PARTIAL CHARACTERIZATIONS OF COMPLETELY NONDETERMINISTIC STOCHASTIC PROCESSES

P. Bloomfield, N. P. Jewell

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STOCHASTIC PROCESSES

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Peter Bloomfield*
Nicholas P. Jewell**
Princeton University

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A B S T R A C T

A discrete weakly stationary Gaussian stochastic process

$X(t): t \leq 0 \Rightarrow 0$

is completely nondeterministic if no non-trivial set from the

$\sigma$-algebra generated by $\{X(t): t > 0\}$ lies in the $\sigma$-algebra generated

by $\{X(t): t < 0\}$. Levinson and McKean essentially showed

that a necessary and sufficient condition for complete non-

determinism is that the spectrum of the process is given by

$|h|^2$

where $h$ is an outer function in the Hardy space, $H^2$, of the

unit circle in $\mathbb{C}$ with the property that $h/|h|$ uniquely deter-

mines the outer function $h$ up to an arbitrary constant. In

this paper we consider several characterizations of complete non-
determinism in terms of the geometry of the unit ball of the

Hardy space $H^1$ and in terms of Hankel operators, and pose an

open problem.
1. INTRODUCTION

In [10] Sarason defines a property of a discrete weakly stationary Gaussian stochastic process, \( \{x(t)\} \), which he called complete nondeterminism. This condition is that no set from the future of the process (i.e. the \( \sigma \)-algebra generated by the random variables \( x(t) \) for \( t>0 \)) lies in the past (i.e. the \( \sigma \)-algebra generated by \( x(t) \) for \( t<0 \)), except for null sets and the complements of null sets. In the spectral representation this condition becomes the following. Let \( m \) be the spectral measure of the process and let \( P \) denote the span in \( L^2(m) \) of the exponentials \( e^{in\theta} \) with \( n\leq 0 \) where functions are defined on \( \mathbb{T} \), the unit circle in \( \mathbb{C} \). Let \( F \) denote the span in \( L^2(m) \) of the exponentials \( e^{in\theta} \) with \( n>0 \). Then complete nondeterminism is equivalent to the condition that \( Pn\cap F=\{0\} \). It is clear that this condition reflects a certain kind of independence (in a statistical sense) of the past, \( P \), and the future, \( F \).

It is of interest to characterize those measures \( m \) on \( \mathbb{T} \) which lead to completely nondeterministic (cnd) processes. In [10] a necessary and sufficient condition for complete nondeterminism was stated as the measure \( m \) being absolutely continuous with respect to Lebesgue measure, \( d\theta \), with \( \log \frac{dm}{d\theta} \) integrable. Unfortunately this characterization is incorrect. In [8, p.105] Levinson and McKean essentially describe a partial characterization of cnd processes which we discuss.
in Section 3. This paper continues an investigation into the problem of characterizing spectral measures of cnd processes.

In Section 2 we examine the relationship between complete nondeterminism and some other familiar kinds of independence of $P$ and $F$.

In Section 3 we restate the question in several ways which yield partial answers in terms of exposed points of the unit sphere of $H^1$ and certain Hankel operators.

The complete characterization of complete nondeterminism in terms of the spectral distribution function remains open and seems to be a hard question.

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2. **COMPLETE NONDETERMINISM**

A Gaussian process is called **deterministic** if its past determines the future, i.e., for each \( t > 0 \), \( x(t) \) is measurable with respect to the past. This is translated in the spectral representation to the property that \( P = L^2(\mathcal{M}) \). A necessary and sufficient condition for this to occur is that \( \log \frac{d\mathcal{M}}{d\theta} \) be not integrable. Conversely the process is **indeterministic** if \( \log \frac{d\mathcal{M}}{d\theta} \) is integrable. A stronger restriction than indeterminism is that the process is **purely indeterministic** or **regular**. This is an asymptotic independence condition which, in the spectral representation, is equivalent to \( \bigcap_{F_k = \{0\}} \) where \( F_k \) is the span in \( L^2(\mathcal{M}) \) of the exponentials \( e^{in\theta} \) with \( n > k \). This condition is often referred to by saying that the process has trivial remote future. Results of Szego [11], Kolmogorov [5] and Krein [6] show that \( \{x(t)\} \) is regular if and only if \( \mathcal{M} \) is absolutely continuous with respect to Lebesgue measure and \( \log \frac{d\mathcal{M}}{d\theta} \) is integrable. First we give an example of a process which is regular but not completely nondeterministic, thereby showing that the characterization in [10] is incorrect. First we establish some notation. \( L^1(\text{resp.} L^2) \) is the space of integrable (resp. square integrable) functions on \( T \). \( L^\infty \) is the space of essentially bounded functions on \( T \). We shall often regard functions in \( L^1 \) as extended harmonically into the open unit disc \( D = \{z : |z| < 1\} \) by means of Poisson's formula. We let \( H^1 \) denote those functions in \( L^1 \) which have analytic...
extensions into the disc. We define $H^2$ and $H^\infty$ similarly. 
$H^2$ is a Hilbert space with orthonormal basis \{$z^n$: n=0,1,2,...\}. For standard results on the Hardy spaces we refer to [4].

For a regular process we can write $dm = wde = |H|d\theta = |h|^2d\theta$
where $H$ is an outer function in $H^1$ and $h$ is an outer function in $H^2$.

**Proposition 1.** There is a regular process which is not completely nondeterministic.

**Proof.** Let $w(e^{i\theta}) = |1+e^{i\theta}|^2 = |1+z|^2$ and put $dm = wde$. Since $\log |1+z|^2 \in L^1$ this process is regular. However $(1+z)^{-1} \in P \cap F$. This follows since $1+z$ is outer. For we have

$$\lim_{n \to \infty} \int_{\mathbb{T}} |1-p_n(1+z)|^2 dz = 0$$
for some sequence $p_n$ of polynomials in $z$.

hence

$$\int_{\mathbb{T}} (1+z)^{-1} - z p_n |1+z|^2 dz \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \int_{\mathbb{T}} (1+z)^{-1} - z p_n |1+z|^2 dz \to 0 \quad \text{as } n \to \infty ;$$

i.e. $(1+z)^{-1} \in F$.

Similarly

$$\int_{\mathbb{T}} (1+z)^{-1} - \bar{p}_n |1+z|^2 dz$$

$$= \int_{\mathbb{T}} |1-p_n(1+z)|^2 dz \to 0 \quad \text{as } n \to \infty ;$$
i.e. $(1+z)^{-1} \in P$. 
We next obtain a simple necessary and sufficient condition for complete nondeterminism. It is straightforward to see that if $m$ is singular with respect to Lebesgue measure then $P \cap F \neq \{0\}$. This, together with earlier comments means that in considering cnd processes we can restrict our attention to regular processes.

We wish to rephrase our question in terms of $L^2$ rather than $L^2(m)$. We have $dm = |h|^2 d\theta$. Consider the mapping $T : L^2(m) \to L^2$ given by $Tf = hf$. It is easily verified that $T$ is an isometry of $L^2(m)$ onto $L^2$. Also $T$ maps $F$ onto $H_0^2 = \{f \in H^2 : f(0) = 0\}$, and $T$ maps $P$ onto $(h/\bar{h})H^2$ where $\bar{H}^2 = \{f : f \in H^2\}$.

**Proposition 2.** A process is not cnd if and only if $h/\bar{h} = a(F/F)$ where $F \in H^2$ is outer and $a$ is inner with $a(0) = 0$.

**Proof.** Using the isometry $T$ we see that $P \cap F \neq \{0\}$ if and only if there are non-zero functions $g_1, g_2$ in $H^2$ such that $zg_1 = (h/\bar{h})\bar{g}_2$

$$\iff z(g_1/h) = (\bar{g}_2/\bar{h}) \text{ and } z(g_2/h) = (\bar{g}_1/\bar{h})$$

$$\iff z(g_1 + g_2)/h = (\bar{g}_1 + \bar{g}_2)/\bar{h}.$$  

Hence $P \cap F \neq \{0\}$ if and only if there exists a function $G \in H^2$ such that $zG/h = \bar{G}/\bar{h}$. If we use the inner-outer factorization of $G$ then this equality becomes
\[ z \phi F/h = \phi F/h \] where \( \phi \) is inner and \( \Phi \in H^2 \) is outer.

\[ \Rightarrow \quad h/\Phi = \alpha(F/\Phi) \text{ and } \alpha(0) = 0. \]

Conversely \( h/\Phi = \alpha(F/\Phi) \), \( \alpha(0) = 0 \)

\[ \Rightarrow \quad (h/\Phi)F = z(\beta F) \text{ where } \alpha = z\beta \]

\[ \Rightarrow \quad PnF\#\{0\} \text{ by the above.} \]

The same reasoning yields the following result for \( k \geq 1 \):

\[ PnF_k\#\{0\} \iff h/\Phi = \alpha(F/\Phi) \text{ where } \Phi \in H^2 \text{ is outer and } \alpha \]

is inner with \( \alpha \) having a zero at the origin of order at least \( k \).

Another strictly stronger property than regularity is that of minimality. Introduced by Kolmogorov [5] this property says that a process is minimal if the value of the random variable \( x(0) \) cannot be predicted without error from the values of the random variables \( \{x(t): t \neq 0\} \). In other words a process is not minimal if it is possible to perfectly interpolate any value of the process from knowledge of the remaining values of the process. Kolmogorov [5] proved that a process is minimal if and only if \( w^{-1} \) is in \( L^1 \).

It is immediately of interest to examine the relationship between minimal processes and completely nondeterministic processes.
**Proposition 3.** If the process \( \{x(t)\} \) is minimal then it is completely nondeterministic. On the other hand there exist completely nondeterministic processes which are not minimal.

**Proof.** Suppose \( \{x(t)\} \) is minimal. Then by Kolmogorov's theorem \( h^{-1}e^{H^2} \). Using Proposition 2 we argue by contradiction. For suppose \( \{x(t)\} \) is not completely nondeterministic. Then \( (h/\overline{h}) = a(f/\overline{f}) \) where \( f \) is outer and \( a \) is inner with \( a(0)=0 \). This equality implies \( \overline{f}/\overline{h} = a(f/h) \). The LHS is in \( \overline{H^1} \) and the RHS is in \( H_0^1 \) which forces both sides to be zero and thus \( f=0 \) which is a contradiction. This proves the first statement of the proposition. An example of a process which yields the second statement is given by \( w = |1+z| \). In this case \( h=(1+z)^{1/2} \) and \( h/\overline{h} = z^{1/2} \). By Kolmogorov's criterion this process is not minimal. On the other hand suppose \( h/\overline{h} = a(f/\overline{f}) \) for \( f \) outer, \( a \) inner with \( a(0)=0 \). Then

\[
    z^{1/2} = a(f/\overline{f}) = z\phi(f/\overline{f}) \quad \text{with} \quad \phi \text{ inner}
\]

\[
    \Rightarrow z^{1/2} \phi f = \overline{f}
\]

\[
    \Rightarrow z(\phi f)^2 = (\overline{f})^2.
\]

The LHS is in \( H_0^1 \) and the RHS is in \( \overline{H^1} \). Again this forces both sides to be zero and hence \( f=0 \) which gives a contradiction. Thus the process with \( w = |1+z| \) is completely nondeterministic.

Let \( P_k \) be the span in \( L^2(m) \) of the exponentials \( e^{in\theta} \) with \( n \leq k \). A minimal process is one for which the function 1
does not belong to the closed linear span of \( P_1 \) and \( F_1 \) i.e. 
\[ 1 \notin P_1 + F_1 . \]
There is a similar restatement of the condition of complete nondeterminacy. Let 
\[ P_1 + F_1 = \{ f \in L^2(m) : f = g + h \text{ with } g \in P_1, h \in F_1 \} . \]

**Proposition 4.** A regular process is completely nondeterministic if and only if 
\[ 1 \notin P_1 + F_1 . \]

**Proof.** Assume \( f \) is a non-zero element of \( P \cap F \). Then, for some \( k \geq 1 \), \( f \in F_k \) but \( f \notin F_{k+1} \) (since \( \bigcap_{k=1}^{\infty} F_k = \{ 0 \} \)). Hence 
\[ f = a e^{ik\theta} + f_1 \text{ where } a \neq 0 \text{ and } f_1 \in F_{k+1} . \]
This implies 
\[ e^{ik\theta} = (f - f_1)/a \in P + F_{k+1} . \]
\[ \implies 1 \in (e^{-ik\theta} P) + F_1 \subseteq P_1 + F_1 . \]

Conversely assume that \( 1 \in P_1 + F_1 \). Then \( 1 = f_1 + f_2 \) with \( f_1 \in P \), 
\( f_2 \in F \). Hence 
\[ e^{i\theta} f_1 = e^{i\theta} - e^{i\theta} f_2 \in F . \]
But 
\[ e^{i\theta} f_1 \in P . \]
Hence 
\[ e^{i\theta} f_1 \in P \cap F . \]

We complete this section by establishing a simple sufficient condition for \( P \cap F \) to be non-trivial.

**Proposition 5.** Suppose that \( w = |p|^2 w_1 \), where \( p \) is a trigonometric polynomial of degree \( n \) with all its zeros in the closed unit disc, and \( w_1 \in L^1 \). Then 
\[ P \cap F \neq \{ 0 \} . \]

**Proof.** We show that \( 1/\overline{p} \in P \cap F \).
(i) $1/\bar{p} \in P$: without loss of generality we can assume that

$$1/\bar{p} = \prod_{j=1}^{m} (1-\bar{z}/\bar{\zeta_j})^{-n_j}$$

where $|\zeta_j| \leq 1$.

Now $(1-\bar{z}/\bar{\zeta_j})^{-n_j}$ can be approximated by polynomials in $\bar{z}$ in $L^2(m)$. In fact

$$\int T |(1-\bar{z}/\bar{\zeta_j})^{-n_j} - (1 + m^{-1}(\bar{z}/\bar{\zeta_j}) + \frac{m-2}{m}(\bar{z}/\bar{\zeta_j})^2 + \ldots + 1/m(\bar{z}/\bar{\zeta_j})^{m-1}) |^2 \omega(\theta) d\theta$$

$$= \int T |1 - [(1-\bar{z}/\bar{\zeta_j})(1 + m^{-1}(\bar{z}/\bar{\zeta_j}) + \ldots + 1/m(\bar{z}/\bar{\zeta_j})^{m-1})]^{n_j} |^2 \omega_2(\theta) d\theta$$

where $\omega_2 = \omega/(1-\bar{z}/\bar{\zeta_j})^{n_j}$.

$$= \int T [1 - 1/m(\bar{z}/\bar{\zeta_j} + \ldots + (\bar{z}/\bar{\zeta_j})^{m-1})]^{n_j} |^2 \omega_2(\theta) d\theta$$

$$\to 0 \text{ as } m \to \infty \text{ by Lebesgue's dominated convergence theorem.}$$

Hence $1/\bar{p} \in P$.

(ii) $1/\bar{p} \in F_n$: $1/\bar{p} = z^n/z^n\bar{p} = z^n/q_n$ where $q_n = z^n\bar{p}$ is also a polynomial of degree $n$ in $z$. The same construction as in (i) shows that $1/q_n$ can be approximated by polynomials in $z$ in $L^2(m)$. Hence $1/q_n \in F_0$ and $1/\bar{p} \in F_n$. 

Remark. This proposition implies that if we restrict our attention to cnd processes then the strong mixing condition implies the property that \( P \) and \( F_1 \) be at positive angle; (see [3], [10, p.77] for definitions). For if the angle between \( P \) and \( F_n \) is converging to \( \pi/2 \) as \( n \to \infty \) then, for some \( k \), \( P \) and \( F_k \) are at a positive angle which implies by [3] that \( w = |p|^2 w_1 \) for some trigonometric polynomial \( p \) where \( w_1 \) is the spectrum of a process for which \( P \) and \( F_1 \) are at a positive angle. If the process is cnd then Proposition 5 implies that \( p \) must have zero degree. In general the strong mixing condition does not imply that \( P \) and \( F_1 \) are at positive angle (e.g. take \( h = 1 + z \)).
3. EXPOSED POINTS OF THE BALL IN $H^1$ AND HANKEL OPERATORS

It is well known (see [7]) that the extreme points of the unit ball of $H^1$ are given by the outer functions $F$ in $H^1$ with $\|F\|_1 = 1$. It is also well known that an $H^1$ function $F$ of unit norm is not determined by its argument.

In [8, p.205] Levinson and McKean showed that for continuous processes the dimension of $\mathcal{P} \cap F_0 = 1$ if and only if $h/\overline{h}$ determines the outer function $h$ up to a constant. In this section we consider this approach which is closely related to the results of Section 2 and consider this characterization in geometrical terms.

In their study of extremum problems in $H^1$ deLeeuw and Rudin introduced the following sets of $H^1$ functions indexed by unimodular $L^\infty$ functions. Let $\phi \in L^\infty$ with $|\phi| = 1$ almost everywhere and define

$$S_\phi = \{FeH^1 : \|f\|_1 = 1, \frac{F}{|F|} = \phi \text{ almost everywhere}\}.$$ 

Geometrically $S_\phi$ is the intersection of the ball of $H^1$ and the hyperplane $\{FeH^1 : \int \phi F d\theta = 1\}$ and so $S_\phi$ is a convex set (which may be empty, in general). When $S_\phi$ contains exactly one function $F$, the hyperplane touches the ball of $H^1$ only at $F$ which means that $F$ is an exposed point of the ball of $H^1$. (In fact the definition of $S_\phi$ we have given corresponds to $S^-_\phi$ as defined by deLeeuw and Rudin.)
Proposition 6. Let \( w = |H| = |h|^2 \). Without loss of generality assume that \( \omega = 1 \). The following statements are equivalent:

1. \( \{x(t)\} \) is completely nondeterministic
2. \( S_{h/\overline{h}} \) contains exactly one function
3. \( h^2 = \overline{h} \) is an exposed point of the unit ball in \( H^1 \).

Proof. Note that \( S_{h/\overline{h}} \) always contains \( h^2 \), so that our comments above show the equivalence of (2) and (3). Now suppose that \( \{x(t)\} \) is not completely nondeterministic. By Proposition 2 \( h/\overline{h} = \alpha(F/F) \) where \( \alpha \) is inner and \( \alpha(0) = 0 \) and \( F \in H^2 \) is outer. Hence \( \frac{\alpha F^2}{|\alpha F^2|} = \frac{F^2}{|F^2|} = \frac{F}{\overline{F}} = h/\overline{h} \).

Thus a positive multiple of \( \alpha F^2 \) is in \( S_{h/\overline{h}} \). But \( \alpha(\alpha F^2) \neq h^2 \) for any \( \alpha > 0 \) since \( \alpha \) has a zero at the origin. Hence \( S_{h/\overline{h}} \) contains more than one function. Conversely suppose \( S_{h/\overline{h}} \) contains more than one function. Then, by Theorem 9 of [7] \( S_{h/\overline{h}} \) contains a function \( f \) with \( f(0) = 0 \). Write \( f = bF^2 \) where \( b \) inner, \( b(0) = 0 \), and \( F \in H^2 \) is outer. Now \( \overline{f} \in S_{h/\overline{h}} \) implies that \( h/\overline{h} = bF/\overline{F} \) which, by Proposition 2, shows that \( \{x(t)\} \) is not completely nondeterministic.

A similar result is given in the following proposition for \( k > 1 \).

Proposition 7. \( \cap F_k \neq \{0\} \) if and only if there is a function \( f \in S_{h/\overline{h}} \) where \( f \) has \( k \) zeros (counting multiplicities) in the open unit disc.
Proof. By Proposition 2 \( \mathcal{P} \mathcal{F}_k \neq \{0\} \) implies that \( h/F = z^k \phi(F/F) \) where \( \phi \) is inner and \( FcH^2 \) is outer. As in the proof of Proposition 6 it follows that \( z^k \phi F^2 \in S_h/F \).

Conversely if \( f \in S_h/F \) and \( f(z_1) = f(z_2) = \ldots = f(z_k) = 0 \) where \( z_j \in \mathbb{D} (1 \leq j \leq k) \) then it is easy to verify that a positive multiple of \( g(z) = z^k f(z) \prod_{j=1}^{k} (z - z_j)^{-1} (1 - \overline{z_j} z)^{-1} \) is in \( S_h/F \).

Factorize \( g \) as \( g = z^k bF \) where \( b \) is inner and \( FcH^2 \) is outer. Since \( ag \in S_h/F \) for some \( a > 0 \) it follows that \( h/F = z^k bF/F \) showing that \( \mathcal{P} \mathcal{F}_k \neq \{0\} \).

Note that Proposition 6 yields the version of the Levinson and McKean result as applied to cnd processes: namely, a process is cnd if and only if \( \arg(h/F) \) is the argument of a unique \( H^1 \) function.

Since we have expressed the characterization of completely nondeterministic processes in terms of an extremum problem it is not surprising that there is a version of the problem in terms of the norms of Hankel operators which are closely related to extremum problems on \( H^1 \).

Let \( P \) be the orthogonal projection of \( L^2 \) onto \( H^2 \). Recall that the Hankel operator with symbol \( \phi \in L^\infty \) is the bounded operator from \( H^2 \) to \( L^2 \otimes H^2 \) defined by

\[
H_\phi(f) = (I - P)(\phi f) \quad (f \in H^2).
\]

The norm of \( H_\phi \) is given by \( \|H_\phi\| = d(\phi, H^\infty) = \inf_{f \in H^\infty} \|\phi - f\|_\infty \).
It is straightforward to show from first principles that the process \( \{x(t)\} \) is completely nondeterministic if and only if \( H_{\phi}/H \) attains the norm of 1 (on the unit sphere of \( H^2 \)).

In fact more is true.

In [1] it is essentially shown that \( H_{\phi} \) attains its norm on the unit sphere on \( H^2 \) if and only if \( \phi = f + \lambda \psi \) where \( f \in H^\infty \), \( \lambda > 0 \) and \( |\psi| = 1 \) a.e on \( T \) with \( S_{\overline{\psi}} \) containing more than one function. Also if \( ||\phi||_\infty = 1 \) then \( H_{\phi} \) attains the norm 1 if and only if \( |\phi| = 1 \) a.e on \( T \) and \( S_{\overline{\phi}} \) contains more than one function [1]. There is another result of this type which does not seem to have appeared in the literature.

**Proposition 8.** \( ||H_{\phi}|| < ||\phi||_\infty \) \( \Rightarrow \) \( \phi = f + \lambda \psi \) where \( f \in H^\infty \), \( \lambda > 0 \) and \( |\psi| = 1 \) a.e on \( T \) with \( S_{\overline{\psi}} \) containing exactly one function.

**Proof.** Without loss of generality we assume that \( ||\phi||_\infty = 1 \). Suppose \( ||H_{\phi}|| < 1 \). Then by [2] there exists \( \psi \in L^\infty \) such that

(i) \( \phi - \psi \in H^\infty \) and (ii) \( \psi = F/|F| \) for some \( F \in H^1 \), \( F \neq 0 \). Now (i) \( \Rightarrow \) \( H_{\psi} = H_{\phi} \) and so \( ||H_{\psi}|| < 1 \). So there exists \( g \in H^\infty \) such that

\( ||(F/|F|) - g||_\infty = a < 1 \) which gives that \( |\arg(gF)| < \pi/2 \). Hence \( (gF)^{-1} \in H^1 \) (since \( gF \neq 0 \) on \( D \) and if \( G \) is analytic on \( D \) and \( |\arg G| < \pi/2 \) then \( G \in H^p \) for all \( p < \pi/2 \)). Thus \( g(gF)^{-1} \in H^1 \) \( \Rightarrow \) \( F^{-1} \in H^1 \). Now \( F/|F| = \overline{\psi} \) so that a positive multiple of \( F \) is in \( S_{\overline{\psi}} \). Then \( F^{-1} \in H^1 \) implies that \( S_{\overline{\psi}} \) contains one and only one function (if \( GeS_{\overline{\beta}} \) and \( G^{-1} \in H^1 \) then \( S_{\beta} = \{G\} \) - See [7, Theorem 8] and use the fact that positive \( H^{1/2} \) functions are constant).
Note however that $S_{H\psi}$ containing exactly one function does not necessarily imply that $\|H\psi\| < \|\psi\|_{\infty}$. For example if 
$h = (1+z)^{1/2}$, and we take $\psi = \overline{h}/h$ it can be shown that 
$\|H\psi\| = 1$ but, as we saw in the proof of Proposition 3, $|h|^2$ corresponds to a cnd process so that $S_{h/\overline{h}} = \{h^2\}$. 
4. $P$ and $F_k$: AN OPEN QUESTION

There is an interesting set of results in [9] which describes the relationship between minimal processes and those processes which may not be minimal but, for some fixed $k$, do not allow perfect interpolation of $k$ "missing" values of the process.

Call a process $k$-minimal if the $k$ functions $1, e^{i\theta}, \ldots, e^{(k-1)i\theta}$ do not all belong to the closed linear span of $P_1$ and $F_k$. The extension of Kolmogorov's result given in [9] is that a process with spectrum $w$ is $k$-minimal if and only if there exists a polynomial $p(e^{i\theta})$ such that $\int \frac{|p(e^{i\theta})|^2}{w(e^{i\theta})} \, d\theta < \infty$

where the degree of the polynomial $p$ is strictly less than $k$ and we may assume that the zeros of $p$ are all on $T$. Thus if $w$ is the spectrum of a $k$-minimal process then $w = |p|^2w_1$ where $w_1$ is the spectrum of a minimal process and $p$ is a trigonometric polynomial with degree $< k$ and zeros all on $T$.

It would be satisfying to have a similar theory relating processes for which $PnF_k \neq \{0\}$ with completely nondeterministic processes. We know that $PnF_k \neq \{0\}$ if and only if $h/F = \alpha(F/F)$ where $F \in \mathbb{H}^2$ is outer and $\alpha$ is inner with a zero at the origin of multiplicity at least $k$.

The fact that $p/p = \lambda z^k$ for some constant $\lambda$ with $|\lambda| = 1$ when $p$ is a trigonometric polynomial of degree $k$, together with our comments above also shows that if a process is $k$-minimal then $P F_k \neq \{0\}$. We also remark that for a cnd process $k$-minimal processes are automatically minimal by the same reasoning as in the remark following Proposition 5.
Proposition 9. Suppose $P\cap F_k=\{0\}$ but $P\cap F_{k-1}\neq\{0\}$. Then $h/\overline{h} = \lambda z^k F/F$ where $F$ is outer and $|\lambda|=1$.

Proof. $P\cap F_{k-1}\neq\{0\}$ implies that $h/\overline{h} = z^{k-1} \alpha(F/F)$ where $F \in H^2$ is outer, $\alpha$ is inner. Suppose that $\alpha$ is non-constant. Then we can find constants $a, b$ such that $0 \neq aF + b(\alpha F) \in H^2_0$. Then

$z^{k-1}(aF+b\alpha F)/h = (a\overline{aF}+b\overline{F})/\overline{h}$.

The LHS $\in F_k$ and the RHS $\in P$. Hence $P\cap F_k\neq\{0\}$. This contradiction implies that $\alpha$ is a constant $\lambda$.

The result we are aiming for is that if $P\cap F_k=\{0\}$ then $w = |p|^2 w_1$ where $p$ is a trigonometric polynomial of degree $\leq k$ with all its zeros on $T$ and $w_1$ is the spectrum of a completely nondeterministic process. (We already know by Proposition 5 that for such a process $P\cap F_j\neq\{0\}$ if $j$ is the degree of $p$.)

Proposition 10 provides a partial answer. First recall that $h$ is a strong outer function [7] if $h/(z-\lambda) \notin H^2$ for all $\lambda \in T$.

For simplicity we will simply look at processes for which $P\cap F_1\neq\{0\}$ and $P\cap F_2=\{0\}$.

Proposition 10. The following are equivalent

(i) $w = |p|^2 w_1$ where $p$ is a trigonometric polynomial of degree 1 with its zero on $T$ and $w_1$ is the spectrum of a cnd process.

(ii) $P\cap F_2=\{0\}$ and $h$ is not strong outer.
Proof. Suppose (i). Suppose $h/H = z^2 a(F/F)$ for a inner, $F \in H^2$, outer. But $|h|^2 = |p|^2 |h_1|^2 \Rightarrow h = ph_1 \Rightarrow h/H = zh_1/H_1$

$\Rightarrow h_1/H_1 = za(F/F)$,
contradicting the fact that $|h_1|^2$ corresponds to a cnd process. Trivially (i) $\Rightarrow h$ is not strong outer.

Conversely suppose (i) does not hold, i.e. $h \not\equiv p\lambda h_1$ for any trigonometric polynomial $p\lambda = z^{-\lambda} (\lambda \in \mathbb{T})$ and outer function $h_1$ corresponding to a cnd process. Then either $h/p\lambda \not\in H^2$ for all $\lambda \in \mathbb{T}$ i.e. $h$ is strong outer or $h = p\lambda h_1$ but $h_1$ does not correspond to a cnd process, i.e. $h_1/H_1 = za(F/F)$ for a inner, $F \in H^2$ outer. Thus $h/H = z(h_1/H_1) = z^2 a(F/F)$, i.e. $P\not\in F^2 = \{0\}$.

The missing link that we require leads us to suspect that $h$ strong outer together with $P\not\in F_1^1 = \{0\} \Rightarrow P\not\in F_2^2 = \{0\}$. In fact we make the following conjecture.

**Conjecture 1.** $h$ strong outer, $P\not\in F_1^1 = \{0\} \Rightarrow P\not\in F_k^k = \{0\}$ for all $k \geq 1$.

We finish by translating this conjecture into the language of Section 3. In [7] it was shown that $S_{h/H} = \{h^2\}$ implies that $h$ is strong outer. The following conjecture would tell us that if $h$ strong outer does not imply $S_{h/H} = \{h^2\}$ then $S_{h/H}$ must contain many functions.

**Conjecture 2.** Suppose $S_\phi$ contains more than one function, one of which is strong outer. Then $S_\phi$ contains a function with an inner factor which is not a finite Blaschke product.
If Conjecture 2 is correct then so is Conjecture 1 for the following reason. Suppose \( h \) is strong outer and \( PnF_1 \neq \{0\} \). Then \( h^2 \in S_{h/F} \) and so if Conjecture 2 is correct \( S_{h/F} \) contains a function with an inner factor which is not a finite Blaschke product. [7, Lemma 4.6] shows that this gives a function \( g \in S_{h/F} \) with infinitely many zeros in \( D \). Proposition 7 then shows that \( PnF_k \neq \{0\} \) for all \( k \geq 1 \). Note that it is easy to construct examples of processes for which \( PnF_k \neq \{0\} \) for all \( k \geq 1 \). In fact by the reasoning above \( w = |1+k|^2 \) gives such an example if \( k \) is an inner function which is not a finite Blaschke product.
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### Title

**PARTIAL CHARACTERIZATIONS OF COMPLETELY NONDETERMINISTIC STOCHASTIC PROCESSES**

### Authors

Peter Bloomfield  
Nicholas P. Jewell

### Performing Organization Name and Address

Department of Statistics  
Princeton University  
Princeton, NJ 08544

### Controlling Office Name and Address

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### Abstract

A discrete weakly stationary Gaussian stochastic process \( \{x(t)\} \), is completely nondeterministic if no non-trivial set from the \( \sigma \)-algebra generated by \( \{x(t) : t > 0\} \) lies in the \( \sigma \)-algebra generated by \( \{x(t) : t < 0\} \). In [8], Levinson and McKean essentially showed that a necessary and sufficient condition for complete nondeterminism is that the spectrum of the process is given by \( |h|^2 \) where \( h \) is an outer function in the Hardy space, \( H^2 \), of the unit circle in \( \mathbb{C} \) with the property that \( h/h^* \) uniquely.

(Over)
determines the outer function $h$ up to an arbitrary constant. In this paper we consider several characterizations of complete non-determinism in terms of the geometry of the unit ball of the Hardy space $H^1$ and in terms of Hankel operators, and pose an open problem.