{1cm} (3a)

or

\[ W_{II}(w) = \frac{\Phi_y(w)}{\Phi_{n1}(w)\Phi_{n2}(w) + \Phi_y(w)[(\Phi_{n1}(w) + a^2\Phi_{n1}(w)) + a^2\Phi_y(w)]} \]  \hspace{1cm} (3b)

With a finite number of samples, one spectral density estimate is the periodogram. A good spectral estimate is obtained by averaging several periodogram from segments, denoted as \( M \), that provides the best spectral density estimate and the time-delay estimate.

For deterministic signals, the resulting optimum filters are in terms of signal and noise spectral densities.

\[ W_I(w) = \frac{\Phi_y(w)}{\Phi_{n1}(w)\Phi_{n2}(w) + \Phi_y(w)[(\Phi_{n1}(w) + a^2\Phi_{n1}(w))]} \]  \hspace{1cm} (4a)

or

\[ W_I(w) = \frac{\Phi_y(w)}{\Phi_{z1}(w)\Phi_{z2}(w) - a^2\Phi_y^2(w)} \]  \hspace{1cm} (4b)

the difference between \( W_I \) and \( W_{II} \) is due to the variance \( a^2\Phi_y^2 \) of the signal which is negligible for small \( a \) and/or low signal-to-noise spectra.
Otherwise, both filters are similar where the first term within brackets in the denominator is due to \( \phi_{1n1}(w) \) and the second term \( \phi_c(w) \). Hassab and Boucher\(^4\) have examined the behavior of every window in the presence of a strong spectral peak in \( \phi_y(w) \). With a no noise case \( W_1(w) \) is undefined and \( W_{II}(w) \) is reduced to \( \frac{1}{\phi_y(w)} \). The expected generalized correlator output becomes

\[
Z_1(w)Z_2(w) = a\phi_y(w)e^{-j\omega T}
\]
divided by \( \phi_y(w) \) (since the window \( W_{II}(w) = \frac{1}{\phi_y(w)} \) then, we get the term of \( e^{-j\omega T} \), the time delay estimation. Hence in this case we use \( W_{II}(w) \) to process the signal of sinusoid signal.

In the case of signal unknown Burg's method is used to estimate the \( \phi(y), \phi_z(w) \) and \( \phi_{z2}(w) \).

**MAXIMUM ENTROPY SPECTRUM ESTIMATION: Burg's Algorithm**

The maximum entropy method (MEM) using the Burg's algorithm was applied by Ulrych\(^5\), who showed the resemble resolution properties of this approach. In applying the concept of maximum entropy to spectral analysis\(^6\) we begin with the relationship between the entropy (strictly speaking, the entropy rate for an infinite process) and the spectral density \( S(f) \) of a stationary Gaussian process,

\[
H = -\frac{1}{4f_N} \int_{-f_N}^{f_N} \log S(f) df \quad (5)
\]

where \( f_N \) is the Nyquist frequency.

Rewriting (5) in terms of the autocorrelation \( \phi(k) \) of the process gives

\[
H = \frac{1}{4f_N} \int_{-f_N}^{f_N} \log \left( \sum_{-\infty}^{\infty} \phi(k) \exp(-2j\pi fk\Delta t) \right) df \quad (6)
\]

where \( \Delta t \) is the uniform sampling rate. Maximizing (6) with respect to the unknown \( \phi(k) \) with the constraint that \( S(f) \) must also be consistent with the known autocorrelations \( \phi(0), \phi(1), \ldots, \phi(m-1) \) results in the MEM spectral estimate.

This estimate expresses maximum uncertainty with respect to the unknown information but is consistent with the known information. The variational procedure
The estimate expresses maximum uncertainty with respect to the unknown information but is consistent with the known information. The variational procedure leads to the expression for the MEI spectral density \[ P_M(f) = \frac{P_M}{f_N \left| 1 + \sum_{i=1}^{M-1} r_i \exp(-j2\pi f_i t) \right|^2} \] where \( P_M \) is a constant and \( r_i \) are prediction error coefficients that are determined from the data.

The chief shortcoming of the MEI spectral estimates has been the lack of a quantitative method of determining the length of the prediction error filter \( r(t) \) in (7). Recent work of AR processes appears to overcome this problem.

The algorithm of Burg's method is considered a wide-sense stationary Gaussian process \( x(t) \) of zero mean and duration \( T \) seconds. The detail of this algorithm is in reference [11].

III. Experimental Study

(A) Signal known:

The random signal is generated by passing white noise through a recursive filter with the impulse response \( B_n \exp(-B_n) \). The Gaussian white noise at the outputs of two sensors are independent and have been generated by Monte Carlo's method. An implementation of the generalized correlator is executed for each set of 128 data points obtained from the sensor outputs. The time delay between the sensor elements is arbitrary and is selected as 16 & 10 units with different cases. Here to compare the time delay results from different signal-to-noise ratios and to determine the optimum \( M \).

When the signal \( y(t) \) is random with known statistics plus the Gaussian noise with zero mean and variance \( \sigma^2 \), the signal-to-noise ratios of -10dB, 0dB, and 10dB are chosen with results given in Figs. 2-5 with \( B=3, 33, \tau=16 \). Figs. 7-11 with \( B=17, \tau=10 \).

The sinusoidal signal \( y(t) \) is known and plus the Gaussian noise, we have the results shown in Figs. 12-14.
(B) Signal unknown:

From the parameter estimator [8] which with the addition of noise in the output of the AR model is

\[ Z(n) = y(n) + n(n) \]

Hence, the model for the sinusoids in white noise problem contains poles and zeros. In using AR parameter estimation methods that assume all-pole model is difficulty. [9][10] It is well known that the least squares estimate of coefficient is biased in this case, and the bias is caused by the expected value of \( E[Z^T]y \) (W=-2+\( \epsilon \)). Corresponding to the FPE we can set \( F_y(w) \) by subtracting \( F_n(w) \) with FPE.

\[ F_y(w) = \text{Power spectrum} \]

Then, use this power spectrum as the window parameter of Hassab & Boucher optimum window, and we can estimate the time delay from a generalized correlator.

Similar to the known signal case when y(t) is random with unknown statistics plus the Gaussian noise with zero mean and variance, the signal-to-noise ratios of -10dB, 0dB, and 10dB are chosen with results given in Figs. 15-17 with \( B=3, 33, T=16 \). In Figs. 16 with \( B=17, T=10 \).

The sinusoidal signal y(t) is unknown and plus the unknown white Gaussian noise, we have the results shown in Figs. 25-27 with \( T=16 \).
IV. Summary

The time delay estimation from cross-correlator without window exhibits a thresholding effect as the signal-to-noise ratio decreases. In which we get the false peak. [11]

After using the window and averaging the periodgrams of the data segments the thresholding effect can be avoided then the maximum M is chose. The unknown signal case before we compute the $\tilde{F}(w)$ the Akaike FPE criterion must be used to decide the order of the prediction error filter coefficients. (See Table 1.)

The relation of SNR and M can be found in Fig. 2 from the Fig. we found M is increased when SNR is decreased.

In the random signal case B=1.33 and B=17 are chosen. Here we can find B=17 has the small power compared to B=3.33 and the small $F(7)$ is enough to detect the time-delay with SNR=40dB.

The key of time delay estimation of unknown signal is the estimation of $\tilde{F}_y(w)$, $\tilde{F}_{z_1}(w)$, and $\tilde{F}_{z_2}(w)$. If we can estimate the power spectrum of $y(t)$ from the sensor outputs, then we must have a good estimate using the above method.
References


Figure Captions

Fig. 1 Block diagram of the generalized cross-correlator.

Fig. 2a Without window output for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 2b Window function of W, for M=1, T=16, SNR=10dB.

Fig. 2c With window of W, for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 3a Without window output for T=16, SNR=0dB, M=1, the result of time-delay =16.

Fig. 3b Window function of W for M=1, T=16, SNR=0dB.

Fig. 3c With window of W for M=1, T=16, SNR=0dB, the result of time-delay =16.

Fig. 4a The same as Fig. 3a, but M=3, the result of time-delay =16.

Fig. 4b Window function of W for M=3, T=16, SNR=0dB.

Fig. 4c With window of W for M=3, T=16, SNR=0dB, the result of time-delay =16.

Fig. 5a Without window output for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 5b Window function of W for M=1, T=16, SNR=10dB.

Fig. 5c With window of W for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 6a Without window output for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 6b With window of W for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 7a Without window output for M=1, T=16, SNR=10dB, the result of time-delay =16.

Fig. 7b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 8a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 8b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 9a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 9b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 10a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 10b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 11a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 11b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 12a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 12b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 13a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 13b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 14a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 14b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 15a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 15b Window function of W for M=1, T=16, SNR=10dB.

Fig. 15c With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 16a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 16b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 17a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 17b The same as Fig. 17a, but M=7, the result of time delay =16.

Fig. 17c The same as Fig. 17a, but M=7, the result of time delay =16.

Fig. 18a Without window output for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 18b With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.

Fig. 18c With window of W for M=1, T=16, SNR=10dB, the result of time delay =16.
Fig. 19a With window, the same as 18-c, but M=7, the result of time delay \( \tau = 11 \).

Fig. 19b The same as 19-a, but M=15, the result of time delay \( \tau = 14 \).

Fig. 20a Without window of unknown signal for \( B=17, SNR=10dB, \tau=10, M=1 \), the result of time delay \( \tau = 10 \).

Fig. 20b With window of unknown signal for \( B=17, SNR=10dB, \tau=10, M=1 \), the result of time delay \( \tau = 10 \).

Fig. 21a Without window of unknown signal, for \( B=17, SNR=0dB, \tau=10, M=1 \), the result of time delay \( \tau = 10 \).

Fig. 21b With window of unknown signal, for \( B=17, SNR=0dB, \tau=10, M=1 \), the result of time delay \( \tau = 10 \).

Fig. 22a Without window of unknown signal, for \( B=17, SNR=-10dB, M=1 \), the result of time delay \( \tau = 13 \).

Fig. 22b With window of unknown signal, for \( B=17, SNR=-10dB, \tau=10, M=1 \), the result of time delay \( \tau = 10 \).

Fig. 23a Without window of unknown signal, for \( B=17, SNR=-20dB, \tau=10, M=1 \), the result of time delay \( \tau = 15 \).

Fig. 23b With window of unknown signal, for \( B=17, SNR=-20dB, \tau=10, M=1 \), the result of time delay \( \tau = 10 \).

Fig. 24a Without window of unknown signal for \( B=17, SNR=-30dB, \tau=10, M=1 \), the result of time delay \( \tau = 6 \).

Fig. 24b Without window of unknown signal for \( B=17, SNR=-30dB, \tau=10, M=7 \), the result of time delay \( \tau = 22 \).

Fig. 24c With window of unknown signal for \( B=17, SNR=-30dB, \tau=10, M=7 \), the result of time delay \( \tau = 12 \).

Fig. 25a Without window of sine wave of unknown signal, for \( \tau = 16, M=1 \), the result of time delay \( \tau = 16 \), SNR=5dB.

Fig. 25b With window of sine wave of unknown signal, for \( \tau = 16, M=1 \), the result of time delay \( \tau = 16 \), SNR=5dB.

Fig. 26a With window of unknown signal M=1, the result of time delay \( \tau = 15 \).

Fig. 26b With window M=3, the result of time delay \( \tau = 16 \), (SNR=5dB).

Fig. 27a With window for \( SNR=-15dB, \tau=16, M=3 \), the result of time delay \( \tau = 14 \).

Fig. 27b Without window for \( SNR=-15dB, M=3, \tau=16 \), the result of time delay \( \tau = 12 \).

Fig. 28 The relationship of random signal case (\( b = \exp(-8t) \), known signal)

Note: Fig. 0 is the random signal case with \( B=3,33 \) and signal is known.
Fig. 1
Burr's MEM (Noiseless.)

Fig. O-a

(10 dB)

Fig. O-b

(0 dB)

Fig. O-c
Fig. 0-d

(-10dB)

Fig. 0-e

Akaike Criterion

chose the order p

(0dB)
Fig. 0-f

Fig. 0-g

(0dB)

(10dB)
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<th>SNR</th>
<th>dB</th>
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<td>0.00000000 00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>0.14653313 00</td>
<td>0.21</td>
</tr>
<tr>
<td>3</td>
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<td>0.18</td>
</tr>
<tr>
<td>4</td>
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<td>0.27</td>
</tr>
<tr>
<td>5</td>
<td>0.14587360 00</td>
<td>0.18</td>
</tr>
<tr>
<td>6</td>
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<td>0.18</td>
</tr>
<tr>
<td>7</td>
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</tr>
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</tr>
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Table 1

The random signal of $\beta=0.35$ ($y(t)=\exp(-\beta t)$)
Fig. 2.3 The relationship of signal-to-noise ratio (SNR) and segment N.
Reference A

(A) Burg's Recursive Algorithm

Let \( X(t) \) denote a sample function of this process, which is sampled uniformly at the rate \( 1/T \) to produce a time series consisting of \( N \) samples, as shown by

\[
\{X(n)\} = \{X(1), X(2), \ldots, X(N)\}
\]

where \( X(n) \) denotes the \( n \)th sample of the series.

The autocorrelation function of the time series \( X(n) \) for a time lag of \( mT \) seconds and time index \( n \) is defined by

\[
R_x(n, m) = E\{X(n)X^*(n-m)\}
\]

where \( E[\cdot] \) denotes the expectation, and the asterisk denotes the complex conjugate operation. With the random process \( X(t) \) assumed wide-sense stationary, the autocorrelation function \( R_x(n, m) \) dependent only on the time lag \( mT \), as shown by

\[
R_x(n, m) = R_x(m)
\]

Also, the autocorrelation function exhibits conjugate symmetry, i.e., \( R_x(m) \) is a Hermitian function

\[
R_x(-m) = R_x^*(m)
\]

Suppose that the time series \( \{X(n)\} \) is applied to a linear digital filter of impulse response \( \{h(n)\} \) of order \( P \). The filter produces an output time series \( \{y(n)\} \) that is designed to approximate a desired time series \( \{d(n)\} \) as shown in Figure A1, and the resultant error time series is

\[
e(n) = d(n) - y(n)
\]

\[\text{Fig. A1}\]
For a special case of a filter designed to predict the value of a random process $X(t)$, one time unit ahead by using the present and past values of the time series $X(n)$, i.e.

$$d(n) = X(n+1)$$

A prediction-error filter is introduced. Fig. A2 shows the functional relationship between the linear prediction filter, characterized by the impulse response $\{h(n)\}$, and the prediction filter, characterized by the impulse response $\{w(n)\}$.

![Diagram showing the relationship between the linear predictive filter and prediction error filter.]

Fig. A2: Illustrating the relationship between the predictive filter and prediction-error filter.

The prediction-error filter equation expressed in matrix form is given by [12]

$$
\begin{bmatrix}
R_X(0) & R_X(-1) & \cdots & R_X(1-M) \\
R_X(1) & R_X(0) & \cdots & R_X(2-M) \\
\vdots & \vdots & \ddots & \vdots \\
R_X(M-1) & R_X(M-2) & \cdots & R_X(0)
\end{bmatrix}
\begin{bmatrix}
1 \\
W(1) \\
W(2) \\
\vdots \\
W(M)
\end{bmatrix}
= 
\begin{bmatrix}
P_M \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

(A6)
where

$$P_M = \sum_{K=0}^{M-1} W_M(K) R_X(-K)$$

is the prediction-error power.

A recursive algorithm developed by Burg [13] for solving the set of prediction-error filter equation is available.

Suppose that we know the solution to the set of $p$ equations pertaining to a prediction filter of order $p$, as shown by (A6). We can take the complex conjugate of both sides of (A6) and recognize that:

1. The prediction-error power $P_M$ is a real quantity.
2. The autocorrelation matrix is Hermitian, i.e.

$$R_X^*(m) = R_X(-m)$$

We thus obtain

$$
\begin{pmatrix}
R_X(0) & R_X(-1) & \cdots & R_X(N-1) \\
R_X(-1) & R_X(0) & \cdots & R_X(N-2) \\
\vdots & \vdots & \ddots & \vdots \\
R_X(1-N)R_X(2-N) & \cdots & R_X(0)
\end{pmatrix}
\begin{pmatrix}
1 \\
W_{M-1}(-1) \\
\vdots \\
W_{M-1}(M-2)
\end{pmatrix}
= 
\begin{pmatrix}
P_M \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

(A7)

(A7) can be rearranged as

$$
\begin{pmatrix}
R_X(0) & R_X(-1) & \cdots & R_X(1-N) \\
R_X(1) & R_X(0) & \cdots & R_X(2-N) \\
\vdots & \vdots & \ddots & \vdots \\
R_X(1-N)R_X(2-N) & \cdots & R_X(0)
\end{pmatrix}
\begin{pmatrix}
W_{M-1}(-1) \\
W_{M-1}(M-2) \\
\vdots \\
1
\end{pmatrix}
= 
\begin{pmatrix}
P_M \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

(A8)

(A6) pertains to a prediction-error filter of order $p$ operated $p\neq M$.
in the forward direction. On the other hand, (A6) pertains to a prediction-error filter of the same order except that is operated in the backward direction. This is illustrated in Fig. A3

Fig. A3: Illustrating: (a) forward and (b) backward prediction error filtering

Combining (A6) and (A8) to expand the number of prediction error filter equations by one is as follows:

\[
\begin{bmatrix}
R_X(0) & R_X(-1) & \cdots & R_X(1-k) & R_X(-M) \\
R_X(1) & R_X(0) & \cdots & R_X(2-k) & R_X(1-k) \\
R_X(1-M) & R_X(1-M-1) & \cdots & R_X(1) & R_X(0)
\end{bmatrix}
\begin{bmatrix}
1 \\
W_{n-1}(1) \\
\vdots \\
W_{n-1}(N-1) \\
1
\end{bmatrix}
\begin{bmatrix}
0 \\
W_{F-1}(-1) \\
\vdots \\
W_{F-1}(N-1) \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_M \\
0 \\
\vdots \\
0 \\
\Delta_M
\end{bmatrix}
+ W_P(N)
\]

(A9)
However, for the corresponding prediction-error filter of order \((m+1)\) we have

\[
\begin{bmatrix}
R_X(0) & R_X(-1) & \cdots & R_X(1-m) & R_X(-M) \\
R_X(1) & R_X(0) & \cdots & R_X(2-m) & R_X(1-M) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_X(M-1) & R_X(M-2) & \cdots & R_X(0) & R_X(-1) \\
R_X(M) & R_X(M-1) & \cdots & R_X(1) & R_X(0)
\end{bmatrix}
\begin{bmatrix}
1 \\
W_M(1) \\
\vdots \\
W_M(M-1) \\
W_M(M)
\end{bmatrix}
= 
\begin{bmatrix}
P_{M+1} \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\]  

\[(A10)\]

Comparing with (A9) and (A10), we deduce that

\[
W_M(k) = W_{M-1}(k) + W_M(N)W_{M-1}(M-k) ; \quad k=0,1,\ldots,M \quad (A11)
\]

\[
P_{M+1} = P_M + W_M(N)\Delta_M^* \quad (A12)
\]

and

\[
0 = \Delta_M^* + W_M(H)P_H \quad (A13)
\]

In the recursive formula of (A11), note that for all values of \(M\)

\[
W_M(k) = \begin{cases} 
1 & \text{for } k=0 \\
0 & \text{for } k>0
\end{cases}
\]

(A13) will always have a solution provided that \(P_M>0\), thus using (A15) to eliminate \(\Delta_M^*\) from (A12), we get

\[
P_{M+1} = P_M \left[ 1 - \left| W_M(N) \right|^2 \right]
\]

By analogy with the transmission of power through a terminated two-port network, we may view \(W_M(N)\) as a "reflection coefficient".
COMPUTATIONAL PROCEDURE

Summarizing the steps in the recursive procedure involved in calculating the prediction-error filter coefficients and related values of the spectral density and autocorrelation function, they are

1. For the given time series \( X(n) \), \( n=1, 2, \ldots, N \) and \( M=0 \).
   Calculate the power
   \[
   P_1 = \frac{1}{N} \sum_{n=1}^{N} X(n) X^*(n)
   \]
   and calculate the reflection coefficient and the prediction-error power by
   \[
   w_1 = \frac{-2 \sum_{n=1}^{N-1} X^*(n) X(n+1)}{\sum_{n=1}^{N-1} \left| X(n) \right|^2 + \left| X(n+1) \right|^2}
   \]
   \[
   p_2 = p_1 \left[ 1 - \left| w_1(n) \right|^2 \right]
   \]

2. Increment \( N \) by 1, and calculate the next values:
   \[
   p_{M}(n) = p_{M-1}(n) + w_{M-1}(n) q_{M-1}(n)
   \]
   \[
   q_{M}(n) = q_{M-1}(n+1) + w_{M}(n) p_{M-1}(n+1)
   \]
   where
   \[
   p_0(n) = X(n)
   \]
   \[
   q_0(n) = X(n+1)
   \]

3. Calculate the reflection coefficient
   \[
   w_{M}(n) = \frac{-2 \sum_{n=1}^{N-M} p_{M-1}(n) q_{M-1}(n)}{\sum_{n=1}^{N-M} \left| p_{M-1}(n) \right|^2 + \left| q_{M-1}(n) \right|^2}
   \]
and the corresponding value of the error prediction power

\[ P_{M+1} = P_M \left[ 1 - |W_M(M)|^2 \right] \]

(4) Calculate the remaining coefficients of the prediction error filter by

\[ W_M(k) = W_{M-1}(k) + W_p(k) W^*_{M-1}(M-K) \quad ; \quad k=1,2,\ldots,k-1 \]

(5) Repeat steps (3) to (4) for each value of \( M \) up to the optimum value \( p(=M) \) for which the final prediction-error

\[ (PPE)_M = \frac{(E+M+1)}{(N-M-1)} P_{M+1} \]

is a minimum.

(6) We now have all the quantities required for calculating the extrapolated value of the autocorrelation function.

\[ R_X(M) = -W_\tau(M) P_M - \sum_{k=1}^{M-1} W_{M-1}(k) R_X(M-K) \]

and for calculating the spectral density estimate

\[ \hat{S}_X(f) = \frac{P_{M+1}}{2B \left| 1 + \sum_{k=1}^{M} W_k(k) \exp(-2\pi k f / T_s) \right|^2} \]

where \( T_s \) is the sampling period and \( X(t) \) is the band-limited to \(-B,B\).

When the process \( X(t) \) is sampled at the Nyquist rate, we have \( T_s=1/2B \)

The computation is thereby completed.
TIME DELAY ESTIMATION: FOR KNOWN & UNKNOWN SIGNALS

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Time delay estimation using Hassab-Boucher windows is extensively examined by computer simulation for both known and unknown (random) signals. Optimum number of data segments is determined for the best result in time delay estimate. For the unknown signal case, the Burg's maximum entropy spectral estimation algorithm is used to estimate the signal spectral density used in the design of Hassab-Boucher windows.