Weibull Accelerated Life Tests When There are Competing Causes of Failure

by

John P. Klein
Department of Statistics, Ohio State University

Asit P. Basu
Department of Statistics, University of Missouri-Columbia

Technical Report 104
Department of Statistics

June 1981

Mathematical Sciences
Weibull Accelerated Life Tests When There Are Competing Causes of Failure.

John P. Klein
Asit P. Basu

Department of Statistics
University of Missouri-Columbia
Columbia, MO. 65211

Office of Naval Research
Department of the Navy
Arlington, VA. 22217

Approved for public release; distribution unlimited

Accelerated life testing of a product under more severe than normal conditions is commonly used to reduce test time and costs. Data collected at such accelerated conditions are used to obtain estimates of the parameters of a stress translation function. This function is then used to make inference about the product's life under normal operating conditions.

We consider the problem of accelerated life tests when the product of interest is a p component series system. Each of the components is assumed to have an independent Weibull time to failure distribution with different shape para-
20. Parameters and different scale parameters which are increasing functions of the stress. A general model is used for the scale parameter which includes the standard engineering models as special cases. This model also has an appealing biological interpretation.

Maximum likelihood estimators of the component parameters are obtained for type I, type II, and progressive censoring. These estimators are used to obtain estimators of the probability of component and system survival under normal operating conditions. This method is illustrated by an example.
WEIBULL ACCELERATED LIFE TESTS WHEN THERE ARE COMPETING CAUSES OF FAILURE

John P. Klein
Department of Statistics
The Ohio State University
Columbus, Ohio 43210

Asit P. Basu
Department of Statistics
University of Missouri
Columbia, Missouri 65211

Key Words and Phrases: safe dose levels, competing risks, accelerated lifetests, Hartley and Sielkin model.

ABSTRACT

Accelerated life testing of a product under more severe than normal conditions is commonly used to reduce test time and costs. Data collected at such accelerated conditions are used to obtain estimates of the parameters of a stress translation function. This function is then used to make inference about the product's life under normal operating conditions.

We consider the problem of accelerated life tests when the product of interest is a p component series system. Each of the components is assumed to have an independent Weibull time to failure distribution with different shape parameters and different scale parameters which are increasing functions
of the stress. A general model is used for the scale parameter which includes the standard engineering models as special cases. This model also has an appealing biological interpretation.

Maximum likelihood estimators of the component parameters are obtained for type I, type II, and progressive censoring. These estimators are used to obtain estimators of the probability of component and system survival under normal operating conditions. This method is illustrated by an example.

1. INTRODUCTION

Accelerated life testing of a product is often used to obtain information on its performance under normal use conditions. Such testing involves subjecting test items to conditions more severe than encountered in the item's everyday use. This results in decreasing the item's mean life and leads to shorter test times and reduced experimental costs. In engineering applications accelerated conditions are produced by testing items at higher than normal temperature, voltage, pressure, load, etc. In biological applications accelerated conditions arise when large doses of a chemical or radiological agent are given. In both cases the data collected at the high stresses is used to extrapolate to some low stress level where testing is not feasible.

Several authors have considered the problem of analyzing accelerated life tests when the product has only a single mode of failure. Nelson (1974a) has a bibliography of applications and analysis of each tests. Mann, Schafer, and Singpurwalla (1974) derive least squares and maximum likelihood estimators of model parameters when the underlying failure distribution is exponential. Nelson (1972c) describes a graphical solution to this problem.
Nelson (1970) derives graphical, maximum likelihood and least squares estimators of model parameters when the underlying distribution is Weibull. Meeker and Nelson (1974a and 1974b) derive maximum likelihood estimators of the model parameters in the Weibull case when the data is type I or type II censored. They also discuss the optimal strategy for designing such tests. Mann (1972) has also discussed optimal design strategy. Tolerance bounds for the Weibull model are discussed in Mann (1978).

Several papers have been written on analyzing accelerated life tests when the failure distribution is normal or log normal. In a series of papers Nelson (1971, 1972a and 1972b) has considered maximum likelihood, least squares, and graphical estimation procedures for an Arrhenius model when all failure times are known. For this model it is assumed that mean of the log failure time is linear in the stress, and that the variance is independent of the stress. Nelson and Hahn (1972, 1973) derive best linear unbiased estimators of the regression parameters of this model for type II censored samples. Kielpinski and Nelson (1975) discuss maximum likelihood estimation procedures for this model when the sample is type I censored.

Several papers have been written on analyzing accelerated life tests when more than one failure mode is present. Here failures can occur from any one of k independent causes. Sample information consists of a failure time for each item and the cause of failure. Assuming that for a given stress V each failure mode follows an independent log normal distribution with parameters \( \mu_i(V) = \alpha_i + \beta_i V \), and \( \sigma_i^2 \) constant with respect to V, \( i = 1, \ldots, k \), Nelson (1973) obtains graphical estimates of \( \alpha_i \) and \( \beta_i \) when there is no censoring. For this model, Nelson (1974b) obtains maximum likelihood estimates of \( \alpha_i \), \( \beta_i \) and \( \sigma_i^2 \).
Klein and Basu (1980a) have considered the above problem when the component lifetimes are exponentially distributed and the data is type I, type II or progressively censored. Klein and Basu (1980b) have described the analysis of accelerated life tests of series systems when the component failure times follow a Weibull distribution with common shape parameter. The object of this paper is to consider the case when the shape parameters are different.

In section 2 we present the model to be used for accelerated life tests in the competing risks framework. In section 3 we use this model to analyze accelerated life tests where there are competing causes of failure and the data is type I, type II, or progressively censored. Finally, in section 4 an example is presented.

2. THE MODEL

The problem considered in the sequel is as follows. Consider a p component system with component lifetimes $X_1, X_2, \ldots, X_p$. Suppose that under normal stress conditions these components have long lifetimes making testing at such conditions unfeasible. To reduce test time and cost, s stresses, $V_1, \ldots, V_s$ are selected and a life test is conducted at constant application of the selected stress. We wish to use this information to make inference about the component lifetimes under normal stress conditions.

Consider the following model introduced by Klein and Basu (1980b) elsewhere.

At a stress $V_i$, $i = 1, \ldots, s$ assume that the $j^{th}$ component has a hazard rate given by

$$h_j(x, V_i; \alpha_j, \beta_j) = g_j(x, \alpha_j)\lambda_j(V_i, \beta_j)$$

(2.1)

$i = 1, \ldots, s$ \hspace{1cm} $j = 1, \ldots, p$.

For $g_j(x, \alpha_j)$ a Weibull form is assumed, that is
\[ g_j(x, \alpha_j) = \alpha_j^{x - 1}, \alpha_j > 0, t > 0. \quad (2.2) \]

The \( \alpha_j \)'s may vary from component to component to allow for differences in component reliability.

For \( \lambda_j(V, \beta_j) \) we assume a model of the form

\[ \lambda_j(V, \beta_j) = \exp \left( \sum_{k=0}^{\infty} \beta_{jk} \theta_{jk}(V) \right). \quad (2.3) \]

where \( \theta_{j0}(V) = 1 \) and \( \theta_{j1}(V), ..., \theta_{jk}(V) \) are \( k_j \) non-decreasing functions of \( V \). The \( \theta_{j}(\cdot) \)'s may differ from one component to another.

This model includes the standard models, namely, the power rule with \( \lambda_j(V, \beta_j) = \beta_j V^{\beta_j} \); the Arrhenius reaction rate model with \( \lambda_j(V, \beta_j) = \exp(\beta_{j0} - \beta_{j1}/V) \); and the Eyring model for a single stress with \( \lambda_j(V, \beta_j) = V^{\beta_j} \exp(\beta_{j0} - \beta_{j2}/V) \) as special cases.

The model also can be derived from the interpretation of the effects of a carcinogen on a cell as proposed by Armitage and Doll (1961). For details see Klein and Basu (1980b). To produce cancer in a single cell, \( k \) independent events must occur. The effect of an increased dose of a carcinogen is to increase the rate at which these \( k \) events occur. If, for the \( j \)th disease, this increase is of the form \( \exp(\beta_{j\ell} \theta_{j\ell}(V)) \) for \( \ell = 1, ..., k_j \) the model (2.3) is obtained. If this increase is assumed linear the model of Hartley and Sielkin (1977) is obtained. Thus the model of Hartley and Sielkin is a first order Taylor Series approximation to (2.3) when \( \theta_{j\ell}(V) = V \) for \( \ell = 1, ..., k_j \).

Consider an accelerated life test conducted at constant applications of \( s \) stress level, \( V_1, ..., V_s \). Let \( X_{11}, X_{12}, ..., X_{1p} \) denote the component lifetimes of the \( p \) component
series system put on test at stress $V_i$. Assume that the component lifetimes are independent. We are not allowed to observe $X_{i1}, \ldots, X_{ip}$ directly but, instead, we observe $Y_i = \min(X_{i1}, \ldots, X_{ip})$ and an indicator variable which describes which of the $p$ components is the minimum. We shall use the method of maximum likelihood to estimate $\alpha_j$ and $\beta_j = (\beta_{j0}, \ldots, \beta_{jk_j})$, $j = 1, \ldots, p$ for various censoring schemes.

3. ESTIMATION OF PARAMETERS

3.1 Type I censoring

For this censoring scheme $n_i$ items are put on test at stress $V_i$, $i = 1, \ldots, s$. The $i$th system on test at stress $V_i$ is tested until it fails or until some fixed time $\tau_{i\ell}$ at which it is removed from the study. The $\tau_{i\ell}$'s may vary from item to item to allow for staggered entry into the study. Let $r_i$ be the number of systems which fail prior to their censoring time at stress $V_i$. Let $r_{i\ell}$ denote the number of these whose failure was caused by failure of the $j$th component. Let $X_{i\ell}$ denote the failure time of the $r_{i\ell}$ systems whose failure is due to failure of component $j$. For convenience let $Y_i$, $i = 1, \ldots, s$, $\ell = 1, \ldots, r_i$ denote the failure times of the $r_i$ systems regardless of the cause of failure.

The overall log likelihood can be written as

$$\ln L = \sum_{j=1}^{p} \ln L_j \tag{3.1.1}$$

where

$$\ln L_j = \sum_{i=1}^{s} \sum_{\ell=1}^{r_{ij}} \ln \left( \sum_{k=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V_i) \right) + r_{ij} \ln \alpha_j + (\alpha_j - 1) \sum_{\ell=1}^{r_{ij}} \ln X_{i\ell}$$

$$- T_i(\alpha_j) \exp \left( \sum_{\ell=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V_i) \right), \quad j = 1, \ldots, p \tag{3.1.2}$$
where
\[
T_i(\alpha_j) = \sum_{k=1}^{r_i} \alpha_j^n_{i-r_i} + \sum_{k=1}^{r_i} \tau_{i,k}^*, i = 1, \ldots, s
\]  
(3.1.3)

with \( \tau_{i,k}^* \), \( i = 1, \ldots, s \), \( k = 1, \ldots, n_i-r_i \) the censoring times of the \( n_i-r_i \) systems removed from the accelerated life test.

When all items on test at stress \( V_i \) have a common censoring time, \( \tau_i \), then
\[
T_i(\alpha_j) = \sum_{k=1}^{r_i} \alpha (n_i-r_i) \tau_i.
\]

The likelihood equations, which must be solved numerically for the maximum likelihood estimators, \( \hat{\alpha}_j \), \( \hat{\beta}_{j0}, \hat{\beta}_{j1}, \ldots, \hat{\beta}_{jk_k} \), are:
\[
0 = \frac{\delta \ln L_j}{\delta \alpha_j} = \frac{s}{\sum_{i=1}^{s} \left[ \frac{r_{ij}}{\alpha_j} + \sum_{k=1}^{r_{ij}} \ln x_{i,k}^{(1)} - T_i(\alpha_j) \exp \left( \sum_{k=0}^{k_j} \beta_{j0}^{\alpha_j} \theta_{jk}(V_i) \right) \right],}
\]
\[
j = 1, \ldots, p
\]  
(3.1.4)

where
\[
T_i^{(1)}(\alpha_j) = \sum_{k=1}^{r_i} \alpha_j^n_{i-r_i} + \sum_{k=1}^{r_i} \tau_{i,k}^* \ln \tau_{i,k}^*, i = 1, \ldots, s.
\]  
(3.1.5)

and
\[
0 = \frac{\delta \ln L_j}{\delta \beta_{j0}} = \frac{s}{\sum_{i=1}^{s} r_{ij} \theta_{ju}(V_i) - T_i(\alpha_j) \theta_{ju}(V_i) \exp \left( \sum_{k=0}^{k_j} \beta_{j0} \theta_{jk}(V_i) \right),}
\]
\[
j = 1, \ldots, p \quad u = 0, \ldots, k_j
\]  
(3.1.6)
The second partial derivatives of the log likelihood are

\[
- \frac{\delta^2 \ln L_j}{\delta \alpha_j^2} = \sum_{i=1}^{s} \frac{r_{ij}}{\alpha_j^2} - \lambda_{ij} \tau_i^{(2)}(\alpha_j), \quad j = 1, \ldots, p. \tag{3.1.7}
\]

\[
\tau_i^{(2)}(\alpha_j) = \sum_{k=1}^{p} \gamma_{ik} (\ln Y_{ik})^2 +
\]

\[
\frac{\delta^2 \ln L_j}{\delta \alpha_j \delta \beta_{ju}} = \sum_{i=1}^{s} \lambda_{ij} \theta_{ju} (V_i) T_i^{(1)}(\alpha_j)
\]

\[
j = 1, \ldots, p \quad u = 0, \ldots, k_j. \tag{3.1.9}
\]

and

\[
\frac{\delta^2 \ln L_j}{\delta \beta_{ju} \delta \beta_{jw}} = \sum_{i=1}^{s} \lambda_{ij} \theta_{ju} (V_i) \theta_{jw} (V_i)
\]

\[
w = 1, \ldots, k_j \quad u = 0, \ldots, k_j. \tag{3.1.10}
\]

To find the information matrix let

\[
C_{i\ell} = \begin{cases} 1 & \text{if } Y_{i\ell} \leq \tau_{i\ell}, \ i = 1, \ldots, s, \ \ell = 1, \ldots, n_i \\ 0 & \text{otherwise} \end{cases} \tag{3.1.11}
\]

and define

\[
n_{ij} = P(C_{i\ell} = 1) = 1 - \exp \left( - \sum_{j=1}^{p} \lambda_{ij} \tau_{i\ell} \right),
\]

\[
i = 1, \ldots, s \quad \ell = 1, \ldots, n_i. \tag{3.1.12}
\]

The conditional density function of \( Y_{i\ell} \) given \( C_{i\ell} = 1 \) is
\[ f(y_{i\ell}|c_{i\ell} = 1) = \begin{cases} \frac{\sum_{j=1}^{p} \alpha_j \lambda_{i\ell}^j y^{\alpha_j - 1}}{n_{i\ell}} \exp \left( - \sum_{j=1}^{p} \lambda_{i\ell} y^{\alpha_j} \right), & y < \tau_{i\ell} \\ 0 & \text{otherwise.} \end{cases} \]  

(3.1.13)

Now

\[ E(T_i(\alpha_j)) = E \left( \sum_{\ell=1}^{n_i} c_{i\ell} y_{i\ell}^{\alpha_j} + \sum_{\ell=1}^{n_i} (1 - c_{i\ell}) \tau_{i\ell} \alpha_j \right) \]

\[ = \sum_{\ell=1}^{n_i} \int_0^{\tau_{i\ell}} y^{\alpha_j} \left( \sum_{m=1}^{p} \alpha_m \lambda_{i\ell} y^{\alpha_m - 1} \right) \exp \left( - \sum_{m=1}^{p} \lambda_{i\ell} y^{\alpha_m} \right) \, dy \\
+ \sum_{\ell=1}^{n_i} \tau_{i\ell} \alpha_j \exp \left( - \sum_{m=1}^{p} \lambda_{i\ell} \tau_{i\ell}^{\alpha_m} \right) \]

i = 1, \ldots, p \quad j = 1, \ldots, s. \quad (3.1.14)

where the integral must be evaluated numerically. Similarly,

\[ E(T_{i}^{(1)}(\alpha_j)) = \]

\[ = \sum_{\ell=1}^{n_i} \int_0^{\tau_{i\ell}} y^{\alpha_j \lambda_{i\ell} n_{i\ell} \tau_{i\ell}} \left( \sum_{m=1}^{p} \alpha_m \lambda_{i\ell} y^{\alpha_m - 1} \right) \exp \left( - \sum_{m=1}^{p} \lambda_{i\ell} \tau_{i\ell}^{\alpha_m} \right) \, dy \\
+ \sum_{\ell=1}^{n_i} \tau_{i\ell} \lambda_{i\ell} n_{i\ell} \tau_{i\ell}^{\alpha_m} \exp \left( \sum_{m=1}^{p} \lambda_{i\ell} \tau_{i\ell}^{\alpha_m} \right), \quad (3.1.15) \]

and
\[ E(T(2)(a_j)) = \]
\[ \sum_{\ell=1}^{n_i} \int_0^{\tau_{i\ell}} \alpha_j (\alpha m)^2 \left( \sum_{m=1}^{\alpha m} \alpha_m \lambda_{im} \alpha_{m-1} \right) \exp \left( - \sum_{m=1}^{\alpha m} \lambda_{im} \alpha_m \right) dy \]
\[ + \sum_{\ell=1}^{n_i} \tau_{i\ell} (\lambda n \tau_{i\ell})^2 \exp \left( \sum_{m=1}^{\alpha m} \lambda_{ij} \alpha_m \right), \]
\[ i = 1, \ldots, s \quad j = 1, \ldots, p. \quad (3.1.16) \]

Now the probability that the \( \ell \)th system fails prior to time \( \tau_{i\ell} \) due to failure of cause \( j \) at stress \( i \) for \( i = 1, \ldots, s \), \( j = 1, \ldots, p \), \( \ell = 1, \ldots, n_i \) is

\[ \psi_{ij\ell} = \int_0^{\tau_{i\ell}} \lambda_{ij} \alpha_j^{-1} \exp \left( - \sum_{m=1}^{\alpha m} \lambda_{im} \alpha_m \right) du \quad (3.1.17) \]

which must be evaluated numerically. Hence

\[ E(r_{ij}) = \sum_{\ell=1}^{n_i} \psi_{ij\ell}. \quad (3.1.18) \]

The asymptotic covariance matrix can now be obtained by using (3.1.18), (3.1.16), (3.1.15), and (3.1.14) to calculate the expected values of (3.1.7), (3.1.9), and (3.1.10). An estimator of this matrix can be obtained by substituting \( \alpha_j \), and \( \hat{\beta}_j \) in the appropriate expressions.

### 3.2 Type II censoring.

For this censoring scheme \( n_i \) systems are put on test at each of the \( s \) stress levels and testing continues until a preassigned number \( r_{ij} \) have failed at which time testing is stopped. Suppose that \( r_{ij} \) systems fail due to failure of the \( j \)th component, \( j = 1, \ldots, p. \) Let \( X_{ij1}, \ldots, X_{ijr_{ij}} \) denote the failure time of those \( r_{ij} \) systems at stress \( V_i \) whose failure was caused by
failure of the jth component, i = 1, ..., s, j = 1, ..., p, 
\varepsilon = 1, ..., r_i. Let \( Y_{i1}(\varepsilon), ..., Y_{i(r_i)} \) denote the ordered failure times of the \( r_i \) systems observed to fail at stress \( i \), regardless of the mode of failure.

One can show that the likelihood of interest is given by (3.1.1) and (3.1.2) with

\[
T_i(\alpha_j) = \sum_{\varepsilon=1}^{r_i} Y_{i\varepsilon}(\varepsilon) + (n_i - r_i)Y_{i(r_i)}^{a_j},
\]

\( i = 1, ..., s \) \( j = 1, ..., p. \) (3.2.1)

The likelihood equations are given by (3.1.4) and (3.1.5) with

\[
T_i^{(1)}(\alpha_j) = \sum_{\varepsilon=1}^{r_i} \alpha_j Y_{i\varepsilon}(\varepsilon) + (n_i - r_i)Y_{i(r_i)}^{a_j},
\]

\( i = 1, ..., s \) \( j = 1, ..., p. \) (3.2.2)

The matrix of second partial derivatives is given by (3.1.7) to (3.1.9) with

\[
T_i^{(2)}(\alpha_j) = \sum_{\varepsilon=1}^{r_i} \alpha_j \ln Y_{i\varepsilon}(\varepsilon)^2 + (n_i - r_i)Y_{i(r_i)}^{a_j} \ln Y_{i(r_i)}^{a_j},
\]

\( j = 1, ..., p \) \( i = 1, ..., s. \) (3.2.3)

To find the information matrix note that the density of \( Y_{i\varepsilon} \) is

\[
f(y_{i\varepsilon}) = \frac{n_i!}{(\varepsilon - 1)! (n_i - \varepsilon)!} \left( \sum_{j=1}^{p} \lambda_{ij} \alpha_j y_{i\varepsilon}^{\alpha_j - 1} \right)^{\varepsilon - 1} \left[ 1 - \exp \left( - \sum_{j=1}^{p} \lambda_{ij} y_{i\varepsilon}^{\alpha_j} \right) \right]^{n_i - \varepsilon} \exp \left[ -(n_i - \varepsilon + 1) \left( \sum_{j=1}^{p} \lambda_{ij} y_{i(k)}^{\alpha_j} \right) \right],
\]

\( 0 \leq y_{i\varepsilon} \leq \infty, \)

\( \varepsilon = 1, ..., n_i \) \( i = 1, ..., s. \) (3.2.4)
Hence,

\[
E(T_i(\alpha_j)) = 
\]

\[
\sum_{j=1}^{r_i} \int_0^\infty y_i(\varepsilon)^j f(y_i(\varepsilon)) dy_i(\varepsilon) + 
\]

\[
(n_i - r_i) \int_0^\infty y_i(r_i)^j f(y_i(r_i)) dy(r_i),
\]

(3.2.5)

\[
E(T_i^{(1)}(\alpha_j)) = 
\sum_{j=1}^{r_i} \int_0^\infty y_i(\varepsilon)^j (n_y(\varepsilon))^j f(y_i(\varepsilon)) dy_i(\varepsilon)
\]

\[
+ (n_i - r_i) \int_0^\infty y_i(r_i)^j (n_y(r_i))^j f(y_i(r_i)) dy(r_i),
\]

(3.2.6)

\[
E(T_i^{(2)}(\alpha_j)) = 
\sum_{j=1}^{r_i} \int_0^\infty y_i(\varepsilon)^j (n_y(\varepsilon))^j f(y_i(\varepsilon)) dy_i(\varepsilon)
\]

\[
+ (n_i - r_i) \int_0^\infty y_i(r_i)^j (n_y(r_i))^j f(y_i(r_i)) dy(r_i),
\]

(3.2.7)

These integrals must be evaluated numerically. Now with \(r_i\) fixed, \((r_{i1}, ..., r_{ip})\) has a multinomial distribution with parameters \(n_{ij}\) given

\[
\Pi_{ij} = \int_0^\infty \lambda_{ij} \alpha_j \exp(- \sum_{k=1}^p \lambda_{ik} \alpha_k) d\alpha_j,
\]

(3.2.8)

so

\[
E(r_{ij}) = r_i \Pi_{ij}.
\]

The information matrix can be obtained by using (3.2.5), (3.2.6), (3.2.7), and (3.2.8) to calculate the expected values of (3.1.7), (3.1.9), and (3.2.10). An estimate of this matrix can be obtained by substituting \(\hat{\alpha}_j\) and \(\hat{\beta}_j\) in the appropriate expressions.
3.3 Progressive censoring

For this censoring scheme \( \bar{N}_i \) items are put on test at the \( i \)th stress level. Let \( \tau_{i1}, \ldots, \tau_{iM_i} \) be fixed censoring times at which a fixed number of items, \( c_{i1}, \ldots, c_{iM_i} \), are removed from the test. At time \( \tau_{iM_i} \) either a fixed number \( c_{iM_i} \) items are removed from the test or testing is terminated with a random number, \( c_{iM_i} \) systems still functioning. Assume that \( N_i \) is sufficiently large to allow removal of the required number of items.

Let \( n_i = N_i - \sum_{k=1}^{M_i} c_{ik} \) be the number of items which are observed to fail and let \( Y_{i1}, \ldots, n_i \) denote their failure times. That is, \( Y_{i1}, \ldots, Y_{in_i} \) are the \( n_i \) system failure times regardless of the mode of failure. Suppose that \( r_{ij} \) of the \( n_i \) failures were caused by failure of the \( j \)th component, and the respective failure times are \( X_{ij1}, \ldots, X_{ijr_{ij}} \), \( i = 1, \ldots, s \), \( j = 1, \ldots, p \).

The likelihood function of interest is given by (3.1.1) which can be factored into component likelihoods as in (3.1.2) with

\[
T_i(a_j) = \sum_{j=1}^{M_i} \frac{\alpha_j}{\alpha_j} + \sum_{k=1}^{n_i} c_{ik} \tau_{ik}, \quad j = 1, \ldots, p, \quad i = 1, \ldots, s.
\]

The likelihood equations are as in (3.1.4), (3.1.5) with

\[
\tau_i^{(1)}(a_j) = \sum_{k=1}^{n_i} Y_{j1} + \sum_{k=1}^{M_i} \tau_{kj} c_{ik} \ln(\tau_{ik}), \quad j = 1, \ldots, p, \quad i = 1, \ldots, s.
\]

The matrix of second partial derivatives are as in (3.1.6),
To calculate $E(T_i(\alpha_j))$, $E(T_i^{(1)}(\alpha_j))$ and $E(T_i^{(2)}(\alpha_j))$ consider any of the $s$ stress levels. The $n_i$ observed failures, $Y_{ik}$, have the following survival function

$$F_i(y) = \exp\left(-\sum_{j=1}^{p} \lambda_{ij} y^j\right), \quad y \geq 0, \quad i = 1, \ldots, s.$$  

(3.3.4)

Let $f_{i\ell}$ denote the number of failures in the interval $[\tau_{i,\ell-1}, \tau_{i\ell})$, $\ell = 1, \ldots, M_i + 1$ where $\tau_{iM_i+1} = \infty$. Define $F_{i\ell} = F_i(\tau_{i\ell})$ and $F_{i\ell} = 1 - F_{i\ell}$, and let $U_{i\ell k}$, $k = 1, \ldots, f_{i\ell}$ denote the failure times of the $f_{i\ell}$ items which fail in this interval.

Cohen (1963) shows that

$$E(f_{i\ell}) = \begin{cases} \text{NF}_{i\ell} \text{ for } \ell = 1 \\ (N - \sum_{k=1}^{M_i} c_{ik} F_{i\ell}) (F_{i\ell} - F_{i\ell-1}) \text{ for } \ell = 1, \ldots, M_i + 1 \\ \end{cases}$$

(3.3.5)

if $c_{M_i}$ is fixed and for $\ell = 1, \ldots, M_i$ if $c_{iM_i}$ is random.

Now

$$E\left(\sum_{k=1}^{f_{i\ell}} U_{i\ell k}^{\alpha_j}\right) = E\left(\sum_{k=1}^{f_{i\ell}} U_{i\ell k}^{\alpha_j} f_{i\ell}\right)$$

$$= E(f_{i\ell}) E(U_{i\ell k}^{\alpha_j} | \tau_{i\ell} = U_{i\ell k} < \tau_{i\ell})$$

$$= E(f_{i\ell}).$$
\[
\int_{F_{1\ell}}^{\tau_{1\ell}-1} a_j \left( \prod_{q=1}^{P} \alpha_q \lambda_{i q} \alpha_{q-1} \right) \exp \left( - \prod_{q=1}^{P} \lambda_{i q} \alpha_{q} \right) \frac{du}{(F_{1\ell} - F_{1\ell-1})}
\]

for \( \ell = 1, \ldots, M_i \)  

(3.3.6)

where \( E(f_i) \) is given in (3.3.5) and the integral must be evaluated numerically, with similar expressions for

\[
E \left( \sum_{k=1}^{P} U_{i k} \ln U_{i k} \right) \text{ and } E \left( \sum_{k=1}^{P} U_{i k} (\ln U_{i k})^2 \right). \]

Using these expressions when a fixed number of items are removed at time \( \tau_{i M_i} \), we have

\[
E(T_i(a_j)) = N_i \int_0^{\infty} u^j \left( \sum_{k=1}^{P} \alpha_k \lambda_{i k} \alpha_{k-1} \right) \exp \left( - \sum_{k=1}^{P} \lambda_{i k} \alpha_k \right) \frac{du}{\prod_{k=1}^{P} \alpha_k \lambda_{i k} u^{\alpha_{k-1}} - \sum_{\ell=1}^{M_i} c_{1\ell} \int_{\tau_{1\ell}}^{F_{1\ell}} F_{1\ell}^{-1}}.
\]

(3.3.7)

\[
E(T_i^{(1)}(a_j)) = N_i \int_0^{\infty} u^j \left( \sum_{k=1}^{P} \alpha_k \lambda_{i k} \alpha_{k-1} \right) \exp \left( - \sum_{k=1}^{P} \lambda_{i k} \alpha_k \right) du + \sum_{\ell=1}^{M_i} c_{1\ell} \tau_{1\ell}^{a_j},
\]

(3.3.8)
and

\[ E(T_1^{(2)}(\alpha_j)) = \]

\[ N_1 \int_0^\infty u^j (\ln u)^2 \left( \sum_{k=1}^P \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left( - \sum_{k=1}^P \lambda_{ik} u^{\alpha_k} \right) du \]

\[ \quad - \sum_{\ell=1}^{M_1} \frac{c_{i\ell}}{F_{i\ell}} \int_{\tau_{i\ell}}^\infty u^j (\ln u)^2 \left( \sum_{k=1}^P \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left( - \sum_{k=1}^P \lambda_{ik} u^{\alpha_k} \right) du \]

\[ \quad + M_1 c_{iM_1} \int_{\tau_{iM_1}}^\infty u^j (\ln u)^2 \left( \sum_{k=1}^P \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left( - \sum_{k=1}^P \lambda_{ik} u^{\alpha_k} \right) du \]

When all testing stops at time \( \tau_{iM_1} \) there are

\[ c_{M_1} = N_1 - \sum_{\ell=1}^{M_1} c_{i\ell} - \sum_{\ell=1}^{M_1} f_{i\ell} \text{ items removed from test. Thus} \]

\[ E(r_{M_1}) = F_{iM_1} (N - \sum_{i=1}^{M_1} \frac{c_{i\ell}}{F_{i\ell}}). \]

And, here,

\[ E(T_1(\alpha_j)) = N_1 \{ \int_0^{\tau_{iM_1}} u^j \left( \sum_{k=1}^P \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left( - \sum_{k=1}^P \lambda_{ik} u^{\alpha_k} \right) du \]

\[ \quad + \int_{\tau_{iM_1}}^{\tau_{iM_1}^+} u^j \left( \sum_{k=1}^P \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left( - \sum_{k=1}^P \lambda_{ik} u^{\alpha_k} \right) du \]

\[ \quad - \sum_{\ell=1}^{M_1} \frac{c_{i\ell}}{F_{i\ell}} \int_{\tau_{i\ell}}^{\tau_{iM_1}} u^j \left( \sum_{k=1}^P \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left( - \sum_{k=1}^P \lambda_{ik} u^{\alpha_k} \right) du \]
\[ M_{1}^{-1} \sum_{j=1}^{M_{1}-1} \frac{F_{iM_{1}} \alpha_{j}}{\tau_{iM_{1}}}, \quad (3.3.10) \]

\[ E(T_{i}^{(1)}(a_{j})) = \]

\[ N_{i} \left( \int_{0}^{\tau_{iM_{1}}} u \alpha j \ln u \left( \sum_{k=1}^{p} \alpha \lambda_{ik} \right) \exp \left( - \sum_{k=1}^{p} \lambda_{ik} u^k \right) \right) \]

\[ \frac{\alpha_{j}}{\tau_{iM_{1}}} \sum_{l=1}^{M_{1}-1} \frac{c_{lM_{1}}}{F_{iM_{1}}} \int_{\tau_{iM_{1}}}^{\tau_{iM_{1}}} \ln u \left( \sum_{k=1}^{p} \alpha \lambda_{ik} \right) \exp \left( - \sum_{k=1}^{p} \lambda_{ik} u^k \right) \]

\[ \cdot \exp \left( - \sum_{k=1}^{p} \lambda_{ik} u^k \right) \] + \[ F_{iM_{1}} \tau_{iM_{1}} \ln u \left( \ln \tau_{iM_{1}} \right) \] \[ + \sum_{j=1}^{M_{1}-1} \frac{c_{jM_{1}}}{F_{iM_{1}}} \ln u \left( \ln \tau_{iM_{1}} \right) \]

\[ (3.3.11) \]

and

\[ E(T_{i}^{(2)}(a_{j})) = \]

\[ N_{i} \left( \int_{0}^{\tau_{iM_{1}}} u \alpha j \left(\ln u \right)^2 \left( \sum_{k=1}^{p} \alpha \lambda_{ik} \right) \exp \left( - \sum_{k=1}^{p} \lambda_{ik} u^k \right) \right) \]

\[ \frac{\alpha_{j}}{\tau_{iM_{1}}} \sum_{l=1}^{M_{1}-1} \frac{c_{lM_{1}}}{F_{iM_{1}}} \left[ \int_{\tau_{iM_{1}}}^{\tau_{iM_{1}}} \left(\ln u \right)^2 \left( \ln \tau_{iM_{1}} \right) \right] \]

\[ + \int_{\tau_{iM_{1}}}^{\tau_{iM_{1}}} u \alpha j \left(\ln u \right)^2 \left( \sum_{k=1}^{p} \alpha \lambda_{ik} \right) \exp \left( - \sum_{k=1}^{p} \lambda_{ik} u^k \right) \] \[ \sum_{j=1}^{M_{1}-1} \frac{c_{jM_{1}}}{F_{iM_{1}}} \left(\ln u \right)^2 \]

\[ (3.3.12) \]

Also when \( c_{iM_{1}} \) is random, \( n_{i} \) is a random variable with mean
Substituting these expectations in the appropriate places in
equations (3.1.7) to (3.1.10) yields the asymptotic covariance
matrix of \((\hat{\beta}_{j0}, \ldots, \hat{\beta}_{jk}, \hat{\alpha}_j)\) for this censoring scheme.

3.4 Estimation of Parameters at the Usage Stress

Let \(\hat{\alpha}_j, \hat{\beta}_{j0}, \ldots, \hat{\beta}_{jk} \) be the maximum likelihood estimators
of \(\alpha_j, \beta_{j0}, \ldots, \beta_{jk}, \) \(j = 1, \ldots, p\) obtained from an accelerated
life test as described in section 3.1 to 3.3. Let \(\Sigma_j\) denote the
covariance matrix of \((\hat{\beta}_{j0}, \ldots, \hat{\beta}_{jk}, \hat{\alpha}_j)\). Recall \(\Sigma_j\) is of the
form

\[
\Sigma_j = \begin{pmatrix}
\Sigma(j,j) & \Sigma(j,\alpha) \\
\Sigma^T(j,\alpha) & \sigma^2_{\alpha,\alpha}
\end{pmatrix}
\]

(3.4.1)

where \(\Sigma(j,j)\) is the covariance matrix of \((\hat{\beta}_{j0}, \ldots, \hat{\beta}_{jk})\), \(\Sigma(j,\alpha)\)
is the vector of covariances between \(\hat{\alpha}_j\) and \((\beta_{j0}, \ldots, \beta_{jk})\)
\((\hat{\beta}_{j0}, \ldots, \hat{\beta}_{jk})\) and \(\sigma^2_{\alpha,\alpha}\) is the variance of \(\hat{\alpha}_j\). Let \(\hat{\Sigma}_j\) be an
estimator of \(\Sigma_j\). Let \(V_u\) denote the design stress of the system.

For sufficiently large sample sizes the vector
\((\hat{\beta}_{j0}, \ldots, \hat{\beta}_{jk}, \hat{\alpha}_j)\) is approximately normal with mean vector
\((\beta_{j0}, \ldots, \beta_{jk}, \alpha_j)\) and covariance matrix \(\Sigma_j\). Following
Thomas, Bain and Antle (1969) we recommend sample sizes of at
least a hundred at each stress level.

At stress \(V_u\) the maximum likelihood estimator of the scale
parameter of the \(j^{th}\) components time to failure distribution,
\(\lambda_{ju}\), is
\[ \hat{\lambda}_{ju} = \exp \left( \sum_{k=0}^{k_j} \hat{\beta}_{jk} \theta_{jk}(V_u) \right), \quad j = 1, \ldots, p. \]  

(3.4.2)

For sufficiently large sample sizes \( \hat{\lambda}_{ju} \) has a log normal distribution with mean \( \lambda_{ju} \) and variance \( \sigma^2_{ju} \) given by

\[ \sigma^2_{ju} = (1, \theta_{j1}(V_u), \ldots, \theta_{jk_j}(V_u)) \Sigma_{jj}(1, \theta_{j1}(V_u), \ldots, \theta_{jk_j}(V_u))^T. \]

(3.4.3)

Hence a reduced biased estimator of \( \lambda_{ju} \) is

\[ \check{\lambda}_{ju} = \hat{\lambda}_{ju} \exp(-\hat{\sigma}^2_{ju}/2), \quad j = 1, \ldots, p \]  

(3.4.4)

where \( \hat{\sigma}^2_{ju} \) is obtained by replacing \( \Sigma_{jj} \) by \( \hat{\Sigma}_{jj} \) in (3.4.3). If \( \Sigma_{jj} \) were known the mean squared error of \( \hat{\lambda}_{ju} \) is \( \lambda^2_{ju}(\exp(\sigma^2_{ju})-1) \) which is always smaller than \( \lambda^2_{ju}(\exp(2\sigma^2_{ju})-2 \exp(\sigma^2_{ju})+1) \), the mean squared error of \( \check{\lambda}_{ju} \). An asymptotic \((1 - \alpha) \times 100\%\) confidence interval for \( \lambda_{ju} \) is given by

\[ (\hat{\lambda}_{ju} \exp(-Z_{1-\alpha/2} \exp(\hat{\sigma}^2_{ju})), \check{\lambda}_{ju} \exp(Z_{1-\alpha/2} \hat{\sigma}^2_{ju})), \quad j = 1, \ldots, p. \]

(3.4.4)

The maximum likelihood estimator of the \( j^{th} \) component cumulative hazard rate at stress \( V_u \) and time \( t \), \( \Lambda_{ju}(t) \) is given by

\[ \hat{\Lambda}_{ju}(t) = t \hat{\lambda}_{ju}, \quad j = 1, \ldots, p, \quad t > 0. \]

(3.4.5)

This estimator is also biased. An asymptotically unbiased estimator of \( \Lambda_{ju}(t) \) is

\[ \check{\Lambda}_{ju}(t) = \hat{\Lambda}_{ju}(t) \exp(-\check{\sigma}^2_{j}(t)/2), \quad j = 1, \ldots, p, \quad t > 0 \]

(3.4.6)

where

\[ \check{\sigma}^2_{j}(t) = (1, \theta_{j1}(V_u), \theta_{jk_j}(V_u), \& t) \]

\[ \check{\Sigma}_{j}(1, \theta_{j1}(V_u), \ldots, \theta_{jk_j}(V_u), \& t)^T. \]

(3.4.7)
This estimator also has smaller mean squared error than \( \hat{\Lambda}_{ju}(t) \).

Asymptotic \((1 - \alpha) \times 100\%\) confidence intervals for \( \Lambda_{ju}(t) \) are given by

\[
(\hat{\Lambda}_{ju}(t)\exp(-Z_{1-a/2}\hat{\sigma}_{j}(t)), \hat{\Lambda}_{ju}(t)\exp(Z_{1-a/2}\hat{\sigma}_{j}(t)),
\]
\[ j = 1, \ldots, p \quad t > 0. \tag{3.4.8}
\]

The maximum likelihood estimator of the \( j^{th} \) components survival function at time \( t \) and stress \( V_u \) is given by

\[
\hat{F}_{ju}(t) = \exp(-\hat{\Lambda}_{ju}(t)), \quad j = 1, \ldots, p \tag{3.4.9}
\]

Approximate \((1 - \alpha) \times 100\%\) confidence intervals for the \( j^{th} \) components survival function at time \( t \) and stress \( V_u \) are given by

\[
(\hat{F}_{ju}(t)\exp(Z_{1-a/2}\hat{\sigma}_{j}(t)), \hat{F}_{ju}(t)\exp(-Z_{1-a/2}\hat{\sigma}_{j}(t)),
\]
\[ j = 1, \ldots, p \quad t > 0. \tag{3.4.10}
\]

Let \( \kappa \) be a subset of \( 1, \ldots, p \) of cardinality \( k \). We are interested in obtaining estimators of

\[
F_u^{(\kappa)}(t) = \prod_{j \in \kappa} F_{ju}(t), \tag{3.4.11}
\]

the survival function of an item which can fail only from the failure of components indexed by elements of \( \kappa \). When \( \kappa = \{1, \ldots, p\} \) then (3.4.1) is the overall system survival function. When \( \kappa \) is a proper subset of \( \{1, \ldots, p\} \) then (3.4.1) represents the survival function of a system which has been redesigned so that those components indexed by \( \kappa^c \) are extremely reliable. Clearly, the maximum likelihood estimator of \( F_u^{(\kappa)}(t) \) is

\[
\hat{F}^{(\kappa)}(t) = \prod_{j \in \kappa} \hat{F}_{ju}(t). \tag{3.4.12}
\]

An approximate \((1 - \alpha) \times 100\%\) confidence interval for \( F_u^{(\kappa)}(t) \) is
\[(1 - 1) = P(\hat{F}_{ju}(t) \leq \hat{F}_{ju}(t) \leq \hat{F}_{ju}(t)) \text{ for } j \in K.\]

Since \((\hat{F}_{ju}(t), \hat{F}_{ju}(t))\) are asymptotically independent for \(j \neq j'\),

\[(1 - 1) = P(\hat{F}_{j_u}(t) \leq F_{j_u}(t) \leq \hat{F}_{j_u}(t) \text{ for all } j \in K)\]

\(\leq P(\prod_{j \in K} \hat{F}_{j_u}(t) \leq \prod_{j \in K} \hat{F}_{j_u}(t)).\)

### 3.5 Dependent Risks

In sections 3.1 - 3.4 it was assumed that the component lifetimes were independent. This assumption may be relaxed by considering a fatal shock model. For simplicity we shall illustrate this model for the bivariate case.

Let \(U_1, U_2, U_{12}\) be independent Weibull random variables with shape parameters \(\alpha_1, \alpha_2, \alpha_{12}\) and scale parameters \(\lambda_1(V, \beta_1), \lambda_2(V, \beta_2), \lambda_{12}(V, \beta_{12})\) in an environment characterized by a constant application of a stress \(V\). Here \(U_1\) represents the time until a shock destroys the first component only, \(U_2\) the time until a shock destroys the second component, and \(U_{12}\) the time until a shock destroys both components. If \((X_1, X_2)\) represent the component lifetimes then, clearly, \(X_1 = \min(U_1, U_{12})\), and \(X_2 = \min(U_2, U_{12})\). The component survival functions are not Weibull, but are given by

\[F_j(t; V) = \exp(-\lambda_j(V, \beta_j)t^{\alpha_j}) - \lambda_{12}(V, \beta_{12})t^{\alpha_{12}},\]

\[j = 1, 2 \quad t \geq 0.\]
An accelerated life test can be conducted as before where now the parameters of interest are $\alpha_1$, $\alpha_2$, $\alpha_1 2$, $\beta_1$, $\beta_2$, $\beta_1 2$. The results of section 3.4 can be used to obtain estimators of the component survival function under normal conditions.

4. EXAMPLE

As an example of these procedures we shall consider an example given in Nelson (1974a). The problem is to analyze an accelerated life test conducted on Class-H insulation systems for electric motors. There are three possible types of insulation failures corresponding to distinct parts of the insulation system, namely Turn, Phase, and Ground. The failure cause is determined by an engineering examination of the failed motor.

The purpose of the experiment is to estimate the average life of such insulation systems at a design temperature of $180^\circ$ C. A median life of 20,000 hours is necessary for the satisfactory performance of these insulation systems. To reduce test time and cost an accelerated life test was conducted at 4 accelerated temperatures, namely, $190^\circ$ C, $220^\circ$ C, $240^\circ$ C, and $2600^\circ$ C.

The accelerated life test was conducted by putting 10 motors on test at each of the 4 stress levels. Motors were run until they failed, then the cause of failure was found and isolated and motors were run until a second failure occurred. The results of this study are reported in Nelson (1974a). The data followed a $\log_{10}$ normal distribution so the Weibull theory results do not apply.

To illustrate the results of the previous section Nelson's example is reproduced by simulating the life test using a Weibull model with shape parameter 1 for each failure cause. The shift parameters are chosen by fitting an Arrhenius Reaction Rate model to the estimated component medians obtained by Nelson. The model is
Table 4.1
Life Test Data Simulated From Nelson's Example

<table>
<thead>
<tr>
<th>Motor</th>
<th>180 Degrees Failure Time</th>
<th>180 Degrees Failure Cause</th>
<th>190 Degrees Failure Time</th>
<th>190 Degrees Failure Cause</th>
<th>220 Degrees Failure Time</th>
<th>220 Degrees Failure Cause</th>
<th>240 Degrees Failure Time</th>
<th>240 Degrees Failure Cause</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5606.0781 Turn</td>
<td>1628.8145 Ground</td>
<td>344.1240 Phase</td>
<td>557.4395 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4905.0859 Turn</td>
<td>1097.6609 Turn</td>
<td>761.8518 Phase</td>
<td>156.9030 Phase</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2871.9370 Phase</td>
<td>630.0374 Phase</td>
<td>1562.7520 Turn</td>
<td>906.5212 Phase</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2762.9712 Ground</td>
<td>1520.8772 Ground</td>
<td>276.9924 Ground</td>
<td>61.1210 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3413.8027 Turn</td>
<td>708.5212 Phase</td>
<td>482.2432 Turn</td>
<td>773.3906 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6321.7617 Turn</td>
<td>205.9655 Phase</td>
<td>213.3295 Turn</td>
<td>148.7977 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4847.3906 Turn</td>
<td>185.6579 Turn</td>
<td>1434.3723 Turn</td>
<td>41.1974 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2690.2847 Turn</td>
<td>434.2930 Turn</td>
<td>1486.6152 Turn</td>
<td>787.6323 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>38.9871 Phase</td>
<td>1938.7297 Phase</td>
<td>1355.4917 Turn</td>
<td>224.2534 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2358.2275 Phase</td>
<td>3093.8237 Turn</td>
<td>1374.0374 Phase</td>
<td>405.3303 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>3755.4043 Ground</td>
<td>1171.8782 Ground</td>
<td>725.5413 Turn</td>
<td>1071.6702 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>4898.8477 Phase</td>
<td>1108.7510 Ground</td>
<td>917.9756 Ground</td>
<td>407.1978 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>3900.2310 Turn</td>
<td>27.5321 Turn</td>
<td>2970.2925 Ground</td>
<td>306.0037 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>1196.4922 Turn</td>
<td>1428.3220 Turn</td>
<td>609.9128 Turn</td>
<td>422.7825 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>6000.6875 Ground</td>
<td>263.7917 Phase</td>
<td>89.8835 Turn</td>
<td>178.5943 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1645.1550 Ground</td>
<td>1113.6123 Turn</td>
<td>741.6179 Turn</td>
<td>588.4976 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>4021.5698 Ground</td>
<td>965.0088 Turn</td>
<td>706.0217 Ground</td>
<td>301.6204 Phase</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>2643.6931 Turn</td>
<td>49.1324 Phase</td>
<td>347.2078 Turn</td>
<td>14.8288 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>4760.6289 Turn</td>
<td>350.6594 Turn</td>
<td>238.5782 Phase</td>
<td>1315.0032 Ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1621.5481 Turn</td>
<td>2026.7441 Phase</td>
<td>1001.3643 Ground</td>
<td>90.0674 Turn</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[
\lambda_i(V; \beta_j) = \exp(\beta_{j0} + \beta_{j1} \theta_j(V)), \quad j = 1, 2, 3 \tag{4.1}
\]

where \( \theta_j(V) = -1000/V \) for \( j = 1, 2, 3 \) and \( V \) is the temperature in degrees absolute. The absolute temperature is 273.16 plus the centigrade temperature. The values of \((\beta_{j0}, \beta_{j1}), j = 1, 2, 3\) are as follows:

<table>
<thead>
<tr>
<th>Table 4.2 True Values of (\beta_0, \beta_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Turn</strong></td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>Turn</td>
</tr>
<tr>
<td>Phase</td>
</tr>
<tr>
<td>Ground</td>
</tr>
</tbody>
</table>

Twenty Weibull observations were generated at each of the four stress levels. The data are in Table 4.1.

A Newton-Raphson procedure was used to solve the likelihood equations. The integrals in (3.1.14), (3.1.15), (3.1.16), and (3.1.17) were evaluated using a repeated seven point Gauss-Laguerre formula. The maximum likelihood estimates are as follows:

**TURN:** \( \hat{\alpha} = 1.0099, \hat{\beta}_0 = 6.1363, \hat{\beta}_1 = 6.9390 \)

**PHASE:** \( \hat{\alpha} = 0.9993, \hat{\beta}_0 = 3.5831, \hat{\beta}_1 = 6.0272 \)

**GROUND:** \( \hat{\alpha} = 1.0395, \hat{\beta}_0 = 7.9788, \hat{\beta}_1 = 8.2679 \).

The estimated covariance matrices are:

\[
\hat{\Sigma}_{\text{TURN}} = \begin{pmatrix}
9.8252 & 5.3254 & 0.1178 \\
5.3254 & 3.1306 & 0.1283 \\
0.1178 & 0.1283 & 0.0188
\end{pmatrix}
\]

\[
\hat{\Sigma}_{\text{PHASE}} = \begin{pmatrix}
21.7658 & 11.6281 & 0.2217 \\
11.6281 & 6.6803 & 0.2442 \\
0.2282 & 0.2442 & 0.0353
\end{pmatrix}
\]

\[
\hat{\Sigma}_{\text{GROUND}} = \begin{pmatrix}
18.2378 & 9.8173 & 0.1856 \\
9.8173 & 5.6782 & 0.2033 \\
0.1856 & 0.2033 & 0.0299
\end{pmatrix}
\]
Using these estimates, at a design stress of 180% the estimates of the probability of component survival at a mission time of 20,000 hours is .1020 for turn failures, .3023 for phase failures, and .3574 for ground failures. 90% confidence intervals for the probability of component survival at 20,000 hours and a temperature of 1800 C are:

- TURN - (.0187, .2701),
- PHASE - (.0749, .5756),
- GROUND - (.1191, .6080).

Using equations (3.4.12) and (3.4.13) the maximum likelihood estimate and 90% confidence interval for system reliability at 20,000 hours and a temperature of 1800 C are .0110 and (.000027, .1402). Similarly, a 90% confidence interval for a redesigned system in which turn failures cannot occur is (.0043, .4024).

We note that the above confidence intervals are suspect due to the relatively small sample sizes and are provided here only to illustrate this procedure.

ACKNOWLEDGEMENT

This research was supported in part by the Office of Naval Research Grant No. N00014-78-C-0655.

BIBLIOGRAPHY


